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A Semantics for Hyperintensional Belief Revision Based on Information Bases

Abstract. I propose a novel hyperintensional semantics for belief revision and a corresponding system of dynamic doxastic logic. The main goal of the framework is to reduce some of the idealisations that are common in the belief revision literature and in dynamic epistemic logic. The models of the new framework are primarily based on potentially incomplete or inconsistent collections of information, represented by situations in a situation space. I propose that by shifting the representational focus of doxastic models from belief sets to collections of information, and by defining changes of beliefs as artifacts of changes of information, we can achieve a more realistic account of belief representation and belief change. The proposed dynamic operation suggests a non-classical way of changing beliefs: belief revision occurs in non-explosive environments which allow for a non-monotonic and hyperintensional belief dynamics. A logic that is sound with respect to the semantics is also provided.

Keywords: Belief revision, Situation semantics, Hyperintensional logic, Dynamic epistemic logic, Dynamic doxastic logic, Non-monotonic logic, Non-classical logics, Non-classical semantics, Epistemic logic, Non-monotonic reasoning.

1. Introduction

The doxastic models constructed here are primarily aimed at reducing some of the idealisations that are common in the belief revision literature. These include the assumptions that the beliefs of an agent are closed under logical inference rules, i.e., an agent believes, or is committed to believe, the logical implications of her beliefs; it is therefore assumed that the agents believe all logical truths. More radically, it is sometimes assumed that the information of an agent, which is the foundation of her beliefs, is complete in the sense that it says something about every aspect of the world, a feature which

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possibly passes on to her beliefs.¹ I will also challenge the idea that the primary elements of doxastic representations are the belief sets, and that belief changes take place directly on these entities. I show that by shifting the representational focus one step back to the possibly inconsistent and incomplete collections of information (which are not necessarily accepted as beliefs), and by defining changes of beliefs as artifacts of the changes of information, we can achieve a more realistic account of belief representation and of belief change.²

The AGM belief revision, so called after the names of the authors, and fully formulated in Alchourrón et al. [1], has revolutionised the belief revision literature by introducing a fully formed theory that combines non-monotonic reasoning and belief change. It has significantly influenced the succeeding works on belief revision especially in terms of the modelling idealisations mentioned above. In particular, it is assumed in the AGM theory that belief sets, before and after the revisions, are closed under classical logical implication (by the closure postulate). However, considering the deductively closed belief sets as the primary elements of doxastic representations has been criticised as they appear to be too large to be the direct objects of belief change (see [7], pp. 41–57). Together with the AGM success postulate, the closure principle suggest that only one form of inconsistency is allowed within the AGM framework, and that is triviality. In particular, a belief set becomes trivial, i.e., it implies everything in the language, when it is revised with an inconsistent sentence. In addition, the AGM theory allows merely derived beliefs, that are, the beliefs that entirely depend on other beliefs, to be retained even after their bases are eliminated from the belief set (by the recovery postulate). Last but not least, the AGM change operators treat classically logically equivalent sentences in an equal manner, hence suggesting an intensional theory of belief revision (by the preservation postulate). The AGM belief revision has been represented in the object language in several works, establishing the Dynamic Epistemic Logic (DEL), or as it sometimes called the Dynamic Doxastic Logic (DDL) tradition (see [23] and [25] for the first examples of modelling the AGM belief change within the

¹As noted, this is rather a radical assumption even among the highly idealised theories of belief revision. Most standard models are still able to represent belief revision via incomplete information.

²The idea that changes of belief should take place on entities that are significantly smaller than belief sets is one of the primary motivations behind the theories of base-generated revisions. Rott and Hansson may be named as leading figures in the construction of these theories in the philosophical belief revision literature [11, 21].

object language, and [24] for the full axiomatisation of the theory. See also [26] and [28]).

The theories of *base-generated revisions*, put forward by Rott and Hansson [11, 21], revoke some of these idealisations established in the AGM paradigm. Base-generated revision models are built on possibly inconsistent sets of beliefs that are not necessarily closed under logical implication, which are called the *belief bases*. Inferential closure of a belief base still serves as the set of sentences an agent is committed to believe.³ Most importantly, the changes of belief take place on the belief bases. This approach allows, for instance, agents to enjoy inconsistent pieces of information without having trivial belief sets (via the failures of various logical closure principles). It is also possible that agents reject making revisions with inconsistent information (by the limited success postulate in [21]). Moreover, because the recovery postulate is not satisfied, the base-generated revisions require that merely derived beliefs are not kept for their own sakes after their support belief is contracted, i.e., deleted from the belief set. They offer a much more fine-grained modelling of belief revision than the AGM models are capable of. For instance, the syntactic structures of the belief bases matter in determining their dynamic aspects. However, they still exemplify an intensional way of changing beliefs. Overall, the dynamic operators of base-generated revisions are more general than the AGM change operators (e.g., while the AGM belief revision operations do not allow iterated revisions, the base-generated revision operators do; see Fermé and Hansson [7] for a comparison of the two approaches), and they are more suitable to establish a ground for a less idealised theory. There is, to my knowledge, not much work on the representation of base-generated revision operators in the object language.⁴ As the present work shares many motivations and mechanisms with such revision theories, it can be seen as a step forward to formulation of them in the object language.

In the following, I propose a hyperintensional version of base-generated revisions, formalised in the object language following the DEL tradition. Hyperintensionality in the belief revision context means that (classically) logically equivalent content may point out to different change policies of the belief sets, depending on how they are represented in a model. For instance,

³This claim is accurate only for Rott's theory. Hansson does not assume the closure of belief sets under logical implications.

⁴Emiliano Lorini's works on epistemic logic with belief bases has been brought to my attention as one of the projects in this area, as they exploit the belief bases for an epistemic logic of implicit and explicit belief [16].

all (classical) logical tautologies and semantic or mathematical truths are intensionally equivalent (e.g., the following sentences are pairwise intensionally equivalent: “either $x = y$ or $x \neq y$ ”, “all husbands are married”, ‘every integer is the sum of four squares’). This means that, within frameworks which are insensitive to hyperintensionality, if an agent comes to believe that all husbands are married (she may have just learned the meaning of the word ‘husband’), she also comes to believe that every integer is the sum of four squares. This however is a very unrealistic form of belief dynamics, since learning a piece of vocabulary does not necessarily provide the means to grasp a mathematical truth. A specific form of hyperintensional belief revision has recently been investigated by Berto and Özgün [3, 20]. They argue that hyperintensionality occurs in belief revision due to subject matter sensitivity. Berto and Özgün obtain the hyperintensionality results without disowning classical logic, hence the works are very significant in integrating the *framing effects*, or hyperintensionality, into the classical approaches of logic of belief revision. On the other hand, I will depend on a much weaker logic and a non-standard semantics. The framework I introduce here does not only yield a non-classical way of revising one’s beliefs, rather it affects the handling of the information whether or not it is accepted as belief, both in static and dynamic scenarios.

The representation of doxastic states primarily based on possibly incomplete and inconsistent collections of information is motivated by the assumption that prior to a belief set, an agent possesses possibly incomplete and inconsistent information about the world. These collections of information are formalised in the models via situations in a situation space. The use of situations to represent incomplete sets of data can be traced back to the *Situation Semantics* developed by Barwise and Perry [2], while more recent examples of their use can be found in Fine [8] and in Leitgeb [12]. In the following framework, the situations are characterised by a valuation function which maps them to the literals in the language, and by a fusion function which structures the situation space as a join semi-lattice. The fusion function represents also the dynamic dispositions of an agent by specifying the possible ways of expanding her information. To the point of (static) characterisation of the situations and the situation space, the models are based on the *HYPE semantics* proposed by Leitgeb [12].

Over and above exploiting the situation semantics, the doxastic models introduced here include (uniform) preference orderings between sets of situations. A preference ordering represents an agent’s epistemic preferences over various collections of information. It determines which parts of the available information are accepted as beliefs by the agent. The current formalisation

of epistemic preferences diverges from the most common examples in the literature due to formal concerns which will be pointed out in the next section. Outside of this framework, they are usually defined between sets of worlds (see Grove's seminal work [10] in which he generalises the epistemic preference relations, and the primary works of the DEL tradition such as van Benthem [26]), or they are defined between possible worlds (e.g, in van Benthem and Liu [27], where the authors model preference change).

I propose a model-shifting dynamic operation (together with its representation in the object language) to model belief revision via new information. This operation is carried out on the possible collections of information that an agent has, rather than on the respective belief sets. The changes in the belief sets follow from the changes of these collections. The dynamic operation preserves the structure of the situation space along with the valuations, however it alters the preference ordering and the information of the agent. Keep in mind that this is rather a simple revision operation, as most of the complexity of reasoning is sustained by the static belief modality. Supported by the non-classical features of the static models, the revision operation suggests a non-classical way of changing beliefs. In particular, belief revision occurs in a non-explosive environment which also allows for a non-monotonic and hyperintensional belief dynamics. While the belief sets that are formed within this framework are not necessarily closed under logical implication, the proposed system of belief revision is still subject to some idealising assumptions. Most importantly, it is assumed that the belief set of an agent is always consistent, and this is manifested by the consistency axiom for belief. The proposed system satisfies the axiom schemas of cumulative transitivity (cut) and cautious monotony, which are usually desired as common properties of various non-monotonic logics [9]. In this paper I will only deal with the revisions of belief sets, i.e., adding new beliefs to a belief set. I will leave the construction of belief contraction models for future work.

I will start the next section with a detailed definition of the static aspect of my doxastic models, they will be called the *belief base models*. I will expand these models with a revision operation in Section 2.2. The expanded models will be called *belief base revision models*. In Section 3, I will give a sound axiomatisation of the logic of belief change as determined by the belief base revision models. Section 4 will be about some (negative) principles of belief base revision, such as non-monotonicity, non-explosiveness and hyperintensionality. Section 5 will be a conclusion of what is presented throughout the paper.

2. Semantics

2.1. Preliminaries

I start with the static belief base models with the intention of expanding them to revision models later on. A static belief base model represents the doxastic state of a single agent at a time. In this paper, a (static) doxastic state is the space of possible belief states of an agent, represented by a logical space of situations that is structured with a parthood ordering between the situations and an epistemic preference ordering between the sets of situations. The models diverge from Leitgeb's [12] by the added preorder, which is an epistemic preference ordering on sets of situations.

The following set of formulas specifies the language for the static belief base models; we will call it L_B . The modal operator B is the static doxastic (belief) operator:

- $\phi ::= AT \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \rightarrow \phi \mid B\phi$.
- \top (always true) and \perp ($\neg\top$) (always false).
- $L_{prop} \subseteq L_B$ is the sublanguage of L_B in which the modal operator B does not occur.
- $l \subseteq L_B$ is the set of literals for L_B such that $l = \{p, \bar{p}, q, \bar{q}, \dots\}$. (\bar{p} is used to denote $\neg p$ and $\bar{\bar{p}}$ denotes p again).

DEFINITION 1. A *belief base model* is a tuple $M = \langle S, V, \circ, \perp, \leq \rangle$ such that

- S is a non-empty and finite situation space, and the situations are denoted by " s_i " with or without the subscript.
- V is a mapping from S to the power set $\mathcal{P}(l)$ of literals. I will also speak of the set of literals that a situation s is mapped to (denoted by $V(s)$) as the local content of that situation.
- \circ is a binary partial function from $S \times S$ to S that satisfies the following conditions:
 - If $s \circ s'$ is defined, it is required that $V(s \circ s') \supseteq V(s) \cup V(s')$
 - $s \circ s$ is always defined and is equal to s (idempotence)
 - if $s' \circ s$ is defined, then $s \circ s'$ is also defined, and $s' \circ s = s \circ s'$ (commutativity)
 - if $(s \circ s') \circ s''$ is defined, then $s \circ ((s \circ s') \circ s'')$ is defined, and $s \circ ((s \circ s') \circ s'') = (s \circ s') \circ s''$
- \perp is a binary symmetric relation on S that satisfies the following conditions:

- For all $v \in l$, if $v \in V(s)$ and $\neg v \in V(s')$ then $s \perp s'$
- if $s \circ s''$ is defined and $s' \circ s'''$ is defined, if $s \perp s'$ then $s \circ s'' \perp s' \circ s'''$
- For all $s \in S$, there is a unique $s^* \in S$ (the star image of s) such that
 - $V(s^*) = \{\bar{v} | v \notin V(s)\}$
 - $s^{**} = s$
 - $s^* \not\leq s$
 - if $s' \not\leq s$ then $s' \circ s^*$ is defined, and $s' \circ s^* = s^*$
 - \leq is a total (transitive, reflexive and connected) preorder on $\mathcal{P}(S)$. For all $A, B \subseteq S$, if $A \leq B$ we say that A is at least as preferred as B .

In the above definition I use v, \bar{v} as metavariables for the literals of the language. The literals that a situation is mapped to via V represent the atomic pieces of information. The models allow mapping of a situation to a set of contradictory literals, such as $\{p, \bar{p}\}$. Such situations are called *glutty* situations. This assumption allows us to represent real world scenarios where agents have contradictory information about their world. It is also usually the case that the agents have incomplete information about the world. We permit the representation of such scenarios by allowing the existence of *gappy* situations: a situation s is gappy iff for some $p \in l$, neither p nor its negation is in $V(s)$.

Via the fusion function, the situations may overlap with each other, be part of other situations, or be the product of two or more situations fused together. The fusion function determines a partial order on the situations which structures the situation space in a join semi-lattice. In the rest of the paper I will call the following a *parthood ordering*:

DEFINITION 2. Given a belief base model M on a situation space S and the situations s, s' in S , $s' \sqsubseteq s$ iff $s \circ s' = s$.

If $s' \sqsubseteq s$, we say that the situation s' is part of the situation s . The parthood ordering is reflexive, transitive and antisymmetric. For the proof see Leitgeb [12]. In this framework, the parts of the situations are as important as their local contents in their characterisation. That is, situations with the same local content are not necessarily identical, they can be distinct situations in virtue of their parts.⁵

The \perp relation is an incompatibility relation between the situations. The incompatibility of two situations may become manifest through contradictory literals (p, \bar{p}) in their local content. The star operation is known from

⁵An example of such cases can be found at the end of the section, see example 5.

the relevance logic (see [6]). It is however not a primitive element of the models. Its existence depends on the assumption that the models are rich enough to include s^* whenever they include s . The star image gives the largest compatible situation for each situation in a model. Its existence means that the ideal agents are capable of expanding their information to a maximally consistent collection, within the limits of the language. For a detailed discussion of the formal aspects of the star image see Leitgeb [12].

The preorder \leq represents the epistemic preference ordering between sets of situations. An epistemic preference ordering represents the agent's dispositions for making rational selections among collections of information. Defining the preference ordering on sets of situations, rather than on situations, simplifies the models significantly. Reflexivity and transitivity are common characteristics of orderings which are used for making rational choices. We furthermore stipulate that this is a connected order. When applied to real agents, a connected preference ordering means the agents always prefer some collections of information over the others. On the technical aspect, it allows us to avoid cases where the agents have access to some information yet fail to form beliefs because of their (lack of) preferences. Lastly, the preference ordering in a belief base model is not relativized to the situations. We represent the shifts in epistemic preferences of the agents only via the model-shifting dynamic operations.

EXAMPLE 1. *The first example is a static belief base model (pictured in Figure 1) which displays the basic principles we stated for the construction of such a model. We start with specifying the language. Let $l = \{p, \bar{p}, q, \bar{q}, t, \bar{t}, r, \bar{r}\}$ be the set of literals of interest for this example. We will construct a model M with only six situations, hence let $S = \{0, 1, 2, 3, 4, 5\}$. Let the valuation of the situations in S be as follows: $V(0) = \emptyset$, $V(1) = \{p, \bar{q}, t, r\}$, $V(2) = \{p, q, \bar{t}, r\}$, $V(3) = \{p, q, \bar{q}, t, \bar{t}, r\}$, $V(4) = \{p, r\}$, $V(5) = \{p, \bar{p}, q, \bar{q}, t, \bar{t}, r, \bar{r}\}$. Let the fusion function of M be as follows: $0 \circ 0 = 0$, $0 \circ 4 = 4$, $0 \circ 1 = 1$, $0 \circ 2 = 2$, $0 \circ 3 = 3$, $0 \circ 5 = 5$, $4 \circ 4 = 4$, $4 \circ 1 = 1$, $4 \circ 2 = 2$, $4 \circ 3 = 3$, $4 \circ 5 = 5$, $1 \circ 1 = 1$, $1 \circ 2 = 3$, $1 \circ 3 = 3$, $1 \circ 5 = 5$, $2 \circ 2 = 2$, $2 \circ 3 = 3$, $2 \circ 5 = 5$, $3 \circ 3 = 3$, $3 \circ 5 = 5$, $5 \circ 5 = 5$. Let the incompatibility relation in the model be determined via the literals. Thus we have $1 \perp 2$, $1 \perp 3$, $1 \perp 5$, $2 \perp 1$, $2 \perp 3$, $2 \perp 5$, $3 \perp 1$, $3 \perp 2$, $3 \perp 3$, $3 \perp 5$, $4 \perp 5$, $5 \perp 1$, $5 \perp 2$, $5 \perp 3$, $5 \perp 4$, $5 \perp 5$. So, the following holds for the star images of the situations: $0^* = 5$, $1^* = 1$, $2^* = 2$, $3^* = 4$, $4^* = 3$, $5^* = 0$. Let the preference ordering \leq_M be such that for all $A, B \subseteq S$ it holds that $A \leq_M B$, i.e., all sets of situation in M are preferred equally.*

Recall that situations in a belief base model represent collections of information, and a belief base model represents the doxastic state of an agent. That

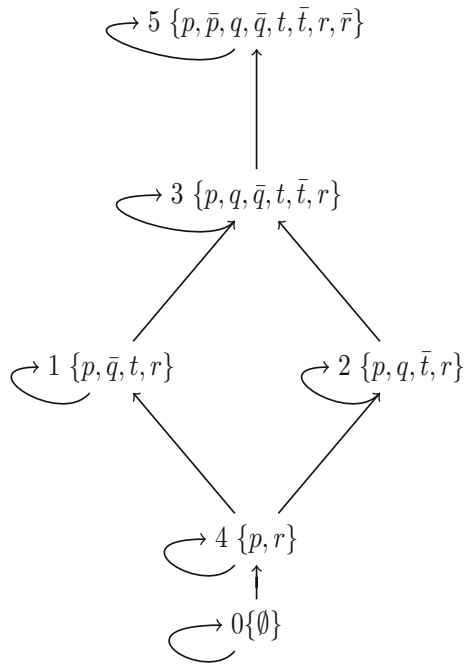


Figure 1. A fraction of a belief base model. The nodes represent the situations in the situation space S , and the arrows represent the parthood ordering on S . The ordering should be read as transitively closed. For instance, given that there is an arrow from 1 to 3 and from 3 to 5, it is assumed that there is an arrow from 1 to 5

is, the situations stand for the collections of information about the world, the agent possibly possesses. An important notion throughout this paper is the *information base* of an agent. An information base consists of a set of situations in the situation space, structured by the epistemic preference ordering and the parthood ordering, and which has an upper bound with respect to the latter. Consider the set of situations $H = \{0, 1, 4\}$ from the above example. The situation 1 is the upper bound of this set according to the parthood ordering of the model since $1 \circ (0 \circ 4) = 1$. The set of situations $H = \{0, 1, 4\}$ is also ordered by the epistemic preference ordering \leq_M . Let us then call the corresponding information base \mathcal{H} . If \mathcal{H} is the current information base of the agent, we say that their total information is given in the situation 1 since by the model assumptions of the fusion function, the local content (i.e., the propositional or non-belief content) of the situation

I includes the local contents of its parts.⁶ In the proposed framework, the upper bound of an information base is also where the beliefs of the agent is located, given that information base. To keep things simple, I will often say that an information base \mathcal{H} is determined by a situation s , if s is the upper bound of the set of situations in the information base according to the parthood ordering. If I intend to address only the (structured) information of the agent at time t , I will talk about the information base at time t . If, on the other hand, I mean to refer to the beliefs of the agent together with their information, I will talk about their *belief state* at time t . In this sense, I will also say that the belief state of the agent at time t is determined by the situation s that is the upper bound of the set of situations in the relevant information base. Note that, as a possible location of the total information and beliefs of an agent, each situation s in a belief base model determines a possible information base, hence a possible belief state of the agent.

As mentioned at the beginning of this section, the doxastic state of the agent includes possible belief states, which may fall outside of their belief state at time t . A belief base model may then include situations that are not parts of the information base or the belief state of the agent at time t , since an information base does not necessarily exhaust the situation space. It might be that only some of the situations in the model are available to the agent through information growth, while the others are not. In particular, the situations which are located above the information base of the agent according to the parthood ordering partly determine the dynamics of their belief state at time t : they indicate which collections of information are possibly available to the agent via information growth. As for the other situations included in a model which are not parts of the agent's current information base, and which are not available to the agent via information growth (in virtue of the partialness of the fusion function), these situations allow us to make hypothetical cases about what the agent would accept as beliefs, and how they would change their beliefs accordingly, if their information base is such and such. On the most basic case, we can see which pieces of information they cannot learn, by virtue the situations which are not connected to their current information base via the parthood ordering.⁷

⁶Note that the belief content of a situation may not include the belief contents of its parts. This is a symptom of the non-monotonicity of the logic of belief revision determined by the proposed framework. In particular, we will see that it is only the propositional content that is preserved through information growth, i.e., up the parthood ordering.

⁷These hypothetical cases are limited by how the information is represented in the models in terms of the situations that include the information and the parthood ordering between these situations.

One of the aims of this framework is to model the assignment of consistent belief sets to possibly inconsistent information bases. Hence, the agents do not necessarily believe everything in their information base. A consistency aiming, cautious process for determining a belief set starts with identifying the *consistent parts of an information base*. Informally, (consistent) parts of an information base amounts to the (consistent) chunks of an agents total information.⁸ Formally, a consistent part of an information base is a pairwise consistent set of situations (i.e., for all situations s, s' in said set, it holds that $s \not\perp s'$) *within* the information base. We will also resort to a maximality principle while identifying these parts in order to keep the amount of information loss at a minimum in the transition from information bases to the belief sets. *Maximally consistent parts of an information base* are its consistent parts that cannot be expanded within the information base by the addition of more situations without breaking the pairwise consistency. When there are multiple maximally consistent parts of an information base, we will make use of the preference ordering to mark off the *best* maximally consistent parts of an information base.

The pieces of information which are given in *all* of the best maximally consistent parts of an information base will constitute the belief set for that base. This definition indicates that the belief operator is a box-like modal operator, hence posing another layer of maximality.⁹ The following definition formally specifies the consistency, maximality and preference requirements mentioned above. The satisfaction clause for modal formulas resorts heavily to this definition. The clauses in definition 4 are based on the HYPE logic [12], except for the modal clause.

⁸We talk about grouping the information of an agent only in order to form consistent chunks. One might think that reasoning also involves the parting of the information with respect to subject matter. However plausible this assumption might be, in this paper we focus on a very simple model of reasoning, working with relatively small collections of information. However, a framework which involves grouping of information based on topic might open up the discussion to another form of hyperintensional belief revision.

⁹In this framework, the use of maximally consistent parts of an information base transforms the common diamond-like modality of belief into a box-like modality, while producing similar semantical and logical results. For instance, Bílková, Majer, and Peliš [4] propose a diamond-like knowledge operator in a framework developed with similar motivations of reducing the idealisation of reasoning in epistemic settings. Their framework is also based on structures such as situations - called (partial) information states and a (parthood-like) ordering on them. However, they impose a *mutual* consistency requirement while identifying the consistent parts of an information state, instead of maximal consistency. The box-like belief operator is also a reminiscent of the inference operator of Rott's base-generated revision system [21].

DEFINITION 3. Given a belief base model M on a situation space S ,

- A situation $s \in S$ is consistent (in M) iff $s \not\perp s$. Otherwise it is inconsistent.
- A set of situations $A \subseteq S$ is consistent (in M) iff for all $s, s' \in A$, $s \not\perp s'$. Otherwise it is inconsistent.
- A set of situations $A \subseteq S$ is *maximally consistent with respect to a situation* $s \in S$ (in M) iff A is consistent, for all $s' \in A$ it holds that $s' \sqsubseteq s$, and for all $s'' \in S$ if $s'' \sqsubseteq s$ and $s'' \notin A$ it holds that $A \cup \{s''\}$ is inconsistent.
- The *best sets of situations in a set* $I \in \mathcal{P}(S)$ (in M) are given by the following: $\min_{\leq_M}(I) = \{A \in I \mid \forall B \in I, A \leq_M B\}$.
- The *best of a situation* s (in M) is given by the following: $Best_M(s) = \min_{\leq_M}(\{A \subseteq S \mid A \text{ is maximally consistent w.r.t. } s\})$.

DEFINITION 4. Given a belief base model M on a situation space S , for all $s \in S$, the satisfaction clauses for formulas of L_B are as follows:

- $s \models v$ iff $v \in V(s)$
- $s \models \neg v$ iff $\neg v \in V(s)$
- $s \models \neg\phi$ iff for all s' , if $s' \models \phi$ then $s \perp s'$ ¹⁰
- $s \models \phi \wedge \psi$ iff $s \models \phi$ and $s \models \psi$
- $s \models \phi \vee \psi$ iff $s \models \phi$ or $s \models \psi$
- $s \models \phi \rightarrow \psi$ iff for all s' , if $s \circ s' = s'$ and $s' \models \phi$ then $s' \models \psi$
- $s \models B\phi$ iff for all $A \in Best_M(s)$, there is $s' \in A$, $s' \models \phi$
- $s \models \top$

The satisfaction of the biconditional is as usual: $s \models \phi \leftrightarrow \psi$ iff $s \models \phi \rightarrow \psi$ and $s \models \psi \rightarrow \phi$. We read $s \models \phi$ as saying that *the situation s satisfies ϕ* . When there is need for specifying the models, we write $s \models_M \phi$ and say that *the situation s satisfies ϕ in the model M* .

Based on the definitions above, an agent's belief set consists of the sentences that are satisfied by all of the best maximally consistent sets of situations (by some situation in these sets), which are parts of her information base. Formally, the proposed belief modality is a reminiscence of the (non-monotonic) partial-meet operations used to define the AGM contraction and revision, as well as the base-generated revisions and contractions by Hansson and Rott (see [1] for the partial-meet contraction and revision operations, and see [11] and [21] for their application on possibly non-closed

¹⁰Lemma 8 in Leitgeb [12] shows that $s \models \bar{v}$ iff for all $s' \in S$, if $s' \models v$ then $s \perp s'$. The lemma is also satisfied in my framework.

sets; for a more general discussion of partial-meet consequence relations and non-monotonicity see [17]).

An important feature of the proposed belief modality is that its objects are the collections of information, represented by the situations, rather than the pieces of information (whereas in the above mentioned applications of similar inference operations, the objects are singleton sentences). This, for instance, makes the following scenario possible. Suppose that in a belief base model, a piece of information ϕ is only satisfied in a ψ -situation, while ϕ is not logically entailed by ψ . That is, the information that ϕ is available to the agent only with the additional information that ψ . Suppose all ψ -situations contradict with the current belief state of the agent. Hence, it might be the case that ϕ is not accepted as belief only because the collection of information of which it is a part of (the ψ -theory) is refuted. The intuition here is that the circumstances surrounding a piece of information matters. Acquiring pieces of information in isolation from other pieces of information mostly occurs in idealised situations. Usually, the agents are confronted with possibly incomplete and inconsistent theories about the world, and it is not always reasonable to believe only a part of a refuted theory on the basis that that particular part is not directly refuted. Some pieces of information stand and fall together. For instance, consider reading a certain newspaper. Suppose you are heavily set on your belief that any piece of information given in this paper is highly doubtful, and generally incorrect. Thus, when encountered with a piece of information ϕ , which looks reasonable, due to the non-logical circumstances around this piece of information, such as other information that comes along with it, you do not accept it as a belief.¹¹

We can now define a unique belief set for each situation $s \in S$:

DEFINITION 5. Given a belief base model M on a situation space S , and $s \in S$, K_s is the set of beliefs satisfied by s :

$$K_s = \{\phi \in L_B \mid s \models B\phi\}$$

¹¹To see the formal possibility of such scenarios, consider a belief base model M on the situation space $S = \{1, 2, 3, 4\}$, on a language whose literals are $l = \{p, \bar{p}, q, \bar{q}, s, \bar{s}\}$. Let $V(1) = \{p, q, \bar{q}\}$, $V(2) = \{p, s\}$, $V(3) = \{p, q, \bar{q}, s\}$, $V(4) = \{p, s, \bar{s}\}$. Let $(1 \circ 2) \circ 3 = 3$, $2 \circ 4 = 4$, $1 \circ 1 = 1$, $2 \circ 2 = 2$, $3 \circ 3 = 3$, $4 \circ 4 = 4$. Finally, let $1 \perp 1, 3 \perp 3, 4 \perp 4, 1 \perp 3, 2 \perp 4, 3 \perp 4$. Thus, it holds that $1^* = 4$ and $2^* = 3$. Given that the current information of the agent is given in the situation 1, the agent does not believe that p (their belief set is empty) although p is among the information of the agent. However, if the current information of the agent is given in the situation 3, they believe that p . This is because, in the former case, the information that p is available only as part of an inconsistent theory. Whereas, in the latter case, it is also available as part of a consistent and unrefuted theory.

We conclude the setting of the belief base models with the following definition.

DEFINITION 6. Logical consequence and truth in belief base models are defined as usual:

- $\phi_1, \dots, \phi_n \models \psi$ iff for all models $M = \langle S, V, \circ, \perp, \leq \rangle$, for all situations $s \in S$, if $s \models \phi_1, \dots, \phi_n$ then $s \models \psi$.
- $\models \psi$ iff for all models $M = \langle S, V, \circ, \perp, \leq \rangle$, for all situations $s \in S$, $s \models \psi$.

EXAMPLE 2. *In the second example we check the belief set for the situation 3 in the above model. Suppose the agent, whose total information is given in the situation 3 in M , is investigating the responsible person for the robbery of a very valuable book from a personal library. Hence, we can state that her information base is determined by the situation 3 in M , (and equally by the set of situations $\{0, 1, 2, 3, 4\}$ in S and \leq_M). She then has the information that the butler has a key to the library (p) and that there are only two keys to the library (r) (e.g., $4 \models p \wedge r$). She also has the information that the maid has a key to the library (t), and that if the maid has a key then gardener does not have a key to the library (e.g., $1 \models t \rightarrow \neg q$); but also that the gardener has a key to the library (q), and if the gardener has a key then the maid does not have a key to the library (e.g., $2 \models q \rightarrow \neg t$). Therefore, her information about who possess a key to the library is contradictory. We now check which part of this information she accepts as beliefs given (the relevant fraction of) her doxastic state is represented by the model M .*

We start with identifying the consistent parts of the agent's information base. In M , there are two maximally consistent sets of situations w.r.t. the situation 3; these are the sets $\{0, 1, 4\}$ and $\{0, 2, 4\}$. Based on the preference ordering of M , we have $\{0, 1, 4\} \leq_M \{0, 2, 4\}$ and $\{0, 2, 4\} \leq_M \{0, 1, 4\}$. So, both sets are among the best of the situation 3: $Best_M(3) = \{\{0, 1, 4\}, \{0, 2, 4\}\}$. By the satisfaction clause for the belief formulas, it holds that $3 \models B(p \wedge r) \wedge B((q \wedge \neg t) \vee (t \wedge \neg q))$. Therefore, the agent believes that there are only two keys to the library and the butler has a key to the library. She also believes that either the maid or the gardener has a key, but not both.

In the rest of this section, I make some observations concerning the belief base models and the belief sets determined via these models.

LEMMA 7. (The implication lemma) *Given a belief base model M on a situation space S , and $\phi, \psi \in L_B$, for all $s \in S$, if $s \models \phi \rightarrow \psi$ then it holds that if $s \models \phi$ then $s \models \psi$.*

PROOF. The lemma follows from the idempotence of \circ and the satisfaction clause for \rightarrow . ■

LEMMA 8. (Persistency for non-modal formulas) *Given a belief base model M on a situation space S , for all $s \in S$ and for all $\phi \in L_{prop}$, if $s \models \phi$ and $s \circ s' = s'$, then $s' \models \phi$.*

PROOF. See Lemma 9 in [12, p. 31]. ■

OBSERVATION 9. (Non-persistency of modal formulas) *The modal formulas of the language L_B are not necessarily persistent through parthood ordering in a belief base model M .*

PROOF. See the model in example 1. It holds that $1 \models Bt$ since $Best_M(1) = \{\{0, 4, 1\}\}$ and $1 \models t$. It also holds that $1 \circ 3 = 3$, however $3 \not\models Bt$. ■

LEMMA 10. *Given a belief base model M on a situation space S , for all $s \in S$ it holds that $Best_M(s)$ is non-empty.*

PROOF. Let M be an arbitrary belief base model on a situation space S , and let s be an arbitrary situation in this model. Suppose for some $s' \in S$ with $s' \not\sqsubseteq s$ it holds that $s' \sqsubseteq s$. That is, s has some consistent parts. So, it follows that there is an $A \subseteq S$, such that A is maximally consistent w.r.t. s and $s' \in A$. By the assumptions for the preference ordering, either $A \in Best_M(s)$ or there is an $A' \subseteq S$ such that A' is maximally consistent w.r.t. s and $A' \leq_M A$ (and $A \not\leq_M A'$). In the latter case however, it holds that $A' \in Best_M(s)$. Therefore, if s has some consistent part, it cannot be the case that $Best_M(s) = \emptyset$. Now suppose for all $s' \sqsubseteq s$ it is the case that $s' \perp s'$. That is, s has no consistent parts. In this case, it follows from the definition 3 that $Best_M(s) = \{\emptyset\}$. Hence, $Best_M(s)$ has the empty set of situations as its unique member. Since M and s are arbitrary, this holds for all belief base models. ■

LEMMA 11. *Given a belief base model M on a situation space S , for all $s \in S$ and for all $\phi \in L_B$, if $s \not\sqsubseteq s$, then $s \models \phi$ iff $s \models B\phi$.¹²*

¹²Another observation, which might be worrying to some, follows from this lemma in combination with the satisfaction clause for disjunction. The present lemma predicts that an agent whose total information is consistent, believes a disjunction iff she believes one of the disjuncts, provided that both disjuncts does not involve a conditional subformula. One might however think that an agent can be in a consistent doxastic state and believe, for instance, that the butler has a key to the library or the maid has a key to the library, without being decidedly opinionated about either of the disjuncts. Hence, the current system puts the criteria for an agent to believe a disjunction without necessarily

PROOF. Let M be an arbitrary belief base model on a situation space S , and let $s \in S$ an arbitrary situation, such that $s \not\sqsubseteq s$. The proof for the direction from left-to-right is simple. Suppose $s \models \phi$. By assumptions for the incompatibility relation, since $s \not\sqsubseteq s$, it follows for all s', s'' in S that if $s \circ (s' \circ s'') = s$ then $s' \not\sqsubseteq s''$. Hence, there is a unique set $A \in \text{Best}_M(s)$, such that for all $s' \sqsubseteq s$ it holds that $s' \in A$. Therefore, since $s \sqsubseteq s$, it follows that $s \in A$, and since $s \models \phi$, it holds that $s \models B\phi$ as desired.

For the direction from right-to-left, we prove by cases. For the first case suppose $\phi \in l$ when l is the set of literals in L_B . So, suppose $s \models B\phi$. Hence, for all $A \in \text{Best}_M(s)$ there is an $s' \in A$, such that $s' \models \phi$. Pick an arbitrary $A \in \text{Best}_M(s)$ and an arbitrary $s' \in A$ with $s' \models \phi$. Since $s' \circ s = s$, by the persistency lemma it follows that $s \models \phi$ as desired.

For the second case, suppose ϕ is in the form $\neg\psi$ for some $\psi \in L_B$. So, suppose $s \models B\neg\psi$. Thus, for all $A \in \text{Best}_M(s)$ there is an $s' \in A$, such that $s' \models \neg\psi$. Pick an arbitrary $A \in \text{Best}_M(s)$ and an arbitrary $s' \in A$ with $s' \models \neg\psi$. Consider an arbitrary $s'' \in S$, such that $s'' \models \psi$. By the clause for negation, it follows that $s' \perp s''$. Since $s \circ s' = s$, by the assumptions for the incompatibility relation, it holds that $s \perp s''$. Since s'' is arbitrary, again by the clause for negation, it follows that $s \models \neg\psi$.

For the third case, suppose ϕ is in the form $\psi \rightarrow \chi$ for some $\psi, \chi \in L_B$. So, suppose $s \models B(\psi \rightarrow \chi)$. So, for all $A \in \text{Best}_M(s)$ there is an $s' \in A$, such that $s' \models \psi \rightarrow \chi$. Pick an arbitrary $A \in \text{Best}_M(s)$ and an arbitrary $s' \in A$ with $s' \models \psi \rightarrow \chi$. For reductio, suppose $s \not\models \psi \rightarrow \chi$. Hence, there is an $s'' \in S$, such that $s \circ s'' = s''$, $s'' \models \psi$ but $s'' \not\models \chi$. By the assumptions for the fusion function, it also holds that $s' \circ s'' = s''$. So, it is the case that $s' \not\models \psi \rightarrow \chi$. However, this contradicts with the assumption that $s' \models \psi \rightarrow \chi$. Therefore, it follows that $s \models \psi \rightarrow \chi$.

For the fourth case, suppose ϕ is in the form $B\psi$ for some $\psi \in L_B$. So, suppose $s \models BB\psi$. Thus, for all $A \in \text{Best}_M(s)$ there is an $s' \in A$, such that $s' \models B\psi$. Pick an arbitrary $A \in \text{Best}_M(s)$ and an arbitrary $s' \in A$ with $s' \models B\psi$. It follows that, for all $A' \in \text{Best}_M(s')$ there is an $s'' \in A'$ such

Footnote 12 continued

believing one of the disjuncts as her having contradictory information (not necessarily directly about either of the disjuncts). While the justification of this situation is not very clear, one should keep in mind that completely consistent information bases is the most idealised scenario in this framework. Moreover, an agent can still believe a disjunction (when both disjuncts involve some conditional sub-formula) without believing one of the disjuncts in case she has inconsistent ways of expanding her information base, i.e., if there are inconsistent collections of information (situations) with which she can combine (fuse) her current information base.

that $s'' \models \psi$. By the assumptions for the fusion function, it also holds that $s'' \circ s = s$. Since $s \not\leq s$, there is a unique set $A \in Best_M(s)$ and $s'' \in A$ (see the proof for the left-to-right direction). Therefore, $s \models B\psi$ as desired.

The cases for formulas in the form of conjunctions and in the form of disjunctions follow by induction using the above cases. Since M and s are arbitrary, this holds for all belief base models. ■

LEMMA 12. *Given a belief base model M on a situation space S , for all $s \in S$ and for all $\phi \in L_B$, if $s \models B\phi$ then $s \models \phi$.*

PROOF. Let M be an arbitrary belief base model on a situation space S , let $s \in S$ an arbitrary situation. The proof is then similar to the proof of right-to-left direction of lemma 11. We only modify the case for the formulas in the form $B\psi$ for some $\psi \in L_B$. We show that if $s \models BB\psi$ then $s \models B\psi$. So, suppose $s \models BB\psi$. Thus, for all $A \in Best_M(s)$ there is an $s' \in A$, such that $s' \models B\psi$. Pick an arbitrary $A \in Best_M(s)$ and an arbitrary $s' \in A$ with $s' \models B\psi$. Thus, for all $A' \in Best_M(s')$ there is an $s'' \in A'$, such that $s'' \models \psi$. Consider an arbitrary $A' \in Best_M(s')$ and an arbitrary $s'' \in A'$ with $s'' \models \psi$. We know by the assumption for the fusion function that $s'' \circ s = s$. For reductio, suppose for some $B \in Best_M(s)$, $s'' \notin B$. So, by the maximality of $Best(s)$, it follows that there is a $u \in B$, such that $u \perp s''$. By the assumptions for the incompatibility relation, it follows that also $s' \perp u$ (since $s'' \circ s' = s'$). However, since $A \in Best_M(s)$ and $s' \in A$ are arbitrary, it follows that B is inconsistent. Since this contradicts with our model assumptions, it should be the case that $s'' \in B$ for all $B \in Best_M(s)$. Therefore, $s \models B\psi$. Since M and s are arbitrary, this holds for all belief base models. ■

LEMMA 13. *Given a belief base model M on a situation space S , for all situations $s \in S$, K_s is always consistent, i.e., for all $\phi \in L_B$, it cannot be the case that $s \models B\phi \wedge B\neg\phi$ or that $s \models B(\phi \wedge \neg\phi)$.*

PROOF. Let M be an arbitrary belief base model on a situation space S , let $s \in S$ an arbitrary situation. First we show that for all $\phi \in L_B$, $s \not\models B\phi \wedge B\neg\phi$. We proceed by induction on the complexity of the formulas of L_B . The base case is when $\phi \in l$ with l is the set of literals for L_B . For reductio, assume $s \models Bp \wedge B\neg p$. So, it is the case that for all $A \in Best_M(s)$, there is a situation u in A , such that $u \models p$, and also a situation u' in A , such that $u' \models \neg p$. Since $u \perp u'$, it follows that A is inconsistent. Hence, a contradiction follows from the requirement of the models that for all $B \in Best(s)$ it holds that B is consistent. Therefore, it cannot be the case that $s \models Bp \wedge B\neg p$.

Next we prove for the formulas in the form $\neg\psi$ for some $\psi \in L_B$. So, for reductio assume $s \models B\neg\psi \wedge B\neg\neg\psi$. So, it is the case that for all $A \in Best_M(s)$, there is a situation u in A , such that $u \models \neg\psi$, and also a situation u' in A , such that $u' \models \neg\neg\psi$. Pick an arbitrary $A \in Best_M(s)$ and arbitrary $u, u' \in A$ with $u \models \neg\psi$ and $u' \models \neg\neg\psi$. By the satisfaction clause for negation (since $u' \models \neg\neg\psi$) it follows that for all $s' \in S$, if $s' \models \neg\psi$ then $u' \perp s'$. So it follows that $u' \perp u$ and that A is inconsistent. Hence, a contradiction follows from the requirement of the models that for all $B \in Best(s)$ it holds that B is consistent. Therefore, it cannot be the case that $s \models B\neg\psi \wedge B\neg\neg\psi$. The rest of the cases can be proved easily by the above cases. I leave them for the reader.

To show that for all $\phi \in L_B$, $s \not\models B(\phi \wedge \neg\phi)$, we again use proof by induction. I only state the base case and leave the rest of the cases out for space issues. The base case is when $\phi \in l$ with l is the set of literals for L_B . For reductio, assume $s \models B(p \wedge \neg p)$. So, it is the case that for all $A \in Best_M(s)$, there is a situation u in A , such that $u \models p \wedge \neg p$. So, it follows that $u \perp u$ and A is inconsistent. Hence, a contradiction follows from the requirement of the models that for all $B \in Best(s)$ it holds that B is consistent. Therefore, it cannot be the case that $s \models B(p \wedge \neg p)$. ■

OBSERVATION 14. *There is a belief base model M on a situation space S , and a situation $s \in S$, such that $s \models B\phi \wedge \neg B\phi$.*

PROOF. Let M be a belief base model constructed on the situation space $S = \{1, 2, 3\}$, and on a language L_B with the literals $l = \{p, \bar{p}\}$. Let $V(1) = \{p, \bar{p}, q\}$, $V(2) = \{p, q\}$, $V(3) = \{q\}$. Let $1 \circ 1 = 1, 2 \circ 2 = 2, 3 \circ 3 = 3, 2 \circ 3 = 2, 1 \circ 2 = 1, 1 \circ 3 = 1, (2 \circ 3) \circ 1 = 1$; and let the incompatibility relation as the following: $1 \perp 1, 1 \perp 2, 2 \perp 1$. So, it follows that $1 = 3^*, 2 = 2^*, 3 = 1^*$. Finally, let \leq_M is such that for all $A, B \subseteq S$, $A \leq_M B$.

We show that $1 \models Bp \wedge \neg Bp$. There is a unique maximally consistent set of situations w.r.t. 1, that is the set $\{2, 3\}$. By the connectivity of the preference ordering it holds that $Best_M(1) = \{\{2, 3\}\}$. Since $2 \models p$, it holds that $1 \models Bp$. It also holds that $2 \models Bp$: since $Best_M(2) = \{\{2, 3\}\}$ and $2 \models p$. However, since $Best_M(3) = \{\{3\}\}$ and $3 \not\models p$, it follows that $3 \not\models Bp$. Since $1 \perp 1$ and $1 \perp 2$, and since 1 and 2 are all and only situations which satisfy Bp , by the satisfaction clause for negation, it holds that $1 \models \neg Bp$. Therefore, by the satisfaction clause for conjunction, $1 \models Bp \wedge \neg Bp$. ■

These lemmas and observations will be used to simplify some proofs in the rest of the paper. They are more important however as indicators of some of the consequences of the framework. The first lemma shows that the proposed

conditional (\rightarrow) appeals to the intuition of conditional reasoning.¹³ Lemma 8 and Observation 9 reflect the non-monotonic nature of beliefs, even though the non-belief content is persistent through information growth. Lemma 10 shows how the trivial, or inconsistent belief sets are blocked by the models since an empty $Best_M(s)$ for a situation s in a model M would lead to a trivial belief set which is equal to the language L_B . Lemma 11 says that when the total information of an agent is consistent (in itself), the agent believes every part of her information base. The equality of the information and the beliefs is the ideal belief state in the proposed frameworks. Lemma 12 says on the other hand, regardless of the consistency of information, the agent believes only what is part of her information. Lemma 13 indicates the consistency of beliefs as a strong property of the proposed framework. Lemma 14 comments on the previous one stating that “believe that ϕ ” and “not believe that ϕ ” are not contradictory. What the latter means may present a lengthy discussion, I only want to highlight that “not believe that ϕ ” is not same as “believing that not ϕ ”.

2.2. Belief Revision

Recall the agent, the investigator from the examples in the previous section. Suppose initially she believes that the butler has a key to the library (p), that there are only two keys that could open the library (r), and also that the maid has a key to the library (t). Hence, we assume that her information base is determined by the situation 1 in the model M in example 1. Suppose she thereafter learns that the gardener may have stolen the maid’s key ($(q \wedge \neg t) \vee (t \wedge \neg q)$). In this section, I will show how an agent should revise her beliefs with new information, within a dynamic framework that will be constructed based on the belief base models of the previous section. In this context, revision means adding new beliefs to a belief set while preserving its internal consistency.¹⁴ We will see that as the preservation of the consistency of the belief sets is already achieved by the static aspect of the models, the dynamic part covers the expansion of the information base with the new

¹³The conditional (\rightarrow) is however stronger than the common intuition when the other direction is considered, the two expressions “ $s \models \phi$ then $s \models \psi$ ” and “ $s \models \phi \rightarrow \psi$ ” are not equivalent.

¹⁴Other common forms of belief change are belief contraction and belief expansion. Belief expansion is the operation of adding new beliefs without a concern for restoring the consistency of the new belief set. The current framework does not allow the construction of inconsistent belief sets (see lemma 13). Belief contraction is the operation of eliminating some of the beliefs from a belief set. I leave the formalisation of belief contraction via belief base models for future work.

information and in particular how the preference ordering is affected during this expansion.

We construct the *belief base revision models* by expanding the static belief base models with a revision operation. This operation is a relation from a situation in a belief base model and a formula in the language to a new situation in a new belief base model. In particular, the situation which determines the initial information base of an agent is expanded to another situation to incorporate the new information. In this way, a shift occurs to a new information base. At the same time, the epistemic preference ordering of the agent changes to ensure that the new information is accepted as a new belief, hence the shift to a new model. In this transition, the structure of the initial belief base model is preserved except for the preference ordering. Since the situation space of the initial model is among what is preserved, the existence of the expanded information base is a precondition for belief base revision.

Various forms of revising beliefs can be defined which differ in terms of the severity and the range of effect, particularly on the preference ordering (see [22] for various ways of changing beliefs within the DDL framework). I will focus on a single option in this paper. The revision operation will be represented in the object language with a pair of dynamic modal operators. Crucially in this setting, there may not be a unique revised model as the result of a revision. Therefore, I will present a box-like revision operator and a diamond-like revision operator for the belief base revision operation. Hence, we expand the language L_B with the following types of formulas, to the new language L_D :

$$[\phi]\psi \mid \langle \phi \rangle \psi.$$

The box-like dynamic operator means that the right-hand-side (sub) formula is satisfied by *all* of the revised models after the revision with the left-hand-side (sub)formula, and the diamond-like dynamic operator means that the right-hand-side formula is satisfied by *some* of the revised models.¹⁵

¹⁵The literature on indeterministic belief change focuses on approaches of belief revision that allows revisions to result in multiple new models. The approach is motivated by the idea that there may be more than one admissible way of changing ones beliefs, none of whom necessarily a better option than the others. Indeterministic belief change is also referred to as *relational belief change*. For various motivations leading to the investigation of relational belief change operations and indeterministic belief change see Doyle [5], Lindström and Rabinowicz [14] and Lindström and Rabinowicz [15], and the discussion by Olsson [19]. Hansson states that indeterministic belief change confirms most of the results and expectations from deterministic models [11].

I now present a *pre-model*, which will be specified to the belief base revision models, and the semantic expansion to the semantics of belief base models.

DEFINITION 15. A *pre-model* is a tuple $M^P = \langle S, V, \circ, \perp, \leq, R \rangle$, such that

- $M = \langle S, V, \circ, \perp, \leq \rangle$ is a belief base model and
- R is the relation from a triple $\langle \phi, M, s \rangle$ to a set of pairs $\langle M', s' \rangle$, such that
 - $\phi \in L_D$, M is a belief base model, such that $M = \langle S, V, \circ, \perp, \leq \rangle$, and $s \in S$
 - M' is a belief base model, such that $M' = \langle S, V, \circ, \perp, \leq' \rangle$, and $s' \in S$.¹⁶

DEFINITION 16. Given a pre-model M^P and the belief base models M, M' on a situation space S , for all $s \in S$, the satisfaction clauses for the revision operators in L_D are as follows:

- $s \models_M [\phi]\psi$ iff $\forall \langle M', s' \rangle : \langle \phi, M, s \rangle R \langle M', s' \rangle \implies s' \models_{M'} \psi$.
- $s \models_M \langle \phi \rangle \psi$ iff $\exists \langle M', s' \rangle : \langle \phi, M, s \rangle R \langle M', s' \rangle$ and $s' \models_{M'} \psi$.

I will introduce some specifications for the revision operation introduced above, in order to obtain the final structure of the belief base revision models. Some new terminology will be used in the definition of the specified revision operation. Given a pre-model and $\phi \in L_D$, a *basic ϕ -situation* is a situation which satisfies ϕ (a *ϕ -situation*) and which do not have any parts other than itself which are also ϕ -situations. We use this terminology to mark the smallest ϕ -situations in a model. We will take these situations as the unique sources of the new information. I propose this restriction in line with the well-known *minimal change principle*: while revising (and contracting) a belief set, the changes that occur in the new belief set shall be minimal. That is, one should only add (or delete) the beliefs which are necessary for the intended change to be successful.

DEFINITION 17. Given a pre-model $M^P = \langle S, V, \circ, \perp, \leq, R \rangle$ where $M = \langle S, V, \circ, \perp, \leq \rangle$, and a situation $s \in S$, for all $\phi \in L_D$, s is a *basic ϕ -situation* (in M) iff $s \models_M \phi$ and for all $s' \sqsubseteq s$, if $s' \neq s$ it holds that $s' \not\models_M \phi$.

¹⁶The current definition of the revision operation mixes syntactic aspects of a model with the semantics. This is a choice I made, in order to have a general revision operation rather than a family of revision operations, indexed to the formulas of the language, e.g. $\mathcal{R} = \{R'_\phi : \phi \in L_D\}$. Application of the latter formulation could also be considered, however it is likely to generate differences in the logic of belief revision, in particular when nested or iterated revisions are in question.

DEFINITION 18. A belief base revision model is a tuple $M^D = \langle S, V, \circ, \perp, \leq, R \rangle$, such that

- $M = \langle S, V, \circ, \perp, \leq \rangle$ is a belief base model and
- R is the relation from a triple $\langle \phi, M, s \rangle$ to a set of pairs $\langle M', s' \rangle$ determined uniquely by the following:
 - $\phi \in L_D$, M is a belief base model, such that $M = \langle S, V, \circ, \perp, \leq \rangle$, and $s \in S$
 - M' is a belief base model, such that $M' = \langle S, V, \circ, \perp, \leq' \rangle$, and $s' \in S$
 - $\exists t \in S$, such that t is a basic ϕ -situation in M and it holds that $(s \circ t) \circ s' = s'$ such that, there is no $s'' \in S$ with $s'' \neq s'$, $s'' \circ s' = s'$, and $(s \circ t) \circ s'' = s''$
 - for all $A, B \subseteq S$, if there is a $u \in A$ with $u \models_M \phi$ and for all $u' \in B$, $u' \not\models_M \phi$, then $A \leq_{M'} B$, and if for all $u \in A$, $u \not\models_M \phi$ and there is a $u' \in B$ with $u' \models_M \phi$, then $A \not\leq_{M'} B$; otherwise $A \leq_{M'} B$ iff $A \leq_M B$.

The existence requirement in the above definition state that there is a basic ϕ -situation (t) whose fusion with the initial situation (s) is defined, and s' is the lowest in the parthood ordering which includes the fusion $s \circ t$. Hence, s' is the smallest situation that we can pick as the revised situation. I will talk about the revision operation in the following manner: if R determines a relation from the triple $\langle \phi, M, s \rangle$ to the pair $\langle M', s' \rangle$, I write $\langle \phi, M, s \rangle R \langle M', s' \rangle$. When I do not need to refer to the models in a revision, I will also use the following notation: $sR^\phi s'$.

OBSERVATION 19. The revision operation R is a partial operation.

PROOF. Consider a belief base revision model $M^D = \langle S, V, \circ, \perp, \leq, R \rangle$ on a situation space $S = \{s, s'\}$ and on the language L_D whose literals are limited to $\{p, \bar{p}\}$. Suppose $V(s) = \{p\}$ and $V(s') = \{\bar{p}\}$. Let $M = \langle S, V, \circ, \perp, \leq \rangle$, the fusion function \circ be empty and let for all $A, B \subseteq S$ it holds that $A \leq_M B$. Therefore, $s \perp s'$ holds in M^D , and $s^* = s$, $s'^* = s'$. Suppose we want to revise s with the sentence $\neg p$. Since there is no basic $\neg p$ -situation in S whose fusion with s is defined, a revised situation cannot be determined in S . Therefore, R can not be executed on the triple $\langle \neg p, M, s \rangle$. ■

EXAMPLE 3. For the first revision example, we go back to the model presented in example 1 and the investigation on the robbery. In order indicate how the preference order is affected by iterated revisions, I include in this example multiple revision processes. Suppose, at the beginning, the investigator is ensured by the owners (only) that there are only two keys to the

library, and the butler has one of those. We assume at this point her information base is determined by the situation 4 in the model M . For the current example, we assume a different preference ordering for the model M from the one given in example 1. Since “there are only two keys to the library” and “the butler has a key to the library” constitute the only information the agent has so far, we assume that her preferences are such that all sets of situations which include a situation of $(p \wedge r)$ are preferred over the ones which does not include such a situation. For simplicity, we assume the rest of the preference ordering is plain, such that all sets of situations are preferred equally. Suppose at the first phase of the investigation she is informed that the second key is held by the maid (t). Let us see how she should revise her beliefs accordingly, based on the belief base revision framework.

We start with identifying the basic t -situations: the situation 1 is the unique basic t -situation in S . Since $1 \circ 4 = 1$ holds in M , the information base of the agent after the expansion is determined by the situation 1. Her epistemic preference order is then adjusted so the sets of situations which include a t -situation are preferred over the ones which do not include any t -situations, and the sets of situations which do not include any t -situations are no longer preferred over the ones which include some t -situations. The rest of her preferences remain as in the beginning. Let us call this new model with the revised preference ordering M' . For instance, when we take some singleton sets of situations in S , the following holds: $\{1\} \leq_{M'} \{4\} \leq_{M'} \{2\} \leq_{M'} \{4\} \leq_{M'} \{0\}$ while $\{4\} \not\leq_{M'} \{1\}$ and $\{0\} \not\leq_{M'} \{4\}$.

The revised belief set is then determined by the situation 1 in M' . There is a unique maximally consistent set of situations w.r.t. 1: $\text{Best}_{M'}(1) = \{\{0, 4, 1\}\}$. It follows that $1 \models_{M'} B(p \wedge t) \wedge B(r \wedge \neg q)$. Therefore, after the revision, the agent believes that the butler and the maid has the only two keys to the library, while the gardener does not have a key. As M' is the unique model for this revision, it follows that $4 \models_M [t](B(p \wedge t) \wedge B(r \wedge \neg q))$.

Suppose at the second phase of the investigation, the agent is told of the owners suspicions about whether or not the gardener stole the maid's key. She then wants to revise her belief set with the information that either the maid has the second key or the gardener has it. Given that her current doxastic model is represented at M' , there are two ways she can use this information to change her beliefs. That is because both 1 and 2 are basic $(q \vee t)$ -situations in M' . Let us call the revised models obtained by expanding her information base with the situation 1 and with the situation 2, M^1 and M^2 respectively.

At M^1 , her expanded information base is again determined by the situation 1 since $1 \circ 1 = 1$. According to the revised preference ordering, the sets of

situations which include a $(q \vee t)$ -situation are preferred over the ones which do not include any $(q \vee t)$ -situations, and the sets of situations which do not include any $(q \vee t)$ -situations are no longer preferred over the ones which include some $(q \vee t)$ -situations. The rest of the preferences remain as in M' . Note that with the revision, since all t -situations are also $(q \vee t)$ -situations, they remain minimal in the preference ordering on the subsets of S . However, since all q -situations are also $(q \vee t)$ -situations, they also move among the most preferred in M^1 . There is however a unique maximally consistent set of situations w.r.t. 1, thus, $Best_{M^1}(1) = \{\{0, 4, 1\}\}$. So, it is the case that $1 \models_{M^1} (B(p \wedge r) \wedge B(q \vee t)) \wedge B(t \wedge \neg q)$.

At M^2 , her expanded information base is determined by the situation 3 since $1 \circ 2 = 3$. According to the revised preference ordering, the sets of situations which include a $(q \vee t)$ -situation are preferred over the ones which do not include any $(q \vee t)$ -situations, and the sets of situations which do not include any $(q \vee t)$ -situations are no longer preferred over the ones which include some $(q \vee t)$ -situations. The rest of the preferences remain as in M' . (Hence the preference ordering of M^2 is identical to that of M^1 .) There are two maximally consistent sets of situations w.r.t. the situation 3, these are the sets $\{0, 1, 4\}$ and $\{0, 2, 4\}$. Based on the revised preference ordering \leq_{M^2} , we have that $\{0, 1, 4\} \leq_{M^2} \{0, 2, 4\} \leq_{M^2} \{0, 1, 4\}$ since both 1 and 2 are $(q \vee t)$ -situations. Hence, $Best_{M^2}(3) = \{\{0, 1, 4\}, \{0, 2, 4\}\}$. So, it follows that $3 \models_{M^2} B(p \wedge r) \wedge B(q \vee t)$. Therefore, after revising her beliefs, the agent still believes that there are only two keys to the library and the butler has a key to the library, however she no longer believes that the maid has a key to the library. She also does not believe that the gardener has a key to the library, while she believes that either one of them has the second key.

We express this indeterministic way of changing beliefs with the help of the diamond-like revision operators in the language: $1 \models_{M'} [q \vee t](B(p \wedge r) \wedge B(q \vee t)) \wedge \langle q \vee t \rangle B(t \wedge \neg q)$. That is, after revising her beliefs with the disjunction, the agent believes the disjunction $(q \vee t)$, while believing neither q nor t , and there is a way of changing her beliefs in which she also comes to believe that t and also $\neg q$ as a result (as in the revised model M^1).

EXAMPLE 4. The following example again shows how indeterministic belief change is interpreted within my framework. Consider a belief base revision model $M^D = \langle S, V, \circ, \perp, \leq, R \rangle$ based on the situation space $S = \{1, 2, 3, 4\}$. Let the literals of the language L_D be limited to $l = \{p, \bar{p}, q, \bar{q}\}$. Let the valuation of the situations in S be as the following: $V(1) = \emptyset, V(2) = \{p, \bar{q}\}, V(3) = \{p, q\}, V(4) = \{p, \bar{p}, q, \bar{q}\}$, and let the following fusion functions be defined on S : $1 \circ 1 = 1, 1 \circ 2 = 2, 1 \circ 3 = 3, 1 \circ 4 = 4, 2 \circ 2 = 2, 2 \circ 3 =$

$4, 2 \circ 4 = 4, 3 \circ 3 = 3, 3 \circ 4 = 4, 4 \circ 4 = 4$. Let the incompatibility relation to be given by the literals, such that $2 \perp 3, 2 \perp 4, 3 \perp 4$ and $4 \perp 4$. So, the following holds: $1 = 4^*, 2 = 2^*, 3 = 3^*, 4 = 1^*$. Let $M = \langle S, V, \circ, \perp, \leq \rangle$. Finally, let the preference ordering \leq_M be such that for all $A, B \subseteq S$ it holds that $A \leq_M B$.

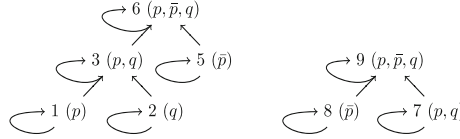
Suppose the information base of the agent is determined by the situation 1 in M . Since $Best_M(1) = \{\{1\}\}$, at this point she only has beliefs in the form of $B(\phi \rightarrow \psi)$. Suppose she learns that p . We show how she should revise her beliefs accordingly. Given the model M , there are two basic p -situations: 2 and 3. Hence, there are two ways she can revise her belief set. Either she expands her information base with the situation 2, or with the situation 3. So, it follows that there are two revised models based on the model M , call them M^2 and M^3 respectively. At each model, the preference ordering of the agent shifts from the preference ordering of the model M in the way that all sets of situations which include a p -situation are strictly preferred over all sets of situations which do not include a p -situation.

At M^2 , her (new) information base is determined by the situation 2. There is a unique maximally consistent set of situations w.r.t. 2, so, $Best_{M^2}(2) = \{\{1, 2\}\}$. It follows that $2 \models_{M^2} (Bp \wedge B\neg q) \wedge B(q \vee \neg q)$. At M^3 , her (new) information base is determined by the situation 3. Since is a unique maximally consistent set of situations w.r.t. 3, $Best_{M^3}(3) = \{\{1, 3\}\}$. It follows that $3 \models_{M^3} (Bp \wedge Bq) \wedge B(q \vee \neg q)$. Therefore, it holds that $1 \models_M [p](Bp \wedge B(q \vee \neg q)) \wedge (\langle p \rangle Bq \wedge \langle p \rangle B\neg q)$. After the revision, the agent believes that p and that $(q \vee \neg q)$. However, it is indetermined whether she believes that q or that $\neg q$.

Before moving to the last example of belief base revision, I want to note that belief revision is not always successful in the current framework. That is, it not necessarily the case that an information piece ϕ is accepted as a belief after the revision with ϕ . In fact, neither $\models [\phi]B\phi$ nor $\models \langle \phi \rangle B\phi$ are valid in this framework. After the revision of a belief set with ϕ , it is accepted as a belief iff there is some consistent part of the revised information base in which ϕ is satisfied by a situation. Moreover, if ϕ is a contradictory sentence in the form $\psi \wedge \neg\psi$ for some $\psi \in L_B$, the revision is bound to be unsuccessful.

EXAMPLE 5. *The last example indicates that representation of information is more fine grained in the belief base revision models than it is in the traditional belief change models. The models allow the existence of multiple situations mapped to the same set of propositional letters since we do not identify the situations with their local content. That means, these situations may differ in terms of their dynamic aspects although their local*

contents are the same. Consider the following fractions of belief base revision models; the nodes represent the situations in the situation space with their local content given in parenthesis, and the arrows represent the parthood ordering, the parthood ordering should be read as transitively closed.



The local contents of the situations 3 and 7 are equal, although their parts differ. If the belief set satisfied by the situation 3 is revised with the information that $(\neg p)$, the revised belief set would still include q , whereas the same revision on the belief set satisfied by the situation 7 would see both p and q being eliminated from the new belief set.

3. Logic

The logic of the belief base revision models restricted to the sublanguage L_{prop} is the logic of HYPE.¹⁷

THEOREM 20. *The following list of axioms and rules is sound for the system of belief base revision (for all formulas $\phi \in L_D$, if $\vdash \phi$ then $\models \phi$):*

- (MP) $\phi, \phi \rightarrow \psi \vdash \psi$
- (Cont) $\frac{\vdash \phi \rightarrow \psi}{\vdash \neg\psi \rightarrow \neg\phi}$
- (1) $\vdash \top$
- (2) $\vdash \phi \rightarrow \phi$
- (3) $\vdash (\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$
- (4) $\vdash \phi \wedge \psi \rightarrow \phi$
- (5) $\vdash \phi \wedge \psi \rightarrow \psi$
- (6) $\vdash \phi \rightarrow \phi \vee \psi$
- (7) $\vdash \psi \rightarrow \phi \vee \psi$
- (8) $\vdash (\phi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \vee \psi \rightarrow \chi))$
- (9) $\vdash \phi \wedge (\psi \vee \chi) \leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$
- (10) $\vdash \phi \vee (\psi \wedge \chi) \leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \chi)$

¹⁷A sound and complete axiom system for the HYPE logic is given in Leitgeb [12].

- (11) $\vdash \phi \rightarrow \neg\neg\phi$
- (12) $\vdash \neg\phi \vee \neg\psi \rightarrow \neg(\phi \wedge \psi)$
- (13) $\vdash \neg\phi \wedge \neg\psi \leftrightarrow \neg(\phi \vee \psi)$
- (14) $\vdash (\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$
- (15) $\vdash (\phi \rightarrow (\phi \rightarrow \psi)) \rightarrow (\phi \rightarrow \psi)$
- (16) $\vdash B\phi \rightarrow BB\phi$ (*Positive Introspection*)
- (17) $\vdash B\phi \vee B\psi \rightarrow B(\phi \vee \psi)$ (*Disjunctive closure*)
- (18) $\vdash B(\phi \wedge \psi) \rightarrow B\phi \wedge B\psi$ (\wedge *distribution*)
- (19) $\vdash [\phi]B\chi \rightarrow [\phi]B(\psi \vee \chi)$ (*Disjunction*)
- (20) $\vdash [\phi]B(\psi \wedge \chi) \rightarrow [\phi]B\psi$ (*Simplification*)
- (21) $\vdash B\phi \wedge B\neg\phi \rightarrow \perp$ (*Consistency 1*)
- (22) $\vdash B(\phi \wedge \neg\phi) \rightarrow \perp$ (*Consistency 2*)
- (CM) $[\phi]B\psi, [\phi]B\chi \vdash [\phi \wedge \psi]B\chi$
- (cut) $[\phi]B\psi, [\phi \wedge \psi]B\chi \vdash [\phi]B\chi$

PROOF. To prove soundness, I demonstrate the detailed proofs for selected axiom schemas only. Let $M^D = \langle S, V, \circ, \perp, \leq, R \rangle$ be an arbitrary belief base revision model on a situation space S , let $M = \langle S, V, \circ, \perp, \leq \rangle$ and let s be an arbitrary situation in S .

- (11) For reductio, assume $s \not\models \phi \rightarrow \neg\neg\phi$. So, there is an $s' \in S$ with $s' \circ s = s'$ and $s' \models \phi$, but $s' \not\models \neg\neg\phi$. Hence, there is an $s'' \in S$ with $s'' \models \neg\phi$ and it holds that $s' \not\leq s''$. However, since $s' \models \phi$, it should be the case that $s' \perp s''$. Hence we have a contradiction. Therefore, $s' \models \neg\neg\phi$, and it follows that $s \models \phi \rightarrow \neg\neg\phi$. Since s and M^D are arbitrary, $\models \phi \rightarrow \neg\neg\phi$ is true in all belief base revision models.
- (16) Let arbitrary $s' \in S$ with $s' \circ s = s'$. Suppose $s' \models B\phi$. So, for all $A \in Best_M(s')$, there is an $s'' \in A$ with $s'' \models \phi$. Consider an arbitrary $A \in Best_M(s')$ and an arbitrary $s'' \in A$ with $s'' \models \phi$. Since A is consistent, it follows that $s'' \not\leq s''$. By the lemma 11, it holds that $s'' \models B\phi$. Since A and s'' are arbitrary, it holds that for all $A' \in Best_M(s')$ there is a $u \in A'$ with $u \models B\phi$. Therefore, $s' \models BB\phi$. Since s' is arbitrary, it follows that $s \models B\phi \rightarrow BB\phi$. Since s and M^D are arbitrary, $\models B\phi \rightarrow BB\phi$ is true in all belief base revision models.

- (17) Let arbitrary $s' \in S$ with $s' \circ s = s'$. Suppose $s' \models B\phi \vee B\psi$. So, either $s' \models B\phi$ or $s' \models B\psi$. Suppose the former. Hence, for all $A \in Best_M(s')$, there is an $s'' \in A$ with $s'' \models \phi$. Pick an arbitrary $A \in Best_M(s')$ and an arbitrary $s'' \in A$ with $s'' \models \phi$. By the satisfaction clause for disjunction, it holds that $s'' \models \phi \vee \psi$. Since A and s'' are arbitrary, it follows that for all $A' \in Best_M(s')$ there is a $u \in A'$ with $u \models \phi \vee \psi$. Therefore, $s' \models B(\phi \vee \psi)$. Similarly, if $s' \models B\psi$, it follows that $s' \models B(\phi \vee \psi)$. Since s' is arbitrary, it follows that $s \models B\phi \vee B\psi \rightarrow B(\phi \vee \psi)$. Since s and M^D are arbitrary, $\models B\phi \vee B\psi \rightarrow B(\phi \vee \psi)$ is true in all belief base revision models.
- (18) Let arbitrary $s' \in S$ with $s' \circ s = s'$. Suppose $s' \models B(\phi \wedge \psi)$. So, for all $A \in Best_M(s')$, there is an $s'' \in A$ with $s'' \models \phi \wedge \psi$. Pick an arbitrary $A \in Best_M(s')$ and an arbitrary $s'' \in A$ with $s'' \models \phi \wedge \psi$. It follows that $s'' \models \phi$ and also $s'' \models \psi$. Since A and s'' are arbitrary, $s' \models B\phi \wedge B\psi$. Since s' is arbitrary, it follows that $s \models B(\phi \wedge \psi) \rightarrow B\phi \wedge B\psi$. Since s and M^D are arbitrary, $\models B(\phi \wedge \psi) \rightarrow B\phi \wedge B\psi$ is true in all belief base revision models.
- (CM) Let M' be a belief base model and let s' be a situation in S with $\langle M, s, \phi \rangle R \langle M', s' \rangle$. Suppose $s \models_M [\phi]B\psi$. Since all $(\phi \wedge \psi)$ -situations are ϕ -situations, and since $s \models_M [\phi]B\psi$, it holds that the set of pairs of models and situations obtained from revising s in M with $(\phi \wedge \psi)$ constitutes a subset of the set of pairs of models and situations obtained from revising s in M with ϕ (that is, if that all sets of situations in M' which include some ϕ -situations are preferred to the sets of situations in M' which do not include any ϕ -situations entails that it is already the case that all sets of situations in $Best_{M'}(s')$ also include some ψ -situations then, provided that there are any, the maximally consistent sets of situations under s' with some $(\phi \wedge \psi)$ -situations are already among the $Best_{M'}(s')$). Therefore, if $s \models_M [\phi]B\chi$, it holds that $s \models_M [\phi \wedge \psi]B\chi$. Since s and M^D are arbitrary, the CM rule holds in all belief base revision models.
- (Cut) Let M' be a belief base model and let s' be a situation in S with $\langle M, s, \phi \rangle R \langle M', s' \rangle$. Suppose $s \models_M [\phi]B\psi$. So, it holds that $s \models_M [\phi](\phi \wedge \psi)$. Moreover, if all sets of situations in M' which include some ϕ -situations are preferred to the sets of situations in M' which do not include any ϕ -situations, then it is the case that all sets of situations in $Best(s')$ also include some ψ -situations. Hence, these

are exactly the maximally consistent sets of situations under s' with some $(\phi \wedge \psi)$ -situation, provided that there are any. Therefore, the set of pairs of models and situations obtained from revising s in M with ϕ constitutes a subset of the set of pairs of models and situations obtained from revising s in M with $(\phi \wedge \psi)$. Since $s \models_M [\phi \wedge \psi]B\chi$ holds, all members of $Best_{M'}(s')$ include also some χ -situations. Therefore, $s \models_M [\phi]B\chi$. Since s and M^D are arbitrary, the cut rule holds in all belief base revision models. ■

The MP rule and the axiom schema 2 follows from the idempotence assumption of \circ and the satisfaction clause for \rightarrow . The validities of axiom schemas 3, 8, 12 - 15 can easily be shown via reductio ad absurdum. The validity of the axiom schema 19 follows from 17 and that of 20 follows from 18. For the proofs of 21 and 22 see lemma 13. The rest can be proved using only the satisfaction clauses. In the above theorem, when it is possible, I stated the claims which include the belief modality and the dynamic operators in the form of axiom schemas rather than as rules. (For instance, the rule for the positive introspection would be $B\phi \vdash BB\phi$.) By the implication lemma, the proofs for the doxastic axiom schemas entail the proofs for the respective rules.

LEMMA 21. *The following deduction theorem is logically valid in belief base revision models iff the language of the models are restricted to the sublanguage L_{prop} (i.e., when $\phi_1, \dots, \phi_n, \psi, \chi \in L_{prop}$): $\phi_1, \dots, \phi_n, \psi \vdash \chi$ iff $\phi_1, \dots, \phi_n \vdash \psi \rightarrow \chi$.*

PROOF. We use (MP) and the axioms schemas (2) and (3) from theorem 20, together with the schema $\vdash \phi \rightarrow (\psi \rightarrow \phi)$. The latter is valid in the current framework only when the models are restricted to the language L_{prop} . ■

4. More Properties of Belief and Belief Revision

In this section, I state some principles concerning the belief sets and belief revision, which fail in the proposed framework although they are satisfied as axiom schemas or rules in some of the more common and well-known theories in the literature.

THEOREM 22. *The following list of axiom schemas and rules are not valid in belief base (revision) models.*

- (1) $B\phi, B(\phi \rightarrow \psi) \models B\psi$ (Modal modus ponens)

- (2) $B(\phi \rightarrow \psi) \models B\phi \rightarrow B\psi$ (*K-rule*)
 (3) $\phi \rightarrow \psi \models B\phi \rightarrow B\psi$ (*Monotonicity of belief*)
 (4) $B\phi \wedge B\psi \models B(\phi \wedge \psi)$ (*Conjunctive closure*)
 (5) $\neg B\phi \models B\neg B\phi$ (*Negative introspection*)
 (6) $B(\phi \vee \psi) \models B\phi \vee B\psi$ (\vee *distribution*)
 (7) $\frac{\vdash \phi}{\vdash B\phi}$ (*Necessitation*)

schema

PROOF. As a counterexample to the first four principles, consider a belief base revision model $M^D = \langle S, V, \circ, \perp, \leq, R \rangle$ on the situation space $S = \{1, 2, 3, 4, 5, 6\}$, and let literals of the language L_D for the model M^D be $l = \{p, \bar{p}, q, \bar{q}, r, \bar{r}, t, \bar{t}, s, \bar{s}\}$. Let $V(1) = \{r, t\}$, $V(2) = \{p, t\}$, $V(3) = \{p, \bar{p}, q, \bar{q}, r, t, s, \bar{s}\}$, $V(4) = \{p, q, \bar{q}, r, \bar{r}, t, s, \bar{s}\}$, $V(5) = \{p, q, r, t, \bar{t}\}$, $V(6) = \{p, q, r, s, \bar{s}\}$. Let $((1 \circ 2) \circ 6) \circ 3 = 3$, $((1 \circ 2) \circ 6) \circ 4 = 4$, $(1 \circ 2) \circ 5 = 5$, $1 \circ 1 = 1$, $2 \circ 2 = 2$, $3 \circ 3 = 3$, $4 \circ 4 = 4$, $5 \circ 5 = 5$, $6 \circ 6 = 6$, and let the parthood relation be transitively closed on these fusions. Let the incompatibility relation on S be given via the literals. It follows that $1 = 3^*$, $2 = 4^*$, $3 = 1^*$, $4 = 2^*$, $5 = 6^*$, $6 = 5^*$. Let $M = \langle S, V, \circ, \perp, \leq \rangle$ and finally, for all $A, B \subseteq S$, $A \leq_M B$.

- (1) We substitute the ' ϕ ' in the schema with ' p ', and the ' ψ ' in the schema with ' q '. In the model M , it holds that $5 \models Bp$ and $5 \models B(p \rightarrow q)$, but $5 \not\models Bq$.
 (2) We use the same substitution of the formulas. It holds in M , that $5 \models B(p \rightarrow q)$, however $5 \not\models Bp \rightarrow Bq$.
 (3) Similarly, $5 \models p \rightarrow q$, however $5 \not\models Bp \rightarrow Bq$.
 (4) We substitute the ' ϕ ' in the schema with ' p ', and the ' ψ ' in the schema with ' r '. In the model M , it holds that $5 \models Bp \wedge Br$, but $5 \not\models B(p \wedge r)$.

As a counterexample to the remaining principles, consider a belief base model M on the situation space $S = \{1, 2, 3, 4\}$, and let literals of the language L_D for the model M' be $l = \{p, \bar{p}\}$. Let $V(1) = \{p\}$, $V(2) = \{\bar{p}\}$, $V(3) = \{p, \bar{p}\}$, $V(4) = \emptyset$. Let $4 \circ 1 = 1$, $4 \circ 2 = 2$, $4 \circ 3 = 3$, $1 \circ 2 = 3$, $1 \circ 3 = 3$, $2 \circ 3 = 3$, $4 \circ 4 = 4$, $1 \circ 1 = 1$, $2 \circ 2 = 2$, $3 \circ 3 = 3$. Let the incompatibility relation of M be given via the literals, hence $1 \perp 2$, $1 \perp 3$, $2 \perp 3$, $3 \perp 3$. It follows that $1 = 1^*$, $2 = 2^*$, $3 = 4^*$, $4 = 3^*$. Let the preference ordering of M' be such that, for all $A, B \subseteq S$, $A \leq_{M'} B$.

- (5) We substitute the the ‘ ϕ ’ in the schema with ‘ p ’. In the model M' , it holds that $3 \models \neg Bp$, however, $3 \not\models B\neg Bp$
- (6) We substitute the ‘ ϕ ’ in the schema with ‘ p ’, and the ‘ ψ ’ in the schema with ‘ $\neg p$ ’. In the model M' , $3 \models B(p \vee \neg p)$, but $3 \not\models Bp \vee B\neg p$.

The necessitation rule fails when for some belief base model M , for some $s \in S$ it holds that $Best_M(s) = \{\emptyset\}$. That is, when s has no consistent parts. Because in this case, for no $\psi \in L_B$ it holds that $s \models B\psi$. ■

The above principles are stated in the form of rules rather than axioms since by the contraposition of the implication lemma, their failures entail the failures of the respective axiom schemas.

THEOREM 23. *The following list of axiom schemas and rules are not valid in belief base revision models. In the following, \Rightarrow stands for classical logical implication.*

- (1) $\models [\phi]B(\psi \vee \neg\psi)$ (*Excluded middle*)
- (2) $[\phi]B\psi, [\phi]B\chi \models [\phi]B(\psi \wedge \chi)$ (*Adjunction*)
- (3) $[\phi]B(\psi \vee \chi) \models [\phi]B\psi \vee [\phi]B\chi$ (*Disjunction 2*)
- (4) $[\phi]B\psi, [\phi]B(\psi \rightarrow \chi) \models [\phi]B\chi$ (*Closure under belief implication*)
- (5) $[\phi]B\chi \models [\phi \wedge \psi]B\chi$ (*Monotony*)
- (6) $\neg[\phi]B\neg\psi, [\phi]B\chi \models [\phi \wedge \psi]B\chi$ (*Rational monotony*)¹⁸
- (7) $\phi \Rightarrow \psi \models [\phi]B\psi$ (*Intensionality*)
- (8) $[\phi]B\psi, \psi \Rightarrow \chi \models [\phi]B\chi$ (*Right weakening*)
- (9) $[\phi]B\chi, \phi \Leftrightarrow \psi \models [\psi]B\chi$ (*Left logical equivalence*)

PROOF. The invalidity of the first axiom schema follows from the failure of general excluded middle ($\models \phi \vee \neg\phi$). The next three invalidities follow respectively from the failures of conjunctive closure, \vee distribution and modal modus ponens in theorem 22.

To show the invalidity of the remaining principles, we construct the following model. Let $M^D = \langle S, V, \circ, \perp, \leq, R \rangle$ be a belief base revision model on a situation space S and a language L_D . Let the literals of the language be $l = \{p, \bar{p}, q, \bar{q}, r, \bar{r}\}$. Let the situation space be $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$, such that $V(1) = \{p, r\}$, $V(2) = \{q, \bar{r}\}$, $V(3) = \{p, \bar{p}, q, \bar{q}, r, \bar{r}\}$, $V(4) = \{p, q, \bar{q}, r\}$,

¹⁸The first premise of this argument could be read as the satisfaction of the negated formula, or as the non-satisfaction of $[\phi]B\neg\psi$.

$V(5) = \{p, \bar{p}, q, \bar{r}\}$, $V(6) = \emptyset$, $V(7) = \{p, q\}$, $V(8) = \{p, q, r, \bar{r}\}$. Let the following fusions be defined on S : $6 \circ 1 = 1$, $6 \circ 2 = 2$, $6 \circ 7 = 7$, $(1 \circ 7) \circ 4 = 4$, $(2 \circ 7) \circ 5 = 5$, $((1 \circ 2) \circ 7) \circ 8 = 8$, $((4 \circ 5) \circ 8) \circ 3 = 3$, and let the parthood relation be transitively closed on these fusions. Let the incompatibility relation on S be given via the literals. Thus, the following holds: $1 = 4^*$, $2 = 5^*$, $3 = 6^*$, $4 = 1^*$, $5 = 2^*$, $6 = 3^*$, $7 = 8^*$, $8 = 7^*$. Let $M = \langle S, V, \circ, \perp, \leq \rangle$, and finally, for all $A, B \subseteq S$, $A \leq_M B$.

- (5) We substitute the ' ϕ ' in the schema with ' r ', the ' ψ ' in the schema with ' q ', and the ' χ ' in the schema with ' $(r \wedge p)$ '. I will state the proof only for the current item in detail to set an example for the rest. We will show that $6 \models_M [r]B(r \wedge p)$, but $6 \not\models_M [r \wedge q]B(r \wedge p)$. Suppose we revise the situation 6 in M with r . Let M' be the revised model. The revised information base is determined by the situation 1 ($6R^r1$). It holds that $Best_{M'}(1) = \{\{1, 6\}\}$, hence $1 \models_{M'} B(r \wedge p)$. Therefore, $6 \models_M [r]B(r \wedge p)$. Now suppose we revise the situation 6 in M with $(r \wedge q)$. There are two ways to do this since both 4 and 8 in M are basic $(r \wedge q)$ -situations. It suffices to show that in one of the revised models, $B(r \wedge p)$ is not satisfied by the revised information base. Let M^8 be the revised model through situation 8 ($6R^{r \wedge q}8$). It holds that $Best_{M^8}(8) = \{\{1, 6, 7\}, \{2, 6, 7\}\}$, hence $8 \not\models_{M^8} B(r \wedge p)$. Therefore, $6 \not\models_M [r \wedge q]B(r \wedge p)$.
- (6) It suffices that $6 \not\models_M [r]B\neg q$ in addition to what we have shown above. In fact, $6 \models_M \neg[r]B\neg q$ since $[r]B\neg q$ is not satisfied anywhere in the model M' . Hence we comply with the first premise in both forms of its reading.
- (7) We substitute the ' ϕ ' in the schema with ' r ', and the ' ψ ' in the schema with ' $(q \vee \neg q)$ '. It holds that $r \Rightarrow (q \vee \neg q)$, however $1 \not\models_M [r]B(q \vee \neg q)$.
- (8) We substitute ' ϕ ' in the formula with ' r ', ' ψ ' with ' p ', and ' χ ' with ' $q \vee \neg q$ '. It holds that $1 \models_M [r]Bp$, and that $p \Rightarrow (q \vee \neg q)$, however $1 \not\models_M [r]B(q \vee \neg q)$.
- (9) We substitute the ' ϕ ' in the schema with ' $p \vee \neg p$ ', the ' ψ ' in the schema with ' $q \vee \neg q$ ', and the ' χ ' in the schema with ' p '. Thus, it holds that $6 \models_M [p \vee \neg p]Bp$, and $(p \vee \neg p) \Leftrightarrow (q \vee \neg q)$, however $6 \not\models_M [q \vee \neg q]Bp$.

■

I want to conclude this section with some remarks on the system of belief base revision, particularly on the specific form of hyperintensionality manifested in my system and the (lack of) reduction axioms. The last three

axiom schemas of theorem 23 express the influence of classical logic on belief base revision. Their invalidities indicate the hyperintensionality of the revision system. That is, the revision operations do not respect classical logical equivalences. That we formulated these schemas in the metalanguage referring to two different logics marks an important difference between this framework and the framework for hyperintensional belief revision presented by Berto [3]. He formulates the intensionality, right weakening and left logical equivalence rules within his object language. Therefore, the failures of the rules indicate some limitations concerning the influence of his already underlying (classical propositional) logic on belief revision.

My final remark is a brief discussion on why my system does not include reduction axioms of dynamic formulas to static ones. In the proposed framework, this is a particularly challenging task. The formulation of the conditional (\rightarrow) successfully hints at some properties of belief revision as it is a forward looking modality. Belief base revision, in most cases, causes a shift from one situation to another. For instance, in order to revise a situation s with a piece of information ϕ , we move to the situations which expand s with ϕ . These are (some of) the situations that are determined by a conditional on s whose antecedent is ϕ . In this respect, the conditional still underdetermines the situations relevant for the revision since when revising s , belief base revision operations pick the situations which expand s with basic ϕ -situations only. The language however is not rich enough to allow precoding revisions completely. That is because, what is satisfied by a situation depends in part on the preference ordering of the model. It might be the case that, for some ψ in the language, ψ is not satisfied by a ϕ -situation until after the preference ordering of the model is revised. Although this is the case in most belief revision systems which include changing the preferences, in some of these systems, the revised models can be given as sub-models of the original one. Hence, it is possible to give reduction axioms by referring to a relativized version of the original model. The system presented in van Benthem [26] is an example to this sort of preference change.

There is an exception to this hardship by the persistency lemma, which ensures that the propositional content of a situation does not change via model-shifts. So, we can present a reduction axiom only concerning the formulas of L_{prop} : given a belief base revision model $M^D = \langle S, V, \circ, \perp, \leq, R \rangle$ on a situation space S , with $M = \langle S, V, \circ, \perp, \leq \rangle$, for all situations $s \in S$, for all $\phi \in L_B$ and for all $\psi \in L_{prop}$, $s \models_M [\phi]\psi \leftrightarrow (\phi \rightarrow \psi)$. For the base case of the proof, suppose $s \models_M [\phi]p$. Hence, for all $s' \in S$, if $sR^\phi s'$, it holds that $s' \models_M p$. Let arbitrary $s'' \in S$ such that $s' \circ s'' = s''$. By the persistency lemma, also $s'' \models_M p$ holds. By the assumptions for the fusion function, it

holds that $s'' \circ s = s''$. Since s'' is arbitrary, by the satisfaction clause for \rightarrow , it holds that $s \models_M \phi \rightarrow p$. For the other direction, suppose $s \models_M \phi \rightarrow p$. That is, whenever we expand s in M with a (basic) ϕ -situation, satisfaction of p follows in the expanded situation. Hence, for all $s' \in S$, if $sR^\phi s'$ then $s' \models p$. Therefore, $s \models_M [\phi]p$. The validity of the claim for non-atomic propositional formulas of the language can be shown easily by induction on the complexity.

5. Conclusion

I have presented a new hyperintensional semantics for belief revision, which also allows non-monotonic and non-explosive belief revision. The non-classical features of the revision framework principally follow from the underlying non-classical semantics. In particular, we have formalised the dynamics of potentially incomplete and inconsistent collections of information using a form of situation semantics. Adoption of situations as the principle elements of the models separates my framework from the DEL paradigm. At the same time, the introduction of the revision operators in the object language marks the effective difference between my semantics and the base-generated belief revision theories in the literature. Syntactically, the underlying (propositional) logic of belief base revision is significantly weaker than classical logic. This quality allows us to have a much adaptable logic of belief representation and belief dynamics.

There are a number of philosophical issues which did not had enough space in this paper. I believe these are issues that require much more space, so I keep them for future work. I will however briefly list them here since they add significantly to the intuitive motivation of the belief base models presented here. My methodology had been to introduce some structure to the models to overcome some of the idealisations in the literature, at the same time maintaining a realistic and smooth intuition of reasoning and a nice logical system. Hence, I have introduced some non-classical features while developing the belief base revision models. The consequences of these features are specifically, non-monotonicity, indeterminacy of information, hyperintensional sensitivity, and fragmentation of information.

Non-monotonicity of the system is apparent by the items (5) and (6) of the theorem 23. It is the belief modality that I employ here that causes the non-monotonicity of belief. Indeterminacy of information is specifically related to disjunctive beliefs. The item (6) in theorem 22 and its dynamic counterpart, the item (3) in theorem 23 indicate that an agent can believe

a disjunction without necessarily believing one of the disjuncts. However, believing a disjunction without believing one of the disjuncts is possible only if the agent has inconsistent information, not necessarily about the disjunction in question.

Hyperintensionality of the logic of belief base revision is presented via the items (1) and (7–9) in Theorem 23. It is a consequence of the partial content of the situations. Hyperintensional sensitivity has usually been introduced as subject-matter sensitivity. Although I do not mention subject-matters of sentences in this paper, such a reading is also possible. One might say that the content of a situation determines a subject-matter. Hence, although two sentences ϕ and ψ are classically logically equivalent, while ϕ is part of the content or the subject-matter of a situation s , ψ may not be part of the content of the same situation. Thus, an agent, whose current belief state is determined by the situation s may not be aware of the classical logical entailment between the two sentences, and do not necessarily believe the latter on the basis of the former.

Some models of hyperintensional belief revision reject also the principle of disjunctive closure on the grounds of subject-matter inclusion requirement for belief entailment [3]. It follows from disjunctive closure, that if an agent believes that ϕ , they also believe that $\phi \vee \psi$ for any ψ . The subject-matter inclusion requirement is such that, given that a sentence ϕ logically entails a sentence ψ , an agent believes that ψ upon believing that ϕ only if the subject matter of ϕ includes the subject matter of ψ . Briefly, logical entailment of sentences with foreign subject-matters do not carry over to the beliefs. However, disjunctive closure is a valid principle of belief base revision framework that I presented in this work. This is because, although the models are sensitive to the hyperintensional contexts, the requirement for entailment in these contexts is weaker than subject-matter inclusion. In fact, it seems that for logical entailment to carry over to the beliefs of an agent, shared subject-matter between the two sentences suffices. That is, it holds that the agent believes that $\phi \vee \psi$ upon believing that ϕ because the sentence $\phi \vee \psi$ is partly about ϕ . A more in depth discussion of the specific form of hyperintensionality presented here and its relation to other models of hyperintensional contexts is left for future work.

Lastly, one of the most important features of the models is the fragmentation of information, due to the partial fusion function and the partial parthood ordering of the models. The consequences of this structure are presented in the paper as the failures of the principles of conjunctive closure and of the principles of closure under implication, in the items (1)-(4) in theorem 22 and in the items (2) and (4) in theorem 23. These consequences

are quite similar to that of fragmented belief approaches, where it is allowed that the doxastic system of an agent involves different centers of rationality. Thus, the agent's belief state is fragmented such that the agent may believe that ϕ in one fragment and believe that ψ in another, and not be able to put the two beliefs together. These systems allow also contradictory beliefs located in different fragments. The models I present here falls short of admitting the full consequences of fragmented belief, by virtue of the employment of a partial-meet-consequence-like belief modality and a total epistemic preference ordering. Although fragmentation of information is one of the most powerful features of the belief base revision models in terms of its consequences, the subject is far too lengthy to be included in this paper. Hence, I leave the detailed discussion of the subject of fragmented belief and the developments of the belief base revision models in that direction (that is, models with a partial epistemic preference ordering and a belief modality that does not include the meet, i.e., the intersection function) for future work. The construction of belief contraction models and a complete axiom system for belief base revision models are also left for future work.

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