Sven Ove Hansson

A Characterization of Probability-based Dichotomous Belief Revision

Abstract. This article investigates the properties of *multistate top revision*, a dichotomous (AGM-style) model of belief revision that is based on an underlying model of probability revision. A proposition is included in the belief set if and only if its probability is either 1 or infinitesimally close to 1. Infinitesimal probabilities are used to keep track of propositions that are currently considered to have negligible probability, so that they are available if future information makes them more plausible. Multistate top revision satisfies a slightly modified version of the set of basic and supplementary AGM postulates, except the inclusion and success postulates. This result shows that hyperreal probabilities can provide us with efficient tools for overcoming the well known difficulties in combining dichotomous and probabilistic models of belief change.

Keywords: Multistate top revision, Probability revision, Belief change, Quasi-revision, AGM postulates, Infinitesimal probabilities, Hyperreal probabilities.

1. Introduction

Some human beliefs are best described as all-or-nothing phenomena, whereas other beliefs come in degrees. In formal epistemology, an agent's all-or-nothing beliefs are represented by a set of propositions, whereas degrees of beliefs are usually represented by probabilities. Actual epistemic agents have both types of beliefs, and we usually do not find it difficult to combine them or shift between them. However, in formal models the two types of beliefs are notoriously difficult to combine. The most obvious solution is to equate full dichotomous belief in a proposition with assigning probability 1 to it. Unfortunately, this does not work since classical probability theory lacks means to change the probability of a proposition once it has obtained probability 1, whereas any reasonably realistic model of full beliefs will have to contain means for giving them up. Probability limits below 1 give rise to paradoxical results such as the lottery and preface paradoxes ([25], pp. 197–198; [26], esp. pp. 54–68 and 112–158,[28]). More complex criteria,

Presented by **Heinrich Wansing**; *Received* November 23, 2020

https://doi.org/10.1007/s11225-021-09961-2

based on comparisons between the probabilities of believed and non-believed propositions, have also turned out to have implausible properties [16, 35].

This article is part of a project aimed at combining dichotomous and probabilistic beliefs in formal modelling in roughly the same way that we combine them in everyday reasoning: Having full belief in a proposition does not mean that we consider it impossible to doubt it or to give it up. It only means that we currently see no reason to do so. Conversely, we consider the probability that such a proposition is false to be negligible but not nil. Hence, there are empirical propositions which we treat as having probability 0, but we are willing to change our minds and assign a non-zero probability to them if new information gives us reason to do so. Such changes are far from uncommon in real life, and a person unable to change her mind in this way would presumably be considered to be irrational. But as already mentioned, this type of epistemic change is not representable in standard probability theory. Once a proposition has been assigned probability 0 or probability 1, no further revision or series of revisions can change it. One of the major motivations behind the model of probability revision to be discussed here is to solve this problem, and make it possible to give up zero and unit probabilities.

In studies of changes in dichotomous beliefs, a distinction is commonly made between change-recording and knowledge-enhancing beliefs [20, 22]. Suppose that a friend tells you, looking out of the window, that there is a raven in the garden. Believing him, you walk up to the window to have a look at the bird. If you find your friend pointing at a rook, which he has misidentified as a raven, then you perform a knowledge-enhancing belief change (often called revision). The bird is the same, but you now have improved knowledge about it. But suppose instead that when you look out of the window, you see no bird, and your friend tells you that it just flew away. This is a change-recording belief change (often called update).

An analogous distinction can be applied to changes in probabilities. Suppose that a friend announces that she will throw nine dice at one time to see if she can get all sixes. You make a quick calculation and tell her that the probability of getting nine sixes is about one in ten million. She throws the dice and to your amazement she throws nine sixes. You can now perform two types of probability revision. One of these is the rather trivial conclusion from seeing the nine sixes: After the fact, the probability that she *has thrown* nine sixes is 1. This is a change-recording probability revision. The other type of revision is more subtle. What you have seen makes you suspect that the dice are loaded, so that the probability that she *would throw* nine sixes was considerably higher than one in ten million. If you assume that she would only do the trick if it works at least half of the times, you might for instance estimate the probability that she would get nine sixes to be around 50 per cent. The latter probability revision will influence your estimate of the probability that she will get the same outcome if she makes a new throw with the same nine dice, and in consequence your attitude to various bets on the outcome of the next throw will also change. Such knowledge-enhancing probability revisions have a central role in learning. It is for instance the mechanism by which we change our opinion on the probability of nuclear accidents, based on how many such accidents actually happen.

In Hansson [17], a *multistate* model of knowledge-enhancing probability revision was presented. It is based on a two-layered model of the world. On one layer, the world is conceived as being in one of several states, which cannot be observed directly. The other layer consists of observable events, whose probabilities differ between the different states. From our observations we can therefore draw inferences on the probabilities of the states. In the dice-throwing example, we first assumed a state of the world in which the probability of a six was one in six for each of the nine dice. After the first throw, we also considered a state of the world in which all the dice were loaded to yield a six. This is a state that we had previously considered to be negligible, or did not think of. Its probability was then representable as infinitesimally small, which is the reason why it had no influence on our first probability estimate of a throw resulting in nine sixes.

The multistate model of probability revision gives rise to a derived model of dichotomous revision, in which only the full beliefs (propositions with probabilities infinitesimally close to 1) are considered. The present contribution is devoted to an investigation of the properties of this probability-based belief change model.

Section 2 provides some formal preliminaries. Section 3 introduces the multistate model of probability revision and multistate top revision, the dichotomous belief change operation that is based on it. In Section 4, sequential quasi-revision is introduced. It is an AGM-style operation that will be highly useful in our analysis. In Section 5, the properties of multistate top revision are investigated, and its relation to AGM revision is scrutinized. All formal proofs are deferred to an appendix.

2. Formal Preliminaries

Sentences, i.e., elements of the language that express propositions, are represented by lowercase letters ($a, b, \ldots$) and sets of sentences by capital letters

$(A, B, \ldots)$. The object language is formed from atomic sentences with the usual truth-functional connectives: negation ($\neg$), conjunction (&), disjunction ($\vee$), implication ($\rightarrow$), and equivalence ($\leftrightarrow$). $\top$ is a tautology and $\perp$ a logically contradictory sentence.

A Tarskian consequence operation Cn expresses the logic. It satisfies the standard conditions: inclusion ($A \subseteq \mathrm{Cn}(A)$), monotony (If $A \subseteq B$, then $\mathrm{Cn}(A) \subseteq \mathrm{Cn}(B)$) and iteration ($\mathrm{Cn}(A) = \mathrm{Cn}(\mathrm{Cn}(A))$). Furthermore, Cn is supraclassical (if p follows from A by classical truth-functional logic, then $p \in \mathrm{Cn}(A)$) and compact (if $p \in \mathrm{Cn}(A)$, then there is a finite subset A' of A such that $p \in \mathrm{Cn}(A')$), and it satisfies the deduction property ($q \in \mathrm{Cn}(A \cup \{p\})$ if and only if $p \rightarrow q \in \mathrm{Cn}(A)$). $\mathrm{Cn}(\varnothing)$ is the set of tautologies. $X \vdash p$ is an alternative notation for $p \in \mathrm{Cn}(X)$ and $\vdash p$ for $p \in \mathrm{Cn}(\varnothing)$.

A set A of sentences is a (consistent) *belief set* if and only if it is consistent and logically closed, i.e. $A = \mathrm{Cn}(A) \neq \mathrm{Cn}(\{\perp\})$. K denotes a belief set. The conjunction of all elements of a finite set A of sentences is denoted $\&A$, and their disjunction is denoted $\bigvee A$. For any finite set A of sentences, $\mathsf{numb}(A)$ is the number of logically non-equivalent elements of A. For all sets A of sentences and all sentences a, the *remainder set* $A \perp a$ is the set of maximal subsets of A not implying a.

The following notation will be used for finite hyperreal numbers:

Definition 1. (1) The letters s, t, u, v, x, y, and z represent hyperreal numbers (which may be real). The letters δ and ϵ represent numbers that are either 0 or infinitesimal.

(2) The standard (real) part of a finite hyperreal number s is denoted $\mathsf{st}(s)$.

(3) The symbols $\approx$, $\not\approx$, $\lessapprox$, and $\ll$ are used as follows:

$a \approx b$ if and only if $\mathsf{st}(a) = \mathsf{st}(b)$

$a \not\approx b$ if and only if $\mathsf{st}(a) \neq \mathsf{st}(b)$

$a \lessapprox b$ if and only if $a < b$ and $\mathsf{st}(a) = \mathsf{st}(b)$

$a \ll b$ if and only if $\mathsf{st}(a) < \mathsf{st}(b)$

Keisler [21] is an accessible introduction to hyperreal numbers. A very brief introduction to finite hyperreal numbers can be found in Hansson ([17], p. 1024).

3. Multistate Models and Probability Revisions

The multistate model of probability revision [17] is based on a combination of several modifications of the standard Bayesian approach to probability revision, intended to adjust the model to satisfy two major criteria, namely (1) to allow for changes in full beliefs, and (2) to specifically mirror knowledge-enhancing changes ("revisions") rather than change-recording ones ("updates"). The crucial means to satisfy the first of these criteria is to extend the codomain of the probability function from the real-valued interval [0,1] to the hyperreal-valued interval with the same limits. This means that infinitesimal probabilities, i.e. probabilities whose value is larger than 0 but smaller than all positive real numbers, become available. Similarly, probabilities infinitesimally smaller than 1 can be used. These are probabilities that are smaller than 1 but larger than all real numbers smaller than 1.

Infinitesimal probabilities have mostly been used to solve problems arising when classical, real-valued probabilities are assigned to infinite domains [4, 36].[1] Here they will be used as tools to record or memorize the quantitative relations between probabilities that the agent currently treats as negligible. For instance, consider an epistemic agent who currently fully believes in the statement d, "Dar es Salaam is the capital of Tanzania". She has no doubt about it, and expresses the same confidence in this statement as in the statement that Paris is the capital of France. In standard probability theory, we would represent this full belief by assigning probability 1 to d. This has the unfortunate consequence that there is no way to give up the belief. To give it up by Bayesian revision, there would have to be some statement e such that $\mathfrak{p}(d \mid e) \neq 1$, or equivalently $\mathfrak{p}(d\&e)/\mathfrak{p}(e) \neq 1$, which is impossible since it follows from $\mathfrak{p}(d) = 1$ that $\mathfrak{p}(d\&e) = \mathfrak{p}(e)$.

This, however, is easily solved when we have access to hyperreal probabilities.Then we can have $\mathfrak{p}(d) = 1 - \delta$, for some infinitesmal number δ. Now we can let e be the surprising and unforeseen event that the headlines on the front page of *New York Times* refer to some other city (such as Dodoma) as the capital of Tanzania. This was something that our epistemic agent considered to be incredible before it happened, and thus $\mathfrak{p}(e)$ had an infinitesimal prior probability. However, she relies on this newspaper, and therefore she assimilates the new information. The outcome of her Bayesian revision by

[1] Much more commonly, arbitrarily small real-valued probabilities are referred to, see for instance Adams [1] and Pearl [31]. The latter called these probabilities "infinitesimal", but they are not infinitesimal in the sense in which this term is used in nonstandard analysis and the study of hyperreal numbers.

e will be a new probability function $\mathfrak{p}'$ such that $\mathfrak{p}'(d) = \mathfrak{p}(d\&e)/\mathfrak{p}(e)$. Presumably, $\mathfrak{p}(d\&e)$ is even much smaller than $\mathfrak{p}(e)$, and consequently she will give up her previous belief in d in this operation.

We have assumed in this example that an agent has full belief in a statement a if and only if she assigns to it a probability that is at most infinitesimally smaller than 1, that is, if and only if $\mathfrak{p}(a) \approx 1$. This is a plausible approximation, since $\mathfrak{p}(a) \approx 1$ implies that $\mathfrak{p}(a)$ is larger than all real numbers below 1. However, it should be emphasized that infinitesimal numbers are used here only for modelling purposes. Their structure makes them suitable as tools to represent beliefs that are currently but not unchangeably undoubted (probability close to 1) or currently but not unchangeably considered negligible (probability close to 0). This does not imply that probabilistic beliefs have any metaphysical properties imputed to infinitesimal numbers.

In the above example, we applied Bayesian revision to hyperreal probabilities. In other words, we assumed that, when we revise a probability function $\mathfrak{p}$ by a sentence a that has non-zero probability, the outcome will be a new probability function $\mathfrak{p}'$, such that:

$$\mathfrak{p}'(e) = \frac{\mathfrak{p}(e\&a)}{\mathfrak{p}(a)}$$

for all e. However, this has the unfortunate consequence that $\mathfrak{p}'(a) = 1$. If we use this method to revise, then each revision input will be ineradicably inserted into the belief set. Even if we start out with a probability function that allows all empirical beliefs to be revised (i.e., does not assign probability 0 or 1 to any of them), this openness to new information will gradually be lost as we make more and more revisions. To avoid this, and make full use of the openness achieved with hyperreal probabilities, we need to apply some revision method that does not share this disadvantage. For that purpose, we can apply Jeffrey conditionalization ([19], pp. 171–172) in such manner that, when we revise a probability function $\mathfrak{p}$ by a sentence a with non-zero and non-unit probability, the outcome will be a new probability function $\mathfrak{p}'$ defined as follows:

$$\mathfrak{p}'(e) = (1-\delta) \times \frac{\mathfrak{p}(a\&e)}{\mathfrak{p}(a)} + \delta \times \frac{\mathfrak{p}(\neg a\&e)}{\mathfrak{p}(\neg a)}$$

Provided that we always set $\delta > 0$ for revisions by empirical inputs, this revision method will ensure that all beliefs that are given up in a revision process will be accessible for future reinsertion, if the need arises.

Let us now turn to the other major adjustment of traditional probability revision that we set out to achieve, namely to make it specifically model a knowledge-enhancing rather than a change-recording process (revision rather than update). The best way to see what this requires of the model is to consider a couple of examples. We can begin with the dice example from Section 1, in which we first assumed that the true state was b_1, in which the probability that a dice throw yields a six is 1/6. After observing the surprising outcome of a throw with nine dice, we reduced the probability of b_1, and instead assigned a considerable amount of probability to a state b_2 in which the dice are loaded. Changes in other probabilities were a consequence of the changes of the probabilities of these two states. This was a knowledge-enhancing belief change.

For another example, suppose that a large pressure tank has been installed close to a playground in your neighbourhood. Government experts assured that the probability that the tank would explode in its expected lifetime of 50 years is about 1 in 1,000,000,000. Therefore, you did not worry about it. But after less than two months, the tank exploded, luckily in the middle of the night and with no casualties or injuries. Your judgment of the safety situation will then depend on the knowledge-enhancing revision of your probabilistic beliefs ("Given what we now know, how probable was it that the tank would explode?"), which has important implications for future instalments of similar tanks. (The outcome of the change-recording revision, namely that the probability is 1 that the tank exploded, is much less informative for that purpose.) When performing the knowledge-enhancing revision, your focus will be on the situation before the explosion, i.e., on the properties of the tank and in particular on how probable the event that took place was. The crucial issue is: Was the tank in a state with a very low probability of an accident, as you had been assured before, or was it in a state with a considerably higher risk of an accident?

Thus, in both examples, knowledge-enhancing revision has a focus on identifying possible states of the world and their probabilities. To construct a formal framework it is therefore useful to introduce separate representations for such states and for observable events:

Definition 2. [17] Let $\mathfrak{p}$ be a (hyperreal-valued) probability function.[2]

(1) An *observational language* for $\mathfrak{p}$ is a non-empty set L_E of sentences within its domain, such that:

(a) If $a_1, a_2 \in L_E$, then $a_1 \& a_2 \in L_E$.
(b) If $a \in L_E$, then $\neg a \in L_E$.

[2]By this is meant a function with a logical language as its domain and the closed hyperreal interval [0,1] as codomain. L_E and B are subsets of its domain.

(2) A *state catalogue* for $\mathfrak{p}$ is a non-empty set B of atomic sentences within the domain of $\mathfrak{p}$, such that:

(a) If $b \in B$, then $\mathfrak{p}(b) \neq 0$.

(b) If $b_1, b_2 \in B$ and $\nvdash b_1 \leftrightarrow b_2$, then $\mathfrak{p}(b_1 \& b_2) = 0$.

(c) $\sum_{b \in B} \mathfrak{p}(b) = 1$

(3) An observational language L_E for $\mathfrak{p}$ is *logically disjoint from* a state catalogue B for $\mathfrak{p}$ if and only if no logically contingent truth-functional combination of elements of L_E follows logically from some logically contingent truth-functional combination of elements of B.

Probability revision will be expressed in formal notation in much the same way as revision in (dichotomous) belief change theory. When the probability function $\mathfrak{p}$ is revised by the input sentence a, this gives rise to a new probability function, denoted $\mathfrak{p} \star a$. Hence the probability of the state of the world b after revision by a will be denoted $(\mathfrak{p} \star a)(b)$. The major advantage of this notation over the traditional conditional notation $\mathfrak{p}(b \mid a)$ is that iterated revisions (iterated conditionalizations) can be clearly expressed, e.g. $(\mathfrak{p} \star a_1 \star a_2)(b)$. (To facilitate reading of the formulas, boldface brackets are placed around subformulas expressing a revised probability function.)

The following definition of the multistate model summarizes the above considerations:

Definition 3. [17] A *multistate model* of (hyperreal) probability revision is a quadruple $\langle \mathfrak{p}, B, L_E, \star \rangle$. $\mathfrak{p}$ is a (hyperreal-valued) probability function, B a state catalogue for $\mathfrak{p}$, and L_E an observational language for $\mathfrak{p}$ that is logically disjoint from B. $\star$ is a two-place operation of revision for $\mathfrak{p}$ that takes a pair consisting of a number δ with $0 \leq \delta \approx 0$ and a sentence a as input, and produces a new probability function as output. The output is denoted $\mathfrak{p} *_\delta a$. Furthermore, for all $a_1, a_2 \in L_E$, all $b \in B$, and all δ with $0 \leq \delta \approx 0$:

If $\mathfrak{p}(a_1) = 0$ or $\mathfrak{p}(a_1) = 1$, then:

(0) $\mathfrak{p} \star_\delta a_1 = \mathfrak{p}$

If $0 \neq \mathfrak{p}(a_1) \neq 1$, then:

(1) $(\mathfrak{p} \star_\delta a_1)(b) = \dfrac{\mathfrak{p}(a_1 \& b)}{\mathfrak{p}(a_1)} + \delta \times \left(\dfrac{\mathfrak{p}(\neg a_1 \& b)}{\mathfrak{p}(\neg a_1)} - \dfrac{\mathfrak{p}(a_1 \& b)}{\mathfrak{p}(a_1)} \right)$

(2) $(\mathfrak{p} \star_\delta a_1)(a_2 \mid b) = \mathfrak{p}(a_2 \mid b) = \dfrac{\mathfrak{p}(a_2 \& b)}{\mathfrak{p}(b)}$

$$\text{(3)}\ \ (\mathfrak{p} \star_\delta a_1)(a_2) = \sum_{b\in B}\left(\frac{\mathfrak{p}(a_1\&b)}{\mathfrak{p}(a_1)} + \delta\left(\frac{\mathfrak{p}(\neg a_1\&b)}{\mathfrak{p}(\neg a_1)} - \frac{\mathfrak{p}(a_1\&b)}{\mathfrak{p}(a_1)}\right)\right) \times \frac{\mathfrak{p}(a_2\&b)}{\mathfrak{p}(b)}$$

As can be seen from clauses (1) and (2), revisions change the probabilities of the alternative states of the world (elements of B), whereas the states themselves are not changed. For any observable $a \in L_E$ and state $b \in B$, the conditional probability of a given b is taken to be constant.[3] Thus this model is based on the (idealized) assumption that all states, i.e., all elements of the state catalogue B, are well-defined in terms of their observational consequences. The probabilities of elements of B are the primary objects of (knowledge-enhancing) change, and changes in the probabilities of observables are secondary to these primary changes. This is the pattern we saw in the above two examples in which the primary objects of change were the properties of the dice, respectively the pressure tank.

The following definition provides some further specifications:

Definition 4. Let $\langle \mathfrak{p}, B, L_E, \star\rangle$ be a multistate model of probability revision.

(1) It is *finite* if and only if both B and L_E are finite.

(2) It is *orderly* if and only if it holds for all $a \in L_E$ and $b \in B$ that $\mathfrak{p}(a \mid b)$ is a real number.

(3) The model $\langle \mathfrak{p}, B, L_E, \star_0\rangle$ is the *zero restriction* of $\langle \mathfrak{p}, B, L_E, \star\rangle$ if and only if $\star_0$ coincides with $\star$ for the index 0, but takes no other index than $\delta = 0$.

Orderliness, as defined here, implies that non-standard (hyperreal but not real) probabilities are primarily assigned only to elements of B, and not to the event probabilities that are conditional on elements of B (such as $\mathfrak{p}(a \mid b)$). This is plausible, since the only role of infinitesimals in this model is to preserve information about currently unconsidered elements of B for possible future reconsideration. There is therefore no need to assign non-standard values to expressions such as $\mathfrak{p}(a \mid b)$.

We are now ready to introduce the dichotomous belief change operation that can be derived from a multistate model of probability revision. It will be called a *multistate top revision* since the belief set it revises is the "top" of the probability function, namely the set of beliefs to which it assigns probabilities at most infinitesimally smaller than 1.

[3] This is somewhat similar to Kern-Isberner's notion of a c-change, an operation of change in which some conditionals are kept constant [3,23].

Definition 5. Let $\mathfrak{p}$ be a hyperreal probability function. The set

$[\![\mathfrak{p}]\!] = \{a \in L_E \mid \mathsf{st}(\mathfrak{p}(a)) = 1\}$

is the *belief set* generated by $\mathfrak{p}$.

Observation 1. *Let* $\mathfrak{p}$ *be a hyperreal probability function. Then:*

$[\![\mathfrak{p}]\!] = \mathrm{Cn}([\![\mathfrak{p}]\!])$ *(top closure)*

Definition 6. Let K be a consistent belief set and $*$ a sentential operation on K.[4] Then $*$ is a *multistate top revision* on K if and only if there is some multistate model $\langle \mathfrak{p}, B, L_E, \star \rangle$ such that $K = [\![\mathfrak{p}]\!]$ and $K * a = [\![\mathfrak{p} \star_\delta a]\!]$ for all $a \in L_E$ and all δ.[5]

A multistate top revision is *finite* if and only if it is based on some finite multistate model. It is *orderly* if and only if it is based on some orderly multistate model.

The following observation may at first sight seem to undermine our use of infinitesimals.

Observation 2. *Let* $\mathfrak{p}$ *be the probability function of a multistate model* $\langle \mathfrak{p}, B, L_E, \star \rangle$ *of probability revision, such that* B *is finite, and let* $0 \leq \delta \approx 0$. *Then:* $[\![\mathfrak{p} \star_\delta a]\!] = [\![\mathfrak{p} \star_0 a]\!]$.

If it makes no difference for the outcome what index we use for the Jeffrey conditionalization, why not just use standard conditionalization and leave out the infinitesimals? The answer is that the infinitesimals are needed as a "memory function" for future revisions. The effects on the belief set of introducing infinitesimals through Jeffrey conditionalization will not be seen in the revision in which they are introduced, but they can have large impacts in later revisions. To see how this works, consider a simple example in which $B = \{b_1, b_2\}$, $\mathfrak{p}(b_1) = \mathfrak{p}(b_2) = 0.5$, and $\mathfrak{p}(a \mid b_1) = \mathfrak{p}(\neg a \mid b_2) = 1$. Let $0 < \delta \approx 0$ and $0 \leq \delta' \approx 0$. Then $[\![\mathfrak{p} \star_\delta a]\!] = [\![\mathfrak{p} \star_0 a]\!]$, whereas $\neg a \in [\![\mathfrak{p} \star_\delta a \star_{\delta'} \neg a]\!]$ and $a \in [\![\mathfrak{p} \star_0 a \star_{\delta'} \neg a]\!]$. In order to obtain one of the main advantages of the multistate model, namely the revisability of full beliefs in a probabilistic model, revision indices should be infinitesimal rather than 0. However, since we are concerned here only with single-step revision, the index 0 will be used in proofs for technical convenience.

[4] A sentential operation on a belief set K is an operation $\circ$ such that for any input sentence a, it produces a new belief set, denoted $K \circ a$.

[5] In principle, the dichotomous operation should have a δ index in the last equation ($K *_\delta a = [\![\mathfrak{p} \star_\delta a]\!]$ instead of $K * a = [\![\mathfrak{p} \star_\delta a]\!]$). We will see in Observation 2 that this index can be omitted.

4. Sequential Quasi-revision

When we perform knowledge-enhancing probability revision – which is what the multistate model is constructed to mirror – the temporal position of the probability function is left unchanged. In our dice example, the outcome of the knowledge-enhancing revision referred to what the probability of nine sixes was at the same time that the original probability referred to (namely just before the dice were thrown), contrary to the change-recording revision, which involved a shift to a later point in time. In consequence, the derived revision of dichotomous belief – the multistate top revision – should also leave the time perspective unchanged. When we learn that an event a has taken place, we cannot in general draw the conclusion that just before it happened, it was certain to take place. What we can conclude from observing a, however, is that just before a happened, it was not certain that a would not take place (or in other words, we can conclude that a was possible at the time, cf. [27] and [9]).[6] This means that for a knowledge-enhancing revision, it is sensible to weaken the standard success criterion of belief revision, $a \in K * a$ [2], to $\neg a \notin K * a$ (excepting the limiting case in which a is inconsistent and thus $\neg a$ is a tautology). An operation with that success criterion will be called an operation of *quasi-revision*.[7]

The following definition introduces a model of quasi-revision. As will be seen in what follows, it coincides in the finite case with multistate top revision (Definition 6). This equivalence significantly simplifies formal investigations of multistate top revision.

Definition 7. Let K be a consistent belief set in a finite language L_E and $*$ a sentential operation on K. Then $*$ is a *sequential quasi-revision* on K if and only if there is a series $\langle \mathbb{X}_0, \dots, \mathbb{X}_n \rangle$ of sets of consistent belief sets, such that $K \in \mathbb{X}_0$, $K = \bigcap \mathbb{X}_0$, and for all a:

---

[6] Furthermore, our estimate of what the probability was that a would happen will not decrease when we learn that a did in fact take place. In some cases, it can be expected to increase. If an event occurs that was previously considered to be extremely unlikely, then this can be a rational reason to believe that the previous estimate was an underestimate. In other cases, there will be no change. For instance, if you toss an ordinary coin, and it yields heads, this does not make you change your previous belief that the probability that it would yield heads was 0.5. (The probability that it yield*ed* heads is 1.0, but that is the outcome of a change-recording, not a knowledge-enhancing revision.)

[7] The success criterion of revocation ($\dot{-}$) by a is $a \notin K \dot{-} a$ ([15], p. 133). Therefore quasi-revision by a coincides with revocation by $\neg a$.

(1) if $\neg a \in \bigcap(\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_n)$, then $K * a = K$, and

(2) otherwise, $K * a = \bigcap\{X \mid \neg a \notin X \in \mathbb{X}_k\}$, where $\mathbb{X}_k$ is the first element in the sequence $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$ whose intersection does not contain $\neg a$ (i.e. $\neg a \in \bigcap(\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_{k-1})$ and $\neg a \notin \bigcap \mathbb{X}_k$).

A sequential quasi-revision on K is *finite* if and only if it is based on a finite sequence $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$, such that its elements and the elements of its elements are all finite. It is *accumulative* if and only if it is based on some sequence $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$ such that $\mathbb{X}_{k-1} \subseteq \mathbb{X}_k$ for all k with $0 < k \leq n$.

5. Properties of the Probability-based Operation of Belief Revision

Our main result is an equivalence result for multistate top revision in the finite and orderly case:

THEOREM 1. *Let K be a consistent belief set in a finite language L_E, and let $*$ be a sentential operation on K. Then the following conditions are equivalent:*

(1) $*$ is a finite and orderly multistate top revision on K,

(2) $*$ is a finite sequential quasi-revision on K, and

(3) $*$ is a finite and accumulative sequential quasi-revision on K. Furthermore, an operation satisfying these conditions also satisfies the following postulates:

- $K * a = \mathrm{Cn}(K * a)$ *(Closure)*
- *If $\vdash a_1 \leftrightarrow a_2$, then $K * a_1 = K * a_2$. (Extensionality)*
- *$\neg a \notin K * a$ or $K * a = K$. (Relative quasi-success)*
- *If $\neg a \in K * a$, then $\neg a \in K * d$. (Quasi-regularity)*
- *$K * \top = K$ (Tautology inertness)*
- *If $\neg a_1 \notin K * a_1$, $\neg a_2 \notin K * a_2$, and $K * a_1 = K * a_2$, then $K * (a_1 \vee d) = K * (a_2 \vee d)$. (Disjunctive equivalence)*
- *$K * (a_1 \vee a_2)$ is equal to one of $K * a_1$, $K * a_2$, and $K * a_1 \cap K * a_2$. (Disjunctive factoring)*
- *If $\neg a \notin K * d$, then $K * (a \vee d) \subseteq K * a$. (Linearity)*

Closure and *extensionality* are standard properties in the AGM tradition, used in Alchourrón et al. [2] and numerous other publications on belief change.

Relative quasi-success and *quasi-regularity* are similar in structure to two weakenings of *success* that are known from the literature on non-prioritized belief revision, namely *relative success* ($a \in K * a$ or $K * a = K$) and *regularity* (if $a \notin K * a$ then $a \notin K * d$).[8]

Tautology inertness is a weak postulate that also holds in AGM. It is highly plausible; we do not expect the addition of a tautology to an already logically closed belief set to make any difference.

Disjunctive factoring, which holds in full AGM revision and is closely related to the AGM supplementary postulates, seems to have been introduced by Hans Rott ([11], pp. 57 and 244). *Disjunctive equivalence* also holds in full AGM revision, but does not seem to have been mentioned in the literature.[9]

Linearity is probably the postulate in Theorem 1 that is most in need of explanation. If *quasi-regularity* holds, then linearity is equivalent with:

If $\neg a \notin K * d$ and $\neg a \notin K * a$, then $K * (a \vee d) \subseteq K * a$

Let us assume that different quasi-revision outcomes (outcomes of applications of $*$) differ in how "costly" they are in terms of the disadvantages associated with them. We can then expect the epistemic agent to have a preference ordering over the quasi-revision outcomes, and always to choose the most preferred outcome among those that satisfy the purpose of the operation. Since ties have to be excluded, this must be a strict ordering. In quasi-revising by a, the purpose is to open up for a (make sure that a is not held to be impossible). $K * a$, the quasi-revision by a, should therefore be the most preferred way to open up for a. If $\neg a \notin K * d$, then quasi-revising by d is one of the ways to open up for a. This means that the most preferred way to open up for a is at least as preferred as the most preferred way to open up for d. The operation $K * (a \vee d)$ has the purpose to open up for $a \vee d$, which is equivalent with opening up for a or for d.[10] The most preferred way to do so should then quasi-revise by a. Thus we should expect $K * (a \vee d)$ to be a subset of $K * a$.

It may be surprising that the AGM postulate *consistency* (if $a \nvdash \perp$, then $K * a \nvdash \perp$) does not appear in the theorem. But in fact an even stronger postulate of consistency holds. It follows from two of the other postulates:

Observation 3. *Let $*$ be a sentential operation on a consistent belief set K. If it satisfies* closure *and* relative quasi-success*, then it satisfies:*

[8] [14], pp. 417–418. Both postulates also have analogues among the axioms for non-prioritized contraction ([33], p. 54; [8], p. 86).

[9] Possibly the best way to see that *disjunctive equivalence* holds in full AGM revision is to use Grove spheres [12].

[10] Note that a set X of propositions is consistent with $a \vee d$ if and only if it is consistent with a or consistent with d. ($X \cup \{a \vee d\} \nvdash \perp$ if and only if $X \cup \{a\} \nvdash \perp$ or $X \cup \{d\} \nvdash \perp$.)

$K * a \nvdash \bot$ *(strong consistency)*

Closure is incompatible with *quasi-success* ($\neg a \notin K * a$), but it is compatible with the following weakened version:

If $a \nvdash \bot$, then $\neg a \notin K * a$ (*consistent quasi-success*)

The following observation specifies the conditions under which *consistent quasi-success* holds.

Observation 4. *Let $*$ be a sentential operation on a consistent belief set K.*

(1) *If $*$ is a finite and orderly multi-state top revision, based on a zero-restricted multistate model $\langle \mathfrak{p}, B, L_E, \star_0 \rangle$, then $*$ satisfies* quasi-success *if and only if it holds for all $a \in L_E$ that if $a \nvdash \bot$, then there is some $b \in B$ with $\mathfrak{p}(a \mid b) \neq 0$.*

(2) *If $*$ is a finite sequential quasi-revision, based on a series $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$ of sets of consistent belief sets with $K \in \mathbb{X}_0$ and $K = \bigcap \mathbb{X}_0$, then $*$ satisfies* quasi-success *if and only if $\bigcap(\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_n) = \mathrm{Cn}(\varnothing)$.*

The following analogous postulate is also of considerable interest:

If $a \nvdash \bot$, then $a \in K * a$ (*consistent success*)

Observation 5. *Let $*$ be a sentential operation on a consistent belief set K.*

(1) *If $*$ is a finite and orderly multi-state top revision, based on a multistate model $\langle \mathfrak{p}, B, L_E, \star \rangle$, then $*$ satisfies* consistent success *if and only if it holds for all $a \in L_E$ that if $a \nvdash \bot$, then (i) there is some $b \in B$ with $\mathfrak{p}(a \mid b) \neq 0$, and (ii) for all $b \in B$ it holds that $\mathfrak{p}(a \mid b) = 0$, $\mathfrak{p}(a \mid b) = 1$, or $\mathfrak{p}(b)/\mathfrak{p}(a) \approx 0$.*

(2) *If $*$ is a finite sequential quasi-revision, based on a series $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$ of sets of consistent belief sets with $K \in \mathbb{X}_0$ and $K = \bigcap \mathbb{X}_0$, then $*$ satisfies* consistent success *if and only if it holds for all $a \in L_E$ that if $a \nvdash \bot$, then (i) there is some X such that $a \in X \in \mathbb{X}_0 \cup \cdots \cup \mathbb{X}_n$, and (ii) for all $X \in \mathbb{X}_0 \cup \cdots \cup \mathbb{X}_n$, at least one of the following is true: $a \in X$, $\neg a \in X$, or there are $\mathbb{X}_k$ and $\mathbb{X}_m$ with $0 \leq k < m \leq n$, $\neg a \notin \bigcap \mathbb{X}_k$ and $X \in \mathbb{X}_m$.*

The condition that is shown in part (1) of Observation 5 to be equivalent with *consistent success* comes close to saying that if something has happened, then it must have happened in a state of the world in which its probability was 1. This condition can be interpreted as a formal representation of determinism. It implies that all possible states of the world are deterministic, so that all uncertainties about the world concern which of these deterministic states is true.

The *confirmation* postulate ([15], p. 118) holds for multistate top revision:

OBSERVATION 6. *Let $*$ be a finite and orderly multistate top revision on the consistent belief set K in a finite language. Then it satisfies:*

If $a \in K$, then $K * a = K$ (confirmation)

It follows from Definition 6 and Observation 6 that if $a \in [\![\mathfrak{p}]\!]$, then $[\![\mathfrak{p} \star_\delta a]\!] = [\![\mathfrak{p}]\!]$. However, it does not follow in general that if $a \in [\![\mathfrak{p}]\!]$, then $\mathfrak{p} \star_\delta a = \mathfrak{p}$. To the contrary, this only holds in four rather special limiting cases:

OBSERVATION 7. *Let $\langle \mathfrak{p}, B, L_E, \star \rangle$ be a multistate model of probability revision. Then $\mathfrak{p} = \mathfrak{p} \star_\delta a$ if and only if*

(i) $\mathfrak{p}(a) = 0$*, or*

(ii) $\mathfrak{p}(a) = 1$*, or*

(iii) $\mathfrak{p}(a) = 1 - \delta$*, or*

(iv) $\mathfrak{p}(a \mid b) = \mathfrak{p}(a)$ *for all* $b \in B$*.*

As was noted by Gärdenfors ([11], p. 54), in the presence of *closure*, the AGM postulate

If $\neg a \notin K$, then $\text{Cn}(K \cup \{a\}) \subseteq K * a$ (*vacuity*)

is equivalent with the conjunction of the following two conditions:

(i) If $\neg a \notin K$, then $a \in K * a$

(ii) If $\neg a \notin K$, then $K \subseteq K * a$ (*preservation*)[11]

Since (i) follows from *success*, *preservation* can replace *vacuity* in the AGM axiomatization. Whereas (i) does not hold for multistate top revision, three of the eight postulates of Theorem 1 are sufficient for *preservation* to hold:

OBSERVATION 8. *Let $*$ be a sentential operation on the consistent belief set K. If $*$ satisfies* extensionality, tautology inertness, *and* linearity, *then it satisfies* preservation.

As shown by Hans Rott, in the presence of the six basic AGM postulates, the two supplementary AGM postulates, *superexpansion* and *subexpansion*, hold if and only if *disjunctive factoring* holds.[12] However, although multistate top revision satisfies *disjunctive factoring*, it satisfies neither *superexpansion* nor *subexpansion*:

---

[11] The *preservation* postulate was introduced by Gärdenfors ([10], p. 82). For a clarifying discussion, see Rott ([34], p. 109).

[12] This result was first reported by Peter Gärdenfors, with due acknowledgement to Hans Rott ([11], pp. 57, 212, and 244).

Observation 9. *Let $*$ be a finite sequential quasi-revision on the consistent belief set K.*

(1) *It does not hold in general that*

$K * (a_1 \& a_2) \subseteq \mathrm{Cn}((K * a_1) \cup \{a_2\})$ *(superexpansion)*

(b) *It does not hold in general that*

If $\neg a_2 \notin K * a_1$, *then* $\mathrm{Cn}((K * a_1) \cup \{a_2\}) \subseteq K * (a_1 \& a_2)$ *(subexpansion)*

What is lacking here is *success*:

Observation 10. *Let $*$ be a sentential operation on the consistent belief set K. If $*$ satisfies* closure*,* extensionality*,* success *and* disjunctive factoring*, then it satisfies* superexpansion *and* subexpansion*.*

Summarizing the above, the following five postulates are all satisfied both by multistate top revision and by AGM revision:

Closure

Preservation

Extensionality

Consistency

Disjunctive factoring

Due to the above-mentioned derivations of *vacuity* (from *preservation*, *closure*, and *success*), and the two supplementary postulates (from *closure*, *extensionality*, *success* and *disjunctive factoring*; Observation 10), if the *success* postulate is added to the list, then we obtain a set of postulates that contains all the AGM postulates except *inclusion*. The relevance of multistate top revision is enhanced by the fact that the *success* postulate is the most controversial among the AGM revision postulates. Many proposals have been made to avoid *success* while retaining the other main features of AGM.[13] This approach is commonly called "non-prioritized belief revision", since it does not give the input sentence absolute priority over previous beliefs. Most variants of non-prioritized belief revision are all-or-nothing in the sense that either the input sentence is fully assimilated in the resulting new belief set ($a \in K * a$), or it does not at all change the belief set ($K * a = K$). However, variants have also been studied in which only a part of the information contained in the input sentence is accepted in the new belief set

[13]See for instance Chopra et al. [5], Falappa et al. [6], Fermé and Hansson [7], Hansson [13], Hansson et al. [18], Hansson ([15], pp. 117–131), Konieczny and Pino Perez [24], Makinson [29], Mazzieri and Dragoni [30], and Perrotin and Velázquez-Quesada [32].

[7]. Since multistate top revision satisfies *relative quasi-success*, it clearly belongs to the all-or-nothing variant, albeit with *quasi-success* substituted for *success*.

Acknowledgements. This work was supported by the Swedish Research Council, Grant 2020-01460.

Funding Open access funding provided by Uppsala University.

6. Appendix: Proofs

Definition 8. A hyperreal number y is an infinitesimal of the *first order* if and only if $0 \neq y \approx 0$ but there is no z such that $0 \neq z \approx 0$ and $y/z \approx 0$. An infinitesimal y is an infinitesimal of the n^{th} *order*, for some $n > 1$, if and only if:

(1) There is a series $z_1, \ldots, z_{n-1}$ of non-zero hyperreal numbers, such that $z_1 \approx 0$, $z_k/z_{k-1} \approx 0$ whenever $1 < k \leq n-1$ and $y/z_{n-1} \approx 0$, and

(2) There is no series $z'_1, \ldots, z'_n$ of non-zero hyperreal numbers, such that $z'_1 \approx 0$, $z'_k/z'_{k-1} \approx 0$ whenever $1 < k \leq n$ and $y/z'_n \approx 0$.

Lemma 1. *If y and y' are both n^{th} order infinitesimals, then y/y' is a real number.*

Proof of Lemma 1. Suppose that this is not the case. Then $y/y' \approx 0$ or $y'/y \approx 0$. Suppose the former. Then we have a series:

$$z_1 \approx 0,\ z_2/z_1 \approx 0, \ldots y'/z_{n-1} \approx 0,\ y/y' \approx 0$$

so that y is of at least $(n+1)^{\text{th}}$ order, contrary to the assumption. ■

Postulate 1. *The codomain of the probability function* $\mathfrak{p}$ *consists of numbers in the closed hyperreal interval* $[0,1]$ *that are either real or the sum of a real number and an infinitesimal of some finite order.*

Lemma 2. *Let* $*$ *be the multistate top revision that is based on a finite and orderly multistate model* $\langle \mathfrak{p}, B, L_E, \star \rangle$. *Let* $a \in L_E$ *and let* $\mathfrak{p}(a) \neq 0$. *Then* $a \in K * a$ *if and only if it holds for all* $b \in B$ *that* $\mathfrak{p}(a \mid b) = 0$, $\mathfrak{p}(a \mid b) = 1$, *or* $\mathfrak{p}(b)/\mathfrak{p}(a) \approx 0$.

Proof of Lemma 2.

$a \in K * a$

iff $a \in [\![\mathfrak{p} \star_\delta a]\!]$ — Definition 6

iff $a \in [\![\mathfrak{p} \star_0 a]\!]$ — Observation 2

iff $(\mathfrak{p} \star_0 a)(a) \approx 1$ — Definition 5

iff $\displaystyle\sum_{b \in B} \frac{\mathfrak{p}(a\&b)^2}{\mathfrak{p}(a)\mathfrak{p}(b)} \approx 1$ — Definition 3, $\mathfrak{p}(a) \neq 0$

iff $\displaystyle\sum_{b \in B} \frac{\mathfrak{p}(a\&b)}{\mathfrak{p}(a)} - \sum_{b \in B} \frac{\mathfrak{p}(a\&b)^2}{\mathfrak{p}(a)\mathfrak{p}(b)} \approx 0$ — Since $\displaystyle\sum_{b \in B} \frac{\mathfrak{p}(a\&b)}{\mathfrak{p}(a)} = 1$

iff for all $b \in B$: $\displaystyle\frac{\mathfrak{p}(a\&b)}{\mathfrak{p}(a)} - \frac{\mathfrak{p}(a\&b)^2}{\mathfrak{p}(a)\mathfrak{p}(b)} \approx 0$ — $\displaystyle\frac{\mathfrak{p}(a\&b)}{\mathfrak{p}(a)} \geq \frac{\mathfrak{p}(a\&b)^2}{\mathfrak{p}(a)\mathfrak{p}(b)}$ for all $b \in B$, B is finite

iff for all $b \in B$: $\displaystyle\frac{\mathfrak{p}(a\&b)}{\mathfrak{p}(a)} \times \left(1 - \frac{\mathfrak{p}(a\&b)}{\mathfrak{p}(b)}\right) \approx 0$

iff for all $b \in B$: $\displaystyle\frac{\mathfrak{p}(a\&b)}{\mathfrak{p}(b)} \times \frac{\mathfrak{p}(b)}{\mathfrak{p}(a)} \times \left(1 - \frac{\mathfrak{p}(a\&b)}{\mathfrak{p}(b)}\right) \approx 0$

iff for all $b \in B$: $\mathfrak{p}(a \mid b) \approx 0$, $\mathfrak{p}(a \mid b) \approx 1$, or $\mathfrak{p}(b)/\mathfrak{p}(a) \approx 0$

iff for all $b \in B$: $\mathfrak{p}(a \mid b) = 0$, $\mathfrak{p}(a \mid b) = 1$, or $\mathfrak{p}(b)/\mathfrak{p}(a) \approx 0$ — $\mathfrak{p}$ is orderly ■

Proof of Observation 1. [17]: It is sufficient to prove that (1) If $a \in [\![\mathfrak{p}]\!]$ and $a \vdash d$, then $d \in [\![\mathfrak{p}]\!]$, and (2) If $a_1 \in [\![\mathfrak{p}]\!]$ and $a_2 \in [\![\mathfrak{p}]\!]$, then $a_1\&a_2 \in [\![\mathfrak{p}]\!]$.

For (1), let $a \in [\![\mathfrak{p}]\!]$ and $a \vdash d$. Then d is equivalent with $a \vee (d\&\neg a)$, and it follows from the third Kolmogorov axiom that $\mathfrak{p}(a) \leq \mathfrak{p}(a \vee (d\&\neg a)) = \mathfrak{p}(d)$, hence $\mathfrak{p}(d) \approx 1$.

For (2), let $\mathfrak{p}(a_1) = 1 - \delta_1$ and $\mathfrak{p}(a_2) = 1 - \delta_2$. Then $\mathfrak{p}(\neg a_1) = \delta_1$ and $\mathfrak{p}(\neg a_2) = \delta_2$, and:

$\mathfrak{p}(a_1\&a_2) = 1 - \mathfrak{p}(\neg(a_1\&a_2))$

$\mathfrak{p}(a_1\&a_2) = 1 - \mathfrak{p}(\neg a_1 \vee \neg a_2)$

$\mathfrak{p}(a_1\&a_2) \geq 1 - (\mathfrak{p}(\neg a_1) + \mathfrak{p}(\neg a_2)) = 1 - (\delta_1 + \delta_2)$ — Since $\mathfrak{p}(\neg a_1 \vee \neg a_2) \leq \mathfrak{p}(\neg a_1) + \mathfrak{p}(\neg a_2)$

$a_1\&a_2 \in [\![\mathfrak{p}]\!]$ — Since $1 - (\delta_1 + \delta_2) \approx 1$ ■

Proof of Observation 2. $d \in [\![\mathfrak{p} \star_\delta a]\!]$

iff $(\mathfrak{p} \star_\delta a)(d) \approx 1$ Definition 5

iff $\sum_{b \in B} \left(\frac{\mathfrak{p}(a\&b)}{\mathfrak{p}(a)} + \delta \left(\frac{\mathfrak{p}(\neg a\&b)}{\mathfrak{p}(\neg a)} - \frac{\mathfrak{p}(a\&b)}{\mathfrak{p}(a)} \right) \right) \times \frac{\mathfrak{p}(d\&b)}{\mathfrak{p}(b)} \approx 1$ Definition 3

iff $\sum_{b \in B} \frac{\mathfrak{p}(a\&b) \times \mathfrak{p}(d\&b)}{\mathfrak{p}(a) \times \mathfrak{p}(b)} \approx 1$ A finite number of infinitesimal terms omitted

iff $(\mathfrak{p} \star_0 a)(d) \approx 1$ Definition 3

iff $d \in [\![\mathfrak{p} \star_0 a]\!]$ Definition 5 ∎

Proof of Theorem 1. The proof consists of four parts, proving the implications:

(I) from (1) a finite and orderly multistate top revision to (2) a finite sequential quasi-revision,

(II) from (2) a finite sequential quasi-revision to (1) a finite and orderly multistate top revision,

(III) from (2) a finite sequential quasi-revision to (3) a finite and accumulative sequential quasi-revision, and

(IV) from (2) a finite sequential quasi-revision to the postulates.

Part I: from a finite and orderly multistate top revision to a finite sequential quasi-revision

Our starting-point is a finite and orderly multistate top revision $*$ on K, based on a multistate model of probability revision according to Definitions 3, 4, and 6. Due to Observation 2 we can assume that the multistate model is zero-restricted, i.e. it is a model $\langle \mathfrak{p}, B, L_E, \star_0 \rangle$ as described in Definition 4. We are going to construct a sequential quasi-revision $\hat{*}$ on K and show that it coincides with $*$.

Based on Definition 8 and Postulate 1 we construct a series of mutually exclusive "plausibility levels" $\langle B_0, \ldots, B_n \rangle$ for the elements of B, such that:

(1) If $0 \ll \mathfrak{p}(b)$, then $b \in B_0$, and

(2) If $\mathfrak{p}(b)$ is a k^{th} order infinitesimal, then $b \in B_k$.

We construct a set $\overline{\mathbb{X}}$ of belief sets with a one-to-one correspondence with B, such that for each $b' \in B$ there is an element X' of $\overline{\mathbb{X}}$ such that $X' = \{a \mid \mathfrak{p}(a \mid b') = 1\}$. We will refer to such a pair of an element of B and one of $\overline{\mathbb{X}}$ as *corresponding* to each other.

Furthermore, we introduce an ordered partitioning $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$ of $\overline{\mathbb{X}}$. An element X' of $\overline{\mathbb{X}}$ is an element of $\mathbb{X}_k$ if and only if the element b' of B that corresponds to X' is an element of B_k. We further note:

$a \in K$ iff $a \in [\![\mathfrak{p}]\!]$
iff $\mathfrak{p}(a) \approx 1$
iff $\sum_{b \in B} \mathfrak{p}(a \mid b) \times \mathfrak{p}(b) \approx 1$
iff $\sum_{b \in B} \mathfrak{p}(b) - \mathfrak{p}(a \mid b) \times \mathfrak{p}(b) \approx 0$
iff $\sum_{b \in B} \mathfrak{p}(b) \times (1 - \mathfrak{p}(a \mid b)) \approx 0$
iff $\mathfrak{p}(a \mid b) = 1$ for all $b \in B$ with $0 \ll \mathfrak{p}(b)$
iff $\mathfrak{p}(a \mid b) = 1$ for all $b \in B_0$
iff $a \in X$ for all $X \in \mathbb{X}_0$
iff $a \in \bigcap \mathbb{X}_0$
Thus, $K = \bigcap \mathbb{X}_0$.

Finally, we let $\hat{*}$ be the sequential quasi-revision that is based on our series $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$ according to Definition 7.

We are going to show that $K * a = K \hat{*} a$ for all a. There are three cases.

First case, $\mathfrak{p}(a) = 0$: It follows from clause (0) of Definition 3 that $\mathfrak{p} \star_0 a = \mathfrak{p}$, thus $K * a = K$. Since all elements of B have non-zero probability, we can conclude from

$$0 = \mathfrak{p}(a) = \sum_{b \in B} (\mathfrak{p}(b) \times \mathfrak{p}(a \mid b))$$

that $\mathfrak{p}(a \mid b) = 0$ for all $b \in B$, thus $\mathfrak{p}(\neg a \mid b) = 1$ for all $b \in B$, thus $\neg a \in X$ for all $X \in \mathbb{X}_0 \cup \cdots \cup \mathbb{X}_n$. It follows from clause (1) of Definition 7 that $K \hat{*} a = K$.

Second case, $\mathfrak{p}(a) = 1$: It follows from clause (0) of Definition 3 that $\mathfrak{p} \star_0 a = \mathfrak{p}$, thus $K * a = K$. We have:

$\mathfrak{p}(a) = 1$
$\sum_{b \in B} (\mathfrak{p}(b) \times \mathfrak{p}(a \mid b)) = 1$
$\sum_{b \in B} (\mathfrak{p}(b) - (\mathfrak{p}(b) \times \mathfrak{p}(a \mid b))) = 0$ Since $\sum_{b \in B} \mathfrak{p}(b) = 1$
$\sum_{b \in B} \mathfrak{p}(b) \times (1 - \mathfrak{p}(a \mid b)) = 0$
For all $b \in B$: $\mathfrak{p}(a \mid b) = 1$ Since $\mathfrak{p}(b) > 0$ for all $b \in B$
For all $X \in \mathbb{X}_0$: $a \in X$ and $\neg a \notin X$ Definition of $\overline{\mathbb{X}}$
$K \hat{*} a = \bigcap \mathbb{X}_0 = K$ Clause (2) of Definition 7, proof above that $K = \bigcap \mathbb{X}_0$

Third case, $0 < \mathfrak{p}(a) < 1$: We have:

$K * a = [\![\mathfrak{p} \star_0 a]\!]$ Definition 6

$d \in K * a$ iff $(\mathfrak{p} \star_0 a)(d) \approx 1$ Definition 5

$$d \in K * a \text{ iff } \sum_{b \in B} \left(\frac{\mathfrak{p}(a \& b) \times \mathfrak{p}(d \& b)}{\mathfrak{p}(a) \times \mathfrak{p}(b)} \right) \approx 1. \qquad \text{Definition 3} \qquad (1)$$

Let B_k be the first level (level with the lowest index k) in $\langle B_0, \ldots, B_n \rangle$ that contains some b with $\mathfrak{p}(\neg a \mid b) \neq 1$. Then all b'' in $B_0 \cup \cdots \cup B_{k-1}$ have $\mathfrak{p}(a \& b'') = 0$ and thus do not contribute to the sum in Eq. 1. We are going to show that a term in the sum of Eq. 1 does not make a non-infinitesimal contribution to that sum if it comes from some b'' with $b'' \in B_m$ and $k < m$.

Let $b'' \in B_m$, with $k < m$. Let b' be an element of B_k that contributes a non-zero amount to the probability of d, as summed up in Eq. 1. We assume that b'' also contributes a non-zero amount to that sum. Then $\mathfrak{p}(a \& b'')$, $\mathfrak{p}(d \& b'')$, $\mathfrak{p}(a \& b')$ and $\mathfrak{p}(d \& b')$ are all non-zero, and consequently so are $\mathfrak{p}(a \mid b'')$, $\mathfrak{p}(d \mid b'')$, $\mathfrak{p}(a \mid b')$ and $\mathfrak{p}(d \mid b')$. The ratio between the term contributed by b'' and that contributed by b' is:

$$\frac{\mathfrak{p}(a \& b'') \times \mathfrak{p}(d \& b'')}{\mathfrak{p}(a) \times \mathfrak{p}(b'')} \Big/ \frac{\mathfrak{p}(a \& b') \times \mathfrak{p}(d \& b')}{\mathfrak{p}(a) \times \mathfrak{p}(b')}$$

$$= \frac{\mathfrak{p}(a \& b'') \times \mathfrak{p}(d \& b'') \times \mathfrak{p}(b')}{\mathfrak{p}(a \& b') \times \mathfrak{p}(d \& b') \times \mathfrak{p}(b'')}$$

$$= \frac{\mathfrak{p}(a \mid b'') \times \mathfrak{p}(b'') \times \mathfrak{p}(d \mid b'') \times \mathfrak{p}(b'') \times \mathfrak{p}(b')}{\mathfrak{p}(a \mid b') \times \mathfrak{p}(b') \times \mathfrak{p}(d \mid b') \times \mathfrak{p}(b') \times \mathfrak{p}(b'')}$$

$$= \frac{\mathfrak{p}(a \mid b'') \times \mathfrak{p}(d \mid b'')}{\mathfrak{p}(a \mid b') \times \mathfrak{p}(d \mid b')} \times \frac{\mathfrak{p}(b'')}{\mathfrak{p}(b')}$$

Since $*$ is orderly (cf. Definitions 4 and 6), $\mathfrak{p}(a \mid b'')$, $\mathfrak{p}(d \mid b'')$, $\mathfrak{p}(a \mid b')$ and $\mathfrak{p}(d \mid b')$ are all real-valued, and $\mathfrak{p}(b'')/\mathfrak{p}(b')$ is by construction infinitesimal. Therefore, the contribution of b'' to the sum in Eq. 1 is an infinitesimal fraction of that of b'. We conclude that all non-infinitesimal contributions to the sum in Eq. 1 derive from elements of B_k. Since the sum has a finite number of terms, all infinitesimal terms can be eliminated without affecting the standard part of the sum. Thus, Eq. 1 is equivalent with:

$$d \in K * a \text{ iff } \sum_{b \in B_k} \left(\frac{\mathfrak{p}(a \& b) \times \mathfrak{p}(d \& b)}{\mathfrak{p}(a) \times \mathfrak{p}(b)} \right) \approx 1,$$

or equivalently:

$$d \in K * a \text{ iff } \sum_{b \in B_k} \mathfrak{p}(b \mid a) \times \mathfrak{p}(d \mid b) \approx 1, \qquad (2)$$

where B_k is the first level with some b such that $\mathfrak{p}(\neg a \mid b) \neq 1$.

Next we need to show that $\sum_{b\in B_k} \mathfrak{p}(b \mid a) \approx 1$. Since $\sum_{b\in B} \mathfrak{p}(b \mid a) \approx 1$, we can do this by showing that if $b'' \in B_m$ for some $m \neq k$, then $\mathfrak{p}(b'' \mid a)$ is either zero or infinitesimal. If $m < k$, then $\mathfrak{p}(a \mid b'') = 0$, i.e. $\mathfrak{p}(a\&b'')/\mathfrak{p}(b'') = 0$, and consequently $\mathfrak{p}(a\&b'') = 0$ and $\mathfrak{p}(b'' \mid a) = 0$. For the case $k < m$ we note that due to the construction of B_k, there is some $b' \in B_k$ such that $\mathfrak{p}(\neg a \mid b') \neq 1$, thus $\mathfrak{p}(a \mid b') \neq 0$, and since $\mathfrak{p}$ is orderly it follows that $0 \ll \mathfrak{p}(a \mid b')$. We then have:

$$\frac{\mathfrak{p}(b'' \mid a)}{\mathfrak{p}(b' \mid a)} = \left(\frac{\mathfrak{p}(a \mid b'') \times \mathfrak{p}(b'')}{\mathfrak{p}(a)}\right) \Big/ \left(\frac{\mathfrak{p}(a \mid b') \times \mathfrak{p}(b')}{\mathfrak{p}(a)}\right)$$
$$= \frac{\mathfrak{p}(a \mid b'')}{\mathfrak{p}(a \mid b')} \times \frac{\mathfrak{p}(b'')}{\mathfrak{p}(b')}$$

Since $\mathfrak{p}$ is orderly, $\mathfrak{p}(a \mid b'')$ is either 0 or a positive real number. In the former case, $\mathfrak{p}(b'' \mid a)/\mathfrak{p}(b' \mid a) = 0$. In the latter case, it follows from the construction of B_k and B_m that $\mathfrak{p}(b'')/\mathfrak{p}(b') \approx 0$, and consequently $\mathfrak{p}(b'' \mid a)/\mathfrak{p}(b' \mid a) \approx 0$. Thus all terms in $\sum_{b\in B} \mathfrak{p}(b \mid a)$ with $b \notin B_k$ are zero or infinitesimal, and since there is only a finite number of such terms they can all be deleted without effect on the standard part of the sum, and we have:

$$\sum_{b\in B_k} \mathfrak{p}(b \mid a) \approx \sum_{b\in B} \mathfrak{p}(b \mid a) = \sum_{b\in B} \mathfrak{p}(a\&b)/\mathfrak{p}(a) = 1.$$

Inserting this into Eq. 2 we continue:

$$d \in K * a \text{ iff } \sum_{b\in B_k} \mathfrak{p}(b \mid a) \times \mathfrak{p}(d \mid b) \approx \sum_{b\in B_k} \mathfrak{p}(b \mid a)$$
$$d \in K * a \text{ iff } \sum_{b\in B_k} \mathfrak{p}(b \mid a) \times (1 - \mathfrak{p}(d \mid b)) \approx 0$$

$$d \in K * a \,\text{iff for all}\, b \in B_k : \text{ If}\, \mathfrak{p}(b \mid a) \not\approx 0 \,\text{then}\, \mathfrak{p}(d \mid b) \approx 1. \tag{3}$$

Furthermore:

$$\sum\nolimits_{b\in B_k} \frac{\mathfrak{p}(a\&b)}{\mathfrak{p}(a)} \approx 1$$
$$\frac{\sum_{b\in B_k} \mathfrak{p}(a\&b)}{\mathfrak{p}(a)} \approx 1$$
$$\frac{\sum_{b\in B_k} \mathfrak{p}(a \mid b) \times \mathfrak{p}(b)}{\mathfrak{p}(a)} \approx 1$$

Let $B_k^{\downarrow a} = \{b \in B_k \mid \mathfrak{p}(a \mid b) \neq 0\}$. Then

$$\frac{\sum_{b\in B_k^{\downarrow a}} \mathfrak{p}(a \mid b) \times \mathfrak{p}(b)}{\mathfrak{p}(a)} \approx 1 \tag{4}$$

Since all $\mathfrak{p}(b)$ with $b \in B_k$ belong to the same order of infinitesimals, and all $\mathfrak{p}(a \mid b)$ with $b \in B_k^{\downarrow a}$ are positive real numbers, we can conclude from Eq. 4 that all $\mathfrak{p}(b)$ with $b \in B_k$ belong to the same order of infinitesimals as $\mathfrak{p}(a)$. Thus, $\mathfrak{p}(b)/\mathfrak{p}(a)$ is a positive real number for all $b \in B_k$. (Note that $\mathfrak{p}(b) \neq 0$ by definition.) Since $\mathfrak{p}(b \mid a) = (\mathfrak{p}(a \mid b) \times \mathfrak{p}(b))/\mathfrak{p}(a)$, it holds for all $b \in B_k$ that $\mathfrak{p}(b \mid a) \approx 0$ if and only if $\mathfrak{p}(a \mid b) \approx 0$. Since $\mathfrak{p}(a \mid b)$ is by definition real-valued, it follows that for all $b \in B_k$: $\mathfrak{p}(b \mid a) \approx 0$ if and only if $\mathfrak{p}(a \mid b) = 0$. It also follows that $\mathfrak{p}(d \mid b) \approx 1$ if and only if $\mathfrak{p}(d \mid b) = 1$. Thus Eq. 3 is equivalent with the following:

$d \in K * a$ iff for all $b \in B_k$: If $\mathfrak{p}(a \mid b) \neq 0$, then $\mathfrak{p}(d \mid b) = 1$

Equivalently:

$d \in K * a$ iff for all $X \in \mathbb{X}_k$: If $\neg a \notin X$, then $d \in X$,

where $\mathbb{X}_k$ is the first element in $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$ with $\neg a \notin \bigcap \mathbb{X}_k$. Equivalently:

$K * a = \bigcap \{X \mid \neg a \notin X \in \mathbb{X}_k\}$, and

$K * a = K \hat{*} a$

Part II: From a finite sequential quasi-revision to a finite and orderly multistate top revision

Let $*$ be the sequential quasi-revision on K that is based on the series $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$ of finite sets of finite belief sets in L_E, such that the conditions of Definition 7 are satisfied. We are going to construct a finite and orderly multistate top revision operation $\bar{*}$ on K and show that it coincides with $*$.

The construction: We introduce a set B of logical atoms not in L_E, one for each element of $\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_n$, and divide it into the mutually exclusive classes $B_0, \ldots, B_n$ so that each element X' in $\mathbb{X}_k$ has a corresponding element in b' in B_k, and vice versa. Furthermore, we introduce a probability function $\mathfrak{p}$ over the language that is formed by closing $L_E \cup B$ under truth-functional combinations, with the following properties:

(1) For all $a \in L_E$ and $b' \in B$, $\mathfrak{p}(a \mid b') = \dfrac{\mathsf{numb}(\{W \in L_E \perp\!\perp \mid X' \cup \{a\} \subseteq W\})}{\mathsf{numb}(\{W \in L_E \perp\!\perp \mid X' \subseteq W\})}$,

where X' is the set corresponding to b', and[14]

(2) For all k, if $b \in B_k$ then

$$\mathfrak{p}(b) = \frac{t_k}{\mathsf{numb}(B_k)},$$

where $\langle t_0, \ldots, t_n \rangle$ is a series of hyperreal numbers such that if $0 <$

[14]Note that $L_E \perp\!\perp$ is the set of maximal consistent subsets of L_E.

$k \leq n$, then t_k is a k^{th} order infinitesimal (Definition 8), and $t_0 = 1 - (t_1 + \cdots + t_n)$.

Let $\star_0$ be the zero-restricted probability revision based on $\mathfrak{p}$, B, and L_E according to Definitions 3 and 4. This construction satisfies the definition of a finite and orderly multi-state model. Note in particular that $\mathfrak{p}(b') \neq 0$ for all $b' \in B$ and that $\mathfrak{p}(a \mid b')$ is always a real number in the interval $[0, 1]$, as required in Definition 4.

Let $\bar{*}$ be the multistate top revision based on $\langle \mathfrak{p}, B, L_E, \star_0 \rangle$ according to Definition 6. Then $\bar{*}$ is a finite and orderly multistate top revision. We are going to prove that $K\bar{*}a = K * a$ for all $a \in L_E$. There are two cases.

First case, $\neg a \in \bigcap(\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_n)$: It follows from Clause (1) of Definition 7 that $K * a = K$. It follows from our construction of $\mathfrak{p}$ for this part of the proof that $\mathfrak{p}(a \mid b) = 0$ for all $b \in B$, and consequently

$$\mathfrak{p}(a) = \sum_{b \in B} \mathfrak{p}(a \& b) = \sum_{b \in B} (\mathfrak{p}(a \mid b) \times (\mathfrak{p}(b)) = 0$$

Clause (0) of Definition 3 yields $\mathfrak{p} \star_0 a = \mathfrak{p}$. Definition 6 yields $K\bar{*}a = K$. Thus $K\bar{*}a = K * a$.

Second case, $\neg a \notin \bigcap(\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_n)$: Let k be the lowest number such that $\neg a \notin \bigcap \mathbb{X}_k$. Two preparatory steps are needed before we can proceed to the actual proof of this case.

First preparatory step: We are going to show that if $m < k$, then $\mathfrak{p}(a \& b') = 0$ for all $b' \in B_m$.

Let X' be the element of $\mathbb{X}_m$ that corresponds to b'. Then $\neg a \in X'$. It follows from the construction of $\mathfrak{p}$ for this part of the proof that $\mathfrak{p}(a \mid b') = 0$, thus $\mathfrak{p}(a \& b') = \mathfrak{p}(a \mid b') \times \mathfrak{p}(b') = 0$.

Second preparatory step: We are going to show that if $k < m$, then there is some $b'' \in B_k$ such that $\mathfrak{p}(b' \mid a)/\mathfrak{p}(b'' \mid a) \approx 0$ for all $b' \in B_m$.

By assumption, there is some $X'' \in \mathbb{X}_k$ such that $\neg a \notin X''$. Let b'' be the element of B_k that corresponds to X''. It follows from the construction of $\mathfrak{p}$ for this part of the proof that $0 \ll \mathfrak{p}(a \mid b'')$. We then have:

$$\frac{\mathfrak{p}(a \& b')}{\mathfrak{p}(a \& b'')} = \frac{\mathfrak{p}(a \mid b')}{\mathfrak{p}(a \mid b'')} \times \frac{\mathfrak{p}(b')}{\mathfrak{p}(b'')}$$

It follows from the construction of $\mathfrak{p}$ that $\mathfrak{p}(a \mid b')/\mathfrak{p}(a \mid b'')$ is a non-negative real number, whereas $\mathfrak{p}(b')/\mathfrak{p}(b'')$ is infinitesimal. Thus, their product is either 0 or infinitesimal. It follows that $\mathfrak{p}(a \& b')/\mathfrak{p}(a \& b'') \approx 0$ for all $b' \in B_m$. Consequently, $\mathfrak{p}(b' \mid a)/\mathfrak{p}(b'' \mid a) \approx 0$ for all $b' \in B_m$.

The actual proof:

$$d \in K\bar{*}a \text{ iff } \sum_{b \in B} \frac{\mathfrak{p}(a \& b) \times \mathfrak{p}(d \& b)}{\mathfrak{p}(a) \times \mathfrak{p}(b)} \approx 1 \qquad \text{Definitions 3 and 6}$$

$d \in K\bar{*}a$ iff $\sum_{b \in B} \mathfrak{p}(b \mid a) \times \mathfrak{p}(d \mid b) \approx 1$

$d \in K\bar{*}a$ iff $\sum_{b \in B_k} \mathfrak{p}(b \mid a) \times \mathfrak{p}(d \mid b) \approx 1$ The two preparatory steps

$d \in K\bar{*}a$ iff $\sum_{b \in B_k} \mathfrak{p}(b \mid a) \times \mathfrak{p}(d \mid b) \approx \sum_{b \in B_k} \mathfrak{p}(b \mid a)$ The two preparatory steps

$d \in K\bar{*}a$ iff $\sum_{b \in B_k} \mathfrak{p}(b \mid a) \times (1 - \mathfrak{p}(d \mid b)) \approx 0$

$$d \in K\bar{*}a \text{ iff for all } b \in B_k : \text{ If } \mathfrak{p}(b \mid a) \not\approx 0, \text{ then } \mathfrak{p}(\neg d \mid b)) \approx 0 \quad (5)$$

We are now going to show that for for all $b \in B_k$, $\mathfrak{p}(b \mid a) \approx 0$ holds if and only if $\mathfrak{p}(a \mid b) = 0$. Let $b' \in B_k$. For one direction, let $\mathfrak{p}(a \mid b') = 0$, i.e. $\mathfrak{p}(a\&b')/\mathfrak{p}(b') = 0$. It follows that $\mathfrak{p}(a\&b') = 0$, and consequently $\mathfrak{p}(b' \mid a) = 0$. For the other direction, let $\mathfrak{p}(a \mid b') \neq 0$. Due to the construction of $\mathfrak{p}$, $\mathfrak{p}(a \mid b')$ is then a positive real number. It holds for all $b'' \in B_k$ that $0 \ll \mathfrak{p}(b')/\mathfrak{p}(b'')$. From this it follows that if $b', b'' \in B_k^{\downarrow a} = \{b \in B_k \mid \mathfrak{p}(a \mid b) \neq 0\}$, then $0 \ll (\mathfrak{p}(a \mid b') \times \mathfrak{p}(b'))/(\mathfrak{p}(a \mid b'') \times \mathfrak{p}(b''))$, i.e. $0 \ll \mathfrak{p}(a\&b')/\mathfrak{p}(a\&b'')$. It follows from the two preparatory steps that $\sum_{b \in B_k^{\downarrow a}} \mathfrak{p}(a\&b)/\mathfrak{p}(a) \approx 1$. Due to the finite number of terms in this sum we can conclude that $0 \ll \mathfrak{p}(a\&b')/\mathfrak{p}(a)$, i.e. $\mathfrak{p}(b' \mid a) \not\approx 0$, as desired. This concludes our proof that $\mathfrak{p}(b \mid a) \approx 0$ if and only if $\mathfrak{p}(a \mid b) = 0$. We can now use this proof to conclude that Eq. 5 is equivalent with:

$d \in K\bar{*}a$ iff for all $b \in B_k$: If $\mathfrak{p}(a \mid b) \neq 0$, then $\mathfrak{p}(\neg d \mid b) \approx 0$

$d \in K\bar{*}a$ iff for all $b \in B_k$: If $\mathfrak{p}(a \mid b) \neq 0$, then $\mathfrak{p}(\neg d \mid b) = 0$ Construction of $\mathfrak{p}$

$d \in K\bar{*}a$ iff for all $X \in \mathbb{X}_k$: If $\neg a \notin X$, then $d \in X$ Construction of $\mathfrak{p}$

$d \in K\bar{*}a$ iff $d \in \bigcap\{X \mid \neg a \notin X \in \mathbb{X}_k\}$

$K\bar{*}a = K * a$

Part III: from a finite sequential quasi-revision to a finite and accumulative sequential quasi-revision

Let $*$ be a sequential quasi-revision on K, based on some sequence $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$ of finite sets of finite belief sets, as described in Definition 7. Let $\langle \mathbb{X}'_0, \ldots, \mathbb{X}'_n \rangle$ be a sequence such that

(1) $\mathbb{X}'_0 = \mathbb{X}_0$, and

(2) $\mathbb{X}'_k = \mathbb{X}'_{k-1} \cup \mathbb{X}_k$ for all k such that $0 < k \leq n$

Let $*'$ be the sequential quasi-revision on K that is based on $\langle \mathbb{X}'_0, \dots, \mathbb{X}'_n \rangle$. It is clearly a finite and accumulative sequential quasi-revision. We are going to show that $K *' a = K * a$ for all a. There are two cases.

Case 1, $\neg a \in \bigcap(\mathbb{X}_0 \cup \dots \cup \mathbb{X}_n)$: Then $\neg a \in \bigcap(\mathbb{X}'_0 \cup \dots \cup \mathbb{X}'_n)$. It follows from clause (1) of Definition 7 that $K * a = K = K *' a$.

Case 2, $\neg a \notin \bigcap(\mathbb{X}_0 \cup \dots \cup \mathbb{X}_n)$: Let $\mathbb{X}_k$ be the first element in the sequence $\langle \mathbb{X}_0, \dots, \mathbb{X}_n \rangle$ for which $\neg a \notin \bigcap \mathbb{X}_k$. Then $\mathbb{X}'_k$ is the first element in the sequence $\langle \mathbb{X}'_0, \dots, \mathbb{X}'_n \rangle$ for which $\neg a \notin \bigcap \mathbb{X}'_k$. Clause (2) of Definition 7 yields:

$K *' a = \bigcap\{X \mid \neg a \notin X \in \mathbb{X}'_k\}$
$= \bigcap\{X \mid \neg a \notin X \in (\mathbb{X}_0 \cup \dots \cup \mathbb{X}_k)\}$
$= \bigcap\{X \mid \neg a \notin X \in \mathbb{X}_k\}$ Since $\neg a \in \bigcap \mathbb{X}_m$ for all m with $0 \le m < k$
$= K * a$

Part IV: from a finite sequential quasi-revision to the postulates

- $K * a = \text{Cn}(K * a)$ (closure)
- If $\vdash a_1 \leftrightarrow a_2$, then $K * a_1 = K * a_2$ (extensionality)
- $\neg a \notin K * a$ or $K * a = K$ (relative quasi-success)
- If $\neg a \in K * a$, then $\neg a \in K * d$. (quasi-regularity)
- $K * \top = K$ (tautology inertness)

These five postulates all follow directly from the construction.

- If $\neg a_1 \notin K * a_1$, $\neg a_2 \notin K * a_2$, and $K * a_1 = K * a_2$, then $K * (a_1 \vee d) = K * (a_2 \vee d)$. (disjunctive equivalence)

There are two main cases:

Case 1, $\neg d \in K * d$: Then $\neg d \in \bigcap(\mathbb{X}_0 \cup \dots \cup \mathbb{X}_n)$, and it holds for all $X \in \mathbb{X}_0 \cup \dots \cup \mathbb{X}_n$ that $\neg a_1 \& \neg d \in X$ iff $\neg a_1 \in X$, thus $\neg(a_1 \vee d) \in X$ iff $\neg a_1 \in X$, and it follows from Definition 7 that $K * (a_1 \vee d) = K * a_1$. It follows in the same way that $K * (a_2 \vee d) = K * a_2$, and we can conclude that $K * (a_1 \vee d) = K * (a_2 \vee d)$.

Case 2, $\neg d \notin K * d$: Let $\mathbb{X}_k$ be the first element in the sequence $\langle \mathbb{X}_0, \dots, \mathbb{X}_n \rangle$ that has an intersection not containing $\neg(a_1 \vee d)$. We are first going to show that it is also the first element in the sequence that has an intersection not containing $\neg(a_2 \vee d)$. We have:

$\neg(a_1 \vee d) \in \bigcap(\mathbb{X}_0 \cup \dots \cup \mathbb{X}_{k-1})$ and $\neg(a_1 \vee d) \notin \bigcap \mathbb{X}_k$
$(\neg a_1 \& \neg d) \in \bigcap(\mathbb{X}_0 \cup \dots \cup \mathbb{X}_{k-1})$ and $(\neg a_1 \& \neg d) \notin \bigcap \mathbb{X}_k$
$\neg a_1 \in \bigcap(\mathbb{X}_0 \cup \dots \cup \mathbb{X}_{k-1})$ *and* $\neg d \in \bigcap(\mathbb{X}_0 \cup \dots \cup \mathbb{X}_{k-1})$ *and* $(\neg a_1 \notin \bigcap \mathbb{X}_k$ *or* $\neg d \notin \bigcap \mathbb{X}_k)$

$\neg a_2 \in \bigcap(\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_{k-1})$ *and* $\neg d \in \bigcap(\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_{k-1})$ *and* $(\neg a_2 \notin \bigcap \mathbb{X}_k$ *or* $\neg d \notin \bigcap \mathbb{X}_k)$

Since $K * a_1 = K * a_2$

$(\neg a_2 \& \neg d) \in \bigcap(\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_{k-1})$ and $(\neg a_2 \& \neg d) \notin \bigcap \mathbb{X}_k$
$\neg(a_2 \vee d) \in \bigcap(\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_{k-1})$ and $\neg(a_2 \vee d) \notin \bigcap \mathbb{X}_k$

It holds for all $\mathbb{X}_g$ that $\neg(a_1 \vee d) \notin \bigcap \mathbb{X}_g$ if and only if $(\neg a_1 \& \neg d) \notin \bigcap \mathbb{X}_g$, i.e. if and only if $\neg a_1 \notin \bigcap \mathbb{X}_g$ or $\neg d \notin \bigcap \mathbb{X}_g$. Consequently, there are three subcases: (2A) $\neg a_1 \notin \bigcap \mathbb{X}_k$ and $\neg d \notin \mathbb{X}_k$, (2B) $\neg a_1 \notin \bigcap \mathbb{X}_k$, and the first element in the sequence whose intersection does not contain $\neg d$ is $\mathbb{X}_m$, with $k < m$, and (2C) $\neg d \notin \bigcap \mathbb{X}_k$, and the first element in the sequence whose intersection does not contain $\neg a_1$ is $\mathbb{X}_m$, with $k < m$.

Subcase 2A, $\neg a_1 \notin \bigcap \mathbb{X}_k$ and $\neg d \notin \mathbb{X}_k$:

$K * (a_1 \vee d) = \bigcap\{X \in \mathbb{X}_k \mid \neg(a_1 \vee d) \notin X\}$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_1 \& \neg d) \notin X\}$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_1 \notin X) \vee (\neg d \notin X)\}$
$= \bigcap(\{X \in \mathbb{X}_k \mid \neg a_1 \notin X\} \cup \{X \in \mathbb{X}_k \mid \neg d \notin X\})$
$= (\bigcap\{X \in \mathbb{X}_k \mid \neg a_1 \notin X\}) \cap (\bigcap\{X \in \mathbb{X}_k \mid \neg d \notin X\})$
$= (K * a_1) \cap (K * d)$
$= (K * a_2) \cap (K * d)$
$= (\bigcap\{X \in \mathbb{X}_k \mid \neg a_2 \notin X\}) \cap (\bigcap\{X \in \mathbb{X}_k \mid \neg d \notin X\})$
$= \bigcap(\{X \in \mathbb{X}_k \mid \neg a_2 \notin X\} \cup \{X \in \mathbb{X}_k \mid \neg d \notin X\})$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_2 \notin X) \vee (\neg d \notin X)\}$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_2 \& \neg d) \notin X)\}$
$= \bigcap\{X \in \mathbb{X}_k \mid \neg(a_2 \vee d) \notin X\}$
$= K * (a_2 \vee d)$

Subcase 2B, $\neg a_1 \notin \bigcap \mathbb{X}_k$, *and the first element in the sequence whose intersection does not contain* $\neg d$ *is* $\mathbb{X}_m$, *with* $k < m$:

$K * (a_1 \vee d) = \bigcap\{X \in \mathbb{X}_k \mid \neg(a_1 \vee d) \notin X\}$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_1 \& \neg d) \notin X\}$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_1 \notin X) \vee (\neg d \notin X)\}$
$= \bigcap\{X \in \mathbb{X}_k \mid \neg a_1 \notin X\}$ Since $\neg d \in \bigcap \mathbb{X}_k$
$= K * a_1$
$= K * a_2$
$= \bigcap\{X \in \mathbb{X}_k \mid \neg a_2 \notin X\}$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_2 \notin X) \vee (\neg d \notin X)\}$ Since $\neg d \in \bigcap \mathbb{X}_k$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_2 \& \neg d) \notin X)\}$
$= \bigcap\{X \in \mathbb{X}_k \mid \neg(a_2 \vee d) \notin X)\}$
$= K * (a_2 \vee d)$

Subcase 2C, $\neg d \notin \bigcap \mathbb{X}_k$, *and the first element in the sequence whose intersection does not contain* $\neg a_1$ *is* $\mathbb{X}_m$, *with* $k < m$:

$K * (a_1 \vee d) = \bigcap\{X \in \mathbb{X}_k \mid \neg(a_1 \vee d) \notin X\}$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_1 \& \neg d) \notin X\}$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_1 \notin X) \vee (\neg d \notin X)\}$
$= \bigcap\{X \in \mathbb{X}_k \mid \neg d \notin X\}$ Since $\neg a_1 \in \bigcap \mathbb{X}_k$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_2 \notin X) \vee (\neg d \notin X)\}$ Since $\neg a_2 \in \bigcap \mathbb{X}_k$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_2 \& \neg d) \notin X)\}$
$= \bigcap\{X \in \mathbb{X}_k \mid \neg(a_2 \vee d) \notin X\}$
$= K * (a_2 \vee d)$

- $K * (a_1 \vee a_2)$ is equal to one of $K * a_1$, $K * a_2$, and $K * a_1 \cap K * a_2$. (disjunctive factoring)

There are three cases.

Case 1, $\neg a_1 \in K * a_1$ *and* $\neg a_2 \in K * a_2$: It follows from the construction of $*$ that $K * a_1 = K * a_2 = K$. Furthermore, $\neg a_1 \in \bigcap(\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_n)$ and $\neg a_2 \in \bigcap(\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_n)$, thus $\neg a_1 \& \neg a_2 \in \bigcap(\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_n)$, thus $\neg(a_1 \vee a_2) \in \bigcap(\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_n)$, and consequently $K * (a_1 \vee a_2) = K$.

Case 2, $\neg a_1 \in K * a_1$ *and* $\neg a_2 \notin K * a_2$: In this case it holds for all $X \in \mathbb{X}_0 \cup \cdots \cup \mathbb{X}_n$ that $\neg a_1 \in X$, thus $\neg a_1 \& \neg a_2 \in X$ iff $\neg a_2 \in X$, thus $\neg(a_1 \vee a_2) \in X$ iff $\neg a_2 \in X$. It follows from Clause (2) of Definition 7 that $K * (a_1 \vee a_2) = K * a_2$.

Case 3, $\neg a_1 \notin K * a_1$ *and* $\neg a_2 \notin K * a_2$: Let $\mathbb{X}_k$ be the first element in the sequence $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$ whose intersection does not contain $\neg a_1$, and let $\mathbb{X}_m$ be the first element whose intersection does not contain $\neg a_2$. We can assume without loss of generality that $k \leq m$. First suppose that $\neg(a_1 \vee a_2) \notin \bigcap \mathbb{X}_g$ for some $g < k$. Then equivalently $(\neg a_1 \& \neg a_2) \notin \bigcap \mathbb{X}_g$, equivalently: $\neg a_1 \notin \bigcap \mathbb{X}_g$ or $\neg a_2 \notin \bigcap \mathbb{X}_g$, which is impossible since $g < k \leq m$. Thus, $\neg(a_1 \vee a_2) \in \bigcap \mathbb{X}_g$ for all $g < k$.

Next, suppose that $\neg(a_1 \vee a_2) \in \bigcap \mathbb{X}_k$. Then equivalently $(\neg a_1 \& \neg a_2) \in \bigcap \mathbb{X}_k$, thus $\neg a_1 \in \bigcap \mathbb{X}_k$, contrary to our assumption. We have shown that $\neg(a_1 \vee a_2) \in \bigcap(\mathbb{X}_0 \cup \cdots \cup \mathbb{X}_{k-1})$ and $\neg(a_1 \vee a_2) \notin \bigcap \mathbb{X}_k$, from which it follows that $K * (a_1 \vee a_2) = \bigcap\{X \in \mathbb{X}_k \mid \neg(a_1 \vee a_2) \notin X\}$. There are two cases:

Case 3A, $k < m$:

$K * (a_1 \vee a_2) = \bigcap\{X \in \mathbb{X}_k \mid \neg(a_1 \vee a_2) \notin X\}$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_1 \& \neg a_2) \notin X\}$
$= \bigcap\{X \in \mathbb{X}_k \mid \neg a_1 \notin X\}$ Since $\neg a_2 \in \bigcap \mathbb{X}_k$
$= K * a_1$

Case 3B, $k = m$:

$K * (a_1 \vee a_2) = \bigcap\{X \in \mathbb{X}_k \mid \neg(a_1 \vee a_2) \notin X\}$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_1 \& \neg a_2) \notin X\}$
$= \bigcap\{X \in \mathbb{X}_k \mid (\neg a_1 \notin X) \vee (\neg a_2 \notin X)\}$
$= \bigcap(\{X \in \mathbb{X}_k \mid \neg a_1 \notin X\} \cup \{X \in \mathbb{X}_k \mid \neg a_2 \notin X\})$
$= (\bigcap\{X \in \mathbb{X}_k \mid \neg a_1 \notin X\}) \cap (\bigcap\{X \in \mathbb{X}_k \mid \neg a_2 \notin X\})$
$= K * a_1 \cap K * a_2$

- If $\neg a \notin K * d$, then $K * (a \vee d) \subseteq K * a$. (linearity)

There are two main cases.

Case 1, $\neg d \in K * d$: Then $K * a = \bigcap\{X \mid \neg a \notin X \in \mathbb{X}_0\}$. From $\neg d \in \bigcap \mathbb{X}_0$ it follows for all $X \in \mathbb{X}_0$ that $\neg(a \vee d) \in X$ iff $\neg a \in X$, thus $K * (a \vee d) = \bigcap\{X \mid \neg(a \vee d) \notin X \in \mathbb{X}_0\} = \{X \mid \neg a \notin X \in \mathbb{X}_0\} = K * a$.

Case 2, $\neg d \notin K * d$: It follows from $\neg a \notin K * d$ and the construction of $*$ that $\neg a \notin K * a$. Let $\mathbb{X}_k$ be the first element of $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$ such that $\neg a \notin \bigcap \mathbb{X}_k$, and let $\mathbb{X}_m$ be the first element of $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$ such that $\neg d \notin \bigcap \mathbb{X}_m$. It follows from $\neg a \notin K * d$ and Definition 7 that $k \leq m$. There are two subcases, $k = m$ and $k < m$.

Subcase 2A, $k = m$: It holds for all $X \in \mathbb{X}_0 \cup \cdots \cup \mathbb{X}_n$ that $\neg(a \vee d) \notin X$ iff $\neg a \& \neg d \notin X$, iff $\neg a \notin X$ or $\neg d \notin X$. Thus $\mathbb{X}_k$ is the first element of $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$ such that $\neg(a \vee d) \notin \bigcap \mathbb{X}_k$, and consequently:
$K * (a \vee d) = \bigcap\{X \in \mathbb{X}_k \mid \neg(a \vee d) \notin X\}$
$K * (a \vee d) = \bigcap\{X \in \mathbb{X}_k \mid (\neg a \& \neg d) \notin X\}$
$K * (a \vee d) = \bigcap\{X \in \mathbb{X}_k \mid (\neg a \notin X) \vee (\neg d \notin X)\}$
$K * (a \vee d) \subseteq \bigcap\{X \in \mathbb{X}_k \mid \neg a \notin X\}$
$K * (a \vee d) \subseteq K * a$.

Subcase 2B, $k < m$: Just as in the previous subcase, $\mathbb{X}_k$ is the first element of $\langle \mathbb{X}_0, \ldots, \mathbb{X}_n \rangle$ such that $\neg(a \vee d) \notin \bigcap \mathbb{X}_k$. Consequently:
$K * (a \vee d) = \bigcap\{X \in \mathbb{X}_k \mid \neg(a \vee d) \notin X\}$
$K * (a \vee d) = \bigcap\{X \in \mathbb{X}_k \mid (\neg a \& \neg d) \notin X\}$
$K * (a \vee d) = \bigcap\{X \in \mathbb{X}_k \mid (\neg a \notin X) \vee (\neg d \notin X)\}$
$K * (a \vee d) = \bigcap\{X \in \mathbb{X}_k \mid \neg a \notin X\}$ Since $\neg d \in \bigcap \mathbb{X}_k$
$K * (a \vee d) = K * a$ ■

Proof of Observation 3. Suppose to the contrary that $K * a \vdash \perp$. It follows from *relative quasi-success* that $\neg a \notin K * a$ *or* $K * a = K$. Due to *closure*, the former is incompatible with $K * a \vdash \perp$. The latter is incompatible with $K * a \vdash \perp$ since K is by definition consistent. It follows from this contradiction that $K * a \nvdash \perp$. ■

Proof of Observation 4. *Part 1:* From Definitions 3 and 6.

Part 2: From Definition 7. ■

Proof of Observation 5. Part 1: For one direction, let $*$ satisfy *consistent success* and let $a \nvdash \bot$. Then $a \in K * a$, and it follows directly that (i) holds and that $\mathfrak{p}(a) \neq 0$. It follows from Lemma 2 that (ii) is satisfied.

For the other direction, we assume that (i) and (ii) both hold for all a with $a \nvdash \bot$. Let $a \nvdash \bot$. Then it follows from (i) that $\mathfrak{p}(a) \neq 0$ and from Lemma 2 that $a \in K * a$. Thus *consistent success* holds.

Part 2 is left to the reader. ■

Proof of Observation 6. We use the equivalence with finite sequential quasi-revision. It follows from $a \in K$ that $a \in X$ for all $X \in \mathbb{X}_0$, thus $\neg a \notin X$ for all $X \in \mathbb{X}_0$, thus $K * a = \bigcap\{X \mid \neg a \notin X \in \mathbb{X}_0\} = \bigcap \mathbb{X}_0 = K$. ■

Proof of Observation 7. Options (i) and (ii) follow directly from clause (0) of Definition 3, and we can focus on the case when $0 \neq \mathfrak{p}(a) \neq 1$. It follows from Definition 3 that $\mathfrak{p} = \mathfrak{p} \star_\delta a$ if and only if $(\mathfrak{p} \star_\delta a)(b) = \mathfrak{p}(b)$ for all $b \in B$. We have:

for all $b \in B$: $(\mathfrak{p} \star_\delta a)(b) = \mathfrak{p}(b)$

iff for all $b \in B$: $\dfrac{\mathfrak{p}(a\&b)}{\mathfrak{p}(a)} + \delta\left(\dfrac{\mathfrak{p}(\neg a\&b)}{\mathfrak{p}(\neg a)} - \dfrac{\mathfrak{p}(a\&b)}{\mathfrak{p}(a)}\right) = \mathfrak{p}(a\&b) + \mathfrak{p}(\neg a\&b)$ Definition 3

iff for all $b \in B$: $\mathfrak{p}(a\&b) \times \dfrac{1-\delta}{\mathfrak{p}(a)} + \mathfrak{p}(\neg a\&b) \times \dfrac{\delta}{1-\mathfrak{p}(a)} = \mathfrak{p}(a\&b) + \mathfrak{p}(\neg a\&b)$

iff for all $b \in B$: $\mathfrak{p}(a\&b)(1-\delta)(1-\mathfrak{p}(a)) + \delta \times \mathfrak{p}(\neg a\&b)\mathfrak{p}(a) =$
$= \mathfrak{p}(a\&b)\mathfrak{p}(a)(1-\mathfrak{p}(a)) + \mathfrak{p}(\neg a\&b)\mathfrak{p}(a)(1-\mathfrak{p}(a))$

iff for all $b \in B$: $\mathfrak{p}(a\&b)(1-\mathfrak{p}(a)) - \delta \times \mathfrak{p}(a\&b)(1-\mathfrak{p}(a)) + \delta \times \mathfrak{p}(\neg a\&b)\mathfrak{p}(a) =$
$= \mathfrak{p}(a\&b)(1-\mathfrak{p}(a)) - \mathfrak{p}(a\&b)(1-\mathfrak{p}(a))^2 + \mathfrak{p}(\neg a\&b)\mathfrak{p}(a)(1-\mathfrak{p}(a))$

iff for all $b \in B$: $\delta(\mathfrak{p}(\neg a\&b)\mathfrak{p}(a) - \mathfrak{p}(a\&b)(1-\mathfrak{p}(a)) =$
$= (1-\mathfrak{p}(a))(\mathfrak{p}(\neg a\&b)\mathfrak{p}(a) - \mathfrak{p}(a\&b)(1-\mathfrak{p}(a))$

iff for all $b \in B$: $(1-\mathfrak{p}(a)-\delta) \times (\mathfrak{p}(\neg a\&b)\mathfrak{p}(a) - \mathfrak{p}(a\&b)(1-\mathfrak{p}(a)) = 0$

iff for all $b \in B$: $\mathfrak{p}(a) = 1-\delta$ or $\mathfrak{p}(\neg a\&b)\mathfrak{p}(a) = \mathfrak{p}(a\&b)(1-\mathfrak{p}(a))$

iff for all $b \in B$: $\mathfrak{p}(a) = 1-\delta$ or $\mathfrak{p}(\neg a \mid b)\mathfrak{p}(b)\mathfrak{p}(a) = \mathfrak{p}(a \mid b)\mathfrak{p}(b)(1-\mathfrak{p}(a))$

iff for all $b \in B$: $\mathfrak{p}(a) = 1-\delta$ or $(1-\mathfrak{p}(a \mid b))\mathfrak{p}(a) = \mathfrak{p}(a \mid b)(1-\mathfrak{p}(a))$ $\mathfrak{p}(b) \neq 0$

iff for all $b \in B$: $\mathfrak{p}(a) = 1-\delta$ or $\mathfrak{p}(a) - \mathfrak{p}(a)\mathfrak{p}(a \mid b) = \mathfrak{p}(a \mid b) - \mathfrak{p}(a)\mathfrak{p}(a \mid b)$

iff for all $b \in B$: $\mathfrak{p}(a) = 1-\delta$ or $\mathfrak{p}(a \mid b) = \mathfrak{p}(a)$

Thus, (iii) or (iv) must hold. ■

Proof of Observation 8.

$\neg a \notin K$

$\neg a \notin K * \top$ *tautology inertness*

$K * (\top \vee a) \subseteq K * a$ *linearity*

$K * \top \subseteq K * a$ *extensionality*

$K \subseteq K * a$ *tautology inertness* ■

Proof of Observation 9. Let the sequence on which $*$ is based be $\langle \mathbb{X}_1, \mathbb{X}_2 \rangle$, with $\mathbb{X}_1 = \{\mathrm{Cn}(\{\neg a_1 \vee \neg a_2, a_3\})\}$ and $\mathbb{X}_2 = \{\mathrm{Cn}(\{a_1 \& a_2\})\}$. We then have:

$K * a_1 = \mathrm{Cn}(\{\neg a_1 \vee \neg a_2, a_3\})$

$\mathrm{Cn}((K * a_1) \cup \{a_2\}) = \mathrm{Cn}(\{\neg a_1 \& a_2 \& a_3\})$

$K * (a_1 \& a_2) = \mathrm{Cn}(\{a_1 \& a_2\})$,

and neither superexpansion nor subexpansion holds. ■

Proof of Observation 10. [15]: *Part 1, superexpansion:* Let $d \in K * (a_1 \& a_2)$. We are going to show that $d \in \mathrm{Cn}((K * a_1) \cup \{a_2\})$. *Closure* yields $a_2 \rightarrow d \in K * (a_1 \& a_2)$. From *success* follows $a_1 \& \neg a_2 \in K * (a_1 \& \neg a_2)$, and since $a_1 \& \neg a_2 \vdash a_2 \rightarrow d$, *closure* yields $a_2 \rightarrow d \in K * (a_1 \& \neg a_2)$.

It follows from *disjunctive factoring* that $K*(a_1 \& a_2) \cap K*(a_1 \& \neg a_2) \subseteq K*((a_1 \& a_2) \vee (a_1 \& \neg a_2))$. We can therefore conclude from $a_2 \rightarrow d \in K*(a_1 \& a_2)$ and $a_2 \rightarrow d \in K * (a_1 \& \neg a_2)$ that $a_2 \rightarrow d \in K * ((a_1 \& a_2) \vee (a_1 \& \neg a_2))$. *Extensionality* yields $a_2 \rightarrow d \in K * a_1$, thus $d \in \mathrm{Cn}((K * a_1) \cup \{a_2\})$.

Part 2, subexpansion: Let $\neg a_2 \notin K * a_1$. We are going to show that $\mathrm{Cn}((K * a_1) \cup \{a_2\}) \subseteq K * (a_1 \& a_2)$.

Due to *extensionality*, $K * a_1 = K * ((a_1 \& a_2) \vee (a_1 \& \neg a_2))$, thus due to *disjunctive factoring*, $K * a_1$ is equal to one of $K * (a_1 \& a_2)$, $K * (a_1 \& \neg a_2)$ and $K*(a_1 \& a_2) \cap K*(a_1 \& \neg a_2)$. If $K*a_1 = K*(a_1 \& \neg a_2)$, then *success* and *closure* yield $\neg a_2 \in K * a_1$, so that case can be excluded. Two cases remain to be treated:

Case 1, $K * a_1 = K * (a_1 \& a_2)$: Due to *success* and *closure*, $a_2 \in K * (a_1 \& a_2)$, thus $a_2 \in K * a_1$, and:

$\mathrm{Cn}((K * a_1) \cup \{a_2\}) = \mathrm{Cn}(K * a_1)$

$= K * a_1$ *closure*

$= K * (a_1 \& a_2)$ Definition of this case

Case 2, $K*a_1 = K*(a_1 \& a_2) \cap K*(a_1 \& \neg a_2)$: Let $d \in \mathrm{Cn}((K*a_1) \cup \{a_2\})$. We are going to show that $d \in K * (a_1 \& a_2)$:

$d \in \mathrm{Cn}((K * a_1) \cup \{a_2\})$

$d \in \mathrm{Cn}((K * (a_1 \& a_2) \cup \{a_2\})$ $K * a_1 \subseteq K * (a_1 \& a_2)$ in this case

$d \in \mathrm{Cn}(K * (a_1 \& a_2))$ $a_2 \in K * (a_1 \& a_2)$ due to *success* and *closure*

$d \in K * (a_1 \& a_2)$ *closure* ■

[15] This is based on the version in Hansson ([14], pp. 272–273) of Hans Rott's proof, which was reported by Gärdenfors ([11], p. 212).

References

[1] Adams, E.W., Probability and the logic of conditionals, in: J. Hintikka, and P. Suppes (eds.) *Aspects of Inductive Logic*, Norht-Holland, Amsterdam, 1966, pp. 265–316.

[2] Alchourrón, C.E., P. Gärdenfors, and D. Makinson, On the logic of theory change: partial meet contraction and revision functions", *Journal of Symbolic Logic* 50: 510–530, 1985.

[3] Beierle, C., and G. Kern-Isberner, A conceptual agent model based on a uniform approach to various belief operations, in B. Mertsching, M. Hund, and Z. Aziz, (eds.), *KI'09: Proceedings of the 32nd annual German conference on Advances in artificial intelligence*, Springer-Verlag, Berlin, Heidelberg, 2009, pp. 273–280.

[4] Benci, V., L. Horsten, and S. Wenmackers, Infinitesimal probabilities, *British Journal for the Philosophy of Science* 69(2): 509–552, 2018.

[5] Chopra, S., A. Ghose, and T. Meyer, Non-prioritized ranked belief change, *Journal of Philosophical Logic* 32(4): 417–443, 2003.

[6] Falappa, M.A., G. Kern-Isberner, M.D.L. Reis, and G.R. Simari, Prioritized and non-prioritized multiple change on belief bases, *Journal of Philosophical Logic* 41(1): 77–113, 2012.

[7] Fermé, E., and S.O. Hansson, Selective revision, *Studia Logica*, 63(3): 331–342, 1999.

[8] Fermé, E., and S.O. Hansson, Shielded Contraction, in: H. Rott, and M.-A. Williams, (eds.), *Frontiers of Belief Revision*, Kluwer, 2001, pp. 85–107.

[9] Fuhrmann, A., Reflective modalities and theory change, *Synthese* 81: 115–134, 1989.

[10] Gärdenfors, P., Belief Revisions and the Ramsey Test for Conditionals, *Philosophical Review* 95(1): 81–93, 1986.

[11] Gärdenfors, P. *Knowledge in Flux: modeling the dynamics of epistemic states*, MIT Press, 1989.

[12] Grove, A., Two modellings for theory change, *Journal of Philosophical Logic* 17: 157–170, 1988.

[13] Hansson, S.O., Semi-revision, *Journal of Applied Non-Classical Logic* 7(1–2): 151–175, 1997.

[14] Hansson, S.O., A Survey of Non-Prioritized Belief Revision, *Erkenntnis* 50: 413–427, 1999.

[15] Hansson, S.O., *Descriptor Revision. Belief Change Through Direct Choice*, Springer, Cham, 2017.

[16] Hansson, S.O., Review of Hannes Leitgeb, *The stability of belief. How rational belief coheres with probability, Journal of Philosophy* 115(5): 276–280, 2018.

[17] Hansson, S.O., Revising Probabilities and Full Beliefs, *Journal of Philosophical Logic* 49(5): 1005–1039, 2020.

[18] Hansson, S.O., E. Fermé, J. Cantwell, and M. Falappa, Credibility-limited revision, *Journal of Symbolic Logic* 66(4): 1581–1596, 2001.

[19] Jeffrey, R.C., *The Logic of Decision*, 2nd ed., University of Chicago Press, Chicago, 1983.

[20] Katsuno, H., and A.O. Mendelzon, On the difference between updating a knowledge base and revising it, in: Pe. Gärdenfors, (ed.), *Belief Revision*, Cambridge University Press, Cambridge, 1992, pp. 183–203.

[21] Keisler, H.J., *Elementary calculus. An infinitesimal approach*, 2nd ed., Prindle, Weber & Schmidt, Boston, 1986.

[22] Keller, A.M., and M.W. Wilkins, On the use of an extended relational model to handle changing incomplete information, *IEEE Transactions on Software Engineering* 11(7): 620–633, 1985.

[23] Kern-Isberner, G., Linking Iterated Belief Change Operations to Nonmonotonic Reasoning, *Proceedings, Eleventh International Conference on Principles of Knowledge Representation and Reasoning, (KR-08)*, 2008, pp. 166–176.

[24] Konieczny, S., and R. Pino Perez, Improvement operators, in: *Proceedings of the 11th International Conference on Principles of Knowledge Representation and Reasoning (KR-08)*, 2008, pp. 177–186.

[25] Kyburg, H., *Probability and the logic of rational belief*, Wesleyan University Press, Middletown, 1961.

[26] Leitgeb, H., *The stability of belief: how rational belief coheres with probability*, Oxford University Press, Oxford, 2017.

[27] Levi, I., Iteration of conditionals and the Ramsey test, *Synthese* 76: 49–81, 1988.

[28] Makinson, D., The paradox of the preface, *Analysis* 25: 205–207, 1965.

[29] Makinson, D., Screened revision, *Theoria* 63: 14–23, 1997.

[30] Mazzieri, M., and A.F. Dragoni, Ontology revision as non-prioritized belief revision, in: P. Haase, A. Hotho, L. Chen, E. Ong, and P. C. Mauroux, (eds.), *Proceedings of the First International Workshop on Emergent Semantics and Ontology Evolution, ESOE 2007, co-located with ISWC 2007 + ASWC 2007*, Busan, Korea, 2007, pp. 58–69.

[31] Pearl, J., Probabilistic Semantics for Nonmonotonic Reasoning: A Survey, *Proceedings, First International. Conference on Principles of Knowledge Representation and Reasoning*, 1989, pp. 505–516.

[32] Perrotin, E., and F.R. Velázquez-Quesada, A Semantic Approach to Non-prioritized Belief Revision, *Logic Journal of the IGPL*, in press, 2020.

[33] Rott, H., Preferential belief change using generalized epistemic entrenchment, *Journal of Logic, Language and Information* 1(1): 45–78, 1992.

[34] Rott, H., *Change, Choice and Inference: A study of belief revision and nonmonotonic reasoning*, Oxford University Press, Oxford, 2001.

[35] Schurz, G., Impossibility results for rational belief, *Noûs* 53(1): 134–159, 2019.

[36] Skyrms, B., *Causal necessity: a pragmatic investigation of the necessity of laws*, Yale University Press, New Haven, 1980.

S. O. Hansson
Uppsala University: Uppsala Universitet
Uppsala
Sweden
soh@kth.se