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Extending the Lambek Calculus with Classical Negation

Abstract. We present an axiomatization of the non-associative Lambek calculus extended with classical negation for which the frame semantics with the classical interpretation of negation is sound and complete.

Keywords: Lambek calculus, Classical negation, Frame semantics, Sequent calculus.

1. Introduction

This paper deals with the extension of the non-associative Lambek calculus ***NL*** [6] with *classical* negation. Namely, what axioms and rules of inference should be added to ***NL*** to obtain sound and complete frame (ternary relational) semantics (see [3]) with the classical interpretation of negation. That is, a world satisfies the negation of a formula if and only if it does not satisfy the formula itself.

A number of extensions of ***NL*** with negation are known from the literature. However, all of them do not comply with the above semantical approach.

- The extension of ***NL*** with De Morgan negation in [1] defined by the axioms

  $$A \to \neg\neg A$$

  and

  $$\neg\neg A \to A$$

  and the rule of inference

  $$\frac{A \to B}{\neg B \to \neg A}$$

  and denoted by ***NLN***, is too weak. This is because the valid in the above mentioned semantics sequent (in which, as usual, $\neg$ has the highest preference)

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$$(\neg A/B) \cdot B \to \neg(A/B) \cdot B \tag{1}$$

is not derivable in that extension, see Examples 2 and 3 in the end of Section 4.

- The classical non-associative Lambek calculus ***CNL*** in [2] is too strong, because the sequents $\neg A/B \to A\backslash\neg B$ and $A\backslash\neg B \to \neg A/B$ are derivable in ***CNL***, but are not valid.
- The non-associative Lambek calculus with negation and connexive implication in [8] is also too strong, because the sequents

  1. $\neg(A\backslash B) \to A\backslash\neg B$ 2. $A\backslash\neg B \to \neg(A\backslash B)$
  3. $\neg(A/B) \to \neg A/B$ 4. $\neg A/B \to \neg(A/B)$

  are derivable in this calculus, but are not valid.

In this paper we present an axiomatization of ***NL*** extended with negation for which the frame semantics with the classical interpretation of negation is sound and complete. It should be noted however, that our sequent calculus, denoted $\boldsymbol{NLN^+}$, is more expressive. Namely, sequents are expressions of the form $\Gamma \to \Delta$, where Γ and Δ are *finite sets* of formulas constructed from propositional variables by means of the Lambek connectives and negation. Such sequents implicitly contain one external conjunction of the elements of Γ and one external disjunction of the elements of Δ.

The calculus $\boldsymbol{NLN^+}$ is tightly related to ***PNL***—the Lambek calculus extended with classical propositional logic, see [4]. The ***PNL*** counterpart of an $\boldsymbol{NLN^+}$-sequent $\Gamma \to \Delta$ is the ***PNL***-formula $\bigwedge_{C\in\Gamma} C \supset \bigvee_{C\in\Delta} C$. Obviously, they are equivalent with respect to the ternary relational semantics, and it follows from the completeness theorem for $\boldsymbol{NLN^+}$ (Theorem 30) that $\boldsymbol{NLN^+}$ corresponds to a conservative fragment of ***PNL***. However, an axiomatization of $\boldsymbol{NLN^+}$ in its own language is not trivial at all: it involves rather complicated rules in the style of resolution calculus.

The paper is organized as follows. In Sections 2 and 3 we recall ***NL*** and its frame semantics. In Section 4 we state how, in our opinion, the classical negation should behave and present two motivating examples. Then, in Section 5, we recall the definition of ***PNL***, define our sequent calculus $\boldsymbol{NLN^+}$ for the extension of ***NL*** with classical negation, and prove the soundness theorem. The completeness theorem for $\boldsymbol{NLN^+}$ is proved in Section 6. Finally, in Section 7, we replace some of the inference rules of $\boldsymbol{NLN^+}$ with their equivalent alternatives.

2. The Non-associative Lambek Calculus

The language of the non-associative Lambek calculus ***NL*** [6] consists of propositional variables (atomic formulas) and the Lambek connectives $\cdot, \backslash$, and /. Expressions of the form $A \to B$, where A and B are formulas, are called sequents.

The axioms of ***NL*** are sequents of the form

$$C \to C$$

and the rules of inference are

$$(a)\ \frac{A \cdot B \to C}{B \to A\backslash C} \qquad (b)\ \frac{B \to A\backslash C}{A \cdot B \to C} \tag{2}$$

$$(a)\ \frac{A \cdot B \to C}{A \to C/B} \qquad (b)\ \frac{A \to C/B}{A \cdot B \to C} \tag{3}$$

and

$$\frac{A \to B \qquad B \to C}{A \to C}$$

The Pentus interpretation $[\![A]\!]$ of a formula A in the free group generated by propositional variables is as follows, see [7, Section 2.1].

- If A is a propositional variable, then $[\![A]\!] = A$;
- $[\![A \cdot B]\!] = [\![A]\!][\![B]\!]$, $[\![A\backslash B]\!] = [\![A]\!]^{-1}[\![B]\!]$, and $[\![A/B]\!] = [\![A]\!][\![B]\!]^{-1}$.

PROPOSITION 1. [7, Lemma 2.3] *If* $\vdash_{\boldsymbol{NL}} A \to B$, *then* $[\![A]\!] = [\![B]\!]$.

Actually, [7, Lemma 2.3] deals with the *associative* Lambek calculus that is stronger than ***NL***.

3. The Frame Semantics of *NL*

The semantics of ***NL*** we consider here is the frame (or ternary relational) semantics from [3]. Namely, an *interpretation* is a triple $\mathfrak{I} = \langle W, R, V\rangle$, where W is a set of (possible) worlds, R is a ternary (accessibility) relation on W, and V is a (valuation) function from W into sets of propositional variables.

The satisfiability relation $\models$ between worlds in W and formulas is defined as follows. Let $u \in W$.

- If A is a propositional variable, then $\mathfrak{I}, u \models A$, if $A \in V(u)$;
- $\mathfrak{I}, u \models A \cdot B$, if there are $v, w \in W$ such that $R(u, v, w)$ and the following holds: $\mathfrak{I}, v \models A$ and $\mathfrak{I}, w \models B$;

- $\mathfrak{I}, u \models A \backslash B$, if for all $v, w \in W$ such that $R(v, w, u)$ the following holds: $\mathfrak{I}, w \models A$ implies $\mathfrak{I}, v \models B$;
- $\mathfrak{I}, u \models A/B$, if for all $v, w \in W$ such that $R(w, u, v)$ the following holds: $\mathfrak{I}, v \models B$ implies $\mathfrak{I}, w \models A$; and
- $\mathfrak{I}, u \models A \to B$, if $\mathfrak{I}, u \models A$ implies $\mathfrak{I}, u \models B$.

A formula A (a sequent $A \to B$) is *satisfiable*, if $\mathfrak{I}, u \models A$ (respectively, $\mathfrak{I}, u \models A \to B$), for some interpretation $\mathfrak{I} = \langle W, R, V \rangle$ and some $u \in W$. Also, we say that $\mathfrak{I}$ satisfies a formula A (a sequent $A \to B$), denoted $\mathfrak{I} \models A$ (respectively, $\mathfrak{I} \models A \to B$), if $\mathfrak{I}, u \models A$ (respectively, $\mathfrak{I}, u \models A \to B$), for all $u \in W$, and we say that $\mathfrak{I}$ satisfies a set of formulas Γ (a set of sequents $\mathbf{\Sigma}$), denoted $\mathfrak{I} \models \Gamma$ (respectively, $\mathfrak{I} \models \mathbf{\Sigma}$), if $\mathfrak{I}$ satisfies all formulas in Γ (respectively, $\mathfrak{I}$ satisfies all sequents in $\mathbf{\Sigma}$).

Finally, a set of sequents $\mathbf{\Sigma}$ *semantically entails* a sequent $A \to B$, denoted $\mathbf{\Sigma} \models A \to B$, if each interpretation satisfying $\mathbf{\Sigma}$ also satisfies $A \to B$.

This semantics is strongly sound and strongly complete for ***NL***, i.e., for a set of sequents $\mathbf{\Sigma}$, $\mathbf{\Sigma} \vdash A \to B$ if and only if $\mathbf{\Sigma} \models A \to B$, cf. [3, Proposition 1].

4. Extending *NL* with Classical Negation

When extending ***NL*** with classical negation, one would expect the following extension of the frame semantics from Section 3 to be (strongly) sound and (strongly) complete for the extended ***NL***.

- $\mathfrak{I}, u \models \neg A$, if $\mathfrak{I}, u \not\models A$.

EXAMPLE 2. Sequent (1) is valid, i.e., it is satisfied by all interpretations.

Indeed, let $\mathfrak{I} = \langle W, R, V \rangle$ be an interpretation, $u \in W$, and let $\mathfrak{I}, u \models (\neg A/B) \cdot B$. That is, there are worlds $v, w \in W$ such that $R(u, v, w)$ and the following holds: $\mathfrak{I}, v \models \neg A/B$ and $\mathfrak{I}, w \models B$. Then, by definition,

$$\mathfrak{I}, u \models \neg A \tag{4}$$

Since $\mathfrak{I}, w \models B$ and $R(u, v, w)$, for $\mathfrak{I}, u \models \neg(A/B) \cdot B$ it suffices to show that $\mathfrak{I}, v \models \neg(A/B)$, that, by definition, is $\mathfrak{I}, v \not\models A/B$,

To show the latter, assume to the contrary that $\mathfrak{I}, v \models A/B$. Then, since $\mathfrak{I}, w \models B$, $\mathfrak{I}, u \models A$ in contradiction with (4).

A possible candidate for such an extension is Buszkowski's ***NLN*** [1] defined in Section 1. Beside of being of interest in its own right, the ***NLN*** negation is motivated by extensions of categorial grammars. However, as shows the example below, this negation is too weak.

EXAMPLE 3. Sequent (1) is not derivable in $\boldsymbol{NLN}$.[1] For the proof, we extend the Pentus interpretation from Section 2 to negation by

- $[\![\neg A]\!] = [\![A]\!]^{-1}$.

Then the proof of [7, Lemma 2.3] extends to the "associative extension" $\boldsymbol{LN}$ of $\boldsymbol{NLN}$.

From this point, the proof of non-derivability of (1) is immediate, because $[\![(\neg A/B) \cdot B]\!] = [\![A]\!]^{-1}$, whereas $[\![\neg(A/B) \cdot B]\!] = [\![B]\!][\![A]\!]^{-1}[\![B]\!]$.

In Example 12 in Section 5 we present a derivation of (1) in the calculus $\boldsymbol{NLN}^+$ for which the frame semantics with the classical interpretation of negation is sound and complete.

5. The Calculi $\boldsymbol{PNL}$ and $\boldsymbol{NLN}^+$

The language of $\boldsymbol{NLN}^+$ is the language of $\boldsymbol{NLN}$ augmented with $\perp$ (*falsity*). The semantics of $\perp$ is standard:

- $\mathfrak{I}, u \not\models \perp$

and we abbreviate $\neg\perp$ as $\top$.

As we have already mentioned in the introduction, sequents of $\boldsymbol{NLN}^+$ are of the form $\Gamma \to \Delta$, where Γ and Δ are finite sets of formulas and we naturally define the satisfiability relation for such sequents by

- $\mathfrak{I}, u \models \Gamma \to \Delta$, if $\mathfrak{I}, u \not\models A$, for some $A \in \Gamma$, or $\mathfrak{I}, u \models B$, for some $B \in \Delta$.

Satisfiability by an interpretation and semantical entailment extend to $\boldsymbol{NLN}^+$ sequents in a straightforward manner.

In what follows we use the notation below.

Let Γ be a set of formulas and let C be a formula. We define the sets of formulas $C \cdot \Gamma$ and $\Gamma \cdot C$ by

$$C \cdot \Gamma = \{C \cdot A : A \in \Gamma\}$$

and

$$\Gamma \cdot C = \{A \cdot C : A \in \Gamma\}$$

[1]The author is grateful to Wojciech Buszkowski for the proof.

and for two sets of formulas Γ' and Γ'' we define the set of formulas $\Gamma' \cdot \Gamma''$ by

$$\Gamma' \cdot \Gamma'' = \{A \cdot B : A \in \Gamma' \text{ and } B \in \Gamma''\}$$

As usual, for a set of formulas

$$\Theta_1 \cup \cdots \cup \Theta_m \cup \{C_1, \ldots, C_n\}$$

we write

$$\Theta_1, \ldots, \Theta_m, C_1, \ldots, C_n$$

possibly in a different order.

Finally, two- and one-sided many-formula resolutions (12), (13), and (14) below employ the following notation. For a set of "resolution" formulas $\Theta = \{C_1, \ldots, C_n\}$ we denote by $\Theta^{\sim}$ the set of all sets of formulas of the form $\{\widetilde{C}_1, \ldots, \widetilde{C}_n\}$, where $\widetilde{C}_i \in \{\neg C_i, C_i\}$, $i = 1, \ldots, n$.

We precede the definition of $\boldsymbol{NLN}^+$ with some motivating examples. These examples involve ***PNL***—the Lambek calculus extended with classical propositional logic, see [4], that is tightly related to $\boldsymbol{NLN}^+$. The rules of inference of ***PNL*** are rules (2) and (3) of the nonassociative Lambek calculus ***NL*** (where $\rightarrow$ is replaced by $\supset$) and *modus ponens*

$$\frac{A, \quad A \supset B}{B}$$

and the axioms of ***PNL*** are the axioms of classical propositional calculus

$$A \supset (B \supset A)$$
$$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$$
$$(\neg A \supset \neg B) \supset ((\neg A \supset B) \supset A)$$
$$A \supset (B \supset (A \wedge B))$$
$$(A \wedge B) \supset A$$
$$(A \wedge B) \supset B$$
$$A \supset (A \vee B)$$
$$B \supset (A \vee B)$$
$$(A \supset C) \supset ((B \supset C) \supset (A \vee B) \supset C))$$

A (strongly) sound and (strongly) complete semantics of ***PNL*** results in extending the semantics of the Lambek connectives and negation with the ordinary semantics of $\vee$, $\wedge$, and $\supset$. Namely,

- $\mathfrak{I}, u \models A \vee B$, if $\mathfrak{I}, u \models A$ or $\mathfrak{I}, u \models B$;

- $\mathfrak{I}, u \models A \wedge B$, if $\mathfrak{I}, u \models A$ and $\mathfrak{I}, u \models B$; and
- $\mathfrak{I}, u \models A \supset B$, if $\mathfrak{I}, u \not\models A$ or $\mathfrak{I}, u \models B$.

REMARK 4. As it has been already noted in the introduction, the ***PNL*** counterpart of an $\boldsymbol{NLN}^+$-sequent $\Gamma \to \Delta$ is the ***PNL***-formula $\bigwedge_{C\in\Gamma} C \supset \bigvee_{C\in\Delta} C$. Obviously, they are semantically equivalent, and it follows from the completeness theorem for $\boldsymbol{NLN}^+$ (Theorem 30 in Section 6) that $\boldsymbol{NLN}^+$ corresponds to a (strongly) conservative fragment of ***PNL***. Namely,

$$\{\Gamma_i \to \Delta_i : i \in I\} \vdash_{\boldsymbol{NLN}^+} \Gamma \to \Delta$$

if and only if

$$\left\{ \bigwedge_{C\in\Gamma_i} C \supset \bigvee_{C\in\Delta_i} C : i \in I \right\} \vdash_{\boldsymbol{PNL}} \bigwedge_{C\in\Gamma} C \supset \bigvee_{C\in\Delta} C$$

EXAMPLE 5. Let $\Theta = \{C_1, \ldots, C_n\}$.

- If $n = 0$, then $\Theta^\sim = \{\emptyset\}$,
- if $n = 1$, then $\Theta^\sim = \{\{\neg C_1\}, \{C_1\}\}$,
- if $n = 2$, then $\Theta^\sim = \{\{\neg C_1, \neg C_2\}, \{C_1, \neg C_2\}, \{\neg C_1, C_2\}, \{C_1, C_2\}\}$,
- etc.

The ***PNL*** counterpart of a set of formulas $\{\widetilde{C}_1, \ldots, \widetilde{C}_n\} \in \Theta^\sim$ is the conjunction $\bigwedge_{i=1}^{n} \widetilde{C}_i$. Since ***PNL***-formulas $\bigvee_{\widetilde{\Theta}\in\Theta^\sim} \bigwedge_{\widetilde{C}\in\widetilde{\Theta}} \widetilde{C}$ and $\bigwedge_{i=1}^{n} (\neg C_i \vee C_i)$ are (semantically) equivalent, the former formula is equivalent to $\top$. Therefore, the "many-formula" resolution

$$\frac{\{\widetilde{\Theta}, \Gamma \to \Delta : \widetilde{\Theta} \in \Theta^\sim\}}{\Gamma \to \Delta} \tag{5}$$

is sound for the frame semantics.

EXAMPLE 6. If $\Theta \neq \emptyset$, then the ***PNL***-formulas $\top \cdot \top$ and $\bigvee_{\Theta',\Theta''\in\Theta^\sim} \bigwedge_{\substack{C' \in \Theta' \\ C'' \in \Theta''}} C' \cdot C''$ are equivalent, implying the "product" many-formula resolution

$$\frac{\{\Gamma, \Theta' \cdot \Theta'' \to \Delta : \Theta', \Theta'' \in \Theta^\sim\}}{\Gamma, \top \cdot \top \to \Delta} \tag{6}$$

The equivalence of $\top \cdot \top$ and $\bigvee_{\Theta',\Theta'' \in \Theta^{\sim}} \bigwedge_{\substack{C' \in \Theta' \\ C'' \in \Theta''}} C' \cdot C''$ can be shown (semantically in ***PNL***) as follows. Example 5 implies that $\top \cdot \top$ is equivalent to

$$\left(\bigvee_{\Theta' \in \Theta^{\sim}} \bigwedge_{C' \in \Theta'} C' \right) \cdot \left(\bigvee_{\Theta'' \in \Theta^{\sim}} \bigwedge_{C'' \in \Theta''} C'' \right)$$

that, in turn, is equivalent to

$$\bigvee_{\Theta',\Theta'' \in \Theta^{\sim}} \left(\bigwedge_{C' \in \Theta'} C' \right) \cdot \left(\bigwedge_{C'' \in \Theta''} C'' \right)$$

and the latter implies $\bigvee_{\Theta',\Theta'' \in \Theta^{\sim}} \bigwedge_{\substack{C' \in \Theta' \\ C'' \in \Theta''}} C' \cdot C''$.

The converse implication is trivial, because $C' \cdot C'' \supset \top \cdot \top$ is a valid formula.

At last, we are ready to define the calculus ***NLN***$^+$ whose axioms are sequents

$$\bot \rightarrow$$

and

$$C \rightarrow C$$

and the rules of inference are as follows.

$$(a) \quad \frac{A \cdot \Gamma \rightarrow C}{\Gamma \rightarrow A \backslash C} \qquad (b) \quad \frac{B \rightarrow A \backslash C}{A \cdot B \rightarrow C} \tag{7}$$

$$(a) \quad \frac{\Gamma \cdot B \rightarrow C}{\Gamma \rightarrow C / B} \qquad (b) \quad \frac{A \rightarrow C / B}{A \cdot B \rightarrow C} \tag{8}$$

$$(a) \quad \frac{\Gamma, C \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg C} \qquad (b) \quad \frac{\Gamma \rightarrow \Delta, C}{\Gamma, \neg C \rightarrow \Delta} \tag{9}$$

thinnings

$$(a) \quad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, C} \qquad (b) \quad \frac{\Gamma \rightarrow \Delta}{\Gamma, C \rightarrow \Delta} \tag{10}$$

mix

$$\frac{\Gamma' \rightarrow \Delta', C \qquad \Gamma'', C \rightarrow \Delta''}{\Gamma', \Gamma'' \rightarrow \Delta', \Delta''} \tag{11}$$

two-sided many-formula resolution

$$\frac{\Gamma \to \Delta, A \cdot B \quad \{\Theta', A \to A_{\Theta'} \quad \Theta'', B \to B_{\Theta''} \quad \Gamma, \{\Theta', A, A_{\Theta'}\} \cdot \{\Theta'', B, B_{\Theta''}\} \to \Delta \ : \ \Theta', \Theta'' \in \Theta^{\sim}\}}{\Gamma \to \Delta} \tag{12}$$

and two one-sided many-formula resolutions

$$\frac{\{\Theta', A \to A_{\Theta'} \qquad \{\Theta', A, A_{\Theta'}\} \cdot \Gamma'' \to C \ : \ \Theta' \in \Theta^{\sim}\}}{\Gamma'' \to A \backslash C} \tag{13}$$

and

$$\frac{\{\Theta'', B \to B_{\Theta''} \qquad \Gamma' \cdot \{\Theta'', B, B_{\Theta''}\} \to C \ : \ \Theta'' \in \Theta^{\sim}\}}{\Gamma' \to C/B} \tag{14}$$

Rule (12) is read as follows: if the sequent $\Gamma \to \Delta, A \cdot B$ is derivable and, for a set of "resolution" formulas Θ and all pairs of sets $\Theta', \Theta'' \in \Theta^{\sim}$, there exist formulas $A_{\Theta'}$ and $B_{\Theta''}$ such that the sequents $\Theta', A \to A_{\Theta'}$, $\Theta'', B \to B_{\Theta''}$, and $\Gamma, \{\Theta', A, A_{\Theta'}\} \cdot \{\Theta'', B, B_{\Theta''}\} \to \Delta$ are derivable, then the sequent $\Gamma \to \Delta$ is derivable, and similarly for rules (13) and (14).

Remark 7. It seems that the "auxiliary" formulas $A_{\Theta'}$ and $B_{\Theta''}$ in the above rules are essential, because the rule

$$\frac{\Theta' \to A_{\Theta'} \quad \Theta'' \to B_{\Theta''} \quad A_{\Theta'} \cdot B_{\Theta''} \to C}{\Theta' \cdot \Theta'' \to C}$$

is not sound. In ***PNL***, one obtains soundness by replacing $A_{\Theta'}$ and $B_{\Theta''}$ with $\bigwedge_{C \in \Theta'} C$ and $\bigwedge_{C \in \Theta''} C$, respectively. However, we do not have conjunctions in $\boldsymbol{NLN}^{+}$.

In fact, rules (7)(a) and (8)(a) are redundant, because they follow from (13) and (14), respectively. Namely, (7)(a) follows from (13) with the empty Θ and $A_{\Theta'}$ being A. Then the first premise of (13) is the axiom $A \to A$ and the other premise of (13) is the the premise of (7)(a). The proof of (8)(a) from (14) is symmetric. Nevertheless, we left those rules to emphasize the similarity with rules (2)(a) and (3)(a) of ***NL***.

Also, in view of (10), we can, equivalently, replace mix with cut

$$\frac{\Gamma \to \Delta, C \qquad \Gamma, C \to \Delta}{\Gamma \to \Delta}$$

However, using mix reduces the size of the derivation trees in the subsequent Examples 10 and 12 and Proposition 18.

Example 8. Rule (5) is derivable in $\boldsymbol{NLN}^{+}$. The proof is by a straightforward induction on the number of formulas in Θ.

For the basis, the case of the empty Θ is immediate and for $\Theta = \{C_1\}$ we have

$$\dfrac{\dfrac{\Gamma, C_1 \to \Delta}{\Gamma \to \Delta, \neg C_1}\,(9)(a) \qquad \Gamma, \neg C_1 \to \Delta}{\Gamma \to \Delta}\,(11)$$

For the induction step, let Θ and Θ_n be $\{C_1, \ldots, C_n, C_{n+1}\}$ and $\{C_1, \ldots, C_n\}$, respectively. Then, since each set of formulas in $\Theta^{\sim}$ is either of the form Θ, C_{n+1} or of the form $\Theta, \neg C_{n+1}$, where $\Theta \in \Theta_n^{\sim}$, we have

$$\dfrac{\dfrac{\{\Theta, \Gamma, C_{n+1} \to \Delta : \Theta \in \Theta_n^{\sim}\}}{\Gamma, C_{n+1} \to \Delta}\,\text{induction hypothesis} \qquad \dfrac{\{\Theta, \Gamma, \neg C_{n+1} \to \Delta : \Theta \in \Theta_n^{\sim}\}}{\Gamma, \neg C_{n+1} \to \Delta}\,\text{induction hypothesis}}{\Gamma \to \Delta}\,\text{basis}$$

Example 9. The resolution (6) is derivable from two sided many-formula resolution (12) by substituting

- $\Gamma, \top \cdot \top$ for Γ,
- $\top$ for both A and B, and
- $\top$ for all $A_{\Theta'}$ and $B_{\Theta''}$.

Then

- the first premise $\Gamma, \top \cdot \top \to \Delta, \top \cdot \top$ of (12) is derivable, by thinnings, from the axiom $\top \cdot \top \to \top \cdot \top$;
- the second and the third premises $\Theta', \top \to \top$ and $\Theta'', \top \to \top$ of (12) are derivable, by thinnings, from the axiom $\top \to \top$; and
- the last premise $\Gamma, \{\Theta', \top, \top\} \cdot \{\Theta'', \top, \top\} \to \Delta$ of (12) is derivable, also by thinnings, from the corresponding premise $\Gamma, \Theta' \cdot \Theta'' \to \Delta$ of (6).

Example 10. (Cf. [4, Example 6].) The following sequents are derivable in $\boldsymbol{NLN}^+$.

$$(a) \ \ \bot \cdot B \to \qquad\qquad (b) \ \ A \cdot \bot \to$$

We only derive sequent (a). The derivation of sequent (b) is symmetric.

$$\dfrac{\dfrac{\dfrac{\bot \to}{\bot \to \bot / B}\,(10)(a)}{\bot \cdot B \to \bot}\,(8)(b) \qquad \bot \to}{\bot \cdot B \to}\,(11)$$

In Example 12 we show that sequent (1) is derivable in $\boldsymbol{NLN}^+$. For this derivation we need the following derivable rules.

PROPOSITION 11. *Rules*

$$(a)\quad \frac{A \to \Delta'}{A \cdot B \to \Delta' \cdot B} \qquad\qquad (b)\quad \frac{B \to \Delta''}{A \cdot B \to A \cdot \Delta''} \tag{15}$$

are derivable.

PROOF. We only prove rule (b). The proof of rule (a) is symmetric.

Let Δ'' be $C_1, C_2, \ldots, C_n$. We contend that all premises of (12) are derivable in $\boldsymbol{NLN}^+$ with

- Γ being $A \cdot B$,
- Δ being $A \cdot \Delta''$ (i.e., $A \cdot C_1, A \cdot C_2, \ldots, A \cdot C_n$),
- Θ being Δ'' (i.e., $C_1, C_2, \ldots, C_n$),
- $A_{\Theta'}$ being $\top$ for all $\Theta' \in \Theta^{\sim}$, and
- $B_{\Theta''}$ being defined by
  - (i) if $\Theta'' = \{\neg C_1, \neg C_2, \ldots, \neg C_n\}$, then $B_{\Theta''}$ is $\perp$, and
  - (ii) if for some $i = 1, 2, \ldots, n$, $C_i \in \Theta''$, then $B_{\Theta''}$ is $\top$,

Indeed, in such setting, we trivially have the first and the second premises of (12).

In case (i) of the definition of $B_{\Theta''}$, it follows from $B \to C_1, C_2, \ldots, C_n$, by (9)$(b)$, that

$$B, \neg C_1, \neg C_2, \ldots, \neg C_n \to$$

implying, by (10)(a),

$$B, \neg C_1, \neg C_2, \ldots, \neg C_n \to \perp$$

That is, the third premise of (12) is derivable. Then, by Example 10(b) and thinnings, we obtain

$$\Gamma, \{\Theta', A, \top\} \cdot \{\Theta'', B, \perp\} \to \Delta$$

That is, the last premise of (12) is derivable as well.

In case (ii) of the definition of $B_{\Theta''}$, the third premise of (12) is trivially derivable and since $A \cdot C_i$ is in both $\{\Theta', A, \top\} \cdot \{\Theta'', B, \perp\}$ and Δ, from the axiom $A \cdot C_i \to A \cdot C_i$, by thinnings, we obtain the last premise

$$\Gamma, \{\Theta', A, \top\} \cdot \{\Theta'', B, \top\} \to \Delta$$

of (12).

This proves our contention. Now, by (12) and the definitions of Γ and Δ, we have the desired conclusion $A \cdot B \to A \cdot \Delta''$. ■

Example 12. Sequent (1) is derivable in $\boldsymbol{NLN^+}$, see the derivation below.

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\dfrac{\neg A/B \to \neg A/B}{(\neg A/B)\cdot B \to \neg A}\,(8)(b)
\quad
\dfrac{\dfrac{A/B \to A/B}{(A/B)\cdot B \to A}\,(8)(b)}{\neg A, (A/B)\cdot B \to}\,(9)(b)
}{(\neg A/B)\cdot B, (A/B)\cdot B \to}\,(11)
}{(\neg A/B)\cdot B, (A/B)\cdot B \to \bot}\,(10)(a)
}{\neg A/B, A/B \to \bot/B}\,(8)(a)
}{\neg A/B \to \neg(A/B), \bot/B}\,(9)(a)
}{(\neg A/B)\cdot B \to \neg(A/B)\cdot B, (\bot/B)\cdot B}\,(15)(a)
\quad
\dfrac{\dfrac{\bot/B \to \bot/B}{(\bot/B)\cdot B \to \bot}\,(8)(b) \quad \bot \to}{(\bot/B)\cdot B \to}\,(11)
}{(\neg A/B)\cdot B \to \neg(A/B)\cdot B}\,(11)
$$

Theorem 13. (Soundness, cf. [3, Proposition 1].) *If* $\boldsymbol{\Sigma} \vdash \Gamma \to \Delta$*, then* $\boldsymbol{\Sigma} \models \Gamma \to \Delta$.

In particular, $\boldsymbol{NLN^+}$ is consistent. In the next section we show that the frame semantics is also (strongly) complete for $\boldsymbol{NLN^+}$.

Proof of Theorem 13. The proof is by a induction on the derivation length. The basis is immediate and for the induction step we consider only the case of rules (12) and (13).

For rule (12), let $\mathfrak{I} = \langle W, R, V\rangle$ be an interpretation that satisfies all premises of the rule and assume to the contrary that for some world $u \in W$, $\mathfrak{I}, u \not\models \Gamma \to \Delta$. That is, u satisfies all formulas in Γ, but satisfies no formula in Δ (in $\mathfrak{I}$, of course). Then, by the premise $\Gamma \to \Delta, A\cdot B$ and the induction hypothesis, $\mathfrak{I}, u \models A\cdot B$. That is, there exists $v, w \in W$ such that $R(u, v, w)$, $\mathfrak{I}, v \models A$ and $\mathfrak{I}, w \models B$.

Let $\Theta', \Theta'' \in \Theta^\sim$ be such that $\mathfrak{I}, v \models \Theta'$ and $\mathfrak{I}, w \models \Theta''$ and let $A_{\Theta'}$ and $B_{\Theta''}$ be such that $\Theta', A \to A_{\Theta'}$ and $\Theta'', B \to B_{\Theta''}$ are the corresponding premises of the rule. By the induction hypothesis, $\mathfrak{I}, v \models A_{\Theta'}$ and $\mathfrak{I}, w \models B_{\Theta''}$.

Then,

$$\mathfrak{I}, u \not\models \Gamma, \{\Theta', A, A_{\Theta'}\}\cdot\{\Theta'', B, B_{\Theta''}\} \to \Delta$$

which contradicts the induction hypothesis.

For rule (13), let $\mathfrak{I} = \langle W, R, V\rangle$ be an interpretation that satisfies all premises of the rule and assume to the contrary that for some world $u \in W$, $\mathfrak{I}, u \not\models \Gamma'' \to A\backslash C$. That is, $\mathfrak{I}, u \models \Gamma''$, but $\mathfrak{I}, u \not\models A\backslash C$.

By the definition of $\models$, there exists $v, w \in W$ such that $R(v, w, u)$, $\mathfrak{I}, w \models A$, but

$$\mathfrak{I}, v \not\models C \tag{16}$$

Let $\Theta' \in \Theta^\sim$ be such that $\mathfrak{I}, w \models \Theta'$ and let $A_{\Theta'}$ be such that $\Theta', A \to A_{\Theta'}$ is the corresponding premise of the rule. By the induction hypothesis, $\mathfrak{I}, w \models A_{\Theta'}$. Then $\mathfrak{I}, v \models \{\Theta', A, A_{\Theta'}\}\cdot\Gamma''$. Therefore, by the premise

$\{\Theta', A, A_{\Theta'}\} \cdot \Gamma'' \rightarrow C$ and (again) by the induction hypothesis, $\mathfrak{I}, v \models C$ which contradicts (16). ■

COROLLARY 14. ***NLN***$^{+}$ *is a strongly conservative extension of* ***NL****.*

PROOF. This is because the relational semantics is sound for ***NLN***$^{+}$ (Theorem 13) and strongly complete for ***NL*** [3, Proposition 1]. ■

6. Completeness

The proof of the completeness theorem, i.e., that $\mathbf{\Sigma} \models \Gamma \rightarrow \Delta$ implies $\mathbf{\Sigma} \vdash \Gamma \rightarrow \Delta$, is similar to that of [4, Theorem 15]. It follows the standard construction, but is more involved because of the presence of negation and the absence of conjunction/disjunction.

For a set of formulas Θ, we write $\mathbf{\Sigma} \vdash \Theta \rightarrow \Delta$ if for some finite subset Γ of Θ, $\mathbf{\Sigma} \vdash \Gamma \rightarrow \Delta$.

DEFINITION 15. A set of formulas Γ is called $\mathbf{\Sigma}$-*consistent* if $\mathbf{\Sigma} \not\vdash \Gamma \rightarrow$. Otherwise, it is called $\mathbf{\Sigma}$-*inconsistent*.

EXAMPLE 16. The set of formulas $\Gamma, C, \neg C$ is inconsistent because

$$\dfrac{\dfrac{C \rightarrow C}{C, \neg C \rightarrow}\ (9)(b)}{\Gamma, C, \neg C \rightarrow}\ \text{a number of thinnings } (10)(b)$$

In what follows by "maximal" we mean maximal with respect to inclusion.

EXAMPLE 17. Let $\mathbf{\Sigma}$ be a set of sequents, $\mathfrak{I} = \langle W, R, V \rangle$ be an interpretation satisfying $\mathbf{\Sigma}$, and let $u \in W$. Then

$$[u]_{\mathfrak{I}} = \{C : \mathfrak{I}, u \models C\} \tag{17}$$

is a maximal $\mathbf{\Sigma}$-consistent set of formulas.

For the proof of the completeness theorem we need the following well-known auxiliary results.

PROPOSITION 18. *If*

$$\mathbf{\Sigma} \not\vdash \Gamma \rightarrow \Delta \tag{18}$$

then, for each formula C*,*

$$\mathbf{\Sigma} \not\vdash \Gamma, C \rightarrow \Delta \tag{19}$$

or

$$\mathbf{\Sigma} \not\vdash \Gamma, \neg C \rightarrow \Delta \tag{20}$$

Proof. Assume to the contrary that neither (19) nor (20). That is, there are finite subsets Θ' and Θ'' of Γ such that

$$\mathbf{\Sigma} \vdash \Theta', C \to \Delta$$

and

$$\mathbf{\Sigma} \vdash \Theta'', \neg C \to \Delta$$

Then

$$\dfrac{\dfrac{\mathbf{\Sigma} \vdash \Theta', C \to \Delta}{\mathbf{\Sigma} \vdash \Theta' \to \Delta, \neg C}(9)(a) \qquad \mathbf{\Sigma} \vdash \Theta'', \neg C \to \Delta}{\mathbf{\Sigma} \vdash \Theta', \Theta'' \to \Delta}(11)$$

which contradicts (18). ■

Corollary 19. *A $\mathbf{\Sigma}$-consistent set of formulas Γ is maximal $\mathbf{\Sigma}$-consistent if and only if for each formula C, either $C \in \Gamma$ or $\neg C \in \Gamma$.*

Proof. The "if" direction of the corollary is Example 16 and the "only if" direction of the corollary is Proposition 18 with empty Δ. ■

Corollary 20. *Let Γ be a maximal $\mathbf{\Sigma}$-consistent set of formulas. If*

$$\mathbf{\Sigma} \vdash \Gamma \to C \tag{21}$$

then $C \in \Gamma$.

Proof. If $C \notin \Gamma$, then, by Corollary 19, $\neg C \in \Gamma$ which, together with (21), contradicts $\mathbf{\Sigma}$-consistency of Γ. ■

Proposition 21. *If $\mathbf{\Sigma} \not\vdash \Gamma \to \Delta$, then there is a maximal $\mathbf{\Sigma}$-consistent set of formulas Θ including Γ such that $\mathbf{\Sigma} \not\vdash \Theta \to \Delta$.*

Proof. Consider the family $\mathcal{F}$ of sets of formulas defined by

$$\mathcal{F} = \{\Phi : \mathbf{\Sigma} \not\vdash \Phi \to \Delta \text{ and } \Gamma \subseteq \Phi\}$$

This family is not empty, because it contains Γ and, by compactness of $\vdash$, it is inductively ordered by inclusion. Therefore, by Zorn's lemma, $\mathcal{F}$ has a maximal (with respect to inclusion, of course) element Θ. Since $\mathbf{\Sigma} \not\vdash \Theta \to \Delta$, this Θ is $\mathbf{\Sigma}$-consistent. Thus, by Corollary 19, for the proof that Θ is maximal $\mathbf{\Sigma}$-consistent, it suffices to show that for each formula C, either $C \in \Theta$ or $\neg C \in \Theta$, which follows from maximality of Θ in $\mathcal{F}$ and Proposition 18. ■

Theorem 22. *Let Γ be a maximal $\mathbf{\Sigma}$-consistent set of formulas containing $A \cdot B$. Then, there exist a maximal $\mathbf{\Sigma}$-consistent sets of formulas Γ' containing A and Γ'' containing B such that $\Gamma' \cdot \Gamma'' \subseteq \Gamma$.*

For the proof of Theorem 22 we need the lemma below.

LEMMA 23. *Let* Γ *be a* $\mathbf{\Sigma}$*-consistent finite set of formulas such that* $\mathbf{\Sigma} \vdash \Gamma \to A \cdot B$*. Then, for each finite set of formulas* Θ *there exist* $\Theta', \Theta'' \in \Theta^{\sim}$ *such that all three sets of formulas*

$$\begin{array}{ll} \bullet & \Theta', A, \\ \bullet & \Theta'', B, \text{ and} \\ \bullet & \Gamma, \{\Theta', A\} \cdot \{\Theta'', B\} \end{array} \tag{22}$$

are $\mathbf{\Sigma}$*-consistent.*

PROOF. Since Γ is $\mathbf{\Sigma}$-consistent and $\mathbf{\Sigma} \vdash \Gamma \to A \cdot B$, by (the contraposition of) two-sided many-formula resolution (12) with empty Δ, there is a pair of sets of formulas $\Theta', \Theta'' \in \Theta^{\sim}$ such that

$$\mathbf{\Sigma} \not\vdash \Gamma, \{\Theta', A, A_{\Theta'_n}\} \cdot \{\Theta'', B, B_{\Theta''}\} \to \tag{23}$$

for all formulas $A_{\Theta'}$ and $B_{\Theta''}$ satisfying $\mathbf{\Sigma} \vdash \Theta', A \to A_{\Theta''}$ and $\mathbf{\Sigma} \vdash \Theta'', B \to B_{\Theta''}$.

It follows from (23) that $\Gamma, \{\Theta', A\} \cdot \{\Theta'', B\}$ is $\mathbf{\Sigma}$-consistent. For the proof that both sets of formulas Θ', A and Θ'', B are also $\mathbf{\Sigma}$-consistent, assume to the contrary, that $\mathbf{\Sigma} \vdash \Theta', A \to$ or $\mathbf{\Sigma} \vdash \Theta'', B \to$ and define the formulas $A_{\Theta'}$ and $B_{\Theta''}$ by

$$A_{\Theta'} = \begin{cases} \bot, & \text{if } \mathbf{\Sigma} \vdash \Theta', A \to \\ \top, & \text{otherwise} \end{cases}$$

and

$$B_{\Theta''} = \begin{cases} \bot, & \text{if } \mathbf{\Sigma} \vdash \Theta'', B \to \\ \top, & \text{otherwise} \end{cases}$$

Then one of the formulas $A_{\Theta'}$ or $B_{\Theta''}$ is $\bot$, implying, by Example 10 and thinning (10)(b),

$$\mathbf{\Sigma} \vdash \Gamma, \{\Theta', A, A_{\Theta'}\} \cdot \{\Theta'', B, B_{\Theta''}\} \to$$

in contradiction with (23). ■

COROLLARY 24. *Let* Γ *be a* $\mathbf{\Sigma}$*-consistent infinite set of formulas such that* $\mathbf{\Sigma} \vdash \Gamma \to A \cdot B$*. Then, for each finite set of formulas* Θ *there exist* $\Theta', \Theta'' \in \Theta^{\sim}$ *such that all three sets of formulas in (22) are* $\mathbf{\Sigma}$*-consistent.*

PROOF. Let $\Gamma = \{C_1, C_2, \ldots\}$ and let m be such that $\mathbf{\Sigma} \vdash C_1, \ldots, C_m \to A \cdot B$. Let $\Gamma_n = \{C_1, C_2, \ldots, C_m, C_{m+1}, \ldots, C_{m+n}\}$, $n = 1, 2, \ldots$. Then Γ_n is $\mathbf{\Sigma}$-consistent and $\mathbf{\Sigma} \vdash \Gamma_n \to A \cdot B$. Thus, by Lemma 23, there is a pair of sets of formulas $\Theta'_n, \Theta''_n \in \Theta^{\sim}$ such that all three sets of formulas Θ'_n, A, Θ''_n, B, and $\Gamma, \{\Theta'_n, A\} \cdot \{\Theta''_n, B\}$ are $\mathbf{\Sigma}$-consistent.

Since $\Theta^{\sim}$ is finite, for some $\Theta', \Theta'' \in \Theta^{\sim}$ there are infinitely many indices n such that $\Theta'_n = \Theta'$ and $\Theta''_n = \Theta''$. By compactness, all three sets in (22) are $\mathbf{\Sigma}$-consistent. ■

Proof of Theorem 22. Let $C_1, C_2, \ldots$ be a list of all formulas and let $\Theta_n = \{C_1, C_2, \ldots, C_n\}$.

Consider the infinite tree T whose nodes are pairs of sets $(\Theta', \Theta'') \in \Theta_n^{\sim} \times \Theta_n^{\sim}$, $n = 1, 2, \ldots$, such that all three sets of formulas in (22) are $\mathbf{\Sigma}$-consistent and $(\{A_1, \ldots, A_n\}, \{B_1, \ldots, B_n\})$ is a child node of $(\{A_1, \ldots, A_{n-1}\}, \{B_1, \ldots, B_{n-1}\})$, $n = 0, 1, \ldots$, where both $\{A_0\}$ and $\{B_0\}$ are $\emptyset$.

Obviously, the degree of T is at most 4. Since, by Corollary 24, for each $n = 1, 2, \ldots$, there exist sets of formulas $\Theta', \Theta'' \in \Theta_n^{\sim}$ such that all three sets of formulas in (22) are $\mathbf{\Sigma}$-consistent, the tree is infinite. Thus, by Kőnig's infinity lemma ([5]), T has an infinite path. Let Γ' and Γ'' be the unions of the first and the second components of the pairs lying on that path, respectively.

By compactness, both Γ' and Γ'' are $\mathbf{\Sigma}$-consistent and

$$\mathbf{\Sigma} \not\vdash \Gamma, \{\Gamma', A\} \cdot \{\Gamma'', B\} \rightarrow$$

implying that $\Gamma, \Gamma' \cdot \Gamma''$ is $\mathbf{\Sigma}$-consistent,

In addition, by Corollary 19, both Γ' and Γ'' are maximal $\mathbf{\Sigma}$-consistent. Thus, it follows from the same corollary that $A \in \Gamma'$ and $B \in \Gamma''$.

Finally, since Γ is maximal $\mathbf{\Sigma}$-consistent, $\Gamma' \cdot \Gamma'' \subseteq \Gamma$ as well. ■

Theorem 25

(*a*) *If* $\mathbf{\Sigma} \not\vdash \Gamma'' \rightarrow A\backslash C$, *then there exists a maximal* $\mathbf{\Sigma}$*-consistent set of formulas* Γ' *containing* A *such that* $\mathbf{\Sigma} \not\vdash \Gamma' \cdot \Gamma'' \rightarrow C$.

(*b*) *If* $\mathbf{\Sigma} \not\vdash \Gamma' \rightarrow C/B$, *then there exists a maximal* $\mathbf{\Sigma}$*-consistent set of formulas* Γ'' *containing* B *such that* $\mathbf{\Sigma} \not\vdash \Gamma' \cdot \Gamma'' \rightarrow C$.

The proof of Theorem 25 is similar to that of Theorem 22.

Lemma 26

(*a*) *Let* Γ'' *be a finite set of formulas. If* $\mathbf{\Sigma} \not\vdash \Gamma'' \rightarrow A\backslash C$, *then for each finite set of formulas* Θ *there exists a set of formulas* $\Theta' \in \Theta^{\sim}$ *such that* Θ', A *is* $\mathbf{\Sigma}$*-consistent and* $\mathbf{\Sigma} \not\vdash \{\Theta', A\} \cdot \Gamma'' \rightarrow C$.

(*b*) *Let* Γ' *be a finite set of formulas. If* $\mathbf{\Sigma} \not\vdash \Gamma' \rightarrow C/B$, *then for each finite set of formulas* Θ *there exists a set of formulas* $\Theta'' \in \Theta^{\sim}$ *such that* Θ'', B *is* $\mathbf{\Sigma}$*-consistent and* $\mathbf{\Sigma} \not\vdash \Gamma' \cdot \{\Theta'', B\} \rightarrow C$.

Proof. We only prove part (*b*) of the lemma. The proof of part (*a*) is symmetric.

Since $\mathbf{\Sigma} \not\vdash \Gamma' \to C/B$, by (the contraposition of) (14), there exists a set of formulas $\Theta'' \in \Theta^{\sim}$ such that

$$\mathbf{\Sigma} \not\vdash \Gamma' \cdot \{\Theta'', B, B_{\Theta''}\} \to C \tag{24}$$

for all formulas $B_{\Theta''}$ satisfying $\mathbf{\Sigma} \vdash \Theta'', B \to B_{\Theta''}$. In particular, it follows that $\mathbf{\Sigma} \not\vdash \Gamma' \cdot \{\Theta'', B\} \to C$.

To show that Θ'', B is $\mathbf{\Sigma}$-consistent, assume to the contrary, that $\mathbf{\Sigma} \vdash \Theta'', B \to$. Then, by (10)($a$), $\mathbf{\Sigma} \vdash \Theta'', B \to \bot$ and we put $B_{\Theta''}$ to be $\bot$. Then, by Example 10(b) and thinnings, of course, $\vdash \Gamma' \cdot B_{\Theta''} \to$, from which, by thinnings, we obtain $\mathbf{\Sigma} \vdash \Gamma' \cdot \{\Theta'', B, B_{\Theta''}\} \to C$, in contradiction with (24). ■

Corollary 27

(a) *Let Γ'' be an infinite set of formulas. If $\mathbf{\Sigma} \not\vdash \Gamma'' \to A \backslash C$, then for each finite set of formulas Θ there exists a set of formulas $\Theta' \in \Theta^{\sim}$ such that Θ', A is $\mathbf{\Sigma}$-consistent and $\mathbf{\Sigma} \not\vdash \{\Theta', A\} \cdot \Gamma'' \to C$.*

(b) *Let Γ' be an infinite set of formulas. If $\mathbf{\Sigma} \not\vdash \Gamma' \to C/B$, then for each finite set of formulas Θ there exists a set of formulas $\Theta'' \in \Theta^{\sim}$ such that Θ'', B is $\mathbf{\Sigma}$-consistent and $\mathbf{\Sigma} \not\vdash \Gamma' \cdot \{\Theta'', B\} \to C$.*

Proof. We only prove part (b) of the corollary. The proof of part (a) is symmetric.

Let $\Gamma' = \{A_1, A_2, \ldots\}$ and let $\Gamma'_n = \{A_1, A_2, \ldots, A_n\}$, $n = 1, 2, \ldots$. Then $\mathbf{\Sigma} \not\vdash \Gamma'_n \to C/B$ and, by Lemma 26, there exists a set of formulas $\Theta''_n \in \Theta^{\sim}$ such that Θ''_n, B is $\mathbf{\Sigma}$-consistent and $\mathbf{\Sigma} \not\vdash \Gamma' \cdot \{\Theta'', B\} \to C$.

Since $\Theta^{\sim}$ is finite, for some $\Theta'' \in \Theta^{\sim}$ there are infinitely many indices n such that $\Theta''_n = \Theta''$. By compactness, $\mathbf{\Sigma} \not\vdash \Gamma' \cdot \{\Theta'', B\} \to C$. ■

Proof of Theorem 25. We only prove part (b) of the theorem. The proof of part (a) is symmetric.

Let $C_1, C_2, \ldots$ be a list of all formulas and let $\Theta_n = \{C_1, C_2, \ldots, C_n\}$.

Consider the infinite tree T whose nodes are sets $\Theta'' \in \Theta^{\sim}_n$, $n = 1, 2, \ldots$, such that Θ'', B is $\mathbf{\Sigma}$-consistent and $\mathbf{\Sigma} \not\vdash \Gamma' \cdot \{\Theta'', B\} \to C$, and $\{A_1, \ldots, A_n\}$ is a child node of $\{A_1, \ldots, A_{n-1}\}$, where $\{A_0\}$ is $\emptyset$.

Obviously, T is a binary tree. Since, by Corollary 27, for each $n = 1, 2, \ldots$, there exists a set of formulas $\Theta'' \in \Theta^{\sim}_n$ such that Θ'', B is $\mathbf{\Sigma}$-consistent and $\mathbf{\Sigma} \not\vdash \Theta'' \cdot \{\widetilde{\Gamma''}, B\} \to C$, the tree is infinite. Thus, by Kőnig's Infinity Lemma ([5]), T has an infinite path. Let Γ'' be the union of of the sets of formulas lying on that path. By compactness, Γ'', B is $\mathbf{\Sigma}$-consistent and $\mathbf{\Sigma} \not\vdash \Gamma' \cdot \{\Gamma'', B\} \to C$. In addition, by Corollary 19, Γ'' is maximal $\mathbf{\Sigma}$-consistent. Thus, it follows from the same corollary that $B \in \Gamma''$. ■

Definition 28. Let $\mathbf{\Sigma}$ be a set of $\boldsymbol{NLN}^+$ sequents. The $\mathbf{\Sigma}$-*canonical* interpretation $\mathfrak{I}_{\mathbf{\Sigma}} = \langle W_{\mathbf{\Sigma}}, R_{\mathbf{\Sigma}}, V_{\mathbf{\Sigma}} \rangle$ is defined as follows.

- $W_{\mathbf{\Sigma}}$ consists of all maximal $\mathbf{\Sigma}$-consistent sets of formulas;
- $R_{\mathbf{\Sigma}} = \{(\Gamma, \Gamma', \Gamma'') \in W_{\mathbf{\Sigma}}^3 : \Gamma' \cdot \Gamma'' \subseteq \Gamma\}$; and
- $V_{\mathbf{\Sigma}}(\Gamma) = \Gamma \cap \mathcal{P}$, where $\mathcal{P}$ is the set of all propositional variables (atomic formulas).

Theorem 29. *Let $\Gamma \in W_{\mathbf{\Sigma}}$. Then, for each formula C, $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma \models C$ if and only if $C \in \Gamma$.*

Proof. The proof is by induction on the complexity of C. The basis (i.e., the case of a propositional variable) is immediate. The cases of all connectives are treated in the standard manner, but, for the sake of completeness, we consider them below.

• Let C be of the form $\neg A$ and let $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma \models \neg A$. By the induction hypothesis $A \notin \Gamma$. Since Γ is maximal $\mathbf{\Sigma}$-consistent, by Corollary 19, $\neg A \in \Gamma$.

Conversely, let $\neg A \in \Gamma$. Then $A \notin \Gamma$, because Γ is $\mathbf{\Sigma}$-consistent. Thus, by the induction hypothesis, $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma \not\models A$, implying, by definition, $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma \models \neg A$.

• Let C be of the form $A \cdot B$ and let $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma \models A \cdot B$. That is, there are $\Gamma', \Gamma'' \in W_{\mathbf{\Sigma}}$ such that $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma' \models A$, $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma'' \models B$, and $\Gamma' \cdot \Gamma'' \subseteq \Gamma$. By the induction hypothesis, $A \in \Gamma'$ and $B \in \Gamma''$, which, together with $\Gamma' \cdot \Gamma'' \subseteq \Gamma$ imply $A \cdot B \in \Gamma$.

Conversely, let $A \cdot B \in \Gamma$. Then, by Theorem 22, there are $\Gamma', \Gamma'' \in W_{\mathbf{\Sigma}}$ such that $A \in \Gamma'$, $B \in \Gamma''$, and $\Gamma' \cdot \Gamma'' \subseteq \Gamma$. By the induction hypothesis, $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma' \models A$ and $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma'' \models B$, implying, by definition, $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma \models A \cdot B$.

• Let C be of the form $A \backslash B$ and let $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma'' \models A \backslash B$. (We have replaced Γ in the statement of the theorem with Γ'', because, in our notation, the latter is the third component of $R_{\mathbf{\Sigma}}$.) Assume to the contrary that $A \backslash B \notin \Gamma''$. Then, by (the contraposition of) Corollary 20, $\mathbf{\Sigma} \not\vdash \Gamma'' \to A \backslash B$ and, by Theorem 25(a), there exists a maximal $\mathbf{\Sigma}$-consistent set of formulas Γ' containing A such that

$$\mathbf{\Sigma} \not\vdash \Gamma' \cdot \Gamma'' \to B$$

It follows that $\Gamma' \cdot \Gamma'', \neg B$ is $\mathbf{\Sigma}$-consistent. Thus, by Proposition 21, there exists a maximal $\mathbf{\Sigma}$-consistent set of formulas Γ including $\Gamma' \cdot \Gamma'', \neg B$.

By definition, $R_{\mathbf{\Sigma}}(\Gamma, \Gamma', \Gamma'')$ and, by the induction hypothesis, $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma' \models A$ and $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma \models \neg B$. This, however, contradicts our assumption $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma'' \models A \backslash B$.

Conversely, let $A \backslash B \in \Gamma''$ and let Γ and Γ' be maximal $\mathbf{\Sigma}$-consistent sets of formulas such that $\Gamma' \cdot \Gamma'' \subseteq \Gamma$ and $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma' \models A$. We have to show that $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma \models B$.

By the induction hypothesis, $A \in \Gamma'$ which, together with $A\backslash B \in \Gamma''$ and $\Gamma' \cdot \Gamma'' \subseteq \Gamma$, implies $A \cdot (A\backslash B) \in \Gamma$. Since the sequent $A \cdot (A\backslash B) \to B$ is derivable in $\boldsymbol{NL}$ and Γ is maximal $\mathbf{\Sigma}$-consistent, by Corollary 20, $B \in \Gamma$. Thus, by the induction hypothesis, $\mathfrak{I}_{\mathbf{\Sigma}}, \Gamma \models B$.

• The case of / is symmetric to that of \ and is omitted. ■

THEOREM 30 (Completeness) *If* $\mathbf{\Sigma} \models \Gamma \to \Delta$*, then* $\mathbf{\Sigma} \vdash \Gamma \to \Delta$*.*

PROOF. Assume to the contrary that

$$\mathbf{\Sigma} \not\vdash \Gamma \to \Delta$$

Then

$$\mathbf{\Sigma} \not\vdash \Gamma, \neg\Delta \to$$

implying that $\Gamma, \neg\Delta$ is $\mathbf{\Sigma}$-consistent.[2] Therefore, by Proposition 21, there exists a maximal $\mathbf{\Sigma}$-consistent set of formulas Θ including $\Gamma, \neg\Delta$. By Theorem 29 and by the definition of the satisfiability relation, $\mathfrak{I}_{\mathbf{\Sigma}}, \Theta \models \Gamma$, but $\mathfrak{I}_{\mathbf{\Sigma}}, \Theta \not\models C$ for all $C \in \Delta$.

Therefore, by the definition of the satisfiability relation, $\mathfrak{I}_{\mathbf{\Sigma}}, \Theta \not\models \Gamma \to \Delta$ which, together with $\mathfrak{I}_{\mathbf{\Sigma}} \models \mathbf{\Sigma}$, contradicts this theorem prerequisite. ■

REMARK 31. Like in [4, Section 5] it can be shown that the relational semantics of $\boldsymbol{NLN}^+$ possess the finite model property. Thus, $\boldsymbol{NLN}^+$ is strongly decidable.

We conclude this section with the *canonical mapping* of an interpretation satisfying a set of formulas $\mathbf{\Sigma}$ into $\mathfrak{I}_{\mathbf{\Sigma}}$.

DEFINITION 32. Let $\mathbf{\Sigma}$ be a set of sequents and let $\mathfrak{I} = \langle W, R, V \rangle$ be an interpretation satisfying $\mathbf{\Sigma}$. The *canonical mapping* $\iota_{\mathfrak{I}} : W \to W_{\mathbf{\Sigma}}$ is defined by $\iota_{\mathfrak{I}}(u) = [u]_{\mathfrak{I}}$, see (17) for the definition of $[u]_{\mathfrak{I}}$.

Since $\mathfrak{I} \models \mathbf{\Sigma}$, this mapping is well-defined.

COROLLARY 33. *Let* $\mathbf{\Sigma}$ *be a set of sequents,* $\mathfrak{I} = \langle W, R, V \rangle$ *be an interpretation satisfying* $\mathbf{\Sigma}$*, and let* $u, v, w \in W$ *be such that* $R(u, v, w)$*. Then* $R_{\mathbf{\Sigma}}(\iota_{\mathfrak{I}}(u), \iota_{\mathfrak{I}}(v), \iota_{\mathfrak{I}}(w))$*.*

PROOF. Assume $R(u, v, w)$ and let $A \in [v]_{\mathfrak{I}}$ and $B \in [w]_{\mathfrak{I}}$. We have to show that $A \cdot B \in [u]_{\mathfrak{I}}$.

By definition, $\mathfrak{I}, v \models A$ and $\mathfrak{I}, w \models B$, which, together with $R(u, v, w)$, implies $\mathfrak{I}, u \models A \cdot B$. Thus, by definition, we have the desired containment $A \cdot B \in [u]_{\mathfrak{I}}$. ■

[2] Of course, $\neg\Delta = \{\neg C : C \in \Delta\}$.

7. Restating the Resolution Rules

It follows from the proofs of Lemmas 23 and 26 that we may restrict two-sided many-formula resolution (12) and one-sided many-formula resolutions (13) and (14) to the cases in which all $A_{\Theta'}$ and all $B_{\Theta''}$ are in $\{\bot, \top\}$.

Therefore, two-sided many-formula resolution (12) can be equivalently restated as:

If the sequent $\Gamma \to A \cdot B$ *is derivable and for each two elements* Θ' *and* Θ'' *of* $\Theta^{\sim}$ *the sequent in one of the clauses* (i)–(iii) *below is derivable, then the sequent* $\Gamma \to$ *is derivable.*

(i) $\Theta', A \to$, *or*

(ii) $\Theta'', B \to$, *or*

(iii) $\Gamma, \{\Theta', A\} \cdot \{\Theta'', B\} \to$,

cf. (22).

This rule of inference will be referred to as *modified* two-sided many-formula resolution.

Lemma 23 follows immediately form (the contraposition of) this rule. Indeed, assume to the contrary that for any two elements Θ' and Θ'' of $\Theta^{\sim}$,

- $\mathbf{\Sigma} \vdash \Theta', A \to$ or
- $\mathbf{\Sigma} \vdash \Theta'', B \to$ or
- $\mathbf{\Sigma} \vdash \Gamma, \{\Theta', A\} \cdot \{\Theta'', B\} \to$.

Then, by the modified two-sided many-formula resolution, we would have $\mathbf{\Sigma} \vdash \Gamma \to$. This, however, contradicts the $\mathbf{\Sigma}$-consistency of Γ.

On the other hand, the modified two-sided many-formula resolution is derivable from Lemma 23 and, therefore, is derivable in $\boldsymbol{NLN^+}$. For the proof, assume to the contrary that we have all the premises of the modified two-sided many-formula resolution, but the sequent $\Gamma \to$ is not derivable. Then, by Lemma 23, there exist $\Theta', \Theta'' \in \Theta^{\sim}$ such that all three sets of formulas in (22) are Σ-consistent. This, however contradicts derivability of the sequent in one of the above clauses (i)–(iii).

Similarly, the one-sided many-formula resolution (13) can be equivalently restated as:

If for each element Θ' *of* $\Theta^{\sim}$ *the sequent in one of the clauses* (i)–(ii) *below is derivable, then the sequent* $\Gamma'' \to A \backslash C$ *is derivable.*

(i) $\Theta', A \to$, *or*

(ii) $\{\Theta', A\} \cdot \Gamma'' \to C$.

Like in the case of the modified two-sided many-formula resolution, it can be shown that this modified one-sided many-formula resolution is equivalent to Lemma 26(a) that follows immediately form (the contraposition of) this rule.

Indeed, assume to the contrary that for each element Θ' of $\Theta^{\sim}$,

- $\mathbf{\Sigma} \vdash \Theta', A \rightarrow$ or
- $\mathbf{\Sigma} \vdash \{\Theta', A\} \cdot \Gamma'' \rightarrow C$.

Then, by the modified one-sided many-formula resolution (13), we would have $\mathbf{\Sigma} \vdash \Gamma'' \rightarrow A \backslash C$. This, however, contradicts the lemma prerequisite.

Conversely, the above modified one-sided many-formula resolution is derivable from Lemma 26(a). For the proof, assume to the contrary that we have all the premises of that modified resolution, but the sequent $\Gamma'' \rightarrow A \backslash C$. is not derivable. Then, by Lemma 26(a), there exists a $\Theta' \in \Theta^{\sim}$ such that Θ', A is Σ-consistent and $\mathbf{\Sigma} \nvdash \{\Theta', A\}\Gamma'' \rightarrow C$. This, however contradicts derivability of the sequent in one of the above clauses (i) or (ii).

Symmetrically, the one-sided many-formula resolution (14) is restated as:

If for each element Θ'' of $\Theta^{\sim}$ the sequent in one of the clauses (i)–(ii) *below is derivable, then the sequent $\Gamma' \rightarrow C/B$ is derivable.*

(i) $\Theta'', B \rightarrow$, *or*

(ii) $\Gamma' \cdot \{\Theta'', B\} \rightarrow C$.

We collect the above equivalences of the resolution rules in Theorem 34 below. To state this theorem, we need one more bit of notation: we denote by $\boldsymbol{NLN}^+_M$ the calculus resulting from $\boldsymbol{NLN}^+$ in replacing rules (12), (13), and (14) with their modified counterparts.

Theorem 34. *Calculi $\boldsymbol{NLN}^+$ and $\boldsymbol{NLN}^+_M$ are deductively equivalent.*

Even though, using $\boldsymbol{NLN}^+_M$ instead of $\boldsymbol{NLN}^+$ shortens the proofs of Lemmas 23 and 26, the former looks very nontraditional. We believe that the rules (12), (13), and (14) are much easier to comprehend.

We conclude this section with the derivation of rules (12) and (13) in $\boldsymbol{NLN}^+_M$, but with a different set of resolution formulas, cf. Remark 7. The derivations of (14) from its modification is similar. Note that the above derivability is already provided by the completeness theorem for the modified rules (which are equivalent to Lemmas 23 and 26).

So, for rule (12), we shall prove that

$$\Gamma, \neg\Delta \rightarrow \tag{25}$$

is derivable from the premises of (12) by the modified two-sided many-formula resolution. Then the desired sequent $\Gamma \to \Delta$ would follow from (25) by (9) and (11).

The premise $\Gamma, \neg\Delta \to A \cdot B$ of the modified rule follows from the premise $\Gamma \to \Delta, A \cdot B$ of (12) by (9)(b).

Given the other premises of (12), we contend that all the premises of its modification in which the set of resolution formulas is the union of Θ from (12),

$$\Theta_1 = \{A_{\Theta'} : \Theta', A \to A_{\Theta'} \text{ is a premise of } (12)\}$$

and

$$\Theta_2 = \{B_{\Theta''} : \Theta'', B \to B_{\Theta''} \text{ is a premise of } (12)\}$$

are derivable.

Let

$$\Theta'_0, \Theta'_0\prime \in (\Theta \cup \Theta_1 \cup \Theta_2)^{\sim}$$

Then

- $\Theta'_0 = \Theta' \cup \Theta'_1 \cup \Theta'_2$, where $\Theta' \in \Theta^{\sim}$, $\Theta'_1 \in \Theta_1^{\sim}$, and $\Theta'_2 \in \Theta_2^{\sim}$; and
- $\Theta'_0\prime = \Theta'' \cup \Theta''_1 \cup \Theta''_2$, where $\Theta'' \in \Theta^{\sim}$, $\Theta''_1 \in \Theta_1^{\sim}$, and $\Theta''_2 \in \Theta_2^{\sim}$.

Now, if $\neg A_{\Theta'} \in \Theta'_1$, then $\Theta', \Theta'_1, \Theta'_2, A \to$ (that is premise (i) of the modified rule) is derivable from premise $\Theta', A \to A_{\Theta'}$ of (12).

If $\neg B_{\Theta''} \in \Theta''_2$, then $\Theta'', \Theta''_1, \Theta''_2, B \to$ (that is premise (ii) of the modified rule) is derivable from premise $\Theta'', B \to B_{\Theta''}$ of (12);.

Finally, if $A_{\Theta'} \in \Theta'_1$ and $B_{\Theta''} \in \Theta''_2$, then

$$\Gamma, \neg\Delta, \{\Theta', \Theta'_1, \Theta''_2, A\} \cdot \{\Theta'', \Theta'_1, \Theta''_2, A\} \to$$

(that is premise (iii) of the modified rule) is derivable from premise

$$\Gamma, \{\Theta', A, A_{\Theta'}\} \cdot \{\Theta'', B, B_{\Theta''}\} \to \Delta$$

of (12).

Thus, by the modified two-sided many-formula resolution, (25) is derivable as well.

The case of one-sided many formula-resolution (13) and its modification is treated in a similar manner. Given the premises of rule (13), we contend that all the premises of its modification in which the set of resolution formulas is the union of Θ from (13) and

$$\Theta_1 = \{A_{\Theta'} : \Theta', A \to A_{\Theta'} \text{ is a premise of } (13)\}$$

are derivable.

Indeed, each set in $(\Theta \cup \Theta_1)^\sim$ is of the form $\Theta' \cup \Theta'_1$, where $\Theta' \in \Theta^\sim$ and $\Theta'_1 \in \Theta_1^\sim$.

If $\neg A_{\Theta'} \in \Theta'_1$, then $\Theta', \Theta'_1, A \rightarrow$ (that is premise (i) of the modified rule) is derivable from premise $\Theta', A \rightarrow A_{\Theta'}$ of (13) and, if $A_{\Theta'} \in \Theta'_1$, then $\{\Theta', \Theta'_1, A\} \cdot \Gamma'' \rightarrow C$ (that is premise (ii) of the modified rule) is derivable from premise $\{\Theta', A, A_{\Theta'}\} \cdot \Gamma'' \rightarrow C$ of (13).

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