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A Few Notes on Quantum B-algebras

Abstract. In order to provide a unified framework for studying non-commutative algebraic logic, Rump and Yang used three axioms to define quantum B-algebras, which can be seen as implicational subreducts of quantales. Based on the work of Rump and Yang, in this paper we shall continue to investigate the properties of three axioms in quantum B-algebras. First, using two axioms we introduce the concept of generalized quantum B-algebras and prove that the opposite of the category **GqBAlg** of generalized quantum B-algebras is equivalent to the category **LogPQ** of logical pre-quantales, but we can not prove that pre-quantales can be used as the injective objects in **GqBAlg**. Next, we use one axiom to propose the concept of C-algebras and show that a C-algebra is a group if and only if each of its elements is dualizing. Further, by dualizing elements of a C-algebra X, we can define different binary operations on X such that X is a moniod. Finally, we by the Zig–Zag relation discuss some properties of quantum B-algebras.

Keywords: Category, Quantale, Quantum B-algebra, Residuated semigroup, Dualizing element, Zig–Zag relation.

1. Introduction

Quantales as a valued domain have been applied to the study of enriched category, lattice-valued algebra, many-valued topology and quantitative domain (see [4,5,8,9,20,21,24,25]). By definition, a *quantale* is a complete lattice Q with an associative multiplication $\cdot$ that distributes over arbitrary joins, that is,

$$a \cdot (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \cdot b_i) \text{ and } (\bigvee_{i\in I} b_i) \cdot a = \bigvee_{i\in I}(b_i \cdot a) \tag{C1}$$

for all $a, b_i \in Q$ (I is an index set). By the completeness of a quantale Q, the multiplication $\cdot$ gives rise to a pair of binary operations $\rightarrow$ and $\rightsquigarrow$ satisfying

$$x \leq y \rightarrow z \Longleftrightarrow x \cdot y \leq z \Longleftrightarrow y \leq x \rightsquigarrow z \tag{C2}$$

for all $x, y, z \in Q$. Based on the above implicational operators of quantales, quantum B-algebras are introduced axiomatically by Rump and Yang (see

Presented by **Constantine Tsinakis**; *Received* October 1, 2018

Studia Logica (2021) 109: 1423–1440
https://doi.org/10.1007/s11225-021-09953-2

[14,18]), and it is proved that quantum B-algebras can provide a unified semantics for non-commutative algebraic logic. Further, in [18] Rump and Yang showed that the three prototypes of algebraic logic together with all their descendants are special cases of quantum B-algebras in a natural way. Recall that a *quantum B-algebra* is a poset X with two binary operations $\rightarrow$ and $\rightsquigarrow$ satisfying the following three conditions

$$y \leq z \Longrightarrow x \rightarrow y \leq x \rightarrow z \tag{C3}$$

$$x \leq y \rightarrow z \Longleftrightarrow y \leq x \rightsquigarrow z \tag{C4}$$

$$x \rightsquigarrow (y \rightarrow z) = y \rightarrow (x \rightsquigarrow z) \tag{C5}$$

for all $x, y, z \in X$. A quantum B-algebra X is called *commutative* if $x \rightarrow y = x \rightsquigarrow y$ for all $x, y \in X$. X is called *unital* if there exists an element $u \in X$ such that $u \rightsquigarrow x = x = u \rightarrow x$ for all $x \in X$. A *residuated semigroup* (*residuated groupoid*) (see [2]) is a partially ordered semigroup (partially ordered groupoid) $(X, \cdot, \leq)$ with two binary operations $\rightarrow$ and $\rightsquigarrow$ which satisfy (C2), where a groupoid means a set with a multiplication (not necessarily associative). A residuated semigroup is said to be *unital* if it is a monoid with respect to the semigroup multiplication. It is easy to verify that every (unital) residuated semigroup is a (unital) quantum B-algebra.

In fact, there is a close relation between quantales and quantum B-algebras. On one hand, a quantale is a quantum B-algebra and quantales can be used as the injective objects in the category **qBAlg** of quantum B-algebras (see [16]). On the other hand, we can construct a quantale from a quantum B-algebra. To see this, given a quantum B-algebra X, Rump and Yang by upper sets constructed the upper-set quantale (the enveloping quantale) $U(X)$ where the multiplication $\odot$ on $U(X)$ is defined as follows

$$A \odot B := \{x \in X : \exists b \in B, b \rightarrow x \in A\}. \tag{C6}$$

By the upper-set quantale $U(X)$, Rump and Yang proved that the opposite of the category **qBAlg** is equivalent to the category **LogQ** of logical quantales (see [18]). However, for the lower-set lattice $L(X)$, Rump and Han et al. respectively gave some examples to indicate that in general we can not use the implicational operators $\rightarrow$ and $\rightsquigarrow$ to define a multiplication $\cdot$ on $L(X)$ such that $(L(X), \cdot)$ is a quantale (see [7,18]).

In [14], Rump proved that the three axioms (C3),(C4),(C5) in a quantum B-algebra are independent to each other. In this note, based on the work of Rump and Yang, we shall continue to investigate the properties of the axioms (C3),(C4),(C5). First, we use the axioms (C3) and (C4) to define generalized quantum B-algebras and prove that most of the properties of quantum B-algebras can be generalized to generalized quantum B-algebras, but we can not prove that pre-quantales can be used as the injective objects in the category **GqBAlg** of generalized quantum B-algebras. Next, we by the axiom (C5) propose the concept of C-algebras and show that a C-algebra is a group if and only if each of its elements is dualizing. Further, by dualizing elements of a C-algebra X, we can define different binary operations on X such that X is a moniod. Finally, we by the Zig–Zag relation discuss some properties of quantum C-algebras.

For more details on quantum B-algebras, readers please refer to [6,7,12, 14–19,22].

2. Generalized Quantum B-algebras

In this section, we shall introduce the concept of generalized quantum B-algebras and consider the relation between generalized quantum B-algebras and logical pre-quantales. Here we shall use the way similar to that of Rump and Yang in [18]. For reader's convenience, we shall give all the details.

2.1. Logical Pre-quantales

A *pre-quantale* is a complete lattice Q with a multiplication $\cdot$ (not necessarily associative) satisfying (C1) (see [13]). By the completeness of a pre-quantale Q, the multiplication $\cdot$ gives rise to a pair of binary operations $\rightarrow$ and $\rightsquigarrow$ satisfying (C2). Note that the three operations $\rightarrow$, $\rightsquigarrow$ and $\cdot$ determine each other. We shall make use of the derived operations (also called the *inverse residuals*)

$$a \twoheadrightarrow b := \bigwedge\{x \in Q : x \cdot a \geq b\}, a \rightarrowtail b := \bigwedge\{x \in Q : a \cdot x \geq b\}.$$

An element c of a complete lattice L is said to be *supercompact* if for any subset $A \subseteq L$, the inequality $c \leq \bigvee A$ implies that $c \leq a$ for some $a \in A$. The set of supercompact elements of L will be denoted by L^{sc}.

Definition 2.1. Let Q be a pre-quantale. Then a non-zero element $c \in Q$ is called *left* (*right*) *balanced* if it satisfies the left (right) of the equations

$$c \cdot (\bigwedge_{i \in I} a_i) = \bigwedge_{i \in I}(c \cdot a_i), (\bigwedge_{i \in I} a_i) \cdot c = \bigwedge_{i \in I}(a_i \cdot c)$$

for arbitrary $a_i \in Q$. If both equations hold, we call c *balanced.*

More properties of balanced elements can be found in [14,18].

Definition 2.2. A pre-quantale Q is called *logical* if every supercompact element of Q is balanced and Q is algebraic, that is, every $a \in Q$ can be represented as a join $a = \bigvee C$ with $C \subseteq Q^{sc}$.

A *morphism* of logical pre-quantales Q, P is a map $f : Q \to P$ which satisfies the condition that $f(\bigvee A) = \bigvee f(A)$ and $f(\bigwedge A) = \bigwedge f(A)$ for any subset $A \subseteq Q$, and $f(a)f(b) \leq f(ab)$ for all $a, b \in Q$. The category of logical pre-quantales will be denoted by **LogPQ**. In what follows, we shall see that logical pre-quantales are related to the following implicational algebras.

2.2. Generalized Quantum B-algebras

In the following we shall use two axioms to define generalized quantum B-algebras. Actually, we shall see that generalized quantum B-algebras have most of the properties of quantum B-algebras.

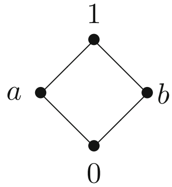
Definition 2.3. A *generalized quantum B-algebra* is a poset X with two binary operations $\to$ and $\rightsquigarrow$ satisfying (C3) and (C4).

Remark 2.4. Any residuated groupoid is a generalized quantum B-algebra. A pre-quantale is a residuated groupoid and hence a generalized quantum B-algebra.

Example 2.5. Let $X = \{0, a, b, 1\}$ be the nonlinear poset with smallest element 0 and greatest element 1. Define a binary operation $\to$ by the below table and put $\rightsquigarrow = \to$.

It is easily checked that X is a commutative generalized quantum B-algebra, but X is not a quantum B-algebra (this is because $1 \to (a \to 0) = 1 \neq a = a \to (1 \to 0)$).

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	b	1	1	1
1	0	1	1	1



Proposition 2.6. *Let $(X, \to, \rightsquigarrow, \leq)$ be a generalized quantum B-algebra. Then for all $x, y, z \in X$ we have*

(1) $y \leq z \Longrightarrow x \rightsquigarrow y \leq x \rightsquigarrow z$;

(2) $y \leq z \Longrightarrow z \rightsquigarrow x \leq y \rightsquigarrow x$;

(3) $y \leq z \Longrightarrow z \to x \leq y \to x$;

(4) $x \leq (x \rightsquigarrow y) \to y, x \leq (x \to y) \rightsquigarrow y$;

(5) $x \to y = ((x \to y) \rightsquigarrow y) \to y, x \rightsquigarrow y = ((x \rightsquigarrow y) \to y) \rightsquigarrow y$*;*

(6) If X *has the smallest element 0, then* $0 \to 0 = 0 \rightsquigarrow 0$ *is the greatest element.*

Proof. It is easy to prove that (C4) can imply (2)–(6). Further, we can show that (C3) and (C4) imply (1). ■

Remark 2.7. It is not difficult to prove that (2),(3),(4) in Proposition 2.6 are equivalent to (C4). Moreover, we can also show that $(X, \to, \rightsquigarrow, \leq)$ be a generalized quantum B-algebra if and only if X satisfies (1)–(4) in Proposition 2.6. Thus, by Proposition 3 in [18] we have that $(X, \to, \rightsquigarrow, \leq)$ be a quantum B-algebra if and only if X satisfies (1)–(4) and the following two conditions $y \to z \leq (x \to y) \to (x \to z)$ and $y \rightsquigarrow z \leq (x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z)$ for all $x, y, z \in X$.

A *morphism* $f : X \to Y$ of generalized quantum B-algebras is a monotonic map which satisfies the following equivalent inequalities

$$f(x \to y) \leq f(x) \to f(y), f(x \rightsquigarrow y) \leq f(x) \rightsquigarrow f(y) \tag{C7}$$

for all $x, y \in X$. The category of generalized quantum B-algebras will be denoted by **GqBAlg**.

Let $(X, \to, \rightsquigarrow, \leq)$ be a generalized quantum B-algebra. Then the set $U(X)$ of upper sets in X is a complete lattice with respect to set-theoretic union. Further, we can verify that $U(X)$ with the multiplication $\odot$ defined by (C6) is a logical pre-quantale. Conversely, given a logical pre-quantale Q, we can show that $(Q^{sc}, \twoheadrightarrow, \rightarrowtail, \geq)$ is a generalized quantum B-algebra (with reverse ordering).

Now, we define two functors

$$U : \mathbf{GqBAlg}^{op} \longrightarrow \mathbf{LogPQ}, V : \mathbf{LogPQ} \longrightarrow \mathbf{GqBAlg}^{op}$$

with $V(Q) = Q^{sc}$. For a morphism $f : X \to Y$ in $\mathbf{GqBAlg}^{op}$ and $A \in U(X)$, we define $U(f)(A) := f^{-1}(A)$, while for a morphism $g : Q \to L$ in **LogPQ**, we define $V(g)$ to be the restriction of $g_\circ$ to L^{sc}, where $g_\circ$ is the left adjoint of g. Using the way similar to Theorem 1 in [18], we can prove the following result.

Theorem 2.8. $\boldsymbol{GqBAlg}^{op}$ *is equivalent to* ***LogPQ***.

Proof. The proof is similar to that of Theorem 1 in [18]. ■

As we mentioned in the introduction, for a given quantum B-algebra X, we in general can not use the implicational operators $\to$ and $\rightsquigarrow$ to define a multiplication $\cdot$ on the lower-set lattice $L(X)$ such that $(L(X), \cdot)$ is a quantale. However, for a given generalized quantum B-algebra X we can define a multiplication $\otimes$ on $L(X)$ as follows

$$\forall A, B \in L(X),\ A \otimes B := \{x \in X : \exists a \in A, b \in B, x \leq a \to b\}$$

or

$$A \otimes B := \{x \in X : \exists a \in A, b \in B, x \leq b \rightsquigarrow a\}.$$

such that $(L(X), \otimes)$ is a pre-quantale. Unfortunately, $(L(X), \otimes)$ is in general not a logical pre-quantale.

In [11,16,26], we have seen that quantales can be used as the injective objects in the categories of quantum B-algebras and partially ordered semigroups. Naturally, we shall consider the question whether pre-quantales can be used as the injective objects in the categories of generalized quantum B-algebras and partially ordered groupoids. First, we shall review the concept of injective objects in a category. Let $\mathcal{C}$ be a category and let $\mathcal{M}$ be a class of morphisms in $\mathcal{C}$. Then an object S in $\mathcal{C}$ is called $\mathcal{M}$-*injective* provided that for any morphism $h\colon A \to B$ in $\mathcal{M}$ and any morphism $f\colon A \to S$ in $\mathcal{C}$ there exists a morphism $g\colon B \to S \in \mathcal{C}$ such that $g \circ h = f$ (see [1]). If there is no confusion for $\mathcal{M}$, an $\mathcal{M}$-injective object is usually called an injective object. Let **POGrpd** denote the category of partially ordered groupoids and their submultiplicative order-preserving maps, where a *submultiplicative map* between partially ordered groupoids is a map $f\colon (A, \cdot, \leq) \to (B, *, \leq)$ such that $f(a) * f(a') \leq f(a \cdot a')$ for all $a, a' \in A$.

Let $(A, \cdot, \leq)$ be a partially ordered groupoid and $a_1, a_2, \cdots, a_n \in A$. Then since $\cdot$ may not be associative the symbol $a_1 a_2 \cdots a_n$ doesn't make sense. If put $n-2$ parentheses to the symbol $a_1 a_2 \cdots a_n$, then we shall work out an element. We assume that there are k_n ways of putting parentheses to the symbol $a_1 a_2 \cdots a_n$, and $\pi_i(a_1 a_2 \cdots a_n)$ denotes the ith way of putting parentheses, where $1 \leq i \leq k_n$. We let $\mathcal{E}$ denote the class of those morphisms $h\colon A \to B$ for which $\pi_i(h(a_1) \cdots h(a_n)) \leq h(a)$ always implies $\pi_i(a_1 \cdots a_n) \leq a$. A morphism $f \in \mathcal{E}$ is called an *embedding*.

Given a partially ordered groupoid $(A, \cdot, \leq)$, we define now a multiplication $\bullet$ on $L(A)$ as follows

$$\forall I, J \in L(A), I \bullet J = \{x \in A : x \leq a \cdot b \text{ for some } a \in I, b \in J\}.$$

It is easy to verify that $(L(A), \bullet)$ is a pre-quantale. To define a map $\mu : A \to L(A)$ by $\mu(a) = \downarrow a$.

Lemma 2.9. *μ is an embedding.*

PROOF. By the case that $\downarrow(a \cdot b) = (\downarrow a) \bullet (\downarrow b)$, we can complete the proof. ■

THEOREM 2.10. *A partially ordered groupoid is injective (with respect to embeddings) if and only if it is a pre-quantale.*

PROOF. Suppose that Q is a pre-quantale, $f : A \to Q$ is a morphism in **POGrpd** and $\varphi : A \to B$ is an embedding. We define now a map $g : B \to Q$ as follows $g(b) = \bigvee\{\pi_i(f(a_1)f(a_2)\cdots f(a_n)) : a_1, a_2, \cdots, a_n \in A \text{such that } \pi_i(\varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)) \leq b\}$. It is easy to show that g is a morphism. When $b = \varphi(a)$, the fact that $\pi_i(\varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)) \leq \varphi(a) \Longrightarrow \pi_i(a_1a_2\cdots a_n) \leq a$ implies $\pi_i(f(a_1)f(a_2)\cdots f(a_n)) \leq f(\pi_i(a_1a_2\cdots a_n)) \leq f(a)$, that is, $g\varphi(a) \leq f(a)$. While the opposite inclusion follows from the fact that $f(a)$ is one of the terms in the sup that defines $g\varphi(a)$.

Conversely, we suppose that A is injective. Then for the embedding $\mu : A \to L(A)$ there exists a morphism $\epsilon : L(A) \to A$ such that $\epsilon \circ \mu = id_A$. Since $(L(A), \bullet)$ is a pre-quantale, using the way similar to Theorem 4.1 in [11], we can prove that A is a pre-quantale. ■

In fact, we want to further prove that every partially ordered groupoid has an injective hull just like partially ordered semigroups (see [23]). However, we failed to do this. Of course, we also hope to prove that pre-quantales can be used as the injective objects in the category **GqBAlg**. Unfortunately, up to now we do not know how to define the embeddings in **GqBAlg** such that pre-quantales are exactly the injective objects.

2.3. Residuated Semigroups

In this subsection we shall consider the condition for generalized quantum B-algebras to be residuated semigroups.

Given a residuated semigroup $(X, \to, \rightsquigarrow, \cdot, \leq)$, it is easy to see that

$$x \to (y \to z) = (x \cdot y) \to z \tag{C8}$$

and

$$x \rightsquigarrow (y \rightsquigarrow z) = (y \cdot x) \rightsquigarrow z \tag{C9}$$

for all $x, y, z \in X$.

PROPOSITION 2.11. *Let $(X, \to, \rightsquigarrow, \cdot, \leq)$ be a residuated groupoid. Then X is a residuated semigroup if and only if X satisfies (C8) or (C9).*

PROOF. Necessity is obvious.

Sufficiency: we only need to prove that $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ for all $a, b, c \in X$.

Assume that X satisfies (C8). Let $x \in X$. Then $(a \cdot b) \cdot c \leq x \Longleftrightarrow a \cdot b \leq c \to x \Longleftrightarrow a \leq b \to (c \to x) = (b \cdot c) \to x \Longleftrightarrow a \cdot (b \cdot c) \leq x$, which implies that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

The case that X satisfies (C9) follows similarly. ■

Remark 2.12. For a given generalized quantum B-algebra $(X, \to, \rightsquigarrow, \leq)$ with a multiplication $\cdot$ satisfying (C8) or (C9), $(X, \to, \rightsquigarrow, \cdot, \leq)$ is not necessarily a residuated semigroup (see Example 2.13).

Example 2.13. Let X be the commutative generalized quantum B-algebra defined in Example 2.5. We define two multiplications $\cdot$ and $*$ on X as follows.

It is easy to check that X satisfies (C8) with respect to $\cdot$ and satisfies (C9) with respect to $*$, but $(X, \to, \rightsquigarrow, \cdot, \leq)$ and $(X, \to, \rightsquigarrow, *, \leq)$ are not residuated semigroups.

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	0	0	1

$*$	0	a	b	1
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
1	0	a	b	1

Remark 2.14. Even if a quantum B-algebra simultaneously satisfies (C8) and (C9) with respect to a multiplication $\cdot$, it is not necessarily a residuated semigroup (see Example 2.15).

Example 2.15. Let $X = \{0, a, b, 1\}$ be the poset defined in Example 2.5. We define now two binary operations $\to$ and $\cdot$ on X as follows.

We put $\rightsquigarrow = \to$. Then it is easy to check that X is a commutative quantum B-algebra and satisfies (C8) and (C9) with respect to $\cdot$, but $(X, \to, \rightsquigarrow, \cdot, \leq)$ is not a residuated semigroup.

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	1	1	1
1	0	1	1	1

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	b	b	b
1	0	1	1	1

If generalized quantum B-algebras satisfy some condition, then we shall reach our desired result. We first recall the concept of positive subsets. A subset A of a generalized quantum B-algebra X is said to be *positive* if A is an upper set and satisfies the condition that $x \rightsquigarrow y \in A \Longleftrightarrow x \leq y \Longleftrightarrow x \to y \in A$ for all $x, y \in X$. A generalized quantum B-algebra is called *positive* if it has a positive subset. In [6], Han et al. proved that a quantum B-algebra X is positive if and only if $U(X)$ is a unital quantale with a multiplication $\odot$. Here, we can also prove that a generalized quantum B-algebra X is positive if and only if $U(X)$ is a unital pre-quantale with a multiplication $\odot$ defined by (C6).

PROPOSITION 2.16. *Let $(X, \rightarrow, \rightsquigarrow, \leq)$ be a positively generalized quantum B-algebra with a multiplication $\cdot$ satisfying $(C8)$ or $(C9)$. Then $(X, \rightarrow, \rightsquigarrow, \cdot, \leq)$ is a residuated semigroup.*

PROOF. Suppose that $(X, \rightarrow, \rightsquigarrow, \leq)$ is a positively generalized quantum B-algebra with a multiplication $\cdot$ satisfying (C8), where A is a positive subset of X.

We first prove that the multiplication $\cdot$ is associative. Let $a, b, c, x \in X$. Then we have

$$\begin{aligned}(a \cdot b) \cdot c \leq x &\Longleftrightarrow ((a \cdot b) \cdot c) \rightarrow x \in A\\ &\Longleftrightarrow (a \cdot b) \rightarrow (c \rightarrow x) \in A\\ &\Longleftrightarrow a \rightarrow (b \rightarrow (c \rightarrow x)) \in A\\ &\Longleftrightarrow a \rightarrow ((b \cdot c) \rightarrow x) \in A\\ &\Longleftrightarrow (a \cdot (b \cdot c)) \rightarrow x \in A\\ &\Longleftrightarrow a \cdot (b \cdot c) \leq x,\end{aligned}$$

which implies that $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Next, we shall prove that X satisfies (C2). Let $a, b, c \in X$. Then we have

$$\begin{aligned}a \cdot b \leq c &\Longleftrightarrow (a \cdot b) \rightarrow c \in A\\ &\Longleftrightarrow a \rightarrow (b \rightarrow c) \in A\\ &\Longleftrightarrow a \leq b \rightarrow c\\ &\Longleftrightarrow b \leq a \rightsquigarrow c.\end{aligned}$$

Thus, $(X, \rightarrow, \rightsquigarrow, \cdot, \leq)$ is a residuated semigroup.

The argument for the case that X satisfies (C9) proceeds similarly. ∎

The inverse of Proposition 2.16 is not necessarily right. In other words, a residuated semigroup is a quantum B-algebra, but it may not be positive (see Example 2.17).

EXAMPLE 2.17. Let $X = \{0, a, b, 1\}$ be the poset defined in Example 2.5. We define now two binary operations $\rightarrow$ and $\cdot$ on X as follows.

We put $\rightsquigarrow = \rightarrow$. Then it is easy to check that X is a residuated semigroup, but it is not positive.

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	0	1
b	0	1	0	1
1	0	1	0	1

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
1	0	a	a	a

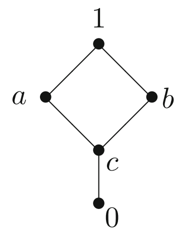
If a generalized quantum B-algebra is unital, then it must be positive. Thus, we have the following result.

Corollary 2.18. *X is a unital generalized quantum B-algebra satisfying $(C8)$ or $(C9)$ if and only if X is a unital residuated semigroup.*

Remark 2.19. There is a generalized quantum B-algebra X on which there does not exist a multiplication $\cdot$ satisfying (C8) or (C9) (see Example 2.20).

Example 2.20. Let $X = \{0, a, b, c, 1\}$ be a poset determined by the figure below. Define a binary operation $\rightarrow$ by the following table and put $\rightsquigarrow = \rightarrow$.

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	c	1	1	1	1
1	0	a	b	c	1



It is easily checked that X is a commutative and unital generalized quantum B-algebra. Since $b \rightarrow (c \rightarrow 0) = a \neq c = c \rightarrow (b \rightarrow 0)$, X is not a quantum B-algebra. By Corollary 2.18, we have that there does not exist a multiplication $\cdot$ on X satisfying (C8) or (C9).

Proposition 2.21. *Let $(X, \rightarrow, \rightsquigarrow, \leq)$ be a generalized quantum B-algebra with a multiplication $\cdot$ satisfying $(C8)$ and $(C9)$. Then $(C_X, \rightarrow, \rightsquigarrow, \leq)$ is a quantum B-algebra, where $C_X = \{a \in X : x \rightarrow (y \rightsquigarrow a) = y \rightsquigarrow (x \rightarrow a)$ for all $x, y \in X\}$.*

Proof. It suffices to prove that C_X is closed under $\rightarrow$ and $\rightsquigarrow$.

Let $a, b \in C_X$. Then for all $x, y \in X$ we have

$$\begin{aligned} x \rightarrow (y \rightsquigarrow (a \rightarrow b)) &\Longleftrightarrow x \rightarrow (a \rightarrow (y \rightsquigarrow b)) \\ &\Longleftrightarrow (x \cdot a) \rightarrow (y \rightsquigarrow b) \\ &\Longleftrightarrow y \rightsquigarrow ((x \cdot a) \rightarrow b) \\ &\Longleftrightarrow y \rightsquigarrow (x \rightarrow (a \rightarrow b)) \end{aligned}$$

and

$$\begin{aligned} x \to (y \rightsquigarrow (a \rightsquigarrow b)) &\Longleftrightarrow x \to ((a \cdot y) \rightsquigarrow b) \\ &\Longleftrightarrow (a \cdot y) \rightsquigarrow (x \to b) \\ &\Longleftrightarrow y \rightsquigarrow (a \rightsquigarrow (x \to b)) \\ &\Longleftrightarrow y \rightsquigarrow (x \to (a \rightsquigarrow b)), \end{aligned}$$

which imply that $a \to b, a \rightsquigarrow b \in C_X$, that is, C_X is closed under $\to$ and $\rightsquigarrow$. ■

3. *C*-algebras

In this section, we shall use the axiom (C5) to define C-algebras and investigate the relation between C-algebras and monoids (groups).

Definition 3.1. A set X with two binary operations $\to$ and $\rightsquigarrow$ is called a *C-algebra* if $\to$ and $\rightsquigarrow$ satisfy the condition (C5).

In fact, the condition (C5) is similar to the axiom (C) that $(x * y) * z = (x * z) * y$ in BCI-algebras (or in BCK-algebras) (see Remark 3 in [10] and Example 2.1 in [14]). So, we call the above algebra satisfying the condition (C5) a C-algebra.

An element d of a C-algebra X is called a *dualizing element* if $(x \to d) \rightsquigarrow d = x = (x \rightsquigarrow d) \to d$ for all $x \in X$. A C-algebra X is said to be *dual* if X has a dualizing element.

Lemma 3.2. *Let $(X, \to, \rightsquigarrow)$ be a dual C-algebra with dualizing element d. Then for all $x, y \in X$ we have*

(1) $x \to y = (y \to d) \rightsquigarrow (x \to d)$;

(2) $x \rightsquigarrow y = (y \rightsquigarrow d) \to (x \rightsquigarrow d)$;

(3) $(x \rightsquigarrow d) \to y = (y \to d) \rightsquigarrow x$;

(4) $x = (d \to d) \rightsquigarrow x,\ x = (d \rightsquigarrow d) \to x$.

Proof. Proof is straightforward. ■

Lemma 3.3. *Let $(X, \to, \rightsquigarrow)$ be a dual C-algebra with dualizing elements d_1, d_2. Then $d_1 \to d_1 = d_2 \to d_2$ and $d_1 \rightsquigarrow d_1 = d_2 \rightsquigarrow d_2$.*

Proof. From Lemma 3.2(4), it follows that $(d_1 \to d_1) \rightsquigarrow (d_2 \to d_2) = d_2 \to d_2$. By Lemma 3.2(2),(4) again, we have that $(d_1 \to d_1) \rightsquigarrow (d_2 \to d_2) = ((d_2 \to d_2) \rightsquigarrow d_1) \to ((d_1 \to d_1) \rightsquigarrow d_1) = d_1 \to d_1$. Thus, we have $d_1 \to d_1 = d_2 \to d_2$.

Similarly, we can show $d_1 \rightsquigarrow d_1 = d_2 \rightsquigarrow d_2$. ■

EXAMPLE 3.4. Let $X = \{a, b, c\}$, $\to$ and $\rightsquigarrow$ be two binary operations on X defined by the table below.

It is easy to verify that $(X, \to, \rightsquigarrow)$ is a dual C-algebra with dualizing element a.

$\to$	a	b	c
a	c	b	b
b	a	b	c
c	b	b	b

$\rightsquigarrow$	a	b	c
a	b	b	b
b	c	b	b
c	a	b	c

Let $(X, \to, \rightsquigarrow)$ be a dual C-algebra with dualizing element d. Then we define a binary operation $\otimes_d$ on X as follows

$$\forall x, y \in X, \quad x \otimes_d y := (x \rightsquigarrow d) \to y.$$

LEMMA 3.5. *Let $(X, \to, \rightsquigarrow)$ be a dual C-algebra with dualizing element d. Then $(X, \otimes_d)$ is a monoid, where d is the unit with respect to $\otimes_d$.*

PROOF. By Lemma 3.2, we see that d is the unit with respect to $\otimes_d$. Next, it suffices to prove that $\otimes_d$ is associative.

Let $x, y, z \in X$. Then by Lemma 3.2 we have

$$\begin{aligned}(x \otimes_d y) \otimes_d z &= ((x \otimes_d y) \rightsquigarrow d) \to z \\ &= (z \to d) \rightsquigarrow (x \otimes_d y) \\ &= (z \to d) \rightsquigarrow ((x \rightsquigarrow d) \to y) \\ &= (x \rightsquigarrow d) \to ((z \to d) \rightsquigarrow y) \\ &= (x \rightsquigarrow d) \to ((y \rightsquigarrow d) \to z) \\ &= (x \rightsquigarrow d) \to (y \otimes_d z) \\ &= x \otimes_d (y \otimes_d z).\end{aligned}$$

Thus, $(X, \otimes_d)$ is a monoid. ■

THEOREM 3.6. *A C-algebra X is a group if and only if X is non-empty and every element of X is dualizing.*

PROOF. Every group X is a C-algebra with two binary operations $x \to y = yx^{-1}$ and $x \rightsquigarrow y = x^{-1}y$. It is easy to verify that every element of X is dualizing.

Conversely, we assume that every element of X is dualizing. By Lemma 3.3, we have that $x \to x = y \to y$ and $x \rightsquigarrow x = y \rightsquigarrow y$ for all $x, y \in X$. Let $u = x \to x$. Then by Lemma 3.2(4) we have $u \rightsquigarrow u = u$, which implies that $x \to x = x \rightsquigarrow x$. From Lemma 3.5, it follows that $(X, \otimes_u)$ is a monoid. It suffices to show that every element $x \in X$ has the left inverse and the right inverse.

Let $x \in X$. Then we have

$$x \otimes_u (x \rightsquigarrow u) = (x \rightsquigarrow u) \to (x \rightsquigarrow u) = u$$

and

$$(x \to u) \otimes_u x = ((x \to u) \rightsquigarrow u) \to x = x \to x = u,$$

which implies that $(X, \otimes_u)$ is a group. ■

REMARK 3.7. In the proof of Theorem 3.6, we can see that $x \to u = x \rightsquigarrow u$. Since every element is dualizing, we by Lemma 3.2 have that $x \otimes_u (x \to y) = (x \rightsquigarrow u) \to (x \to y) = ((x \to y) \to u) \rightsquigarrow x = ((x \to y) \rightsquigarrow u) \rightsquigarrow x = ((u \rightsquigarrow y) \to x) \rightsquigarrow x = y$. Similarly, we have $(x \rightsquigarrow y) \otimes_u x = y$.

Let $(X, \to, \rightsquigarrow)$ be a dual C-algebra with dualizing element d. Then we can define two new binary operations $\odot_d$ and $\oplus_d$ on X as follows

$$x \odot_d y := (x \to (y \to d)) \rightsquigarrow d,$$
$$x \oplus_d y := (x \rightsquigarrow (y \rightsquigarrow d)) \to d.$$

In general, $\odot_d \neq \otimes_d$ and $\oplus_d \neq \otimes_d$. To see this, in Example 3.4, a is a dualizing element. By elementary calculation, we have that $b \odot_a c = c \neq b = b \otimes_a c$ and $c \oplus_a b = a \neq b = c \otimes_a b$.

PROPOSITION 3.8. *Let $(X, \to, \rightsquigarrow)$ be a dual C-algebra with dualizing element d. Then $(X, \odot_d)$ is a monoid with unit $d \rightsquigarrow d$, and $(X, \oplus_d)$ is a monoid with unit $d \to d$.*

PROOF. It follows from Lemma 3.2 that $d \rightsquigarrow d$ is the unit with respect to $\odot_d$ and $d \to d$ is the unit with respect to $\oplus_d$.

Let $x, y, z \in X$. Then we have

$$\begin{aligned}(x \odot_d y) \odot_d z &= ((x \odot_d y) \to (z \to d)) \rightsquigarrow d\\ &= (((x \to (y \to d)) \rightsquigarrow d) \to (z \to d)) \rightsquigarrow d\\ &= (((z \to d) \to d) \rightsquigarrow (x \to (y \to d))) \rightsquigarrow d\\ &= (x \to (((z \to d) \to d) \rightsquigarrow (y \to d))) \rightsquigarrow d\\ &= (x \to (y \to (z \to d))) \rightsquigarrow d\\ &= (x \to (((y \to (z \to d)) \rightsquigarrow d) \to d)) \rightsquigarrow d\\ &= (x \to ((y \odot_d z) \to d)) \rightsquigarrow d\\ &= x \odot_d (y \odot_d z).\end{aligned}$$

Thus, $(X, \odot_d)$ is a monoid.

Similarly, we can prove that $(X, \oplus_d)$ is also a monoid. ■

REMARK 3.9. For a dualizing element d in a C-algebra X, in general $d \to d \neq d \rightsquigarrow d$ and $\odot_d \neq \oplus_d$. For instance, a is a dualizing element in Example 3.4. It is easy to verify that $a \to a = c \neq b = a \rightsquigarrow a$ and $a \odot_a c =$

$c \neq a = a \oplus_a c$. Further, one can prove that since $b \otimes_a a = b = b \otimes_a b$, $a \odot_a a = c = a \odot_a c$ and $a \oplus_a a = a = a \oplus_a b$, $(X, \otimes_a)$, $(X, \odot_a)$ and $(X, \oplus_a)$ are not groups.

PROPOSITION 3.10. *Let $(X, \to, \rightsquigarrow)$ be a dual C-algebra with dualizing element d and satisfy $(C4)$. Then $\odot_d = \oplus_d^{op}$ and $d \to d = d \rightsquigarrow d$.*

PROOF. Let $(X, \to, \rightsquigarrow)$ be a dual C-algebra with dualizing element d and satisfy (C4). Then by Proposition 2.6, we have that $y \rightsquigarrow d \leq x \rightsquigarrow d \iff x \leq y \iff y \to d \leq x \to d$.

For all $x, y, t \in X$, we have

$$\begin{aligned} x \odot_d y \leq t &\iff (x \to (y \to d)) \rightsquigarrow d \leq t \\ &\iff t \to d \leq x \to (y \to d) \\ &\iff x \leq (t \to d) \rightsquigarrow (y \to d) \\ &\iff x \leq y \to t \iff y \leq x \rightsquigarrow t \\ &\iff y \leq (t \rightsquigarrow d) \to (x \rightsquigarrow d) \\ &\iff t \rightsquigarrow d \leq y \rightsquigarrow (x \rightsquigarrow d) \\ &\iff (y \rightsquigarrow (x \rightsquigarrow d)) \to d \leq t \\ &\iff y \oplus_d x \leq t, \end{aligned}$$

which implies $x \odot_d y = y \oplus_d x$, that is, $\odot_d = \oplus_d^{op}$.

By Proposition 3.8 and the above description, we have that $d \to d = (d \rightsquigarrow d) \odot_d (d \to d) = (d \to d) \oplus_d (d \rightsquigarrow d) = d \rightsquigarrow d$. ∎

PROPOSITION 3.11. *X is a dual C-algebra satisfying $(C4)$ if and only if X is a dual residuated semigroup.*

PROOF. Sufficiency is obvious.

Assume that $(X, \to, \rightsquigarrow)$ is a dual C-algebra with dualizing element d. From Proposition 3.8, it suffices to prove that $(X, \to, \rightsquigarrow, \odot_d, \leq)$ satisfies the condition (C2).

Let $x, y, z \in X$. Then by Lemma 3.2 and Proposition 3.10 we have

$$\begin{aligned} x \odot_d y \leq z &\iff (x \to (y \to d)) \rightsquigarrow d \leq z \\ &\iff z \to d \leq x \to (y \to d) \\ &\iff x \leq (z \to d) \rightsquigarrow (y \to d) \\ &\iff x \leq y \to z \\ &\iff y \leq x \rightsquigarrow z. \end{aligned}$$

Thus, $(X, \to, \rightsquigarrow, \odot_d, \leq)$ is a dual residuated semigroup. ∎

COROLLARY 3.12. *X is a dual quantum B-algebra if and only if X is a dual residuated semigroup.*

4. Zig–Zag Relation on Quantum B-algebras

In this section, we shall consider the Zig–Zag relation on a quantum B-algebra.

Definition 4.1. Let X be a poset. Then two elements $x, y \in X$ are said to be *Zig–Zag connected* (or to have the *Zig–Zag relation*) if there is a finite comparable sequence $x = z_0, z_1, \cdots, z_n = y$ in X, that is, z_{i-1} and z_i are comparable (i.e., $z_{i-1} \leq z_i$ or $z_{i-1} \geq z_i$) (see [3,18]). This defines an equivalence relation $\sim$ on X, that is, $x \sim y$ if and only if x and y are Zig–Zag connected.

In [14,18], Rump proved that $\sim$ is a congruence relation on a quantum B-algebra X and $([x] \to [y]) \rightsquigarrow [y] = [x] = ([x] \rightsquigarrow [y]) \to [y]$ for all $[x], [y] \in X/\sim$, where $[x] = \{y \in X : x \sim y\}$. By Theorem 3.6, we have that $(X/\sim, \otimes_U)$ is a group, where $U = [x \to x] = [x \rightsquigarrow x]$.

Let $(X, \to, \rightsquigarrow, \leq)$ be a quantum B-algebra. Then we denote by $A_{a,b}$ the set $\{x \in X : a \leq b \to x\}$. Usually, $A_{a,a}$ is simply written as A_a. In order to investigate the properties of the equivalence class modulo $\sim$, we need to introduce the following concepts.

Definition 4.2. A quantum B-algebra X is called *conditional* if $A_a \neq \emptyset$ for all $a \in X$. A conditional quantum B-algebra X is called *strong* if for any descending chain $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots$ in $A_{a,b}$ it is bounded below in X for all $a, b \in X$.

Remark 4.3. A residuated semigroup is a strongly conditional quantum B-algebra.

Theorem 4.4. *Let X be a conditional quantum B-algebra. Then $a \in X$ is maximal in X if and only if $a = \max[a]$.*

Proof. Sufficiency is easy.

Suppose that $a \in X$ is a maximal element in X. If $\downarrow x \cap \downarrow a \neq \emptyset$, then $x \leq a$. Indeed: Let $y \in \downarrow x \cap \downarrow a$. Then since $A_a \neq \emptyset$ there exists an element $a_0 \in A_a$ such that $a \leq a \to a_0$ which implies $a \leq a \to a_0 \leq y \to a_0$. By the fact that a is maximal, we have that $a = a \to a_0 = y \to a_0 \geq x \to a_0$. Thus, $a = a \rightsquigarrow a_0 \leq (x \to a_0) \rightsquigarrow a_0$, which implies $a = (x \to a_0) \rightsquigarrow a_0 \geq x$.

Let $b \in [a]$. Then we need to prove $b \leq a$. Suppose that in any finite Zig–Zag chain joining a to b there exists a first element a_k such that $a_k \not\leq a$. Then $a_{k-1} \leq a$ and $a_{k-1} \leq a_k$ which imply that $\downarrow a_k \cap \downarrow a \neq \emptyset$. By the above description, we have $a_k \leq a$, contradiction. So, all the elements in the finite Zig–Zag chain are therefore less than or equal to a, which implies $b \leq a$, that is, $a = \max[a]$. ∎

Remark 4.5. Let X be a poset with two maximal elements a, b and $\downarrow a \cap \downarrow b \neq \emptyset$. Then we can not define two binary operations $\rightarrow$ and $\rightsquigarrow$ on X such that $(X, \rightarrow, \rightsquigarrow, \leq)$ is a conditional quantum B-algebra.

Lemma 4.6. *Let $(X, \rightarrow, \rightsquigarrow, \leq)$ be a non-empty quantum B-algebra, $\mathcal{A}, \mathcal{B}, \mathcal{C} \in X/\sim$. If $\mathcal{C} \otimes_U \mathcal{B} = \mathcal{A}$, then $c \rightarrow a \in \mathcal{B}, b \rightsquigarrow a \in \mathcal{C}$ for all $a \in \mathcal{A}, b \in \mathcal{B}, c \in \mathcal{C}$.*

Proof. Let $a \in \mathcal{A}, b \in \mathcal{B}, c \in \mathcal{C}$. Then $[a] = \mathcal{A}, [b] = \mathcal{B}, [c] = \mathcal{C}$. By Remark 3.7, we have that $\mathcal{C} \otimes_U [c \rightarrow a] = [c] \otimes_U ([c] \rightarrow [a]) = [a] = \mathcal{A} = \mathcal{C} \otimes_U \mathcal{B}$. Since $(X/\sim, \otimes_U)$ is a group, we have that $[c \rightarrow a] = \mathcal{B}$, which implies $c \rightarrow a \in \mathcal{B}$. Similarly, we can prove that $b \rightsquigarrow a \in \mathcal{C}$. ■

Proposition 4.7. *Let X be a strongly conditional quantum B-algebra that is non-empty. If X contains a descending chain that is unbounded below, then every class (modulo $\sim$) contains at least one such chain.*

Proof. Let $a_1 \geq a_2 \geq \cdots a_n \geq \cdots$ be a descending chain, unbounded below, in X. Then this chain must be in some class $\mathcal{A}$ modulo $\sim$. We let $\mathcal{B}$ be any class modulo $\sim$. Then there exists a unique class $\mathcal{C}$ modulo $\sim$ such that $\mathcal{C} \otimes_U \mathcal{B} = \mathcal{A}$, and by Lemma 4.6 we see that for every $c \in \mathcal{C}$ we have in $\mathcal{B}$ the descending chain $c \rightarrow a_1 \geq c \rightarrow a_2 \geq c \rightarrow a_3 \geq \cdots$. Suppose that there exists $b \in \mathcal{B}$ such that $c \rightarrow a_n \geq b$ for every n. Then the descending chain $a_1 \geq a_2 \geq \cdots a_n \geq \cdots$ is in $A_{b,c}$. Since X is strongly conditional, we have that the descending chain $a_1 \geq a_2 \geq \cdots a_n \geq \cdots$ is bounded below, contradiction. Thus, $\mathcal{B}$ contains a descending chain that is unbounded below, and since $\mathcal{B}$ is arbitrary the same is true for all classes modulo $\sim$. ■

Proposition 4.8. *Let X be a strongly conditional quantum B-algebra that is non-empty. If X contains an ascending chain that is unbounded above, then every class (modulo $\sim$) contains at least one descending chain that is unbounded below.*

Proof. Let $a_1 \leq a_2 \leq \cdots a_n \leq \cdots$ be an ascending chain, unbounded above, in X. Then this chain must be in some class $\mathcal{A}$ modulo $\sim$. Assume that $\mathcal{B}$ is any class modulo $\sim$. Then there exists a unique class $\mathcal{C}$ modulo $\sim$ such that $\mathcal{A} \otimes_U \mathcal{B} = \mathcal{C}$. By Lemma 4.6 we see that for every $c \in \mathcal{C}$ we have in $\mathcal{B}$ the descending chain $a_1 \rightarrow c \geq a_2 \rightarrow c \geq a_3 \rightarrow c \geq \cdots$. Suppose that there exists $b \in \mathcal{B}$ such that $a_n \rightarrow c \geq b$ for every n, which implies $a_n \leq b \rightsquigarrow c$ and the ascending chain $a_1 \leq a_2 \leq \cdots a_n \leq \cdots$ is bounded above, contradiction. Thus, $\mathcal{B}$ contains a descending chain that is unbounded below. ■

Acknowledgements. The authors gratefully acknowledge the financial support of the National Natural Science Foundation of China

(Grant No. 11971286) and the Fundamental Research Funds for the Central University (GK202101009), and also wish to express their sincere thanks to the anonymous referees for their valuable comments and suggestions which improved the quality of the paper.

References

[1] Adámek, J., H. Herrlich, and G.E. Strecker, *Abstract and concrete categories: The Joy of Cats*, John Wiley & Sons, 1990.

[2] Birkhoff, G., *Lattice theory*, Amer. Math. Soc., New York, Providence, RI, 1940.

[3] Blyth, T. S., *Lattices and Ordered Algebraic Structures*, Springer-Verlag, London, 2005.

[4] Fan, L., A new approach to quantitative domain theory, *Electronic Notes in Theoretical Computer Science* 45: 77–87, 2001.

[5] Han, S.W., and B. Zhao, Q-fuzzy subsets on ordered semigroups, *Fuzzy Sets Syst.* 210: 102–116, 2013.

[6] Han, S.W., X.T. Xu, and F. Qin, The unitality of quantum B-algebras, *Int. J. Theor. Phys.* 57: 1582–1590, 2018.

[7] Han, S.W., R.R. Wang, and X.T. Xu, On the injective hulls of quantum B-algebras, *Fuzzy Sets Syst.* 369: 114–121, 2019.

[8] Hofmann, D., and P. Waszkiewicz, Approximation in quantale-enriched categories, *Topol. Appl.* 158(8): 963–977, 2011.

[9] Höle, U., and T. Kubiak, A non-commutative and non-idempotent theory of quantale sets, *Fuzzy Sets Syst.* 166: 1–43, 2011.

[10] Iséki, K., An algebra related with a propositional calculus, *Proc. Japan Acad.* 42(1): 26–29, 1966.

[11] Lambek, J., M. Barr, J.F. Kennison, and R. Raphael, Injective hulls of partially ordered monoids, *Theory Appl. Categ.* 26: 338–348, 2012.

[12] Pan, F.F., Dual quantum B-algebras, *Soft Compt.* 23: 6813–6817, 2019.

[13] Rosenthal, K.I., *Quantales and their applications*, Longman Scientific & Technical, New York, 1990.

[14] Rump, W., Quantum B-algebras, *Cent. Eur. J. Math.* 11: 1881–1899, 2013.

[15] Rump, W., Multi-posets in algebraic logic, group theory, and non-commutative topology, *Ann. Pure Appl. Logic* 167: 1139–1160, 2016.

[16] Rump, W., The completion of a quantum B-algebra, *Cah. Topol, Géom. Différ. Catég.* 57: 203–228, 2016.

[17] Rump, W., Quantum B-algebras: their omnipresence in algebraic logic and beyond. *Soft Compt.* 21: 2521–2529, 2017.

[18] Rump, W., and Y.C. Yang, Non-commutative logical algebras and algebraic quantales, *Ann. Pure Appl. Logic* 165: 759–785, 2014.

[19] Rump, W., and Y.C. Yang, Hereditary arithmetics, *J. Algebra* 468: 214–252, 2016.

[20] Solovyov, S.A., From quantale algebroids to topological spaces: fixed- and variable-basis approaches, *Fuzzy Sets Syst.* 161: 1270–1287, 2010.

[21] Stubbe, I., Categorical structures enriched in a quantaloid: categories, distributors and functors, *Theory Appl. Categ.* 13: 1–45, 2005.

[22] Xia, C.C., On the finite embeddability properties for quantum B-algebras, *Math. Slovaca* 69: 721–728, 2019.

[23] Xia, C.C., S.W. Han, and B. Zhao, A note on injective hulls of posemigroups, *Theory Appl. Categ.* 7: 254–257, 2017.

[24] Yao, W., A survey of fuzzifications of frames, the Papert-Papert-Isbell adjunction and sobriety, *Fuzzy Sets Syst.* 190: 63–81, 2012.

[25] Zhang, D., An enriched category approach to many valued topology, *Fuzzy Sets Syst.* 158(4): 349–366, 2007.

[26] Zhang, X., and V. Laan, Injective hulls for posemigroups, *Proc. Est. Acad. Sci.* 63: 372–378, 2014.

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