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A Generalization of Monadic n-Valued Łukasiewicz Algebras

Abstract. $\mathcal{M}L_n^m$ of monadic m-generalized Łukasiewicz algebras of order n (or ML_n^m-algebras), namely a generalization of monadic n-valued Łukasiewicz algebras. In this article, we determine the congruences and we characterized the subdirectly irreducible ML_n^m-algebras. From this last result we proved that $\mathcal{M}L_n^m$ is a discriminator variety and as a consequence we characterized the principal congruences. In the last part of this paper we find an immersion of these algebras in a functional algebra and we proved that in the finite case they are isomorphic. This last result allows to show a new functional representation for monadic n-valued Łukasiewicz algebras. Finally, we define the notions of ML_n^m-algebra of fractions and maximal algebra of fractions and we prove the existence of a maximal ML_n^m-algebra of fractions for an ML_n^m-algebra.

Keywords: Monadic n-valued Łukasiewicz algebras, m-generalized Łukasiewicz algebras of order n, Congruences, Subdirectly irreducible algebras, Discriminator variety, Functional representation, ML_n^m-algebra of fractions, Maximal ML_n^m-algebra of fractions.

1. Introduction and Preliminaries

In 1971, Georgescu and Vraciu [17] in a very interesting paper introduced monadic n-valued Łukasiewicz algebras and studied their relationship with monadic Boolean algebras. From the results obtained by these authors, a functional representation theorem for these algebras is deduced without following Halmos's reasoning [18]. These algebras were extensively studied in [4,12,13,19] to mention a few.

On the other hand, in 2001, Almada and Vaz de Carvalho [1] considered a generalization of Łukasiewicz algebras of order n (or L_n-algebras) and they introduced the variety $\mathcal{L}_n^m$ of m-generalized Łukasiewicz algebras of order n in the following way.

An m–generalized Łukasiewicz algebra of order n (or L_n^m–algebra) is an algebra $\langle L, \vee, \wedge, f, \{D_i\}_{1\leq i\leq n-1}, 0, 1\rangle$ of type $(2, 2, 1, \{1_i\}_{1\leq i\leq n-1}, 0, 0)$ which satisfies the following conditions:

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(GL_1) $\langle L, \vee, \wedge, f, 0, 1\rangle$ is a bounded distributive lattice where f is a dual endomorphism satisfying the identity $f^{2m}x = x$,

(GL_2) $D_i(x \wedge \overline{y}) = D_i x \wedge D_i \overline{y},\ 1 \leq i \leq n-1$, where $\overline{z}$ denotes the element $\bigvee_{p=0}^{m-1} f^{2p} z$,

(GL_3) $D_i x \wedge D_j x = D_j x,\ 1 \leq i \leq j \leq n-1$,

(GL_4) $D_i x \vee f D_i x = 1,\ 1 \leq i \leq n-1$,

(GL_5) $D_i f \overline{x} = f D_{n-i} \overline{x},\ 1 \leq i \leq n-1$,

(GL_6) $D_i D_j x = D_j x,\ 1 \leq i, j \leq n-1$,

(GL_7) $x \vee D_1 x = D_1 x$,

(GL_8) $D_i x = D_i \overline{x},\ 1 \leq i \leq n-1$,

(GL_9) $(x \wedge fx) \vee y \vee fy = y \vee fy$,

(GL_{10}) $\overline{x} \leq \overline{y} \vee f D_i \overline{y} \vee D_{i+1} \overline{x},\ 1 \leq i \leq n-2$.

In every L_n^m-algebra the identities listed below are also verified [1,14] and they will be use throughout this paper.

Proposition 1.1. *Let $A \in \mathcal{L}_n^m$. Then it holds:*

(GL_{11}) $D_i(x \vee y) = D_i(x) \vee D_i(y)$, $1 \leq i \leq n-1$,

(GL_{12}) $f^2(D_i(x)) = D_i(x)$, $1 \leq i \leq n-1$,

(GL_{13}) $D_i(x) \wedge f(D_i(x)) = 0$, $1 \leq i \leq n-1$,

(GL_{14}) $f(x) \vee D_1(x) = 1$,

(GL_{15}) $f(\overline{x}) \wedge D_{n-1}(\overline{x}) = 0$,

(GL_{16}) $\overline{x} \wedge D_{n-1}(\overline{x}) = D_{n-1}(\overline{x})$,

(GL_{17}) $D_i(0) = 0$, $1 \leq i \leq n-1$,

(GL_{18}) $D_i(1) = 1$, $1 \leq i \leq n-1$.

(GL_{19}) $f^2\overline{x} = \overline{x}$,

(GL_{20}) $f\overline{x} = \overline{f\overline{x}}$,

(GL_{21}) $f^2\overline{x} = f\overline{f\overline{x}}$,

(GL_{22}) $x \leq y$ *implies* $\overline{x} \leq \overline{y}$,

(GL_{23}) $\overline{x \vee y} = \overline{x} \vee \overline{y}$,

(GL_{24}) $\overline{\overline{x} \wedge \overline{y}} = \overline{x} \wedge \overline{y}$.

(GL_{25}) $\overline{\overline{x}} = \overline{\underline{x}}$, *where* $\underline{z}$ *denotes the element* $\bigwedge_{p=0}^{m-1} f^{2p} z$,

(GL_{26}) $\overline{D_i x} = D_i \overline{x} = D_i x$, $1 \leq i \leq n-1$,

(GL_{27}) $D_i x = D_i y$ *for all* i, $1 \leq i \leq n-1$ *imply* $\overline{x} = \overline{y}$.

From the above results is simple to verify that the operators D_i do not distribute with the infimum and the negation does not commute with them. However, we will see that it is possible to define a quantifier on these algebra such that they constitute a generalization of monadic n-valued Lukasiewicz algebras.

Remark 1.1. Let $A \in \mathcal{L}_n^m$. The set $S(A) = \{x \in A : f^2(x) = x\}$ plays an important role in the study of these algebras. In particular, $S(A)$ is a subalgebra of A and it is the greatest subalgebra of A that belongs to the variety of L_n-algebras. Let us observe that for all $x \in A$, both $\overline{x}$ and $\underline{x}$ belong to $S(A)$ and $\underline{x} \leq x \leq \overline{x}$. Obviously, $x \in S(A)$ if and only if $\underline{x} = x$ (if and only if $\overline{x} = x$). [1, Proposition 2.2].

Remark 1.2. In [1] T. Almada and J. Vaz de Carvalho determine that the variety $\mathcal{L}_n^m$ is semisimple and locally finite, then we conclude that all L_n^m-algebra is a Heyting algebra. On the other hand, A. Iorgulescu in [22], after surveying chronologically several algebras related to logic, she redefined them as particular cases of BCK-algebras (see also [20,21]). Thus, she showed that Hilbert algebras are positive implicative BCK-algebras. Hence, from the above results and taking into account that the implicative reduct of a Heyting algebra is a Hilbert algebra, it follows that every L_n^m-algebra is a positive implicative BCK-algebra.

2. Monadic m-Generalized Łukasiewicz Algebras of Order n

The class of algebras which is of our concern now, rises from m-generalized Łukasiewicz algebras of order n endowed with a unary operation which verifies certain properties.

Definition 2.1. Let $L \in \mathcal{L}_n^m$. An existential quantifier on L is a mapping $\exists : L \to L$ which verifies the identities:

(E_1) $\exists 0 = 0$,

(E_2) $x \wedge \exists x = x$,

(E_3) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$,

(E_4) $\exists D_i(\overline{x}) = D_i(\exists\overline{x})$ for all i, $1 \leq i \leq n-1$,

(E_5) $\exists f^{2m-1}\exists x = f^{2m-1}\exists x$.

DEFINITION 2.2. Let $L \in \boldsymbol{\mathcal{L}}_n^m$. A universal quantifier on L is a mapping $\forall : L \to L$ verifying the following conditions:

(U_1) $\forall 1 = 1$,

(U_2) $\forall x = \forall x \wedge x$,

(U_3) $\forall(x \vee \forall y) = \forall x \vee \forall y$,

(U_4) $\forall D_i(\overline{x}) = D_i(\forall\overline{x})$ for all i, $1 \leq i \leq n-1$,

(U_5) $\forall f^{2m-1}\forall x = f^{2m-1}\forall x$.

Propositions 2.1 and 2.3 summarize the most important properties of both existential and universal quantifiers in L_n^m-algebras which are necessary for further development.

PROPOSITION 2.1. *Let $L \in \boldsymbol{\mathcal{L}}_n^m$ and $\exists$ be an existential quantifier on L. Then the following properties are satisfied:*

(E_6) $\exists 1 = 1$,

(E_7) $\exists\exists x = \exists x$,

(E_8) $x \leq y$ *implies* $\exists x \leq \exists y$,

(E_9) $\exists(\exists x \wedge \exists y) = \exists x \wedge \exists y$,

(E_{10}) $x \in \exists L$ *if and only if* $\exists x = x$,

(E_{11}) $x \in \exists L$ *implies* $f^{2m-j}x \in \exists L$, *for all* j, $1 \leq j \leq 2m$,

(E_{12}) $x, y \in \exists L$ *imply* $x \vee y \in \exists L$,

(E_{13}) $\exists(x \vee y) = \exists x \vee \exists y$,

(E_{14}) $\exists L \lhd L$,

(E_{15}) $x \in B(S(L))$ *implies* $\exists x \in B(S(L))$,

(E_{16}) $x \in S(L)$ *implies* $\exists x \in S(L)$,

(E_{17}) $S(\exists L) = \exists S(L)$.

PROOF. We only prove (E_{11}), (E_{12}), (E_{16}) and (E_{17}).

(E_{11}): By the hypothesis, (E_{10}) and (E_5) we have that $f^{2m-1}x \in \exists L$. This assertion and (E_5) imply that $f^{2m-2}x = f^{2m-1}f^{2m-1}x \in \exists L$. Hence, following an analogous reasoning we conclude that $f^{2m-j}x \in \exists L$, for all j, $1 \leq j \leq 2m$.

(E_{12}): Let $x, y \in \exists L$. Then, by (E_{10}) and (E_5) it follows that $f^{2m-1}(x) \wedge f^{2m-1}(y) \in \exists L$. So, by (E_3) and (E_{11}) we have that $x \vee y = f(f^{2m-1}(x) \wedge f^{2m-1}(y)) \in \exists L$.

(E_{16}): Let us suppose that $\exists x \notin S(L)$. Hence, there is $y \in L \setminus S(L)$ such that $y = \exists x$. By the hypothesis and (E_2) we have that $x < f^2 y$ which implies that $x \leq \underline{y}$. From this assertion and (E_8) we infer that $\exists x \leq \exists \underline{y}$. On the other hand, by (E_{14}) and the fact that $y \in \exists L$ we conclude that $\exists \underline{y} = \underline{y}$. Therefore, $y < \underline{y}$ which is a contradiction.

(E_{17}): Let $x \in \exists S(L)$. Hence, there is $y \in S(L)$ such that $x = \exists y$. Taking into account (E_{16}) we have that $f^2 x = f^2 \exists y = \exists y = x$ and so $x \in S(\exists L)$. The converse follows immediately. ■

Next we show the relationship between existential and universal quantifiers in $\mathcal{L}_n^m$.

Proposition 2.2. *Let $L \in \mathcal{L}_n^m$. If $\exists$ is an existential quantifier on L, then $\forall x = f^{2m-1} \exists f x$ is a universal quantifier.*

Proof. We only prove (U_4). Indeed, taking into account (GL_5), (GL_{20}), (E_4) and (E_{16}) we have that: $\forall D_i \overline{x} = f^{2m-1} D_{n-i} \exists f \overline{x} = D_i f^{2m-1} \exists f \overline{x} = D_i \forall \overline{x}$. ■

Proposition 2.3. *Let $L \in \mathcal{L}_n^m$ and $\forall$ be a universal quantifier on L. Then the following properties hold:*

(U_6) $\forall 0 = 0$,

(U_7) $\forall\forall x = \forall x$,

(U_8) $x \leq y$ implies $\forall x \leq \forall y$,

(U_9) $\forall (x \wedge y) = \forall x \wedge \forall y$,

(U_{10}) $x \in \forall L$ if and only if $x = \forall x$,

(U_{11}) $x \in \forall L$ implies $f^{2m-j} x \in \forall L$, for all j, $1 \leq j \leq 2m$,

(U_{12}) $\forall L \lhd L$.

(U_{13}) $x \in S(L)$ implies $\forall x \in S(L)$.

Proof. It is routine. ■

Proposition 2.4. *Let $L \in \mathcal{L}_n^m$. If $\forall$ is a universal quantifier on L, then $\exists x = f \forall f^{2m-1} x$ is an existential quantifier.*

Proof. Properties (E_1)–(E_3) and (E_5) follow immediately. Therefore, it only remains to prove (E_4). Indeed, bearing in mind (GL_5), (GL_{20}) and (U_4) we have that $\exists D_i \overline{x} = f \forall f^{2m-1} D_i \overline{x} = f \forall D_{n-i} f^{2m-1} \overline{x} = f \forall D_{n-i} \overline{f^{2m-1} \overline{x}} = f D_{n-i} \forall f^{2m-1} \overline{x} = D_i f \forall f^{2m-1} \overline{x} = D_i \exists \overline{x}$. ■

Definition 2.3. Let $L \in \mathcal{L}_n^m$. A monadic m-generalized Łukasiewicz algebra of order n (or ML_n^m-algebra) is a pair $(L, \exists)$, where $\exists$ is an existential quantifier on L or equivalently, is a pair $(L, \forall)$, where $\forall$ is a universal quantifier on L.

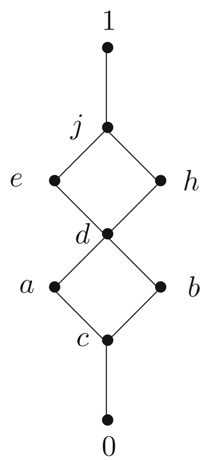
In what follows we will denote by $\mathcal{ML}_n^m$ the variety of ML_n^m-algebras.

Some of the results on ML_n^m-algebras given in this paper were communicated in the meeting indicated in [16].

Let us observe that by defining $\exists x = x$ on every L_n^m-algebra L we have that $(L, \exists) \in \mathcal{ML}_n^m$. In Example 2.1 we show an ML_n^m-algebra where the quantifier is not the trivial one.

Example 2.1. Let us consider the L_5^2-algebra L represented bellow and where the operations f, D_i, $1 \leq i \leq 4$, $\exists$ and $\forall$ are defined as follows:

x	0	a	b	c	d	e	h	j	1
fx	1	h	e	j	d	a	b	c	0
D_1x	0	1	1	1	1	1	1	1	1
D_2x	0	1	1	0	1	1	1	1	1
D_3x	0	0	0	0	0	1	1	1	1
D_4x	0	0	0	0	0	0	0	0	1
$\exists x$	0	d	d	c	d	j	j	j	1
$\forall x$	0	c	c	c	d	d	d	j	1



Furthermore, in this case it holds that $\forall x = f\exists fx$ for all $x \in L$. However, this is not true if we define $\exists x = x = \forall x$. Indeed, $f\exists fa = b \neq a = \forall a$.

Remark 2.1. Let $L \in \mathcal{ML}_n^m$. Bearing in mind the above results we can assert that $\exists$ is an additional closure operator on L [3]. Furthermore, $\mathcal{ML}_n^1$ coincides with the variety of monadic n-valued Łukasiewicz algebras.

Other properties of the quantifiers on ML_n^m-algebras are indicated in Proposition 2.5. The proof of them is an easy exercise.

Proposition 2.5. *Let L be an ML_n^m-algebra. Then the following properties are satisfied:*

(M_1) $\forall\exists x = \exists x$,

(M$_2$ $\exists\forall x = \forall x$,

(M$_3$) $x = \forall x$ *if and only if* $x = \exists x$,

(M$_4$) $\exists(x \wedge \forall y) = \exists x \wedge \forall y$,

(M$_5$) $\forall(x \vee \exists y) = \forall x \vee \exists y$.

Proposition 2.6 determines the relationship between the existential quantifiers and special subalgebras of L_n^m-algebras.

PROPOSITION 2.6. *Let* L *be an* L_n^m*-algebra.*

(a) A subset M *of* L *is of the form* $M = \exists L$ *where* $\exists$ *is an existential quantifier, if and only if the following conditions hold:*

(i) M *is a Moore family of* L,

(ii) M *is a subalgebra of* L,

(iii) for any $a, b \in M$, $a \Rightarrow b$ *exists in* L *and* $a \Rightarrow b \in M$, *where* $x \Rightarrow y$ *stands for the relative pseudocomplement of* x *with respect to* y,

(iv) $D_i(\bigwedge\{z \in \exists L : \overline{x} \leq z\}) = \bigwedge\{z \in \exists L : D_i x \leq z\}$ *for all* i, $1 \leq i \leq n-1$.

(b) When this is the case, $\exists$ *is uniquely determined by* $\exists x = \bigwedge\{z \in M : x \leq z\}$.

PROOF. Let $\exists$ be an existential quantifier on L and $M = \exists L$. Then [3, Theorem 2.4.11]) implies (i) and (b). Besides (iv) follows from (E_4). To prove (ii) we use (E_{10}), which shows that $0 \in \exists L$ by (E_1) and $1 \in \exists L$ by (E_6), while if $x, y \in \exists L$ then $x \wedge y \in \exists L$ and $x \vee y \in \exists L$ follow from (E_3) and (E_{13}) respectively. Furthermore, to prove (iii) suppose that $x, y \in \exists L$ and $x \Rightarrow y \in L$. Then taking into account (E_{10}), (E_3), the fact that $x \Rightarrow y$ is the relative pseudocomplement of x with respect to y, and (E_8) we have that $x \wedge \exists(x \Rightarrow y) = \exists x \wedge \exists(x \Rightarrow y) = \exists(\exists x \wedge (x \Rightarrow y)) = \exists(x \wedge (x \Rightarrow y)) \leq \exists y = y$. From this last assertion and (E_2) we conclude that $x \Rightarrow y = \exists(x \Rightarrow y)$.

Conversely, suppose (i)–(iv) hold and define $\exists$ by (b). Then (E_1) follows from (ii) and (E_4) from (iv) while $\exists$ is a closure operator by (i), therefore (E_2) also holds. This assertion implies that $x \wedge \exists y \leq \exists x \wedge \exists y$. By virtue of (ii) it results that $\exists x \wedge \exists y \in M$. Hence, $\exists(x \wedge \exists y) \leq \exists x \wedge \exists y$. On the other hand, if $k \in M$ verifies that $x \wedge \exists y \leq k$, then $x \leq \exists y \Rightarrow k$. Furthermore, from (iii) we infer that $\exists y \Rightarrow k \in M$. Therefore, $\exists x \leq \exists y \Rightarrow k$ and so, $\exists x \wedge \exists y \leq k$. Thus, $\exists x \wedge \exists y \leq \exists(x \wedge \exists y)$ and consequently, we conclude that (E_3) holds. Finally, taking into account (ii) we have that $f^{2m-1}\exists x \in M$, Hence by (b) $\exists f^{2m-1}\exists x \leq f^{2m-1}\exists x$, and so by (E_2) we obtain (E_5). ■

3. Congruences and Subdirectly Irreducible $\mathbf{ML^{m_n}}$-Algebras

Now, we will describe the congruence lattice of ML_n^m-algebras taking into account the results established in [11].

Let us recall that in any L_n^m-algebra L the congruences were characterized by means of the m-filters of L, that is to say the filters F of L which verify this condition: $x \in F$ implies $fD_1fx \in F$. In what follows, we will denote by $\mathcal{F}_m(L)$ the set of all m-filters of L.

Definition 3.1. Let $L \in \mathcal{ML}_n^m$. An m-filter F of L is monadic (or M-filter) if it verifies this condition: $x \in F$ implies $\forall x \in F$.

We will denote by $\mathcal{F}_M(L)$ the set of all monadic filters of L.

Theorem 3.1. *Let $(L, \exists)$ be an ML_n^m-algebra with more than one element. Then*

(i) $Con(L) = \{R(F) : F \in \mathcal{F}_M(L)\}$, *where* $R(F) = \{(x, y) \in L \times L :$ *there exists* $w \in F$ *such that* $x \wedge D_{n-1}w = y \wedge D_{n-1}w\}$,

(ii) *the lattices $Con(L)$ and $\mathcal{F}_M(L)$ are isomorphic considering the mappings $\theta \longmapsto [1]_\theta$ and $F \longmapsto R(F)$ which are mutually inverse, where $[x]_\theta$ stands for the equivalence class of x modulo θ.*

Proof. It only remains to prove that if $(x, y) \in R(F)$ then $(\exists x, \exists y) \in R(F)$. Suppose that there is $w \in F$ such that $x \wedge D_{n-1}w = y \wedge D_{n-1}w$. Hence, by (U_8), (GL_{11}), (GL_8) and (U_4) we have that $x \wedge \forall D_{n-1}\overline{w} = y \wedge \forall D_{n-1}\overline{w}$. This assertion, (M_4) and (U_4) imply that $\exists x \wedge D_{n-1}\forall\overline{w} = \exists y \wedge D_{n-1}\forall\overline{w}$. Furthermore, taking into account that $w \leq \overline{w}$ and the fact that F is a monadic filter of L we infer that $w_1 = \forall\overline{w} \in F$ and so $(\exists x, \exists y) \in R(F)$, which completes the proof. ■

Next, our attention is focused on characterizing subdirectly irreducible ML_n^m-algebras. To this end recall that a filter F of an ML_n^m-algebra L is a Stone filter of L if it verifies $x \in F$ implies $D_{n-1}x \in F$ [31, p 295]. We will denote by $\mathcal{F}_S(L)$ the set of all Stone filters of L. On the other hand, for each $(L, \exists) \in \boldsymbol{\mathcal{ML}}_n^m$, let us consider the L_n^m-algebra $\exists L$, the monadic Łukasiwicz algebra of order n $(S(L), \exists)$ and the Łukasiewicz algebra of order n $\exists S(L)$. Then by defining the mappings:

$\alpha_1 : \mathcal{F}_M(L) \to \mathcal{F}_m(\exists L),\ \alpha_1(F) = F \cap \exists L,$
$\alpha_2 : \mathcal{F}_M(L) \to \mathcal{F}_{MS}(S(L)),\ \alpha_2(F) = F \cap S(L),$
$\alpha_3 : \mathcal{F}_m(\exists L) \to \mathcal{F}_S(\exists S(L)),\ \alpha_3(F) = F \cap \exists(S(L)),$
$\alpha_4 : \mathcal{F}_{MS}(S(L)) \to \mathcal{F}_S(\exists S(L)),\ \alpha_4(F) = F \cap \exists(S(L)),$

where $\mathcal{F}_{MS}(S(L))$ is the set of the monadic Stone filters of $(S(L), \exists)$, we have the following result:

Theorem 3.2. *Let L be an ML_n^m-algebra. Then the mappings α_1, α_2, α_3 and α_4 are order isomorphisms where $\mathcal{F}_M(L)$, $\mathcal{F}_m(\exists L)$, $\mathcal{F}_{MS}(S(L))$ and $\mathcal{F}_S(\exists S(L))$ are ordered by set inclusion. Besides, $\alpha_3 \circ \alpha_1 = \alpha_4 \circ \alpha_2$.*

Proof. From Remark 1.1, (E_7), (E_9) (E_{10}), (E_{14}), (E_{16}),(E_{17}), (U_{13}), [11, Proposition 2.12], [31, Proposition 3.5], [12, Corollary 3.3] and by applying standard techniques we infer that α_i, $1 \leq i \leq 4$ are isomorphisms. Finally, it is straightforward to show that $\alpha_3 \circ \alpha_1 = \alpha_4 \circ \alpha_2$. ■

Now, by Theorems 3.1 and 3.2 we are ready to characterize subdirectly irreducible ML_n^m-algebras as follows:

Theorem 3.3. *Let $(L, \exists) \in \boldsymbol{\mathcal{ML}}_n^m$. Then the following conditions are equivalent:*

(i) $(L, \exists)$ *is simple,*

(ii) $\exists L$ *is a simple L_n^m–algebra,*

(iii) $\exists S(L)$ *is a simple n-valued Łukasiewicz algebra,*

(iv) $B(\exists S(L)) = \{0, 1\}$

(v) $(L, \exists)$ *is subdirectly irreducible.*

Remark 3.1. Example 2.1 and Theorem 3.3 allow us to assert that it is possible to define more than one existential quantifier on a simple L_n^m-algebra. This shows a fundamental difference with simple L_n-algebras.

As a direct consequence of Theorem 3.3 and from well-know results of universal algebra we conclude that

Corollary 3.1. *Let $\boldsymbol{\mathcal{ML}}_n^m$ is semisimple.*

In order to determine the principal congruences on ML_n^m-algebras in a simple way we will start by pointing out some results established in [31] and [11], which will be fundamental in what follows. M. Sequeira obtained some unpublished results in the context of congruences on algebras of certain subvarieties of Ockham algebras some of which are $\mathcal{K}_{m,0}$. Bearing in mind these notions J. Vaz de Carvalho [31] considered certain elements which we will describe in what follows. Let $L \in \mathcal{L}_n^m$ and $T = \{0, 1, \ldots, m-1\}$. For each $z \in L$ and $s \in \{1, \ldots, m\}$ take

$$q_s z = \bigwedge_{\substack{J \subseteq T \\ |J| = s}} \bigvee_{j \in J} f^{2j} z.$$

where $| X |$ means the cardinal number of the set X.

The same author asserted that it is straightforward to see the following statements.

LEMMA 3.1. [31] *Let $L \in \mathcal{L}_n^m$. Then it holds:*

(i) $q_s z \in S(L)$, $s \in \{1, \dots, m\}$,

(ii) $q_s z \leq q_{s+1} z$, $s \in \{1, \dots, m-1\}$,

(iii) $q_1 z = \underline{z}$ *and* $q_m z = \overline{z}$,

(iv) $z \in S(L)$ *implies* $q_s z = z$, $s \in \{1, \dots, m\}$,

(v) $x \leq z$ *implies* $q_s x \leq q_s z$, $s \in \{1, \dots, m\}$.

On the other hand, in [11] we introduced a new binary operation on L_n^m-algebras which we called weak implication as follows: $x \to y = D_1 f x \vee y$. This implication allowed us to define an element which played a central role to characterize the principal L_n^m-congruences. Let $L \in \mathcal{L}_n^m$ and $a, b \in L$. Then

$$w_{a,b} = \bigwedge_{i=1}^{n-1} ((D_i a \to D_i b) \wedge (D_i b \to D_i a)).$$

LEMMA 3.2. *Let $L \in \mathcal{L}_n^m$ and $a, b \in L$. Then the following properties are satisfied:*

(i) $w_{a,b} = \bigwedge_{i=1}^{n-1} ((f D_i a \wedge f D_i b) \vee (D_i a \wedge D_i b))$,

(ii) $D_j w_{a,b} = w_{a,b}$, *for all* j, $1 \leq j \leq n-1$,

(iii) $w_{a,b} \in S(L)$,

(iv) $D_j(a \wedge w_{a,b}) = D_j(b \wedge w_{a,b})$, *for all* j, $1 \leq j \leq n-1$,

(v) $a \wedge \bigwedge_{s=1}^{m} w_{q_s(a \wedge b), q_s(a \vee b)} = b \wedge \bigwedge_{s=1}^{m} w_{q_s(a \wedge b), q_s(a \vee b)}$,

(vi) $D_{n-1} \bigwedge_{s=1}^{m} \forall w_{q_s(a \wedge b), q_s(a \vee b)} = \bigwedge_{s=1}^{m} \forall w_{q_s(a \wedge b), q_s(a \vee b)}$.

PROOF. From [11, Lemma 2.9] and [15, Proposition 1.3]) it follows (i)-(iv) and (v) respectively.

(vi) Taking into account (U_9), (i) in Lemma 3.1, (U_4) and (GL_2) we have that $D_{n-1} \bigwedge_{s=1}^{m} \forall w_{q_s(a \wedge b), q_s(a \vee b)} = \forall \bigwedge_{s=1}^{m} D_{n-1} w_{q_s(a \wedge b), q_s(a \vee b)}$. Hence, by (ii) and (U_9) we conclude the proof. ∎

Let $L \in \boldsymbol{\mathcal{ML}}_n^m$ and $z \in L$. We will denote by $[z)$ the principal filter of L generated by z (i.e.: $[z) = \{x \in L : z \leq x\}$).

Proposition 3.1. *Let* $L \in \boldsymbol{\mathcal{ML}}_n^m$*, k be a positive integer and* $a_j, b_j \in L$*,* $1 \leq j \leq k$*. Then* $[\bigwedge_{j=1}^{k} \forall w_{a_j,b_j})$ *is an M-filter of L.*

Proof. It is a direct consequence of [11, Proposition 2.6], [11, Proposition 2.10], (U_9) and (U_7). ■

Now, Theorem 3.4 gives us a new characterization of simple ML_n^m-algebras.

Theorem 3.4. *Let* $L \in \boldsymbol{\mathcal{ML}}_n^m$*. Then the following conditions are equivalent:*

(i) *L is simple,*

(ii) $w_{D_1\forall\overline{a},1} = 1$ *for all* $a \in L$ *such that* $\forall a \neq 0$*.*

Proof. (i) ⇒ (ii): Let $F = [w_{D_1\forall\overline{a},1})$. By Remark 1.1, items (iii) and (ii) in Lemma 3.2, (U_8) and (U_4) it follows that F is an M-filter. Hence, by Theorem 3.1 we have that $R(F) \in Con(L)$ and $F = [1]_{R(F)}$ and so, $(w_{D_1\forall\overline{a},1}, 1) \in R(F)$. If $w_{D_1\forall\overline{a},1} = 0$ then $D_1\forall\overline{a} = 0$ which implies by (GL_7) and (GL_8) that $\forall a = 0$ and also that $L/R(F) = L \times L$. Since L is simple, the only congruences on L are the trivial ones. Therefore, $L/R(F) = id_L$ and $w_{D_1\forall\overline{a},1} = 1$ for all $a \in L$ such that $\forall a \neq 0$.

(ii) ⇒ (i): Let $\theta \in Con(L)$, $\theta \neq id_L$. Then, there are $x, y \in L$, $x \neq y$ such that $(x, y) \in \theta = R([1]_\theta)$. By Theorem 3.1 it follows that there is $v \in [1]_\theta$ and $x \wedge D_{n-1}\forall\overline{v} = y \wedge D_{n-1}\forall\overline{v}$. These assertions and (GL_{18}) imply that $\forall\overline{v} \neq 1$. Hence, by (U_{10}) and (U_{12}) we have that $\forall f\forall\overline{v} = f\forall\overline{v} \neq 0$ and so by the hypothesis we conclude that $w_{D_1\forall f\forall\overline{v},1} = 1$. On the other hand, taking into account that $(v, 1) \in \theta$ we infer that $(w_{D_1\forall f\forall\overline{v},1}, 0) \in \theta$. Therefore, $(0, 1) \in \theta$ which implies that $\theta = L \times L$. ■

Next, we will apply the results we developed so far to show that $\boldsymbol{\mathcal{ML}}_n^m$ is discriminator variety. Furthermore, we will determine the principal congruences. It what follows, for each $a, b \in L$ we will denote by $\theta(a, b)$ the principal congruence generated by (a, b).

Recall that the ternary discriminator function t on a set A is defined by the conditions:

$$t(x, y, z) = \begin{cases} z \ if \ x = y, \\ x \ otherwise. \end{cases}$$

A variety $\mathcal{V}$ is a discriminator variety, if it has a polynomial p that coincides with the ternary discriminator function on each subdirectly irreducible member of $\mathcal{V}$; such a polynomial is called ternary discriminator polynomial for $\mathcal{V}$.

Theorem 3.5. *The variety $\boldsymbol{\mathcal{ML}}_n^m$ is a discriminator variety.*

Proof. Let $p(x,y,z) = (\bigwedge_{s=1}^m \forall w_{q_s(x\wedge y), q_s(x\vee y)} \wedge z) \vee (f \bigwedge_{s=1}^m \forall w_{q_s(x\wedge y), q_s(x\vee y)} \wedge x)$. In view of the definition of $w_{a,b}$ for all $a, b \in L$, (GL_{13}) and (U_1) we have that $p(x,y,z) = z$. On the other hand, by Lemma 3.2 (v) we infer that $x \wedge \bigwedge_{s=1}^m \forall w_{q_s(x\wedge y), q_s(x\vee y)} = y \wedge \bigwedge_{s=1}^m \forall w_{q_s(x\wedge y), q_s(x\vee y)}$. Hence, if $x \neq y$ we have that $\bigwedge_{s=1}^m \forall w_{q_s(x\wedge y), q_s(x\vee y)} \neq 1$. Furthermore, by Lemma 3.2 (iii) and (M_3) follow that $\bigwedge_{s=1}^m \forall w_{q_s(x\wedge y), q_s(x\vee y)} \in \exists S(L)$. From this assertion, Lemma 3.2 (vi), the fact that $\exists S(L)$ is an n-valued Łukasiewicz algebra and ([6, Theorem 1.9]) we conclude that $\bigwedge_{s=1}^m \forall w_{q_s(x\wedge y), q_s(x\vee y)} \in B(\exists S(L))$. So, by Theorem 3.3 we have that $\bigwedge_{s=1}^m \forall w_{q_s(x\wedge y), q_s(x\vee y)} = 0$. Therefore, $p(x,y,z) = x$. ■

Corollary 3.2.

(i) $\boldsymbol{\mathcal{ML}}_n^m$ *is arithmetic,*

(ii) *for each* $L \in \boldsymbol{\mathcal{ML}}_n^m$ *and* a, b, c, $d \in L$, *it is verified that* $(c,d) \in \theta(a,b)$ *if and only if* $p(a,b,c) = p(a,b,d)$, *i.e.* $\boldsymbol{\mathcal{ML}}_n^m$ *has equationally definable principal congruences,*

(iii) *every principal congruence on* $L \in \boldsymbol{\mathcal{ML}}_n^m$ *is a factor congruence,*

(iv) *the principal congruences on* $L \in \boldsymbol{\mathcal{ML}}_n^m$ *form a sublattice of the lattice* $Con(L)$,

(v) *each compact congruence on* $L \in \boldsymbol{\mathcal{ML}}_n^m$ *is a principal congruence,*

(vi) *the congruences on each* $L \in \boldsymbol{\mathcal{ML}}_n^m$ *are regular, normal and filtral,*

(vii) $L \in \boldsymbol{\mathcal{ML}}_n^m$ *has the congruence extension property.*

Lemma 3.3 will allow us to determine the principal congruences on ML_n^m-algebras.

Lemma 3.3. *Let* $L \in \boldsymbol{\mathcal{ML}}_n^m$. *Then*

(i) $\theta(a,b) = \theta(\bigwedge_{s=1}^m \forall w_{q_s(a\wedge b), q_s(a\vee b)}, 1)$,

(ii) $[1]_{\theta(a,b)} = [\bigwedge_{s=1}^m \forall w_{q_s(a\wedge b), q_s(a\vee b)})$.

Proof. (i) Bearing in mind the definitions of $q_s z$ and $w_{z,t}$ with $z, t \in L$ and the fact that $(a\wedge b, a\vee b) \in \theta(a,b)$ we infer that $(\bigwedge_{s=1}^m \forall w_{q_s(a\wedge b), q_s(a\vee b)}, 1) \in \theta(a,b)$. On the other hand, $(a \wedge \bigwedge_{s=1}^m \forall w_{q_s(a\wedge b), q_s(a\vee b)}, a) \in \theta(\bigwedge_{s=1}^m \forall w_{q_s(a\wedge b), q_s(a\vee b)}, 1)$ and $(b \wedge \bigwedge_{s=1}^m \forall w_{q_s(a\wedge b), q_s(a\vee b)}, a) \in \theta(\bigwedge_{s=1}^m \forall w_{q_s(a\wedge b), q_s(a\vee b)}, 1)$ and so, from Lemma 3.2 (v) it results that $\theta(a,b) \in \theta(\bigwedge_{s=1}^m \forall w_{q_s(a\wedge b), q_s(a\vee b)}, 1)$. Consequently (i) holds.

(ii) Let $x \in [\bigwedge_{s=1}^m \forall w_{q_s(a\wedge b), q_s(a\vee b)})$. Hence, by Theorem 3.1 and (i) we conclude that $x \in [1]_{\theta(a,b)}$. On the other hand, let $x \in [1]_{\theta(a,b)}$. By virtue of item (ii) in Corollary 3.2 we have that $p(a,b,x) = p(a,b,1)$ and so, $(\bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)} \wedge x) \vee (f \bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)} \wedge a) = (\bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)} \wedge 1) \vee (f \bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)} \wedge a)$. Hence, $((\bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)} \wedge x) \vee (f \bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)} \wedge a)) \wedge \bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)} = ((\bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)} \wedge 1) \vee (f \bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)} \wedge a)) \wedge \bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)}$, which implies by Lemma 3.2 (ii) and [6] that $\bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)} \wedge x = \bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)}$ and so, $\bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)} \leq x$. Therefore, $[1]_{\theta(a,b)} \subseteq [\bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)})$. ■

THEOREM 3.6. *Let $L \in \mathcal{ML}_n^m$, and $a, b \in L$. Then* $\theta(a,b) = \{(x,y) \in L^2 : x \wedge \bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)} = y \wedge \bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)}\}$.

PROOF. From Theorem 3.1 and (ii) in Lemma 3.3 we have that $\theta(a,b) = R([1]_{\theta(a,b)}) = R([\bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)})) = \{(x,y) \in L^2 : x \wedge \bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)} = y \wedge \bigwedge_{s=1}^{m} \forall w_{q_s(a\wedge b),q_s(a\vee b)}\}$. ■

4. Functional Representation for $\mathbf{ML_n^m}$-Algebras

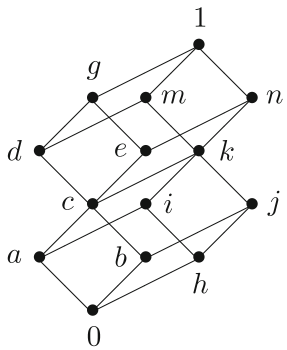
Multipliers have been studied in different branches of mathematic, for example in semilattices and lattices from the point of view of interior operators by Szász [29], Szász and Szendrei [30] and Kolibiar [23], in semigroups by Petrich [26] and Laca and Raeburn [24], in rings by M. Ashraf and A. Shakir ([2]), in f-rings by P. Colville, G. Davis, K. Keimel in [8], in lattices and BCK-algebras by W. H. Cornish in [9,10], in subtraction algebras by Y. H. Yon and K. H. Kim ([32]), and in implicative algebras by J. Cirulis ([7]). On the other hand, in [16] we introduced the concept of multiplier in L_n^m-algebras. In this section, we will extend these last results in order to get a functional representation for monadic L_n^m-algebras, and in particular, for monadic Łukasiewicz algebras of order n.

Let L be an L_n^m-algebra and $I \subseteq L$. Recall that I is a 1-ideal of L if I is an ideal of the lattice L which verifies the condition: $x \in I$ implies $D_1 x \in I$ ([16]).

DEFINITION 4.1. Let $L \in \mathcal{ML}_n^m$. A non-empty set I of L is a q-ideal of L, if I is a 1-ideal of L and verifies the following condition: $x \in I$ implies $\exists x \in I$.

Let us observe that $\{0\}$ and L are q-ideals of L. In what follows we will denote by $\mathcal{I}_q(L)$ the set of all q-ideals of L.

EXAMPLE 4.1. Let L be the L_3^2–algebra whose Hasse diagram and the operations are the following ones:



x	0	a	b	c	d	e	g	h	i	j	k	m	n	1
fx	1	n	m	k	i	j	h	g	e	d	c	a	b	0
D_1x	0	g	g	g	g	g	g	h	1	1	1	1	1	1
D_2x	0	0	0	0	g	g	g	h	h	h	h	1	1	1
$\exists x$	0	a	b	c	d	e	g	h	i	j	k	m	n	1

Then we have that $I = \{0, a, b, c, d, e, g\}$ is a proper q-ideal of L.

Lemma 4.1. *Let $L \in \mathcal{ML}_n^m$ and $I \in \mathcal{I}_q(L)$. If $x \in I$ then $\overline{\exists x} \in I \cap S(\exists L)$.*

Proof. From the hypothesis and Definition 4.1 follow that $D_1 \exists x \in I$. Hence from (Gl_8) and (GL_7) we have that $\overline{\exists x} \in I$, which allows us to conclude the proof. ■

Definition 4.2. Let $L \in \mathcal{ML}_n^m$ and $I \in \mathcal{I}_q(L)$. A q-multiplier on L is a map $h : I \to L$, which verifies the following condition:
$h(e \wedge x) = e \wedge h(x)$, for each $e \in L$ and $x \in I \cap S(\exists L)$.

Let $L \in \mathcal{ML}_n^m$. We will denote by $M_\exists(I, L)$ the set of all q-multipliers having domain $I \in \mathcal{I}_q(L)$ and by $M_\exists(L) = \bigcup_{I \in \mathcal{I}_q(L)} M_\exists(I, L)$ the set of all q-multipliers of L.

Remark 4.1. The maps $\mathbf{0}, \mathbf{1} : L \to L$ defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = x$ for all $x \in L$ are q-multipliers.

Lemma 4.2. *Let $L \in \mathcal{ML}_n^m$ and $I \in \mathcal{I}_q(L)$. If $f : I \to L$ is a q-multiplier of L, then $f(x) \leq x \leq \exists x$ for all $x \in I$.*

Proof. Taking into account Definition 4.2, (E_2) and Lemma 4.1 we have that $f(x) = f(x \wedge \overline{\exists x}) = x \wedge f(\overline{\exists x}) \leq x \leq \exists x$. ■

Lemma 4.3. *Let $L \in \mathcal{ML}_n^m$ and $h, g \in M_\exists(I, L)$. If $h(x) = g(x)$ for all $x \in I \cap S(\exists L)$, then $h(x) = g(x)$ for all $x \in I$.*

Proof. It follows by Lemma 4.1, Definition 4.2 and the hypothesis. ■

Proposition 4.1. *Let $L \in \mathcal{ML}_n^m$, $a \in L$ and $I \in \mathcal{I}_q(L)$. Then, the map $h_a : I \to L$ defined by $h_a(x) = a \wedge x$ for every $x \in I$, is a q-multiplier of L.*

Proof. It is routine. ■

The map h_a defined in Proposition 4.1 is called principal q-multiplier. Besides, if $dom\, h_a = L$ we will denote it by h_a^t.

Definition 4.3. Let $L \in \mathcal{ML}_n^m$. A non-empty set $R \subseteq L$ is regular, if for all $x, y \in L$ such that $x \wedge \overline{r} = y \wedge \overline{r}$ for all $r \in R$, then $x = y$.

We will denote by $\mathcal{R}(L) = \{R \subseteq L$: R is a regular subset of $L\}$. Let us observe that $\exists L \in \mathcal{R}(L)$. More generally, every subset of L which contains 1 is regular.

Lemma 4.4. *If $R, T \in \mathcal{I}_q(L) \cap R(L)$, then $R \cap T \in \mathcal{I}_q(L) \cap R(L)$.*

Proof. It is routine. ■

Proposition 4.2. *Let $L \in \mathcal{ML}_n^m$ and $f_1, f_2 \in M_\exists(L)$ such that $f_1 : I_1 \to L$ y $f_2 : I_2 \to L$. Then, the operations defined as follows belong to $M_\exists(L)$:*

- $f_1 \wedge f_2 : I_1 \cap I_2 \to L$, $(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x)$,
- $f_1 \vee f_2 : I_1 \cap I_2 \to L$, $(f_1 \vee f_2)(x) = f_1(x) \vee f_2(x)$,
- $f_j^* : I_j \to L$, $f_j^*(x) = x \wedge N f_j(D_1 x)$, *for all* $j = 1, 2$,
- $\widetilde{D}_i f_j : I_j \to L$, $\widetilde{D}_i f_j(x) = x \wedge D_i f_j(D_1 x)$, *for all* $j = 1, 2$,
- $\widetilde{\forall} f_j : I_j \to L$, $\widetilde{\forall} f_j(x) = x \wedge \forall f_j(\exists x)$, *for all* $j = 1, 2$.

Proof. Following an analogous reasoning to that of [16, Proposition 5], it only remains to prove that $\widetilde{\forall} f_j$ is a q-multiplier of L. Indeed, let $e \in L$ and $x \in I \cap S(\exists L)$, then by (E_3) we deduce that $\widetilde{\forall} f_j(e \wedge x) = e \wedge x \wedge \forall f_j(\exists x \wedge \exists e)$. Taking into account that $f_j \in M_\exists(I, L)$ and (U_9) we have that $\widetilde{\forall} f_j(e \wedge x) = e \wedge x \wedge \exists e \wedge \forall f_j \exists x$. Thus, by (M_1) we conclude the proof. ■

Proposition 4.3. *Let $\langle L, \vee, \wedge, N, \{D_i\}_{1 \leq i \leq n-1}, 0, 1\rangle$ be an L_n^m-algebra. Then $\langle M_\exists R(L), \vee, \wedge, ^*, \{\widetilde{D}_i\}_{1 \leq i \leq n-1}, \widetilde{\forall}, \mathbf{0}, \mathbf{1}\rangle$ is an ML_n^m-algebra, where $M_\exists R(L) = \{f \in M_\exists(L) : dom\, f \in q\mathcal{I}_1(L) \cap \mathcal{R}(L)\}$ and the operations are the ones defined in Proposition 4.2 and Remark 4.1.*

Proof. Taking into account [16, Proposition 5], Proposition 4.2 and Lemma 4.4, it only remains to prove that $\widetilde{\forall}$ verifies Definition 2.2. We only prove (U_3), (U_4) and (U_5).

(U_3): For all $f_1, f_2 \in M_\exists R(L)$ such that $f_1 : I_1 \to L$, $f_2 : I_2 \to L$ and for all $x \in I_1 \cap I_2$, using in turn (E_7), Lemma 4.2, (U_9), (M_1), (E_2) and (U_3)

we have that

$$\begin{aligned}(\widetilde{\forall}(f_1 \vee \widetilde{\forall} f_2))(x) &= x \wedge \forall(f_1(\exists x) \vee (\exists x \wedge \forall f_2(\exists x)))\\ &= x \wedge \forall(\exists x \wedge (f_1(\exists x) \vee \forall f_2(\exists x)))\\ &= x \wedge \forall(\exists x) \wedge \forall(f_1(\exists x) \vee \forall f_2(\exists x))\\ &= x \wedge (\forall f_1(\exists x) \vee \forall f_2(\exists x))\\ &= (\widetilde{\forall} f_1)(x) \vee (\widetilde{\forall} f_2)(x).\end{aligned}$$

(U_4): For all $f \in M_{\exists}R(L)$, $f : I \to L$ and for all $x \in I \cap S(\exists L)$, taking into account (U_9), (M_1) and (E_2) we get

$$\begin{aligned}(\widetilde{\forall}\widetilde{D}_i\overline{f})(x) &= x \wedge \forall \widetilde{D}_i\overline{f}(\exists x)\\ &= x \wedge \forall(\exists x \wedge D_i\overline{f}(D_1(\exists x)))\\ &= x \wedge \exists x \wedge \forall D_i\overline{f}(D_1(\exists x))\\ &= x \wedge D_i\forall\overline{f}(D_1(\exists x)).\end{aligned}$$

On the other hand, bearing in mind (GL_2), (GL_{26}), (GL_6), (GL_7) and (E_4) we have that

$$\begin{aligned}(\widetilde{D}_i\widetilde{\forall}\overline{f})(x) &= x \wedge D_i\widetilde{\forall}\overline{f}(D_1 x)\\ &= x \wedge D_i(D_1 x \wedge \forall\overline{f}(\exists D_1 x))\\ &= x \wedge D_i D_1 x \wedge D_i\forall\overline{f}(\exists D_1 x)\\ &= x \wedge D_i\forall\overline{f}(\exists D_1 x)\\ &= x \wedge D_i\forall\overline{f}(D_1 \exists x).\end{aligned}$$

From the above assertions it follows that $(\widetilde{\forall}\widetilde{D}_i\overline{f})(x) = (\widetilde{D}_i\widetilde{\forall}\overline{f})(x)$ for all $x \in I \cap S(\exists L)$. Hence, by Lemma 4.3, $(\widetilde{\forall}\widetilde{D}_i\overline{f})(x) = (\widetilde{D}_i\widetilde{\forall}\overline{f})(x)$ for all $x \in I$.

(U_5): For all $f \in M_{\exists}R(L)$, $f : I \to L$ and for all $x \in I \cap S(\exists L)$, using (U_9), (M_1), (E_2), (E_{14}), (M_3), (U_3), we have that

$$\begin{aligned}(\widetilde{\forall}(\widetilde{\forall} f)^{*_{2m-1}})(x) &= x \wedge \forall(\widetilde{\forall} f)^{*_{2m-1}}(\exists x)\\ &= x \wedge \forall(\exists x \wedge N^{2m-1}(\widetilde{\forall} f)(D_1 \exists x))\\ &= x \wedge \forall(\exists x \wedge N^{2m-1}(D_1 \exists x \wedge \forall f(\exists D_1 \exists x))\\ &= x \wedge \forall(\exists x \wedge (N^{2m-1} D_1 \exists x \vee N^{2m-1}\forall f(\exists D_1 \exists x))\\ &= x \wedge \forall(N^{2m-1} D_1 \exists x \vee N^{2m-1}\forall f(\exists D_1 \exists x))\\ &= x \wedge \forall(\forall N^{2m-1} D_1 \exists x \vee N^{2m-1}\forall f(\exists D_1 \exists x))\\ &= x \wedge (\forall N^{2m-1} D_1 \exists x \vee N^{2m-1}\forall f(\exists D_1 \exists x))\\ &= x \wedge (N^{2m-1} D_1 x \vee N^{2m-1}\forall f(\exists D_1 x))\end{aligned}$$

$$= x \wedge N^{2m-1}(D_1 x \wedge \forall f(\exists D_1 x))$$
$$= x \wedge N^{2m-1}(\widetilde{\forall} f)(D_1 x)$$
$$= ((\widetilde{\forall} f)^{*_{2m-1}})(x), \text{for all}\, x \in I \cap S(\exists L).$$

Hence, by Lemma 4.3, $(\widetilde{\forall}(\widetilde{\forall} f)^{*_{2m-1}})(x) = ((\widetilde{\forall} f)^{*_{2m-1}})(x)$ for all $x \in I$. ■

Now, in order to obtain a functional representation for monadic m-generalized Łukasiewicz algebras of order n we define a binary relation ρ on $M_{\exists}R(L)$ as follows:

$$(h_1, h_2) \in \rho \Leftrightarrow h_1(x) = h_2(x) \ \textit{for all} \ x \in \textit{dom}(h_1) \cap \textit{dom}(h_2).$$

Lemma 4.5. *ρ is a congruence on the ML_n^m-algebra $M_{\exists}R(L)$.*

Proof. It is routine. ■

Let $h \in M_{\exists}R(L)$ with $\textit{dom}\, h = I$. Then $[I, h]$ and $qL_{\mathcal{M}}$ denote the congruence class of h relative to ρ and the quotient algebra $M_{\exists}R(L)/\rho$, respectively.

Remark 4.2. Let L be an ML_n^m-algebra and $h \in M_{\exists}R(L)$ with $\textit{dom}\, h = I$. Let us observe that for all $a \in L$ we have that $[L, h_a^t] = [I, h_a]$.

Proposition 4.4. *$qL_{\mathcal{M}}$ is an ML_n^m-algebra, where the operations are defined for all $[I_1, h_1], [I_2, h_2] \in qL_{\mathcal{M}}$ by:*

- $[I_1, h_1] \vee [I_2, h_2] = [I_1 \cap I_2, h_1 \vee h_2]$,
- $[I_1, h_1] \wedge [I_2, h_2] = [I_1 \cap I_2, h_1 \wedge h_2]$,
- $[I_1, h_1]^* = [I_1, h_1^*]$,
- $\widetilde{D_i}[I_1, h_1] = [I_1, \widetilde{D_i} h_1]$ *for each* i, $1 \leq i \leq n-1$,
- $\widetilde{\forall}[I_1, h_1] = [I_1, \widetilde{\forall} h_1]$,
- $\mathbf{0} = [L, \mathbf{0}]$ *y* $\mathbf{1} = [L, \mathbf{1}]$.

Proof. It follows from Proposition 4.3 and Lemma 4.5. ■

Theorem 4.1. *Let $L \in \mathcal{ML}_n^m$ and $\overline{v} : L \to qL_{\mathcal{M}}$ be the function defined by $\overline{v}(a) = [L, h_a^t]$ for all $a \in L$. Then,*

(i) $\overline{v}$ is a monomorphism in $\mathcal{ML}_n^m$,

(ii) $\overline{v}(L) \in \mathcal{R}(qL_{\mathcal{M}})$.

Proof. (i) To prove de injectivity of $\overline{v}$ let us consider $a, b \in L$ such that $\overline{v}(a) = \overline{v}(b)$. Hence, $h_a^t(x) = a \wedge x = b \wedge x = h_b^t(x)$ for every $x \in L$. Therefore, by choosing $x = 1$ we obtain that $a = b$. Now, following an analogous reasoning of [16, Lemma 20], we only show that $\overline{v}(\forall a) = \widetilde{\forall}(\overline{v}(a))$ to complete the proof. Taking into account (U_9), (M_1) and (E_2) we infer that $(\widetilde{\forall} h_a^t)(x) = x \wedge \forall h_a^t(\exists x) = x \wedge \forall(a \wedge \exists x) = x \wedge \forall a \wedge \exists x = x \wedge \forall a = h_{\forall a}^t(x)$. Hence, from this last assertion, the definition of $\overline{v}$ and Proposition 4.4 we have that $\overline{v}(\forall a) = [L, h_{\forall a}^t] = [L, \widetilde{\forall}(h_a^t)] = \widetilde{\forall}[L, h_a^t] = \widetilde{\forall}(\overline{v}(a))$. (ii) Let $[I_1, h_1], [I_2, h_2] \in qL_{\mathcal{M}}$ and suppose that $[L, \overline{h_a^t}] \wedge [I_1, h_1] = [L, \overline{h_a^t}] \wedge [I_2, h_2]$ for all $[L, \overline{h_a^t}] \in \overline{v}(L)$. For each $a \in L$, there exists $K_a \subseteq L \cap I_1 \cap I_2 = I_1 \cap I_2$ such that $\overline{h_a^t}(x) \wedge h_1(x) = \overline{h_a^t}(x) \wedge h_2(x)$ for every $x \in K_a$. Taking into account Proposition 4.2, Proposition 4.1, (GL_{12}) and (GL_7) we have that $\overline{h_a^t}(x) = \bigvee_{p=0}^{m-1}(x \wedge N^{2p} h_a^t(D_1 x)) = x \wedge \bigvee_{p=0}^{m-1} N^{2p} h_a^t(D_1 x) = x \wedge \bigvee_{p=0}^{m-1} N^{2p}(D_1 x \wedge a) = \bigvee_{p=0}^{m-1}(x \wedge D_1 x \wedge N^{2p} a) = \bigvee_{p=0}^{m-1}(x \wedge N^{2p} a) = x \wedge \bigvee_{p=0}^{m-1} N^{2p} a = x \wedge \overline{a}$. From these assertions we infer that $x \wedge \overline{a} \wedge h_1(x) = x \wedge \overline{a} \wedge h_2(x)$. This equality and Lemma 4.2 imply that $\overline{a} \wedge h_1(x) = \overline{a} \wedge h_2(x)$ for every $a \in L$. Hence, taking into account that $L \in \mathcal{R}(L)$ we conclude that $h_1(x) = h_2(x)$. ■

Remark 4.3. Finally, from Theorem 4.1 and Remark 2.1 we obtain a new functional representation for monadic Łukasiewicz algebras of order n, as we previously announced.

Now our attention is focus on define the notions of ML_n^m-algebra of fractions and maximal algebra of fractions where the central role in these constructions is played by the concept of multiplier. Let us recall that the concept of maximal lattice of fractions for a distributive lattice was defined by Schmid in [28] taking as a guide-line the construction of a complete ring of fractions by partial morphisms introduced by Lambek [25]. On the other hand, it is worth mentioning that bearing in mind, that a large part of the researches of the theory of localization and maximal algebras of fraction in different classes of algebras reveal quite similar techniques and results, S. Rudeanu in his important paper [27] gave a unifying approach able to eliminate redundancies. Thus, we will define the following notion.

Definition 4.4. An ML_n^m-algebra L' is called an ML_n^m-algebra of fractions of L if it verifies the following conditions:

(i) L is an ML_n^m-subalgebra of L',

(ii) For every $a', b', c' \in L'$, $a' \neq b'$ there is $e \in L$ such that $\overline{e} \wedge a' \neq \overline{e} \wedge b'$ and $\exists D_1 e \wedge c' \in L$.

We will write $L \preceq L'$ to indicate that L' is an ML_n^m-algebra of fractions for L.

Lemma 4.6. *Let $L \in \mathcal{M}L_n^m$ and $[I,g] \in qL_{\mathcal{M}}$. Then $I \subseteq \{a \in L : \widetilde{\exists}\widetilde{D_1}[L,h_a^t] \wedge [I,g] \in \overline{v}(L)\}$.*

Proof. Using in turn Proposition 4.4, Lemma 4.2, Proposition 4.1, Lemma 4.3, Remark 4.2 and Theorem 4.1 we obtain the proof. ■

Lemma 4.7. *Let $L' \in \mathcal{M}L_n^m$, $L \preceq L'$ and $a' \in L'$. Then $I_{a'} = \{x \in L : \exists D_1 x \wedge a' \in L\} \in \mathcal{I}_q(L) \cap \mathcal{R}(L)$.*

Proof. Let $e \in I_{a'}$. Then by (GL_6), (E_{14}) and (E_4) we have that $\exists D_1 D_1 e \wedge a' \in L$ and so, $\exists D_1 \exists e \wedge a' \in L$. Therefore, $I_{a'} \in \mathcal{I}_q(L)$. To prove that $I_{a'} \in \mathcal{R}(L)$ let $x, y \in L$ such that $x \wedge \overline{e} = y \wedge \overline{e}$ for all $e \in I_{a'}$ and suppose that $x \neq y$. Hence taking into account that $L \preceq L'$, there is $e_1 \in I_{a'}$ such that $x \wedge \overline{e}_1 \neq y \wedge \overline{e}_1$, which is a contradiction. Thus, we conclude that $I_{a'} \in \mathcal{R}(L)$. ■

Lemma 4.8. *For every $[I,g],[J,s],[H,t] \in qL_{\mathcal{M}}$, if $[I,g] \neq [J,s]$ then there is $a_o \in L$ such that $[I,g] \wedge [L,\overline{h_{a_o}^t}] \neq [J,s] \wedge [L,\overline{h_{a_o}^t}]$ and $\widetilde{\exists}\widetilde{D_1}[L,h_{a_o}^t] \wedge [H,t] \in \overline{v}(L)$.*

Proof. Indeed, suppose that for all $a \in L$ we have that $[I,g] \wedge [L,\overline{h_a^t}] = [J,s] \wedge [L,\overline{h_a^t}]$ and so, $(\overline{h_a^t} \wedge g)(x) = (\overline{h_a^t} \wedge s)(x)$ for every $x \in I \cap J$. Consequently, $\overline{a} \wedge x \wedge g(x) = \overline{a} \wedge x \wedge s(x)$. This statement and Lemma 4.2 imply that $\overline{a} \wedge g(x) = \overline{a} \wedge s(x)$ and taking into account that $L \in \mathcal{R}(L)$, we conclude that $g(x) = h(x)$ for every $x \in I \cap J$. Therefore, $[I,g] = [J,s]$ which is a contradiction. On the other hand, as $[H,t] \in L_{\mathcal{M}}$ by Lemma 4.6 we infer that $H \subseteq \{a \in L : \widetilde{\exists}\widetilde{D_1}[L,h_a^t] \wedge [H,t] \in \overline{v}_L(L)\}$. Hence, since $a_o \in L$ we have that $\widetilde{\exists}\widetilde{D_1}[L,h_{a_o}^t] \wedge [H,t] \in \overline{v}_L(L)$. ■

Now, we will prove that for finite ML_n^m-algebras the function $\overline{v}$ is an isomorphism.

Theorem 4.2. *Let L be a finite ML_n^m-algebra. Then $\overline{v}(L) = qL_{\mathcal{M}}$.*

Proof. Let $[I,h] \in qL_{\mathcal{M}}$. Since $\mathbf{0} = \widetilde{\exists}\widetilde{D_1}[L,h_0^t] = [L,h_0^t] \in \overline{v}(L)$ and $[I,h] \wedge \widetilde{\exists}\widetilde{D_1}[L,h_0^t] = [L,h_0^t]$, we have that $B = \{[L,h_x^t] : [I,h] \wedge \widetilde{\exists}\widetilde{D_1}[L,\overline{h_x^t}] \in \overline{v}(L)\} \neq \emptyset$. From the hypothesis B is a finite set and so, it has last element $[L,h_y^t]$.

Suppose that $[I,h] \vee \widetilde{\exists}\widetilde{D_1}[L,h_y^t] \neq \widetilde{\exists}\widetilde{D_1}[L,h_y^t]$. Then by Lemma 4.8 there is $a_0 \in L$ such that $[L,\overline{h_{a_0}^t}] \wedge ([I,h] \vee \widetilde{\exists}\widetilde{D_1}[L,h_y^t]) \neq [L,\overline{h_{a_0}^t}] \wedge \widetilde{\exists}\widetilde{D_1}[L,h_y^t]$ and $\widetilde{\exists}\widetilde{D_1}[L,\overline{h_{a_o}^t}] \wedge [I,h] \in \overline{v}_L(L)$. Hence, from the above assertions it follows that $[L,\overline{h_{a_o}^t}] \in B$ and so $[L,\overline{h_{a_o}^t}] \leq \widetilde{\exists}\widetilde{D_1}[L,h_y^t]$. This statement implies that $[L,\overline{h_{a_0}^t}] = [L,\overline{h_{a_0}^t}] \wedge ([I,h] \vee \widetilde{\exists}\widetilde{D_1}[L,h_y^t]) \neq [L,\overline{h_{a_0}^t}]$, which is a contradiction. Then, $[I,h] = [I,h] \wedge \widetilde{\exists}\widetilde{D_1}[L,h_y^t] \in \overline{v}_L(L)$ and so $qL_{\mathcal{M}} \subseteq \overline{v}(L)$. ■

Theorem 4.3. *Let $L \in \mathcal{M}L_n^m$. Then $L_{\mathcal{M}}$ verifies the following conditions:*

(i) $\overline{v}(L) \preceq qL_{\mathcal{M}}$,

(ii) For every ML_n^m-algebra $L^{'}$ such that $L \preceq L^{'}$, there exists an ML_n^m-monomorphism $u : L^{'} \to L_{\mathcal{M}}$ which induces the canonical monomorphism $\overline{v}_L$ of L into $qL_{\mathcal{M}}$.

Proof. (i) It follows immediately from Lemma 4.8.

(ii) By Lemma 4.7 for each $a' \in L'$ we have that $I_{a'} \in \mathcal{I}_q(L) \cap \mathcal{R}(L)$. Hence, we define the multiplier $h_{a'} : I_{a'} \to L$ by $h_{a'}(x) = a' \wedge x$ and $u : L' \to L_{\mathcal{M}}$ by $u(a') = [I_{a'}, h_{a'}]$ for every $a' \in L'$. Thus, we have that $[L, h^t_{a'}] = [I_{a'}, h_{a'}]$ and $u_{|_L} = \overline{v}_L$. Then, following an analogous reasoning of Theorem 4.1 we conclude that u is a monomorphism. ■

Theorem 4.3 provides the motivation for the following

Definition 4.5. For any ML_n^m-algebra L, $qL_{\mathcal{M}}$ is called a maximal ML_n^m-algebra of fractions of L.

Final Remarks As ML_n^m-algebras for $m = 1$ coincide with monadic n-valued Łukasiewicz algebras, from Definition 4.5, we obtain the maximal algebra of fractions for these last ones. Furthermore, if the quantifier is the trivial one, some of the results that have been shown above coincide with those established by D. Buşneag and F. Chirteş in [5] where they introduced the concept of n–valued Łukasiewicz algebra of fractions and proved the existence of the maximal one. Finally, for $n = 2$ and $m = 1$ we obtain a maximal algebra of fractions for a monadic Boolean algebra.

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References

[1] Almada, T., and J. Vaz-De-Carvalho, A generalization of the Łukasiewicz algebras, *Studia Logica* 69:329–338, 2001.

[2] Ashraf, M., and A. Shakir, On left multipliers and the commutativity of prime rings, *Demonstratio Mathematica* 41:761–761, 2008.

[3] Balbes, R., and Ph. Dwinger, *Distributive Lattices*, Univ. of Missouri Press, Columbia, 1974.

[4] Boicescu, V., A. Filipoiu, G. Georgescu, and S. Rudeanu, *Łukasiewicz-Moisil Algebras*, vol. 49 of *Annals of Discrete Mathematics*, North-Holland, 1991.

[5] Busneag, D., and F. Chirtes, LM_n-algebra of fractions and maximal LM_n-algebra of fractions, *Discrete Mathematics* 296: 143–165, 2005.
[6] Cignoli, R., *Moisil Algebras*, vol. 27 of *Notas de Lógica Matemática*, Univ. Nacional del Sur, Bahía Blanca, 1970.
[7] Cirulis, J., Multipliers in implicative algebras, *Bulletin of the Section of Logic* 15: 152–157, 1986.
[8] Colville, P., G. Davis, and K. Keimel, Positive Derivations on f-rings, *Journal of the Australian Mathematical Society* 23: 371–375, 1977.
[9] Cornish, W. H., The multiplier extension of a distributive lattice, *Journal of Algebra* 32: 339–355, 1974.
[10] Cornish, W. H., A multiplier approach to implicative BCK-algebras, *Mathematics Seminar Notes* 8: 157–169, 1980.
[11] Figallo, A. V., C. Gallardo, and A. Ziliani, Weak implication on generalized Łukasiewicz algebras of order n, *Bulletin of the Section of Logic* 39: 187–198, 2010.
[12] Figallo, A. V., I. Pascual and A. Ziliani, Notes on monadic n-valued Łukasiewicz algebras, *Mathematica Bohemica* 129: 255–271, 2004.
[13] Figallo, A. V., C. Sanza and A. Ziliani, Functional monadic n-valued Łukasiewicz algebras, *Mathematica Bohemica* 130: 337–348, 2005.
[14] Gallardo, C., *Sobre las álgebras de Łukasiewicz $m-$ generalizadas de orden n*, Ph. D. Thesis, Univ. Nac. del Sur, Bahía Blanca, 2016.
[15] Gallardo, C., and A. Ziliani, The L_n^m-propositional calculus, *Mathematica Bohemica*, 140: 11–33, 2015.
[16] Gallardo, C., and A. Ziliani, Localization of m-generalized Łukasiewicz algebras of order n, *Journal of Multiple-Valued Logic and Soft Computing* 27: 21–45, 2016.
[17] Georgescu, G., and C. Vraciu, Algebre Boole monadice si algebre Łukasiewicz monadice, *Studii Cercet. Mat.* 23: 1025–1048, 1971.
[18] Halmos, P. R. *Algebraic Logic*, Chelsea, New York, 1962.
[19] Iorgulescu, A., On the construction of three-valued Łukasiewicz-Moisil algebras, *Discrete Mathematics* 48: 213–227, 1984.
[20] Iorgulescu, A., On BCK Algebras - Part I.a. An Attempt to Treat Unitarily the Algebras of Logic. New Algebras, *Journal of Universal Computer Science* 13: 1628–1654, 2007.
[21] Iorgulescu, A., On BCK Algebras - Part I.b. An Attempt to Treat Unitarily the Algebras of Logic. New Algebras, *Journal of Universal Computer Science* 14: 3686–3715, 2008.
[22] Iorgulescu, A., *Algebras of logic as BCK–algebras*, Academy of Economic Studies Press, Bucharest, 2008.
[23] Kolibiar, M., Bermerkungen über Translationen der Verbände, *Acta Fac. Rerum Nat. Univ. Comenian Math.* 5: 455–458, 1961.
[24] Laca, M., and I. Raeburn, Extending Multipliers from Semigroups, *Proceedings of the American Mathematical Society* 123: 355–362, 1995.
[25] Lambek, J., *Lectures on Rings and Modules*, Blaisdell Publishing Company, 1966.
[26] Petrich, M., The translational hull in semigroups and rings, *Semigroup Forum* 1: 283–360, 1970.

[27] Rudeanu, S., Localizations and fractions in algebra of logic, *Journal of Multiple-Valued Logic and Soft Computing* 16: 467–504, 2010.
[28] Schmid, J., Multipliers on distributive lattices and rings of fractions, *Houston Journal of Mathematics* 6: 401–425, 1980.
[29] Szász, G., Die Translationen der Halbverbände, *Acta Scientiarum Mathematicarum* 17: 165–169, 1956.
[30] Szász, G., and J. Szendrei, Über der Translationen der Halbverbände, *Acta Scientiarum Mathematicarum* 18: 44–47, 1957.
[31] Vaz De Carvalho, J, On the variety of m-generalized Łukasiewicz algebras of order n, *Studia Logica* 94: 291–305, 2010.
[32] Yon, Y. H., and K. H. Kim, Multipliers in subtraction algebras, *Scientiae Mathematicae Japonicae* 73: 117–123, 2011.

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