N. Belnap
T. Müller ⓘ
T. Placek

New Foundations for Branching Space-Times

Abstract. The theory of branching space-times, put forward by Belnap (*Synthese* 92, 1992), considers indeterminism as local in space and time. In the axiomatic foundations of that theory, so-called choice points mark the points at which the (local) possible future can turn out in different ways. Working under the assumption of choice points is suitable for many applications, but has an unwelcome topological consequence that makes it difficult to employ branching space-times to represent a range of possible physical space-times. Therefore it is interesting to develop a branching space-times theory without choice points. This is what we set out to do in this paper, providing new foundations for branching space-times in terms of choice sets rather than choice points. After motivating and developing the resulting theory in formal detail, we show that it is possible to translate structures of one style into structures of the other style and vice versa. This result shows that the underlying idea of indeterminism as the branching of spatio-temporal histories is robust with respect to different implementations, making a choice between them a matter of expediency rather than of principle.

Keywords: Indeterminism, Branching histories, Choice points, Topology, Local Euclidicity, Hausdorff property.

1. Motivating a Formal Theory of Indeterminism in Space-Time

The theory of branching space-times (henceforth BST) considers local indeterminism in space and time. Indeterminism is the thesis that a system has more than one alternative possible future evolution. In BST, the emphasis is on the divergence of alternative evolutions occurring in small regions of space and time that can be idealized to be just point events. From the perspective of BST as put forward by Belnap [1], which we will denote BST_{92}, indeterminateness gives way to determinateness at point events.

A focus on local aspects of indeterminism is by no means peculiar to the BST_{92} analysis of indeterminism. It is quite typical for questions of where and when to come up in indeterministic contexts. We ask, for instance, where and when a person decided on a particular course of action. Or, in a science-related context, we may wonder at which location, and from which instant

on, it has been determined that an electron passing a Stern-Gerlach device would be deflected in a given direction. The naturalness of such questions indicates, we believe, that an adequate analysis of indeterminism needs to pay attention to local details of indeterminism: where, when, and how alternative evolutions diverge. These questions are pressing, no matter whether one prefers to analyze indeterminism in terms of branching possible histories, like in BST_{92}, or in terms of non-overlapping segment-wise isomorphic possible scenarios of the Lewis tradition.

The question of what the branching of possible histories looks like at the local level is difficult to answer. BST_{92}, being a rigorous axiomatic theory, decides the question of how branching occurs via one of its axioms: in BST_{92}, the overlap of two possible histories always has a maximal element. The opposite option would be for the difference of two histories to always have a minimal element. Which option is right? Or maybe this cannot be decided, or an answer to the question can be avoided altogether? The present paper shows how to avoid taking sides, in the following sense. First, we present a number of theorems that show that the second option mentioned, branching of histories without maximal elements, can be worked out in a formally precise way.[1] Second, we show that there is a systematic way of translating branching structures of one kind into branching structures of the other kind. In this sense, we can leave it open what branching is really like. For both steps, the notion of a transition structure, which represents local indeterminism, is crucial.[2]

A key motivation for working out an ecumenical position is that we do not want to press any global argument for preferring one of the two options mentioned above. We acknowledge that there are valid reasons in favor of both. In the original paper developing BST_{92} [1], the decision in favor of maxima in the intersection of histories is commented as follows:

Finally, let me explicitly note that on the present theory [. . .],

a causal origin has always 'a last point of indeterminateness' (the choice point) and never 'a first point of determinateness'. I find the matter puzzling since it's neither clear to me how an alternate theory would work nor clear what difference it makes. [1, 428]

---

[1]Technically, we offer a way of replacing the prior choice principle PCP_{92} of Belnap [1], Definition 6, with a new version, PCP_{NF}, via Definition 18.

[2]Transition structures figure prominently in the work of Rumberg [19, 20]; see also Rumberg and Zanardo [21]. These works focus mainly on semantics and proof theory, and they consider transitions in the simpler framework of so-called branching time, not branching space-times.

This feeling of puzzlement also stands behind some objections to BST_{92}: the objectors ask what the reasons are for assuming one pattern of branching, or they are skeptical whether that pattern is compatible with the physics of space-time.[3] Thus, it seems prudent not to decide the matter of patterns of branching by fiat, at least not in advance of some further considerations of theoretical physics.

A second motivation for this paper stems from topological consequences of preferring one option over the other, and relates to topological requirements standardly imposed on the mathematical structures used in physics for representing space-times, so-called differential manifolds. In this context, it is perhaps worth observing that BST_{92} is not meant to be a theory of physics, for two reasons. On the one hand, its axioms are too frugal to be specific enough for a direct application in physics. On the other hand, BST_{92} explores the combination of space-time and modality, and that combination does not fall into the standard repertoire of physics. Thus, general relativity represents the evolution of *single* space-times, whereas branching structures aim to accommodate *multiple alternative* space-times. In BST_{92} one should thus not expect to find the same representational mathematical structures as in physics, since these structures represent different aspects of spatio-temporal reality. But of course, some way of combining these endeavors is desirable.

Now, a differential manifold used in physics to represent a space-time has two properties that are hard to satisfy in branching structures. First, by definition, a differential manifold is locally Euclidean, which means that each point of the manifold has a neighborhood that can be mapped continuously onto an open subset of $\mathbb{R}^n$ (in realistic applications $n = 4$; see Section 3.2 for the definition). In this way, points in the manifold can be assigned spatio-temporal coordinates via so-called charts. A BST_{92} structure, however, is not locally Euclidean with respect to its natural topology (barring trivial one-history cases). The reason is that a neighborhood of a maximal element in the intersection of two histories cannot be appropriately mapped onto $\mathbb{R}^n$. We thus face a problem when trying to assign spatio-temporal coordinates to the elements of a BST_{92} structure. To address this problem, we will develop a version of BST without maximal elements in the intersection of histories. The natural topology on such structures stands a chance of being locally Euclidean. We are also interested in finding an operation that would

---

[3]See, e.g., Earman [6]. Jeremy Butterfield asked about reasons to assume non-Hausdorff branching and its compatibility with space-time physics already in 2001 (Butterfield, personal communication).

transform BST_{92} structures into structures that are more friendly to local Euclidicity.

The second property that differential manifolds in space-time physics satisfy, but which is typically violated by BST_{92} structures, is a topological separation property known as the Hausdorff property (see again Section 3.2). We will not enforce this feature: the branching structures we develop still violate the Hausdorff property. We will argue, however, that this violation is innocuous, as each separate possible history is Hausdorff in its natural topology. In the same spirit, a manifold representing a single space time of general relativity is Hausdorff. Since we aim at enabling both the assignment of coordinates to events and the representation of alternative spatio-temporal histories, the combination of local Euclidicity and the violation of Hausdorffness on the whole structure seems to be the best result that one can aim to achieve.[4]

The paper is organized as follows. In Section 2, we describe the extant theory of BST_{92} in formal detail. We comment on topological issues in Section 3. In Section 4, we introduce the "new foundations" BST theory, BST_{NF}. In Section 5 we show how the two frameworks are linked, and we prove general translatability results both ways. We conclude in Section 6.

2.   What is Out There: The Formal Framework of BST_{92}

Branching space-times theories have been developed in a number of writings starting with Belnap [1]. The dominant theory that has emerged, BST_{92}, demands that histories branch at choice points, in the following way: any two histories overlap, and their overlap contains at least one maximal element. That decision has the problematic topological consequence mentioned above, which will be avoided by the novel BST theory to be developed in this paper. There is a substantial common core of the two theories, which we present in Section 2.1. In Section 2.2 we provide some general facts about common BST structures. In Section 2.3 we go on to describe the branching of histories in general terms, and in Section 2.4 we give a formal definition of BST_{92}, including its prior choice postulate that demands the existence of choice points.

---

[4]Technically, we are after branching structures that give rise to generalized differential manifolds, where maximal Hausdorff sub-manifolds can be identified with possible histories representing single space-times. In this paper, we are only concerned with the topological structure, deferring a discussion of the differential structure to a separate companion paper.

2.1. The Core Theory of Branching Space-Times

In this section we describe the formal core of *common BST structures* (Definition 2), which is shared by both the established theory of BST_{92} and by the "new foundations" theory, BST_{NF}, that we are motivating and discussing in this paper.[5] Both theories are spelled out in terms of partial orderings, so we provide some pertinent general notions first.

DEFINITION 1. *(Partial order, chain, directed set)* A pair $\langle W, < \rangle$ is a *strict partial ordering* iff $<$ is a relation on the set W that is antisymmetric ($\forall x, y \in W \, [x < y \rightarrow y \not< x]$) and transitive ($\forall x, y, z \in W \, [(x < y \wedge y < z) \rightarrow x < z]$). Note that $<$ is thereby irreflexive ($\forall x \in W \, [x \not< x]$). We use the companion notation $\leqslant$ in the standard way, i.e., $e \leqslant e'$ iff ($e < e' \vee e = e'$). We extend the ordering notation to sets, with the universal reading, that is: for $E, F \subseteq W$ and $e \in W$, we write $E \leqslant e$ iff $\forall x \in E \, [x \leqslant e]$, we write $e \leqslant F$ iff $\forall y \in F \, [e \leqslant y]$, and we write $E \leqslant F$ iff $\forall y \in F \, [E \leqslant y]$.

A set $l \subseteq W$ is a *chain* iff any two of its members are comparable, i.e., for any $x, y \in l$, either $x \leqslant y$ or $y < x$. A set $D \subseteq W$ is *(upward) directed* iff it contains a common upper bound for any two of its members, i.e., D is directed iff for any $x, y \in D$ there is $z \in D$ s.t. $x \leqslant z$ and $y \leqslant z$.

DEFINITION 2. *(Common BST structure)* A *common BST structure* is a pair $\langle W, < \rangle$ that fulfills the following conditions:

1. W is a non-empty set of possible point events.

2. $<$ is a strict partial ordering (Definition 1) denoting precedence on W. A set $h \subseteq W$ is a *history* iff h is maximal w.r.t. to the property of being directed. We write $\mathrm{Hist}(W)$ for the set of all histories in $\langle W, < \rangle$ (omitting "(W)" if the reference is clear), and for $E \subseteq W$, we write $H_{[E]}$ for the set of those histories containing the whole of E, i.e.: $H_{[E]} =_{\mathrm{df}} \{h \in \mathrm{Hist} \mid E \subseteq h\}$. We abbreviate $H_{[\{e\}]}$ as H_e, i.e., $H_e =_{\mathrm{df}} \{h \in \mathrm{Hist} \mid e \in h\}$.

3. W contains no maximal elements: $\forall x \in W \, \exists y \in W \, [x < y]$. Also, W contains no minima elements: $\forall x \in W \, \exists y \in W \, [y < x]$.

4. The ordering $<$ is dense: $\forall x, y \in W \, [x < y \rightarrow \exists z \in W \, [x < z < y]]$.

5. The ordering contains infima for all lower bounded chains: If $l \subseteq W$ is a chain that has a lower bound (for some $e \in W$, $e \leqslant l$), then l has a unique greatest lower bound $\inf l$, which satisfies $\forall x \, [x \leqslant l \rightarrow x \leqslant \inf l$.

---

[5]The common structures of Definition 2 contain one axiom that was not part of the original formal definitions; see note 9 below for details. Published discussions of BST structures also differ with respect to the permissibility of maximal and/or minimal elements in these structures. In this paper, for simplicity's sake, we exclude maxima and minima.

6. The ordering contains history-relative suprema for all upper bounded chains: If $l \subseteq W$ is a chain with an upper bound (for some $e \in W$, $l \leqslant e$), and $h \in \text{Hist}$ is a history for which $l \subseteq h$, then l has a unique smallest upper bound $\sup_h l$ in h:

$$\forall x \ [(x \in h \wedge l \leqslant x) \rightarrow \sup_h l \leqslant x].$$

7. Weiner's postulate: Let $l, l' \subseteq h_1 \cap h_2$ be upper bounded chains in histories h_1 and h_2. Then the order of the suprema in these histories is the same:

$$\sup_{h_1} l \leqslant \sup_{h_1} l' \quad \text{iff} \quad \sup_{h_2} l \leqslant \sup_{h_2} l'.$$

8. Historical connection: Any two histories intersect non-emptily, i.e., for h_1, $h_2 \in \text{Hist}$, we have $h_1 \cap h_2 \neq \emptyset$.

From this list, items 1–6 are motivated by the demand that BST structures be continuous orderings such as employed in space-time theories. Item 7 arises as a technical requirement ruling out unintended ordering structures. Item 8, on the other hand, is philosophically motivated by taking indeterminism to be a feature of Our World.

2.2.   Some Facts About Common BST Structures: Histories and Chains

If our world is indeterministic, then not everything that can happen at all, can happen together—some sets of events are compatible, but others are not. For example, it is possible that it rains in Pittsburgh next week, and it is possible that it rains in Kraków next week, and indeed it is possible that next week, it rains both in Pittsburgh and in Kraków. On the other hand, it is possible that the coin I am about to toss comes up heads, and it is possible that it comes up tails, but it is not possible that it comes up heads and comes up tails. These events are incompatible, or inconsistent. In BST theory, the notion of compatibility is expressed via the definition of a history: histories are maximal sets of compatible events.[6] Formally, as stated in Definition 2(2), a history is a maximal directed set, i.e., a set h maximal w.r.t. the property that for any two events $x, y \in h$, there is some event $z \in h$ such that x and y lie in the past of z. A useful motivation for

---

[6]In some related literature, histories are called "chronicles"; see, e.g., Jacobsen et al. [7]. We stick with the well-established terminology of "history". It is important not to confuse a history in the technical sense of BST theory (a maximal consistent set of events) with the everyday notion of the history of a given event. The latter use is always relational, whereas the BST use is non-relational.

this definition of a history is in terms of the consistency of the past: if there is some possible event z from the perspective of which both x and y have already happened, then x and y are consistent.

Here are some useful facts about histories in common BST structures:

FACT 1. *(Facts about histories)* Let $h, h_1, h_2 \in$ Hist and let $e, e' \in W$. (1) Histories are closed downward: If $e \in h$ and $e' < e$, then $e' \in h$. Thus, if $e' < e$, we have $H_e \subseteq H_{e'}$. (2) The complement of a history is closed upward: If $e \notin h$ and $e < e'$, then $e' \notin h$. (3) No history can be a proper subset of another history: If $h_1 \subseteq h_2$, then $h_1 = h_2$. (4) Any chain is part of some history. In particular, for any $e \in W$, $H_e \neq \emptyset$. (5) As W contains no maximal elements, no history h contains a maximal element either.

PROOF. (1, 2) By maximal directedness. (3) By maximality. (4) Chains are directed, and thus can be extended using the Zorn-Kuratowski lemma. Note that singletons are also chains. (5) For reductio, let e be a maximal element of history h. Then e is the unique maximal element of h. Otherwise there would be $e' \in h$ incomparable with e and an upper bound $e'' \in h$ for e and e', with $e < e''$, so e would not be maximal in h. But then as e is the unique maximal element of h and there is some $e' \in W$ for which $e < e'$ (as W contains no maximal elements), the set $h \cup \{e'\}$ is also directed (e', being the maximum, is a common upper bound for any two elements). Thus h is not maximal directed, i.e., not a history, contrary to our assumption. ∎

As history-relative suprema of chains will play a crucial role in this paper, we provide a number of pertinent definitions and facts.

DEFINITION 3. *(Chains and related sets)* We define the following classes of chains and related sets:

- $\mathscr{C}_e$: the set of chains ending in, but not containing, e. That is:
  $l \in \mathscr{C}_e$ iff l is an upper bounded chain and there is some $h \in$ Hist for which $l \subseteq h$ and $\sup_h l = e$, but $e \notin l$.

- $\mathscr{S}(l)$: the set of all history-relative suprema for an upper bounded chain l:
  $$\mathscr{S}(l) =_{\mathrm{df}} \{s \in W \mid \exists h \in \mathrm{Hist}\,[l \subseteq h \wedge s = \sup_h l]\}.$$

- $\mathscr{P}_e$: the proper past of e:
  $$\mathscr{P}_e =_{\mathrm{df}} \{e' \in W \mid e' < e\}.$$

We establish the following Facts, using the existence of history-relative suprema and the Weiner postulate:

FACT 2. Let $l \subseteq h$, and let $s = \sup_{h'} l$ for some $h' \in$ Hist with $l \subseteq h'$. If we have $s \in h$, then $\sup_h l = s$.

PROOF. Assume that $s \in h$. Observe that $s \in h'$, and that $\{s\}$ is a (trivial) chain with $\sup_{h^*}\{s\} = s$ for any $h^* \in H_s$. We can use the Weiner postulate on the chains l and $\{s\}$. As $\sup_{h'}\{s\} = s = \sup_{h'} l$, we also have to have $\sup_h l = \sup_h\{s\} = s$. ∎

FACT 3. Let l be an upper bounded chain and $h_1, h_2 \in H_{[l]}$, and let

$$\sup_{h_1} l = s_1 \neq s_2 = \sup_{h_2} l.$$

Then there is no history $h \supseteq \{s_1, s_2\}$.

PROOF. Assume otherwise, and let $h \supseteq \{s_1, s_2\}$ for some $h \in$ Hist. We have $l \subseteq h$, since $s_1 \in h$ and $l \leqslant s_1$. By Fact 2, we have both $\sup_h l = s_1$ (as $s_1 \in h$) and $\sup_h l = s_2$ (as $s_2 \in h$). So, contrary to our assumption, $s_1 = s_2$. ∎

Here is another useful fact about suprema of chains: If you remove the endpoint of a maximal upper bounded chain, the history-relative supremum does not change. The same holds for infima.

FACT 4. Suprema and infima of maximal chains are unaffected by removing the supremum or infimum: (1) Let l be a maximal upper bounded chain, and let $h \in$ Hist s.t. $l \subseteq h$. Let $s =_{\mathrm{df}} \sup_h l$. Then for $l' =_{\mathrm{df}} l \setminus \{s\}$, we also have $\sup_h l' = s$. (2) Let l be a maximal lower bounded chain, and let $e =_{\mathrm{df}} \inf l$. Then for $l' =_{\mathrm{df}} l \setminus \{e\}$, we also have $\inf l' = e$.

PROOF. (1) If $s \notin l$, we have $l' = l$, and there is nothing to prove. Otherwise, let $s' =_{\mathrm{df}} \sup_h l'$. Clearly, $l' \leqslant s$, so $s' \leqslant s$ (by the definition of suprema). Now assume for reductio that $s \neq s'$, i.e., $s' < s$. By the construction of l', we then have (*) $\forall x \in l\,[x \neq s \to x \leqslant s']$. By density, there is some $e \in W$ for which $s' < e < s$. By (*), we have $e \notin l$. But then, again by (*), we have that $l^* =_{\mathrm{df}} l \cup \{e\}$ is also a chain with $\sup_h l^* = s$, and $l^* \supsetneq l$. This contradicts the maximality of l. So, we have $s = s'$.

The proof for (2) is exactly parallel to that for (1). ∎

The next Fact shows that the proper past of an event e consists of all the chains ending in, but not containing, e.

FACT 5. For $e \in W$, we have $\mathscr{P}_e = \cup_{l \in \mathscr{C}_e} l$.

PROOF. "$\Leftarrow$" Let $x \in \cup_{l \in \mathscr{C}_e} l$, i.e., $x \in l$ for some $l \in \mathscr{C}_e$, and let $h \in H_e$. As $\sup_h l = e$ and $e \notin l$, we have $l < e$, and thus, $x < e$, i.e., $x \in \mathscr{P}_e$.

"$\Rightarrow$" Let $x \in \mathscr{P}_e$, i.e., $x < e$. Then $\{x, e\}$ is a chain, which by the Zorn-Kuratowski lemma can be extended to a maximal chain l ending in e. Let $h \in H_e$; we have $\sup_h l = e$. By Fact 4, for $l' =_{df} l \setminus \{e\}$ we also have $\sup_h l' = e$, whereby we have some $l' \in \mathscr{C}_e$ for which $x \in l'$. ∎

2.3. Indeterminism as the Branching of Histories

If a common BST structure contains just one history, then it is trivial from the point of view of indeterminism: all events are compatible, and the picture of a world with just one history is the picture of a deterministic world. Since there are multiple histories in any non-trivial BST structure, there are different ways in which these histories can interrelate. A strong intuitive principle is historical connection (Definition 2(8)): The idea is that any two histories should share some common past. In this way, any common BST structure fulfills Lewis's condition of qualifying as "a world", since it is connected by a "suitable external relation", namely, the relation of precedence, $<$ (see [8], 208).

We will later see that historical connection is implied by stronger principles about the interrelation of histories. These so-called prior choice principles (Definitions 6 and 18) make specific demands on the way in which histories branch off one from another. The key decision is what the branching of histories looks like locally: what are the objects at which histories branch? BST_{92} decides for points: histories branch, or remain undivided, at points. With a view to the formal definition of this type of branching in Section 2.4 below, we first provide some essential definitions. We start with the notion of *undividedness*. Let two histories h_1, h_2 share some event $e \in h_1 \cap h_2$. Then they also may or may not share a later event. In the former case, we call the histories *undivided at e*:

DEFINITION 4. (Undividedness) Let $h_1, h_2 \in \text{Hist}$, and let $e \in h_1 \cap h_2$. We say that *h_1 and h_2 are undivided at e ($h_1 \equiv_e h_2$)* iff there is some $e' \in h_1 \cap h_2$ for which $e < e'$.

For any event e, the relation $\equiv_e$ among the set H_e of histories containing e is obviously symmetrical and reflexive, by the form of the definition. We will discuss the issue of transitivity in more detail later on. Our way of enforcing historical connection via prior choice principles will ensure transitivity. Given transitivity, $\equiv_e$ is an equivalence relation on H_e. We use the notation Π_e to indicate the partition of histories from H_e into equivalence classes according to $\equiv_e$, i.e.,

for $H \subseteq H_e$ with $H \neq \emptyset$, we have $H \in \Pi_e \leftrightarrow \forall h_1, h_2 \in H [h_1 \equiv_e h_2]$.

It may be that in fact *all* histories from H_e are undivided at e, i.e., e is not maximal in the intersection of any two histories from H_e. In that case, we have $\Pi_e = \{H_e\}$.

In case two histories h_1, h_2 share an event e but no event later than e, that event e is a maximum in the intersection of the histories $h_1 \cap h_2$. In that case, we say that the histories *split at e*:

DEFINITION 5. *(Splitting at a point; choice point)* Let $h_1, h_2 \in \mathrm{Hist}$, and let $e \in h_1 \cap h_2$. We say that *h_1 and h_2 split at e*, and that *e is a choice point for histories h_1 and h_2* ($h_1 \perp_e h_2$), iff it is not the case that $h_1 \equiv_e h_2$, i.e., iff e is a maximal point in $h_1 \cap h_2$.

The existence of choice points has important implications for the topological properties of the resulting structures, as we noted above and as we will discuss further in Section 3. At this point, we thus reach an important question: do the postulates of a common BST structure decide whether there are choice points? It turns out that the answer is no: we can show that both the existence and the non-existence of choice points are live options for the branching of histories in common BST structures. Consider, thus, the two common BST structures depicted in Figure 1a, b.[7] These structures illustrate the two possibilities for histories to branch in common BST structures, thus picturing the nuclei of, on the one hand, the well-developed theory of BST_{92} (a), and the "new foundations" theory $\mathrm{BST}_{\mathrm{NF}}$ (b), which will be developed in formal detail in Section 4.[8]

We provide a formal definition of these structures, so as not rely solely on pictures. Both structures are defined as quotients of $L_2 =_{\mathrm{df}} \mathbb{R} \times \{1, 2\}$, the double real line, under the equivalence relations $\equiv_a$ and $\equiv_b$, which are defined, respectively, as

$$\langle x, i \rangle \equiv_a \langle x', i' \rangle \Leftrightarrow_{\mathrm{df}} (x = x' \wedge (i = i' \vee x \leqslant 0));$$
$$\langle x, i \rangle \equiv_b \langle x', i' \rangle \Leftrightarrow_{\mathrm{df}} (x = x' \wedge (i = i' \vee x < 0)).$$

These relations differ only in their handling of $x = 0$. The ordering on the quotient structures $M_a =_{\mathrm{df}} L_2 / \equiv_a$ and $M_b =_{\mathrm{df}} L_2 / \equiv_a$ is defined uniformly via

---

[7]These structures have just the bare minimum of complexity to fulfill the axioms of Definition 2 in a non-trivial way: they contain just two histories each. Furthermore, they do not include any spatial extension—so in fact they are so-called branching time structures as well.

[8]The two topological possibilities for branching have been discussed, e.g., by McCall [12], McCabe [11], and Strobach [22, 208].
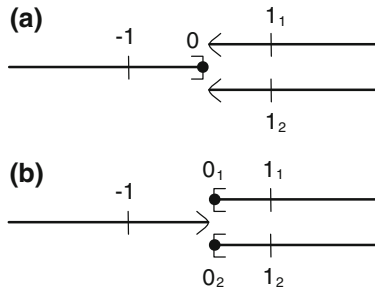
Figure 1. Two simple common BST structures **a** with a choice point and **b** without a choice point. Both (**a**) and (**b**) depict partial orderings in which there are two continuous histories branching at point 0. In (**a**), point 0 is the shared maximum in the intersection of the histories, i.e., a choice point. In (**b**), the intersection of the histories has no maximum, and points 0_1 and 0_2 are different history-relative suprema (minimal upper bounds) of the intersection

$$[\langle x, i\rangle] < [\langle x', i'\rangle] \Leftrightarrow_{\mathrm{df}} (x < x' \wedge [\langle x, i\rangle] = [\langle x, i'\rangle]).$$

It is easy to check that these structures are non-empty partial orderings without maxima or minima that satisfy the density and continuity (infima and suprema) conditions of a common BST structure. The two histories h_1^a, h_2^a in M_a and h_1^b, h_2^b in M_b are, respectively (for γ one of a or b),

$$h_1^\gamma = \{[\langle x, 1\rangle] \in M_\gamma \mid x \in \mathbb{R}\}; \quad h_2^\gamma = \{[\langle x, 2\rangle] \in M_\gamma \mid x \in \mathbb{R}\}.$$

The intersections of these two histories are, respectively, the upper bounded chains

$$l_a =_{\mathrm{df}} h_1^a \cap h_2^a = \{[\langle x, 1\rangle] \in M_a \mid x \leqslant 0\}; \quad l_b =_{\mathrm{df}} h_1^b \cap h_2^b = \{[\langle x, 1\rangle] \in M_b \mid x < 0\}.$$

The difference is this: while the chain l_a in M_a has a maximal element, $[\langle 0, 1\rangle]$, the chain l_b in M_b has no maximal element. That latter chain instead has two different history-relative suprema:

$$\sup_{h_i^b} l_b = [\langle 0, i\rangle], \quad i = 1, 2.$$

2.4. Branching via the Prior Choice Principle of BST₉₂

Having exhibited the two options for fulfilling the common BST axioms in a simple case, we could now enter into a philosophical discussion of what is the right way to go. We refrain from attempting any a priori arguments here. One can give good reasons for both options. Thus, in favor of the existence of choice points, one can argue that a causal account of indeterministic choice

requires a special last element of indecision, and thus, a maximal element of any two branching histories. In favor of the absence of choice points, one can cite issues of uniformity (it is possible to have branching without maxima in a uniform way), or topological aspects of the resulting structures. Both these issues will be discussed in Section 3. In our view, they provide a good motivation for investigating common BST structures without choice points, and this is what we will do in Section 4. As a matter of fact, however, BST theory was developed with the requirement of the existence of choice points, and the resulting axiomatic theory, BST_{92}, has proved to be fruitful for quite a number of applications, e.g., to causation [4], to probability theory [13,25], and to physics [16,17].

So in what follows, we first characterize BST_{92}, which is the BST theory with choice points, in full formal detail. This requires the addition of just a single extra axiom to the basis of common BST structures.[9] As originally described by Belnap [1], the theory posits the axioms of a common BST structure together with the so-called prior choice principle, which we will denote PCP_{92} to indicate its historical origin. Basically, PCP_{92} requires that whenever an event e belongs to one history h_1 but not to another history h_2, these two histories split at a choice point c in the past of e:

$$e \in (h_1 \setminus h_2) \rightarrow \exists c \, [c < e \wedge h_1 \perp_c h_2].$$

It turns out, however, that in order to enforce the transitivity of the relation of undividedness, PCP_{92} needs to be formulated not for points, but for lower bounded chains contained in the difference of two histories, as follows.[10]

DEFINITION 6. (BST_{92} *prior choice principle,* PCP_{92}) A common BST structure $\langle W, < \rangle$ *fulfills the* BST_{92} *prior choice principle* iff it fulfills the following condition: Let $h_1, h_2 \in$ Hist be two histories, and let $l \subseteq (h_1 \setminus h_2)$ be a lower-bounded chain that is contained fully in history h_1 but does not intersect history h_2. Then there is a choice point $c \in h_1 \cap h_2$ s.t. $c < l$ and $h_1 \perp_c h_2$, i.e., c lies properly below l and is a choice point for h_1 and h_2, which is maximal in the intersection of h_1 and h_2.

That definition obviously implies the point version described above, as any singleton $\{e\}$ is a lower-bounded chain. It also ensures historical connection independently of the explicit requirement of Definition 2(8): any two

---

different histories have a non-empty difference (see Fact 1(3)), so that they have to share a choice point.

We can now enter PCP$_{92}$ in its official form as an additional item to our list of axioms for BST$_{92}$:

DEFINITION 7. (BST$_{92}$ *structure*) A BST$_{92}$ *structure* is a common BST structure $\langle W, < \rangle$ (Definition 2) that also fulfills the BST$_{92}$ prior choice principle (Definition 6).

In the next section, we continue our description of BST$_{92}$ with a focus on its topological aspects. In particular, we provide some facts related to local Euclidicity and Hausdorffness. As indicated, these are the features that are essential for relating histories in BST to space-time structures studied in physics.

3. Topological Aspects of BST$_{92}$

In this section we describe the natural topology for common BST structures, the so-called diamond topology, in Section 3.1. We comment on some of the topological features of BST$_{92}$ in Section 3.2. We will return to topological issues for the case of BST$_{NF}$ further down, in Section 4.7.

3.1. General Idea of the Diamond Topology

BST admits a natural topology, introduced by Paul Bartha,[11] which we call the diamond topology. The topology is defined either for W, the base set of a BST structure, or for a given history h. In the definitions below, $MC(e)$ ($MC_h(e)$) stands for the set of maximal chains in W (in h) that contain e.

DEFINITION 8. *(Diamond topology $\mathscr{T}$ on W)* Z is an open subset of W, $Z \in \mathscr{T}$, iff $Z = W$ or for every $e \in Z$ and for every $t \in MC(e)$ there are $e_1, e_2 \in t$ such that $e_1 < e < e_2$ and the diamond $D_{e_1, e_2} \subseteq Z$, where

$$D_{e_1, e_2} =_{\mathrm{df}} \{e' \in W \mid e_1 \leqslant e' \leqslant e_2\}.$$

DEFINITION 9. *(History-relative diamond topologies $\mathscr{T}_h$ on W)* For $h \in \mathrm{Hist}$, Z is an open subset of h, $Z \in \mathscr{T}_h$, iff $Z = h$ or for every $e \in Z$ and for every $t \in MC_h(e)$ there are $e_1, e_2 \in t$ such that $e_1 < e < e_2$ and the diamond $D_{e_1, e_2} \subseteq Z$.

---

[11]Cf. note 26 of Belnap [3].

It is straightforward to check that indeed $\mathscr{T}$ and $\mathscr{T}_h$ are topologies, i.e., both the empty set and the base set (W or h, respectively) are open, the intersection of two open sets is open, and the union of countably many open sets is open. The claim of naturalness is based on the observation that these topologies, if appropriately restricted, coincide with the standard open-ball topology on $\mathbb{R}^n$, and that the notion of convergence they induce coincides with the order-theoretic notions of infima and suprema.[12] The history-relative topologies are the so-called subspace topologies induced by the diamond topology on W, by taking a history as a subspace of W. This means that $A \in \mathscr{T}_h$ iff there is $A' \in \mathscr{T}$ such that $A = A' \cap h$.

In BST_{92}, the global topology and the history-relative topologies have different features. This fact reflects a problem with local Euclidicity, which we discuss next.

3.2.   Properties of the Diamond Topology for BST_{92}

We review here some facts about diamond topologies in BST_{92}, which are proved in Placek et al. [18]. The first observation is that, unless $\langle W, < \rangle$ is a one-history structure, a history h is not open in the global topology $\mathscr{T}$, whereas it is open by definition in its own history-relative topology $\mathscr{T}_h$. Generally, if $A \in \mathscr{T}_h$ and A contains a choice point, then $A \notin \mathscr{T}$, so there is a systematic discrepancy between the global and the history-relative notions of openness. This discrepancy is reflected in a difference with respect to the Hausdorff property, which is defined as follows:

DEFINITION 10. *(Hausdorff property)* A topological space $\langle X, \mathscr{T}(X) \rangle$ is *Hausdorff* iff for any distinct $x, y \in X$ there are disjoint open environments of x and of y, i.e., there are $O_x, O_y \in \mathscr{T}(X)$ for which $O_x \cap O_y = \emptyset$.

Putting aside pathological structures that prohibit the construction of light-cones,[13] it can be proved that the history-relative topologies $\mathscr{T}_h$ on a BST_{92} structure have the Hausdorff property. This fact stands in sharp contrast with the properties of the global diamond topology $\mathscr{T}$: if a BST_{92} structure has more than one history, its global topology is non-Hausdorff (again barring pathological structures). Moreover, non-Hausdorffness is related to the existence of upper-bounded chains that have more than one history-relative supremum. As one might expect, a pair of distinct history-relative suprema of a chain provides a witness for non-Hausdorffness: if any

---

[12]For a discussion of the naturalness of the diamond topology, see Placek et al. [18, §6].

[13]Such pathological BST_{92} structures violate one of the conditions C1–C4 of Placek et al. [18].

two open sets in $\mathscr{T}$ each contain a distinct supremum, they must overlap because they share some final segment of the chain in question.

In physics it is standardly required that individual space-times be Hausdorff (see, e.g., [24], 12). As individual space-times are represented by single histories in a BST_{92} structure, we take the above result as showing that BST_{92} structures are not in tension with the Hausdorffness requirement of space-time physics. The non-Hausdorffness of the global topology of a BST_{92} structure simply reflects the fact that such a structure brings together more than one space-time, explicitly representing a number of alternative spatio-temporal developments.

There is, however, another topological feature of BST_{92} that is highly problematic, viz., an issue with local Euclidicity,[14] which is defined as follows:

DEFINITION 11. *(Local Euclidicity)* A topological space $\langle X, \mathscr{T}(X) \rangle$ is *locally Euclidean of dimension n* iff for every $x \in X$ there is an open neighborhood $O_x \in \mathscr{T}(X)$ and a homeomorphism φ_x that maps O_x onto an open set $R_x \in \mathscr{T}(\mathbb{R}^n)$.

Local Euclidicity is standardly presupposed (often without explicitly mentioning the condition by name) when the notion of a space-time manifold is introduced. On such a manifold, local coordinates are defined via so-called charts (see, e.g., [24], 12f.): at each point of the manifold, it is possible to find a neighborhood that is homeomorphic to some open set of $\mathbb{R}^n$, and the respective mapping induces the coordinates. If a topological space is not locally Euclidean, it is not possible to assign coordinates in this way.

Given the frugality of the axioms, BST_{92} structures come in many varieties. Hence it is not realistic to hope that their global topology will always be locally Euclidean. One can reasonably require, however, that local Euclidicity should transfer from individual histories to the global structure: if each history-relative topology $\mathscr{T}_h$ is locally Euclidean, then the global topology $\mathscr{T}$ should also be locally Euclidean. If we have some collection of physically reasonable space-times, each with an assignment of coordinates, then a BST analysis of indeterminism should not destroy the coordinate assignment. Unfortunately, local Euclidicity does not transfer from the history-relative topologies to the global topology of BST_{92}. A case in point is the simple two-history model of Figure 1a. The overlap of the histories h_1^a and h_2^a has a maximal element $[\langle 0, 1 \rangle] = [\langle 0, 2 \rangle]$. According to the definition of the history-relative diamond topology, the open sets of $\mathscr{T}_{h_i^a}$ $(i = 1, 2)$ are

---

[14]See Müller [15] for a BST-related discussion.

either of the form $\{[\langle x, i\rangle] \mid x \in (c, d)\}$, for some open interval $(c, d) \subseteq \mathbb{R}$, or they are unions of such sets. As every element of the history h_i^a belongs to a set $\{[\langle x, i\rangle] \mid x \in (c, d)\}$, and such a set is trivially homeomorphic to an open interval of $\mathbb{R}$, $\mathscr{T}_{h_i^a}$ is locally Euclidean of dimension 1. On the global topology $\mathscr{T}$ on M_a, however, any open neighborhood of the branching point $[\langle 0, 1\rangle]$ must extend somewhat to the trunk and to both the arms, i.e., it must contain subsets $\{[\langle x, 1\rangle] \mid x \in (c, d)\}$ and $\{[\langle x, 2\rangle] \mid x \in (c, d')\}$ with $c < 0$ and $d, d' > 0$. A fork of that sort, however, cannot be homeomorphically mapped onto an open interval of the real line. Thus, the global topology of M_a is not locally Euclidean, despite the fact that each history-relative topology is. Note that no such problem arises for the structure M_b of Figure 1b, in which the intersection of the two histories does not have a maximum.

4.   New Foundations for BST, Via Transition Structures in BST$_{92}$

Recall that the underlying goal of branching space-times theories is to provide a formal framework for analyzing local indeterminism. We have just seen that one way to achieve that goal, via BST$_{92}$, leads to the failure of local Euclidicity, meaning that there is no way to continuously assign spatio-temporal coordinates to the elements of non-trivial BST$_{92}$ structures. Our priority in constructing a "new foundation" theory is to secure local Euclidicity. On the other hand, a violation of the Hausdorff property as in BST$_{92}$, i.e., confined to the global level of indeterministic structures and not already arising at the level of single histories, seems unproblematic. The task we set ourselves is therefore to develop BST theory in such a way that there are no choice points. The resulting formal theory will then provide generalized manifolds, not required to be Hausdorff. On such manifolds one can do calculus and, more generally, develop some space-time physics.

The perhaps surprising fact is that the sought-for framework is readily available via the transition structure of a BST$_{92}$ structure. More precisely, starting with a BST$_{92}$ structure, we will define the set of its transitions and define an ordering relation on it. The resulting partial order turns out to satisfy all postulates of a common BST structure. However, instead of PCP$_{92}$, it satisfies a different prior choice principle, which, crucially, excludes the existence of maximal elements in the intersection of histories. Quite generally, histories do not split at points, but rather at more complex objects that we call *choice sets*. Given some mild assumptions, the resulting global structures are provably locally Euclidean.

In this section we introduce the BST_{92} notion of a transition (Section 4.1) and show that the full transition structure of a BST_{92} structure satisfies all the postulates of a common BST structure (Section 4.2). The resulting notion of a choice set and the emerging pattern of branching are discussed in Section 4.3. In Section 4.4 we formulate a new prior choice principle, PCP_{NF}, and axiomatize the "new foundations" theory BST_{NF}. In Section 4.5 we prove that the postulates of BST_{NF} are satisfied by the transition structure of a BST_{92} structure. Some further facts about choice sets are established in Section 4.6. The crucial topological results for BST_{NF} are announced and proved in Section 4.7.

4.1. Transitions

The notion of a transition is a powerful tool for discussing indeterminism. Belnap [2] picks up the notion from von Wright [23], adding formal rigor. Generally, a transition is a pair $\langle I, O \rangle$, written $I \rightarrowtail O$, in which I is appropriately prior to O, and O is, in some appropriate sense, an outcome of I. Various notions of transitions are discussed in Belnap [4]. For our purposes we focus on the simplest notion of a transition, a *basic transition*, which in BST_{92} is from a possible point event e to one of the immediate possibilities open at e, i.e., from e to a member of the partition Π_e of the set H_e of histories containing e,

$$\tau = e \rightarrowtail H, \quad e \in W, \ H \in \Pi_e.$$

Basic transitions are divided up into those that witness local indeterminism, and those at which, so to speak, nothing happens. The formal distinction reflects whether or not there are multiple immediate future possibilities open at e, or just one. Thus, at an indeterministic event e (at a choice point), the partition Π_e has more than one member (histories split at e; there are some $h_1, h_2 \in H_e$ for which $h_1 \perp_e h_2$), whereas at a deterministic event e, there is only one immediate possibility for the future, whence for all $h_1, h_2 \in H_e$, we have $h_1 \equiv_e h_2$, and $\Pi_e = \{H_e\}$.

DEFINITION 12. *(Deterministic and indeterministic basic transitions)* A *basic transition* is a pair $\langle e, H \rangle$, written $e \rightarrowtail H$, with $e \in W$ and $H \in \Pi_e$. For $h \in H_e$, we write $\Pi_e \langle h \rangle$ for the member of Π_e that contains h, so that the basic transition $e \rightarrowtail \Pi_e \langle h \rangle$ is from e to that (unique) basic outcome of e that contains h. A basic transition is *indeterministic* iff Π_e has more than one member. On the other hand, if $\Pi_e = \{H_e\}$, then the transition $e \rightarrowtail H_e$ is called *deterministic* or *trivial*.

We denote the set of basic indeterministic transitions of a BST_{92} structure $\langle W, < \rangle$ by $\mathrm{TR}(W)$, and the set of all basic transitions by $\mathrm{TR}_{\mathrm{full}}(W)$.

The set of basic transitions, whether deterministic or indeterministic, admits of a natural partial ordering.

DEFINITION 13. *(Transition ordering)* For $\tau_1 = e_1 \rightarrowtail H_1$, $\tau_2 = e_2 \rightarrowtail H_2$, we say that τ_1 *precedes* τ_2, written $\tau_1 \prec \tau_2$, iff $(e_1 < e_2$ and $H_2 \subseteq H_1)$. The companion non-strict partial ordering is defined via $\tau_1 \preccurlyeq \tau_2 \Leftrightarrow_{\mathrm{df}} (\tau_1 \prec \tau_2 \vee \tau_1 = \tau_2)$.

4.2. Characterizing the Transition Structure of a BST_{92} Structure

We are often interested only in indeterministic transitions, as deterministic transitions make no difference to the branching of histories.[15] In the present context, however, it is important to consider *all* transitions, including the ones that are trivial from the point of view of indeterminism. In a BST_{92} model, we therefore define the *full transition structure* as follows:

DEFINITION 14. *(The full transition structure of a* BST_{92} *structure.)* Let $\langle W, < \rangle$ be a BST_{92} structure. Then we define the full transition structure (including trivial transitions), $\Upsilon(\langle W, < \rangle)$, using the transition ordering $\prec$ from Definition 13, as follows:

$$\Upsilon(\langle W, < \rangle) =_{\mathrm{df}} \langle W', \prec \rangle, \text{ where } \quad W' =_{\mathrm{df}} \{e \rightarrowtail H \mid e \in W, H \in \Pi_e\}.$$

From here on we will denote elements resulting from a transformation with primes.

Having defined the transition structure, we now characterize its properties. It turns out that the full transition structure $\Upsilon(\langle W, < \rangle)$ looks very much like the original BST_{92} structure $\langle W, < \rangle$, except for what happens at the choice points. In fact, we will be able to show that apart from the prior choice postulate, all defining properties of BST_{92}, i.e., the whole list of properties of a common BST structure from Definition 2, continue to hold; see Theorem 1 below. But by failing the PCP_{92}, a transition structure is not a BST_{92} structure. With respect to the choice points, the difference is the following. In BST_{92}, the branching of histories is from a choice point, shared among the histories that branch, to the immediate possibilities for the future at that choice point. There are no first points in these different possible futures, and this fact leads to the failure of local Euclidicity in the global BST_{92} topology (see Section 3.2). In $\Upsilon(\langle W, < \rangle)$, on the other hand, each choice point is replaced by all the transitions that have that choice point as

---

[15]For a study along those lines, see Müller [14].

an initial. Therefore, where in BST_{92} there was a last point that was shared between the different possibilities, in the transition structure there are now multiple first points characterizing these different possibilities, and there is no last shared point any more.[16] In the structures of Figure 1, the move from (a) to (b) exactly corresponds to the move from the BST_{92} structure M_a to its transition structure M_b.[17] For the topological consequences, see Section 3.2 above and Section 4.7 below.

We now show that the common BST properties of Definition 2 still hold for the Υ transform of a BST_{92} structure. As a first easy fact, we note that $\Upsilon(\langle W, < \rangle)$ is a non-empty partial ordering:

FACT 6. Let $\langle W, < \rangle$ be a BST_{92} structure. Then $\langle W', \prec \rangle =_{df} \Upsilon(\langle W, < \rangle)$ is (1) non-empty and (2) a strict partial ordering (3) without maxima or minima.

PROOF. (1) Since W is non-empty, there is some $e \in W$, hence $H_e \neq \emptyset$ by Fact 1(4). So there is a non-empty $H \in \Pi_e$, and hence there exists a transition $e \rightarrowtail H \in W'$.

(2) Since $<$ is irreflexive and asymmetric, $\prec$ is irreflexive and asymmetric as well. For transitivity, let $(e_1 \rightarrowtail H_1) \prec (e_2 \rightarrowtail H_2)$ and $(e_2 \rightarrowtail H_2) \prec (e_3 \rightarrowtail H_3)$. By transitivity of $<$ we have $e_1 < e_3$. Also, from $H_2 \subseteq H_1$ and $H_3 \subseteq H_2$ we have $H_3 \subseteq H_1$ by transitivity of $\subseteq$. Together this establishes $(e_1 \rightarrowtail H_1) \prec (e_3 \rightarrowtail H_3)$.

(3) For no maxima, let $\tau = e \rightarrowtail H \in W'$, and let $h \in H \subseteq H_e$. As W contains no maxima (Definition 2(3)), h contains no maxima either (Fact 1(5)), so there is $e_1 \in h$ for which $e < e_1$. Accordingly we have $\tau' =_{df} e_1 \rightarrowtail \Pi_{e_1}\langle h \rangle \in W'$. It is easy to check that $\tau \prec \tau'$, which establishes that τ is not maximal in W'.

For no minima, similarly, let $\tau = e \rightarrowtail H \in W'$. As W contains no minima, there is $e_1 \in W$ for which $e_1 < e$. Let $h \in H_e$. By downward closure, $e_1 \in h$, i.e., $h \in H_{e_1}$. So there is $\tau' =_{df} e_1 \rightarrowtail \Pi_{e_1}\langle h \rangle \in W'$, and $\tau' \prec \tau$. Thus, τ is not minimal in W'. ∎

The following facts about alternatives to the definition of the transition ordering (Definition 13) will be helpful later on.

FACT 7. Let $\tau_1 = e_1 \rightarrowtail H_1$, $\tau_2 = e_2 \rightarrowtail H_2$ be transitions in a BST_{92} structure $\langle W, < \rangle$. (1) Generally, $\tau_1 \prec \tau_2$ iff ($e_1 < e_2$ and $H_{e_2} \subseteq H_1$) iff

---

[16]This image of fanning out the transitions from a choice point motivates out notation, Υ.

[17]To be precise, the transition structure of M_a is order-isomorphic to M_b. See Section 5 for a formal discussion.

($e_1 < e_2$ and there is some $h \in H_{e_2}$ for which $H_1 = \Pi_{e_1}\langle h \rangle$). (2) If τ_1 is deterministic, then $\tau_1 \prec \tau_2$ iff $e_1 < e_2$. (3) For the non-strict companion order, we have $\tau_1 \preccurlyeq \tau_2$ iff ($e_1 \leqslant e_2$ and $H_2 \subseteq H_1$).

PROOF. (1) We prove the first "iff", from which the second follows immediately. Let $e_1 < e_2$. We have to show that $H_2 \subseteq H_1$ iff $H_{e_2} \subseteq H_1$. As $H_2 \subseteq H_{e_2}$, the "$\Leftarrow$" direction is trivial. For "$\Rightarrow$", assume $H_2 \subseteq H_1$, and let $h \in H_2$ and $h' \in H_{e_2}$. We have $e_2 \in h \cap h'$, which establishes $h \equiv_{e_1} h'$. Since $h \in H_1 \in \Pi_{e_1}$, we have $H_1 = \Pi_{e_1}\langle h \rangle$, and by $h \equiv_{e_1} h'$, $h' \in H_1$ as well. So indeed, $H_{e_2} \subseteq H_1$.

(2) The "$\Rightarrow$" direction is trivial. For "$\Leftarrow$", let $e_1 < e_2$, and assume that τ_1 is deterministic, so that $H_1 = H_{e_1}$. We have to show that $H_2 \subseteq H_1$. By downward closure of histories (Fact 1(1)), we have $H_{e_2} \subseteq H_{e_1}$, and $H_2 \subseteq H_{e_2}$ by definition, so that $H_2 \subseteq H_{e_2} \subseteq H_{e_1} = H_1$. The claim follows by transitivity of $\subseteq$.

(3) "$\Rightarrow$": Assume $\tau_1 \preccurlyeq \tau_2$, i.e., either $\tau_1 \prec \tau_2$ or $\tau_1 = \tau_2$. In the first case, the claim follows from the definition of $\prec$, in the second case the claim is obvious as then, $e_1 = e_2$ and $H_1 = H_2$.

"$\Leftarrow$": Let $H_2 \subseteq H_1$, and let $e_1 \leqslant e_2$. Again there are two cases. If $e_1 = e_2$, then as $H_1, H_2 \in \Pi_{e_1}$ and Π_{e_1} is a partition, $H_2 \subseteq H_1$ implies $H_2 = H_1$, whence $\tau_1 = \tau_2$, establishing the claim. The remaining case, $e_1 < e_2$ and $H_2 \subseteq H_1$, satisfies the definition of $\prec$ exactly as in Definition 13. ∎

In order to prove that $\Upsilon(\langle W, < \rangle)$ is a common BST structure, we need to establish the form that histories, i.e., maximal directed sets, have in that ordering. Their form is quite intuitive, even though it turns out that the proof of that fact is somewhat lengthy. We first establish a useful general fact about directed sets of transitions:

FACT 8. Let $T \subseteq \Upsilon(\langle W, < \rangle)$ be a set of transitions, and let there be $e \in W$ and $H_1, H_2 \in \Pi_e$, $H_1 \neq H_2$, s.t. both $\tau_1 =_{df} e \rightarrowtail H_1$ and $\tau_2 =_{df} e \rightarrowtail H_2$ are members of T. Then T is not directed.

PROOF. Assume otherwise, i.e., assume that there is some $\tau^* = e^* \rightarrowtail H^* \in T$ for which $\tau_1 \preccurlyeq \tau^*$ and $\tau_2 \preccurlyeq \tau^*$. By Fact 7(1), this implies $H_{e^*} \subseteq H_1$ and $H_{e^*} \subseteq H_2$. But as H_1 and H_2 are different elements of the partition Π_e, we have $H_1 \cap H_2 = \emptyset$, contradicting $H_{e^*} \subseteq H_1 \cap H_2$. (Note that $H_{e^*} \neq \emptyset$ by Fact 1(4).) ∎

Now we can tackle the form of histories in $\Upsilon(\langle W, < \rangle)$.

LEMMA 1. *Let $\langle W, < \rangle$ be a BST_{92} structure, and let $\langle W', \prec \rangle =_{df} \Upsilon(\langle W, < \rangle)$. The histories (maximal directed sets) in $\langle W', \prec \rangle$ are exactly the sets*

$$T_h =_{df} \{ e \rightarrowtail \Pi_e\langle h \rangle \mid e \in h \}$$

for h in $\mathrm{Hist}(W)$.

PROOF. First we establish that such sets are indeed histories in $\langle W', \prec \rangle$. Thus, take some $h \in \mathrm{Hist}(W)$, and let $T_h =_{\mathrm{df}} \{e \rightarrowtail \Pi_e \langle h \rangle \mid e \in h\} \subseteq W'$. The set T_h is directed: take $e_1 \rightarrowtail \Pi_{e_1} \langle h \rangle$ and $e_2 \rightarrowtail \Pi_{e_2} \langle h \rangle$ from T_h, whence $e_1, e_2 \in h$. As h is directed, there is $e_3 \in h$ such that $e_1, e_2 \leqslant e_3$. By construction, $e_3 \rightarrowtail \Pi_{e_3} \langle h \rangle \in T_h$. And as to the ordering, $H_{e_3} \subseteq \Pi_{e_1} \langle h \rangle$ because $e_3 \in h$ and $e_1 \leqslant e_3$. Analogously, $H_{e_3} \subseteq \Pi_{e_2} \langle h \rangle$. So indeed (noting Fact 7(1)), $e_i \rightarrowtail \Pi_{e_i} \langle h \rangle \preccurlyeq e_3 \rightarrowtail \Pi_{e_3} \langle h \rangle$ $(i = 1, 2)$, establishing the common upper bound. Moreover, T_h is maximal directed. To prove this, take some $\tau^* \in (W' \setminus T_h)$; this transition has the form $\tau^* = e^* \rightarrowtail H^*$ for some $e^* \in W$, $H^* \in \Pi_{e^*}$. There are two cases.

Case 1: There is some $\tau = e \rightarrowtail \Pi_e \langle h \rangle \in T_h$ for which $e = e^*$, i.e., $e^* \in h$. Then, as $\tau \neq \tau^*$, by Fact 8, $T_h \cup \{\tau^*\}$ cannot be directed.

Case 2: There is no $\tau = e \rightarrowtail \Pi_e \langle h \rangle \in T_h$ for which $e = e^*$, i.e., $e^* \notin h$. Then we have $e^* \in h'$ for a different $h' \in \mathrm{Hist}(W)$, and by the BST_{92} prior choice principle, there is some $c \in h \cap h'$ such that $c < e^*$ and $h \perp_c h'$. As $c \in h$, we have $\tau_c =_{\mathrm{df}} c \rightarrowtail \Pi_c \langle h \rangle \in T_h$. We can now show that $T_h \cup \{\tau^*\}$ is not directed: there can be no common upper bound for τ_c and τ^* in W'. Assume for reductio that there is some $\tau' = e' \rightarrowtail H' \in T_h \cup \{\tau^*\}$ for which $\tau_c \preccurlyeq \tau'$ and $\tau^* \preccurlyeq \tau'$. We can rule out $\tau' = \tau^*$: we have $\tau_c \neq \tau^*$ by $c < e^*$, and $\tau_c \not\prec \tau^*$ as $H_{e^*} \not\subseteq \Pi_c \langle h \rangle$ (as $H_{e^*} \subseteq \Pi_c \langle h' \rangle$). So we must have $\tau' \in T_h$. By the definition of $\prec$, the assumed ordering relations imply $H_{e'} \subseteq \Pi_c \langle h \rangle$ and $H_{e'} \subseteq H^* \subseteq \Pi_c \langle h' \rangle$. But we have $\Pi_c \langle h \rangle \cap \Pi_c \langle h' \rangle = \emptyset$, contradicting Fact 1(4).

So, having shown that the sets T_h are indeed histories in $\langle W', \prec \rangle$, we need to show that *all* histories in $\langle W', \prec \rangle$ are of that form. Thus, let $g \subseteq W'$ be a history in $\langle W', \prec \rangle$, maximal directed w.r.t. $\prec$. By Fact 8, there is no $e \in W$ for which g contains two transitions $e \rightarrowtail H_1$ and $e \rightarrowtail H_2$, $H_1 \neq H_2$, so that we can write

$$g = \{e \rightarrowtail H(e) \mid e \in E\}$$

for some set $E \subseteq W$, where $H(e) \in \Pi_e$. We first show that E is directed: Take $e_1, e_2 \in E$, so that $\tau_i =_{\mathrm{df}} e_i \rightarrowtail H(e_i) \in g$. By directedness of g, there is some $\tau_3 = e_3 \rightarrowtail H(e_3) \in g$ for which $\tau_i \preccurlyeq \tau_3$ $(i = 1, 2)$, which implies $e_3 \in E$, $e_1 \leqslant e_3$, and $e_2 \leqslant e_3$. This proves that E is directed, and therefore there is some history $h \in \mathrm{Hist}(W)$ for which $E \subseteq h$. We now show that for all $e \in E$, we have $h \in H(e)$. Thus, take some $e \in E$, which is the initial of some $\tau = e \rightarrowtail H(e) \in g$. By Fact 6(3) and Fact 1(5), τ cannot be maximal in g, i.e., there is some $\tau' = e' \rightarrowtail H(e') \in g$ for which $\tau \prec \tau'$. This implies

that $H_{e'} \subseteq H(e)$, and as $e' \in E \subseteq h$, we have $h \in H_{e'}$ and therefore also $h \in H(e)$.

As $h \in H(e)$ for all $e \in E$, we have $H(e) = \Pi_e \langle h \rangle$ for all $e \in E$. This implies $g \subseteq T_h$, and by Fact 1(3), we have established $g = T_h$.   ■

Given these facts, we can now prove that switching from a BST_{92} structure to its full transition structure preserves the common BST structure axioms.

THEOREM 1. *Let $\langle W, < \rangle$ be a BST_{92} structure. Its full transition structure $\Upsilon(\langle W, < \rangle)$ is still a common BST structure according to Definition 2.*

PROOF. We need to check that $\langle W', \prec \rangle =_{\mathrm{df}} \Upsilon(\langle W, < \rangle)$ satisfies all the properties (1)–(8) listed in Definition 2.

1. W' is non-empty. See Fact 6(1).

2. $\langle W' \prec \rangle$ is a strict partial ordering. See Fact 6(2).

3. W' contains neither maximal nor minimal elements. See Fact 6(3).

4. $\prec$ is dense.
   Let $(e_1 \rightarrowtail H_1) \prec (e_3 \rightarrowtail H_3)$, which means that $e_1 < e_3$ and $H_3 \subseteq H_1$. By density of $<$, there is $e_2 \in W$ for which $e_1 < e_2 < e_3$. Take some $h \in H_3$, so that $\{e_1, e_2, e_3\} \subseteq h$. Let $H_2 =_{\mathrm{df}} \Pi_{e_2} \langle h \rangle$. We claim that the transition $e_2 \rightarrowtail H_2$ is $\prec$-sliced between the two transitions above. We have to show that $H_2 \subseteq H_1$ and $H_3 \subseteq H_2$. For the former, take some $h_2 \in H_2$; we have $h_2 \equiv_{e_1} h$ as witnessed by e_2. As $H_1 \in \Pi_{e_1}$, therefore, $h_2 \in H_1$ iff $h \in H_1$. Now as $h \in H_3$ and $H_3 \subseteq H_1$, we have $h \in H_1$, so that indeed, $H_2 \subseteq H_1$. The latter claim is established analogously.

5. Any lower bounded chain in $\langle W', \prec \rangle$ has an infimum in $\prec$.
   Let $l' = \{e_i \rightarrowtail H_i \mid i \in K\}$ (K some index set) be a chain that is lower bounded by $e^* \rightarrowtail H^*$. Then the set $l =_{\mathrm{df}} \{e_i \mid i \in K\}$ of initials of l' is a chain lower bounded by e^*, and there is a history $h \subseteq W$ for which $l \subseteq h$. By the BST_{92} postulate of infima, l has an infimum v in $<$. The infimum v gives rise to the transition $v' =_{\mathrm{df}} v \rightarrowtail \Pi_v \langle h \rangle \in W'$. Let $e_i \rightarrowtail H_i \in l'$. We have $v \leqslant e_i$ (as $e_i \in l$), and $H_i \subseteq \Pi_v \langle h \rangle$ because $e_i \in h$ and $v \leqslant e_i$. Thus, $v' \prec (e_i \rightarrowtail H_i)$, so v' is a lower bound of l'. Let now $e \rightarrowtail H$ be any lower bound of l', whence e is a lower bound of l. As v is the infimum of l, we have $e \leqslant v$, and as $l \subseteq h$, we have $H = \Pi_e \langle h \rangle$, which implies $H_v \subseteq H$. Thus $e \rightarrowtail H \preccurlyeq v'$, i.e., v' is indeed the greatest lower bound of l'.

6. Any upper-bounded chain in $\langle W', \prec \rangle$ has a history-relative supremum in each history to which it belongs.

Let the chain l' be upper bounded by u' in $\langle W', \prec \rangle$ and $l' \cup \{u'\} \subseteq h'$ for $h' \in \text{Hist}(W', \prec)$. Given the form of histories in $\text{Hist}(W', \prec)$ (see Lemma 1), $h' = \{e \rightarrowtail \Pi_e\langle h \rangle \mid e \in h\}$ for some $h \in \text{Hist}(W)$. It follows that for the set l of initials of l' and for u the initial of u', $l \cup \{u\} \subseteq h$; additionally, $l \leqslant u$. By the BST_{92} axiom of history-relative suprema, there is a history-relative supremum $s = \sup_h l$ of l in h. Consider now the transition $s' = s \rightarrowtail \Pi_s\langle h \rangle \in h'$. That transition is an upper bound of l': for any $e \rightarrowtail H \in l'$, we have $e \leqslant s$ and $H_s \subseteq H = \Pi_e\langle h \rangle$. Furthermore, s' is the least upper bound (i.e., the supremum) of l' in h': let $s^{*\prime} \in h'$ be an upper bound of l' in h'; by the form of histories, $s^{*\prime} = s^* \rightarrowtail \Pi_{s^*}\langle h \rangle$ with $s^* \in h$. Thus, $s \leqslant s^*$ (as s is the h-relative supremum of l), and as $s', s^{*\prime} \in h'$, by Fact 7(1), we have $s' \preccurlyeq s^{*\prime}$.

Summing up, we have established that for $l' \subseteq h'$ an upper-bounded chain in a W'-history $h' = \{e \rightarrowtail \Pi_e\langle h \rangle \mid e \in h\}$ (where h is the corresponding W-history), the initials satisfy $l \subseteq h$, and there exists the h'-relative supremum

$$\sup_{h'} l' = s \rightarrowtail \Pi_s\langle h \rangle,$$

where $s = \sup_h l$.

7. $\langle W', \prec \rangle$ satisfies the Weiner postulate.

We will employ the claim established at the end of the previous item (6), and the fact that $\langle W, < \rangle$ satisfies the Weiner postulate.

Let $h'_1, h'_2 \in \text{Hist}(W')$, and let $h'_i = \{e \rightarrowtail \Pi_e\langle h_i \rangle \mid e \in h_i\}$ for $h_i \in \text{Hist}(W)$ $(i = 1, 2)$. We consider two chains $l', k' \subseteq h'_1 \cap h'_2$, their respective chains of initials $l, k \subseteq h_1 \cap h_2$, and their history-relative suprema $s'_i = s_i \rightarrowtail \Pi_{s_i}\langle h_i \rangle = \sup_{h'_i}(l')$ and $c'_i = c_i \rightarrowtail \Pi_{c_i}\langle h_i \rangle = \sup_{h'_i}(k')$, where $s_i = \sup_{h_i} l$ and $c_i = \sup_{h_i} k$, for $i = 1, 2$. Suppose that $s'_1 \preccurlyeq c'_1$, i.e., $s_1 \leqslant c_1$ and $\Pi_{c_1}\langle h_1 \rangle \subseteq \Pi_{s_1}\langle h_1 \rangle$ (see Fact 7(3)). By the Weiner Postulate of BST_{92} applied to the chains l and k, from $s_1 \leqslant c_1$ we may infer $s_2 \leqslant c_2$. Note that $s_2 \in h_2$. Hence $\Pi_{c_2}\langle h_2 \rangle \subseteq \Pi_{s_2}\langle h_2 \rangle$. In terms of the transition ordering, this means that $s'_2 \preccurlyeq c'_2$.

8. Historical connection.

Let $h'_1, h'_2 \in \text{Hist}(W')$ be histories, which by Lemma 1 correspond to $h_1, h_2 \in \text{Hist}(W)$. By historical connection for W, there is some $e \in h_1 \cap h_2$, and by no minima, there is some $e^* \in W$ for which $e^* < e$. It follows that $e^* \in h_1 \cap h_2$. Let $\tau =_{\text{df}} e^* \rightarrowtail \Pi_{e^*}\langle h_1 \rangle$. By Lemma 1, we have $\tau \in h'_1$. Now $e > e^*$ is a witness for $h_1 \equiv_{e^*} h_2$, so that $\Pi_{e^*}\langle h_1 \rangle = \Pi_{e^*}\langle h_2 \rangle$, i.e., $\tau \in h'_2$ as well. ∎

4.3. Characterizing the Branching of Histories in Transition Structures

We can now discuss formally what has become of the BST_{92} choice points, and in which sense a prior choice principle still holds in a full transition structure.

It turns out that when characterizing transition structures, the role of a choice point e in BST_{92} is played here by the notion of a choice set $\ddot{e}$, which describes the local point-wise alternatives for e. In introducing choice sets, we build on some notions introduced above in Section 2.2. We work with a common BST structure $\langle W', \prec \rangle$ derived from a BST_{92} structure $\langle W, < \rangle$, i.e., $\langle W', \prec \rangle =_{\mathrm{df}} \Upsilon(\langle W, < \rangle)$. Generic elements of W' are written e, etc., and histories h, etc. The notions of a chain and of a directed set in this section are relative to the transition ordering $\prec$.

DEFINITION 15. *(Choice set)* For $e \in W'$, we define the *choice set based on* e, written $\ddot{e}$, to be the intersection of the sets of suprema of all chains ending in e.[18]

$$\ddot{e} =_{\mathrm{df}} \bigcap_{l \in \mathscr{C}_e} \mathscr{S}(l).$$

We also call the choice set $\ddot{e}$ the set of *local point-wise alternatives* for e. Note that e then counts as an alternative to itself. The related notions of alternative histories and history-wise alternatives are defined via the point-wise alternatives:

DEFINITION 16. *(Alternative histories and local history-wise alternatives)* We define the set of *alternative histories at* $\ddot{e}$, $H_{\ddot{e}}$, and the set of *local history-wise alternatives for* e, $\Pi_{\ddot{e}}$, to be

$$H_{\ddot{e}} = \{h \in \mathrm{Hist} \mid h \cap \ddot{e} \neq \emptyset\}; \qquad \Pi_{\ddot{e}} =_{\mathrm{df}} \{H_s \mid s \in \ddot{e}\}.$$

In order to spell out the variant of the prior choice principle that is appropriate for transition structures, we define two new relations between histories, splitting at a choice set and being undivided at a choice set, written $h_1 \perp_{\ddot{e}} h_2$ and $h_1 \equiv_{\ddot{e}} h_2$, resp., in analogy to the respective BST_{92} notions. In what follows, we will often use two consequences of Fact 3:

---

[18]Note that $\mathscr{C}_e \neq \emptyset$, as W' contains no minima.—Our notation with the double dot over e is meant to be suggestive of a number of different history-relative suprema on top of a chain. Think of Figure 1b rotated counterclockwise by $90°$. Graphically, the analogy with choice partitions for agents, as defined in Belnap et al. [5, Chapter 7C.2], is suggestive. Note, however, that these partitions pertain to branching time, not BST, and that they allow for a coarse-graining of the natural partition of undividedness.

FACT 9. (1) For any $h \in \mathrm{Hist}(W')$ and for any $e \in W'$, we either have $h \cap \ddot{e} = \emptyset$, or $h \cap \ddot{e} = \{e'\}$ for some $e' \in \ddot{e}$, i.e., a choice set has at most one element in common with any history. (2) The set of sets of histories $\Pi_{\ddot{e}}$ partitions $H_{\ddot{e}}$.

PROOF. (1) By Fact 3, no history contains more than one history-relative supremum of any upper bounded chain.

(2) Exhaustiveness is immediate: by definition, $\cup \Pi_{\ddot{e}} = H_{\ddot{e}}$. To prove disjointness of the elements of $\Pi_{\ddot{e}}$, let $H_1, H_2 \in \Pi_{\ddot{e}}$ such that $H_1 \neq H_2$. Then $H_1 = H_{s_1}$ and $H_2 = H_{s_2}$ for two distinct members $s_1, s_2 \in \ddot{e}$. Let $h \in H_1$; by (1), we then have $h \notin H_2$. ∎

DEFINITION 17. Let h_1, h_2 be histories in $\mathrm{Hist}(W')$, and let $e \in W'$. We require as a presupposition for $h_1 \equiv_{\ddot{e}} h_2$ and $h_1 \perp_{\ddot{e}} h_2$ that $h_1, h_2 \in H_{\ddot{e}}$, i.e., $h_1 \cap \ddot{e} \neq \emptyset$ and $h_2 \cap \ddot{e} \neq \emptyset$. Then the relations are defined as follows:

$$h_1 \equiv_{\ddot{e}} h_2 \quad \Leftrightarrow_{\mathrm{df}} \quad h_1 \cap \ddot{e} = h_2 \cap \ddot{e};$$
$$h_1 \perp_{\ddot{e}} h_2 \quad \Leftrightarrow_{\mathrm{df}} \quad h_1 \cap \ddot{e} \neq h_2 \cap \ddot{e}.$$

If $h_1 \perp_{\ddot{e}} h_2$, we say that the choice set $\ddot{e}$ is *a choice set for histories h_1 and h_2*. We can establish the following fact about the interrelation of the relations $\perp_{\ddot{e}}$ and $\equiv_{\ddot{e}}$:

FACT 10. Let $e \in W$ and let $h_1, h_2 \in H_{\ddot{e}}$. Then $\equiv_{\ddot{e}}$ and $\perp_{\ddot{e}}$ are mutually exclusive and jointly exhaustive: we have $h_1 \equiv_{\ddot{e}} h_2$ iff not $h_1 \perp_{\ddot{e}} h_2$.

PROOF. Given the assumptions, we have $h_1 \cap \ddot{e} = \{s_1\}$ and $h_2 \cap \ddot{e} = \{s_2\}$ for some $s_1, s_2 \in \ddot{e}$. Now by our definitions, $h_1 \equiv_{\ddot{e}} h_2$ iff $s_1 = s_2$, and $h_1 \perp_{\ddot{e}} h_2$ iff $s_1 \neq s_2$. These are mutually exclusive and jointly exhaustive alternatives. ∎

FACT 11. (1) The relation $\equiv_{\ddot{e}}$ is an equivalence relation on the set of alternative histories at $\ddot{e}$, $H_{\ddot{e}}$. (2) We have $\Pi_{\ddot{e}} = H_{\ddot{e}}/\equiv_{\ddot{e}}$.

PROOF. (1) Let $h_1, h_2, h_3 \in H_{\ddot{e}}$. By Fact 9(1), we have $h_i \cap \ddot{e} = \{s_i\}$ ($i = 1, 2, 3$) for some $s_1, s_2, s_3 \in \ddot{e}$. We have to establish reflexivity, symmetry and transitivity. Reflexivity and symmetry are trivial. For transitivity, assume $h_1 \equiv_{\ddot{e}} h_2$ and $h_2 \equiv_{\ddot{e}} h_3$, which holds iff $s_1 = s_2$ and $s_2 = s_3$. By transitivity of identity, $s_1 = s_3$, which implies $h_1 \equiv_{\ddot{e}} h_3$. So $H_{\ddot{e}}/\equiv_{\ddot{e}}$ is a partition of $H_{\ddot{e}}$. To see that this is the same partition as $\Pi_{\ddot{e}}$ characterized in Fact 9(2), note that $H_{s_1} = H_{s_2}$ iff $s_1 = s_2$. ∎

4.4. Introducing new Foundations: BST_{NF}

With the required notions at hand, we can now propose a new prior choice principle, PCP_{NF}. That principle is crucial for our new foundations: will later show that PCP_{NF} holds in BST_{92} transition structures, which themselves are not BST_{92} structures (see Lemma 2 below):

DEFINITION 18. (PCP_{NF}) Let $h_1, h_2 \in \mathrm{Hist}(W)$, and let l be a lower bounded chain for which $l \subseteq h_1$ but $l \cap h_2 = \emptyset$. Then there is some $e \in W$ for which $e \leqslant l$ and for which the set $\ddot{e}$ of local alternatives to e satisfies $h_1 \perp_{\ddot{e}} h_2$.

Note the weak relation $e \leqslant l$ in the formulation of PCP_{NF}, in contradistinction to the strict relation in the formulation of the BST_{92} PCP in Definition 6. For example, if l has just one element c such that $c \in h_1 \setminus h_2$ and c is an element of a non-trivial choice set $\ddot{c} \neq \{c\}$, then the choice set for c is just $\ddot{c}$ itself, and $h_1 \perp_{\ddot{c}} h_2$. In such a case we only have the weak ordering relation, $c \leqslant l$.

Having proposed a new prior choice principle, we can now give a full definition of "new foundations" BST, BST_{NF}:

DEFINITION 19. (BST_{NF} *structure*) A partial ordering $\langle W, \leqslant \rangle$ is *a structure of* BST_{NF} iff it is a common BST structure (Definition 2) for which the PCP_{NF} (Definition 18) holds.

BST_{NF} structures, being common BST structures, satisfy historical connection just as BST_{92} structures do. The new PCP_{NF}, however, implies that the branching of histories in BST_{NF} is different from the branching in terms of choice points in BST_{92}: there can be no maximal elements in the intersection of histories in a BST_{NF} structure.

FACT 12. Let h_1, h_2 be two histories in a BST_{NF} structure $\langle W, < \rangle$, $h_1 \neq h_2$. Then $h_1 \cap h_2$, which is non-empty, contains no maximal elements. Accordingly, for any $e \in W$, if $e \in h_1 \cap h_2$, then $h_1 \equiv_e h_2$, where $\equiv_e$ is the BST_{92} notion of undividedness.

PROOF. By historical connection, $h_1 \cap h_2 \neq \emptyset$. Assume for reductio that there is an $e \in h_1 \cap h_2$ that is maximal in $h_1 \cap h_2$, i.e., for all $e' \in W$, if $e < e'$, then $e' \notin h_1 \cap h_2$. By the Zorn-Kuratowski lemma, there is a maximal lower bounded chain $J \subseteq \{x \in h_1 \mid e \leqslant x\}$ in the h_1-future of e. As $e \in J$, we have $\inf J = e$, and by no maxima, $J' =_{df} J \setminus \{e\} \neq \emptyset$. By Fact 4, e is also the infimum of J'. As we have $J' \subseteq h_1 \setminus h_2$, by PCP_{NF} there is $\ddot{c}$ with some $c_1, c_2 \in \ddot{c}$ such that $c_1 \in h_1$, $c_2 \in h_2$, $h_1 \cap \ddot{c} = \{c_1\} \neq \{c_2\} = h_2 \cap \ddot{c}$,

and $c_1 \leqslant J'$. Then c_1 and c_2 cannot share a history (by Fact 9(1)). However, since $e = \inf J'$, by the definition of inifima, $c_1 \leqslant e$, and as histories are downward closed, $c_1 \in h_2$. But we have $c_2 \in h_2$ as well, which contradicts Fact 9(1). ∎

4.5. BST$_{92}$ Transition Structures are BST$_{\mathrm{NF}}$ Structures

We can now show that the full transition structure of a BST$_{92}$ structure is indeed a BST$_{\mathrm{NF}}$ structure. The only thing that is still missing is to show that the new prior choice principle is satisfied. To this end we need an auxiliary fact that shows how BST$_{92}$ choice points give rise to BST$_{\mathrm{NF}}$ choice sets: in the transition structure, a BST$_{92}$ choice point c is replaced by all the basic transitions with initial c, such that these transitions together form a choice set in the resulting structure.

FACT 13. Let $\langle W, < \rangle$ be a BST$_{92}$ structure, and let $h_1 \perp_c h_2$ for $h_1, h_2 \in \mathrm{Hist}(W)$. Then $c'_1 =_{\mathrm{df}} c \rightarrowtail \Pi_c \langle h_1 \rangle$ and $c'_2 =_{\mathrm{df}} c \rightarrowtail \Pi_c \langle h_2 \rangle$ belong to $\Upsilon(\langle W, < \rangle)$ and are elements of a choice set $\ddot{c}$ for which $h'_1 \perp_{\ddot{c}} h'_2$, where $h'_i = \{ e \rightarrowtail \Pi_e \langle h_i \rangle \mid e \in h_i \}$ $(i = 1, 2)$ are the histories in $\Upsilon(\langle W, < \rangle)$ corresponding to h_1 and h_2.

PROOF. Let $\langle W', <' \rangle =_{\mathrm{df}} \Upsilon(\langle W, < \rangle)$, let $h_1, h_2 \in \mathrm{Hist}(W)$, and let $c \in h_1 \cap h_2$ be s.t. $h_1 \perp_c h_2$. By Lemma 1, $h'_i = \{ e \rightarrowtail \Pi_e \langle h_i \rangle \mid e \in h_i \} \in \mathrm{Hist}(W')$ $(i = 1, 2)$. Let $c'_i =_{\mathrm{df}} c \rightarrowtail \Pi_c \langle h_i \rangle$, so that $c'_i \in h'_i$ $(i = 1, 2)$. In order to show that c'_1, c'_2 are elements of a choice set $\ddot{c}$ that fulfills $h'_1 \perp_{\ddot{c}} h'_2$, we need to show that every chain $l' \in \mathscr{C}_{c'_1}$, for which $\sup_{h'_1} l' = c'_1$, has c'_2 as another history-relative supremum, and vice versa. Since W (and thereby W') has no minimal elements, $\mathscr{C}_{c'_1} \neq \emptyset$. Pick an arbitrary chain $l' \in \mathscr{C}_{c'_1}$, and note that it has the form $l' = \{ e \rightarrowtail \Pi_e \langle h_1 \rangle \mid e \in l \}$ for some chain $l \subseteq h_1$, with $c = \sup_{h_1} l$. Since $h_1 \perp_c h_2$, $l \subseteq h_2$ as well, and as $l < c$, we have that for every $e \in l$, $\Pi_e \langle h_1 \rangle = \Pi_e \langle h_2 \rangle$. Hence $l' \subseteq h'_1 \cap h'_2$. It follows that $\sup_{h'_1} l' = c'_1$ and $\sup_{h'_2} l' = c'_2$ (note that the $c'_i \in h'_i$ are upper bounds of l' and that their initial, c, is the h_1- as well as the h_2-relative supremum of l). Since l' is an arbitrary chain in $\mathscr{C}_{c'_1}$, we showed that every chain in $\mathscr{C}_{c'_1}$ has at least two history-relative suprema, c'_1 and c'_2, i.e., there is a choice set $\ddot{c}'$ s.t $\{c'_1, c'_2\} \subseteq \ddot{c}'$. Since $h'_1 \cap \ddot{c}' = c'_1 \neq c'_2 = h'_2 \cap \ddot{c}'$, we have $h'_1 \perp_{\ddot{c}'} h'_2$. ∎

Given this auxiliary fact, we can establish our lemma:

LEMMA 2. *Let $\langle W, < \rangle$ be a BST$_{92}$ structure. Then that structure's full transition structure $\langle W', <' \rangle =_{\mathrm{df}} \Upsilon(\langle W, < \rangle)$ fulfills the new prior choice principle according to Definition 18.*

PROOF. Let l' be a chain in $\langle W', <' \rangle$ that is lower bounded by u', and let $h_1', h_2' \in \text{Hist}(W')$ be such that (†) $l' \subseteq h_1'$ but (‡) $l' \cap h_2' = \emptyset$. By Lemma 1, $h_i' = \{e \rightarrowtail \Pi_e \langle h_i \rangle \mid e \in h_i\}$ for some $h_i \in \text{Hist}(W)$, $i = 1, 2$. By (†) we have that $l' = \{e \rightarrowtail \Pi_e \langle h_1 \rangle \mid e \in l\}$ for a chain $l \subseteq h_1$ that is lower bounded by u, where u is the initial of u'. By (‡), for every $e \rightarrowtail \Pi_e \langle h_1 \rangle \in l'$, either $e \notin h_2$, or ($e \in h_2$ but $\Pi_e \langle h_1 \rangle \neq \Pi_e \langle h_2 \rangle$). There are now three cases, depending on the form of l': (i) l' has a minimal element $v' = v \rightarrowtail \Pi_v \langle h_1 \rangle$, $v \in h_1 \cap h_2$, and $\Pi_v \langle h_1 \rangle = \Pi_v \langle h_2 \rangle$, or (ii) l' has a minimal element $v' = v \rightarrowtail \Pi_v \langle h_1 \rangle$, $v \in h_1 \cap h_2$, but $\Pi_v \langle h_1 \rangle \neq \Pi_v \langle h_2 \rangle$, or (iii) l' either has no minimal element at all, or for the minimal element $v' = v \rightarrowtail \Pi_v \langle h_1 \rangle$ of l', $v \in h_1 \setminus h_2$. Case (i) is impossible as it contradicts (‡). Consider then case (ii): since $h_1 \perp_v h_2$, by Fact 13 the two transitions, $v_1' = v \rightarrowtail \Pi_v \langle h_1 \rangle$ and $v_2' = v \rightarrowtail \Pi_v \langle h_2 \rangle$ are distinct elements of a choice set $\ddot{v}'$ at which histories h_1' and h_2' split, $h_1' \perp_{\ddot{v}'} h_2'$. Furthermore, as v_1' is the minimal element of l', $v_1' \preccurlyeq l'$, as required by PCP_{NF}. Finally, in case (iii), no element $e \in l$ can belong to h_2: if l' has a minimum $v' = v \rightarrowtail \Pi_v \langle h_1 \rangle$ with $v \in h_1 \setminus h_2$, no point above v can belong to h_2 (Fact 1(2)), and in case l' has no minimum at all, the assumption that $e \in l \cap h_2$ implies that there is also some other $e_1 \in l \cap h$ with $e_1 < e$, and hence $e_1 \rightarrowtail \Pi_{e_1} \langle h_2 \rangle \in l' \cap h_2'$, contradicting (‡). Thus, $l \cap h_2 = \emptyset$. Applying the PCP of BST_{92} to the chain $l \subseteq h_1 \setminus h_2$ that is lower bounded by u, we get $v \in W$ such that $v < l$ and $h_1 \perp_v h_2$. Exactly like in case (ii) we thus invoke Fact 13 to produce the sought-for choice set $\ddot{v}'$ containing $v \rightarrowtail \Pi_v \langle h_1 \rangle$ and $v \rightarrowtail \Pi_v \langle h_2 \rangle$, for which $h_1' \perp_{\ddot{v}'} h_2'$. Since $v < l$ and given the form of l', we have $v \rightarrowtail \Pi_v \langle h_1 \rangle \preccurlyeq l'$ as well. ∎

Given this result, we have shown that the full transition structure of a BST_{92} structure is a BST_{NF} structure:

THEOREM 2. *Let $\langle W, < \rangle$ be a BST_{92} structure. Then that structure's full transition structure $\Upsilon(\langle W, < \rangle)$ is a BST_{NF} structure.*

PROOF. From Theorem 1 and Lemma 2. ∎

4.6. Facts About Choice Sets

In this section we prove a few facts related to sets of local point-wise alternatives and sets of local history-wise alternatives, which will also justify our terminology. Our main result is Theorem 3, which states that choice sets fully capture the notion of a local alternative in BST_{NF}.

FACT 14. Let there be h_1, h_2 and $e \in W$ such that $h_1 \perp_{\ddot{e}} h_2$. Then there is no $c < e$ for which $h_1 \perp_{\ddot{c}} h_2$.

PROOF. Assume for reductio that $h_1 \perp_{\ddot{c}} h_2$ for some $c_1 < e$, where $c_1 \in h_1 \cap \ddot{c}$. Then there are $c_1, c_2 \in \ddot{c}$ for which $\ddot{c} \cap h_1 = c_1 \neq c_2 = \ddot{c} \cap h_2$. By $c_1 < e$ there is some $l \in \mathscr{C}_e$ with $c_1 \in l$. By $h_1 \perp_{\ddot{c}} h_2$, we have $l \subseteq h_1 \cap h_2$. But this implies $c_1 \in h_2$, so that $\{c_1, c_2\} \subseteq h_2$, contradicting Fact 9(1). ∎

LEMMA 3. *Let $s \in \ddot{e}$ for some $e \in W$. Then we have $x < s$ iff $x < e$, i.e., $\mathscr{P}_e = \mathscr{P}_s$.*

PROOF. If $e = s$, there is nothing to prove. Thus, assume $e \neq s$.

"⇐": Let $s \in \ddot{e}$, and let $x < e$. By the Zorn-Kuratowski lemma, there is some chain $l \in \mathscr{C}_e$ for which $x \in l$. As $s \in \ddot{e}$, we know that there is some $h \in$ Hist for which $\sup_h l = s$. We cannot have $s \in l$: otherwise, for h' witnessing $\sup_{h'} l = e$, we would have $\{e, s\} \subseteq h'$, contradicting Fact 3. Thus, $l < s$, which implies $x < s$.

"⇒": Let $s \in \ddot{e}$, and let $x < s$. Assume for reductio that $x \not< e$. We first show that under this assumption, x and e cannot share a history. Assume otherwise, and let $h_1 \in H_e \cap H_x$. Let $h_2 \in H_s$. Take some $l \in \mathscr{C}_e$. We have $x \in h_1$, $x \in h_2$ (by downward closure of histories, as $x < s$), and $l \subseteq h_1$ (as $e \in h_1$). Now, as $s \in h_2$ and $s \in \ddot{e}$, we have $l < s$, so by downward closure of histories, $l \subseteq h_2$ as well. Noting that $\sup_{h_2}\{x\} = x < s = \sup_{h_2} l$, the Weiner postulate implies that $\sup_{h_1}\{x\} = x < e = \sup_{h_1} l$, contradicting the assumption that $x \not< e$.

Under our reductio assumption, x and e do not share a history. Choose some $h_1 \in H_e$ and some $h_2 \in H_s$. Since $s \in \ddot{e}$ and $e \neq s$, clearly $h_1 \perp_{\ddot{e}} h_2$. Moreover, by downward closure of histories we have $x \in h_2$, and as $e \in h_1$ and x and e do not share a history, $x \notin h_1$. By PCP$_{\mathrm{NF}}$ applied to $x \in h_2 \setminus h_1$, there is $c \in W$ such that $h_1 \perp_{\ddot{c}} h_2$ and $c \leqslant x$, and hence $c < s$ and $c \in h_2$. And there is $c' \in \ddot{c}$ such that $c' \in h_1$. Picking $I \in \mathscr{C}_e$ and $J \in \mathscr{C}_c$, $I, J \subseteq h_1 \cap h_2$ and observing $c = \sup_{h_2} J < \sup_{h_2} I = s$, the Weiner Postulate implies $c' = \sup_{h_1} J < \sup_{h_1} I = e$. But by Fact 14 then h_1 and h_2 cannot split at $\ddot{c}$, if they split at $\ddot{e}$, where $c' < e$. So, on our reductio assumption $x \not< e$, we have derived a contradiction, which proves that $x < e$. ∎

Given Lemma 3 it is also not difficult to see that for $s \in \ddot{e}$, a chain ends in e iff it ends in s:

FACT 15. Let $s \in \ddot{e}$. Then we have $l \in \mathscr{C}_s$ iff $l \in \mathscr{C}_e$.

PROOF. If $e = s$, there is nothing to prove. Thus, assume $e \neq s$.

"⇐": Given $s \in \ddot{e}$ and $l \in \mathscr{C}_e$, by the definition of $\ddot{e}$ there is some history h for which $\sup_h l = s$, which implies $l \in \mathscr{C}_s$.

"⇒": Let $s \in \ddot{e}$, and let $l \in \mathscr{C}_s$, i.e., $l < s$ and for some $h \in H_s$, $\sup_h l = s$. By Lemma 3, $l < e$. Take some $h' \in H_e$, and pick some $J \in \mathscr{C}_e$, so $J < e$,

and hence $J \subseteq h'$. Then we also have $J < s$ (given $s \in \ddot{e}$), which gives us $J \subseteq h$. We have $\sup_h l = s = \sup_h J$. Thus, by the Weiner postulate, we also have $\sup_{h'} l = \sup_{h'} J = e$, and therefore, $l \in \mathscr{C}_e$.   ∎

It follows that the set $\ddot{e}$ is independent of the witness chosen:

FACT 16. *Let $s \in \ddot{e}$. Then $e \in \ddot{s}$.*

PROOF. Let $s \in \ddot{e}$. We have to show that $e \in \ddot{s}$, i.e., $e \in \mathscr{S}(I)$ for all $I \in \mathscr{C}_s$. Thus consider an arbitrary $I \in \mathscr{C}_s$. By Fact 15, $I \in \mathscr{C}_e$. Now take some $h \in H_e$; we have $\sup_h I = e$, i.e., $e \in \mathscr{S}(I)$. As I was arbitrary, we have $e \in \ddot{s}$.   ∎

LEMMA 4. *We have $s \in \ddot{e}$ iff $\ddot{e} = \ddot{s}$.*

PROOF. "⇐": Immediate, since $s \in \ddot{s}$ by Definition 15.

"⇒": Let $s \in \ddot{e}$. For $s = e$ there is nothing to prove, so we assume $s \neq e$.

"⊆": Let $x \in \ddot{e}$. We have to show that $x \in \ddot{s}$, i.e., that $x \in \mathscr{S}(l)$ for all $l \in \mathscr{C}_s$. Thus, take some $l \in \mathscr{C}_s$. By Fact 15, $l \in \mathscr{C}_e$, and as $x \in \ddot{e}$, we have $x \in \mathscr{S}(l)$.

"⊇": Let $x \in \ddot{s}$. Take some $l \in \mathscr{C}_e$. As above, by Fact 15, $l \in \mathscr{C}_s$, and as $x \in \ddot{s}$, we have $x \in \mathscr{S}(l)$.   ∎

Having prepared the ground, we can now finally justify calling the partition $\Pi_{\ddot{e}}$ the set of local history-wise alternatives: The set of sets of histories $\Pi_{\ddot{e}}$ partitions the set of histories containing $\mathscr{P}_e$. That is, any history containing the whole proper past of e ends up in exactly one of the elements of $\Pi_{\ddot{e}}$.

THEOREM 3. *Let $e \in W$. Then $\Pi_{\ddot{e}}$ partitions $H_{[\mathscr{P}_e]}$, i.e.: (1) For $H_1, H_2 \in \Pi_{\ddot{e}}$, if $H_1 \neq H_2$, then $H_1 \cap H_2 = \emptyset$; (2) $\cup \Pi_{\ddot{e}} = H_{\ddot{e}} = H_{[\mathscr{P}_e]}$.*

PROOF. (1) Let $H_1, H_2 \in \Pi_{\ddot{e}}$. By Fact 9(2), if $H_1 \neq H_2$, then $H_1 \cap H_2 = \emptyset$.

(2) We have to show that $\cup \Pi_{\ddot{e}} = H_{[\mathscr{P}_e]}$. Note that $\cup \Pi_{\ddot{e}} = H_{\ddot{e}}$ by Fact 9(2).

"⊆": Take $h \in \cup \Pi_{\ddot{e}}$, i.e., $h \in H_s$ for some $s \in \ddot{e}$. By the definition of $\mathscr{P}_s$, we have $\mathscr{P}_s \subseteq h$, and by Lemma 3, $\mathscr{P}_e \subseteq h$. Thus, $h \in H_{[\mathscr{P}_e]}$.

"⊇": Take $h \in H_{[\mathscr{P}_e]}$, so that $\mathscr{P}_e \subseteq h$. By Fact 5, for all $l \in \mathscr{C}_e$ we have $l \subseteq h$. Take some $l_0 \in \mathscr{C}_e$, and let $s =_{\mathrm{df}} \sup_h l_0$. We can show that s is the h-relative supremum of any chain from $\mathscr{C}_e$. Fix some $h' \in H_e$. Take any $l \in \mathscr{C}_e$. We have $\sup_{h'} l = e = \sup_{h'} l_0$, and thus by Weiner's postulate we also have $\sup_h l = \sup_h l_0 = s$. Thus we have $s \in \mathscr{S}(l)$ for any $l \in \mathscr{C}_e$, which implies $s \in \ddot{e}$. As $h \in H_s$, we have $h \in \cup \Pi_{\ddot{e}}$.   ∎

The main message of the constructions studied in this section is that some $e \in W$ generate non-trivial choice sets $\ddot{e}$, in the sense that $\ddot{e} \neq \{e\}$. Such a set $\ddot{e}$ indeed consists of local point-wise alternatives to e. We can think of a choice set as a set of "indeterministic transitions", and each choice set induces a set of history-wise alternatives for e, namely $\Pi_{\ddot{e}}$. Finally, PCP$_{NF}$ requires that any two histories split at a choice set. So, in BST$_{NF}$ the basic concepts of branching histories still apply, but in a slightly different way from BST$_{92}$. As we will show now, this has the beneficial topological consequences announced earlier.

4.7. The Diamond Topology in BST$_{NF}$

In this section we return to the motivation of this paper, that is, the idea that a framework for local indeterminism should preserve local Euclidicity: if each history (space-time) is locally Euclidean of dimension n, then the global topology should be locally Euclidean of dimension n as well. As we saw in Section 3.2, the diamond topology on BST$_{92}$ structures does not preserve local Euclidicity when moving from the history-relative topologies to the global topology. In contrast, we can prove that the diamond topology on BST$_{NF}$ structures preserves local Euclidicity. Working towards Theorem 4 about the preservation of local Euclidicity, we first need an auxiliary lemma, which is also of interest of its own. Recall the disturbing feature of BST$_{92}$ discussed in Section 3.2: a set that is open in a history-relative topology need not be open in the corresponding global topology. In fact, no such set is open in the global topology if it contains a choice point. The Lemma below states that the opposite holds for the diamond topology on BST$_{NF}$ structures:

LEMMA 5. *Let a BST$_{NF}$ model $\langle W, \leqslant \rangle$ be given, let $h \in \text{Hist}(W)$, and let $Z \subseteq W$ be such that $Z \in \mathscr{T}_h$, i.e., Z is an open set w.r.t. the history-relative topology $\mathscr{T}_h$. Then $Z \in \mathscr{T}$, i.e., Z is also open w.r.t. the global topology on W.*

PROOF. Let $Z \in \mathscr{T}_h$ for some $h \in \text{Hist}$. Let $e \in Z$, and let $t \in MC(e)$. In order to establish the openness of Z w.r.t $\mathscr{T}$, we need to show that there is an e-centered diamond with vertices on t wholly contained in Z. The openness of Z w.r.t. $\mathscr{T}_h$ gives us such a diamond for any $t_h \in MC_h(e)$, but not necessarily for our given $t \in MC(e)$.

We show that the given t has a segment both below and above e that is contained in some $t_h \in MC_h(e)$, proceeding in two steps. First, we claim that t contains some $e' \in h$ for which $e' > e$. Assume otherwise, i.e., the chain $t^+ =_{df} \{e^* \in t \mid e^* > e\}$ contains no element of h. Note that by

construction, $\inf t^+ = e$. As t^+ is a chain, it is directed, and thus wholly contained in some history h_2. Pulling these facts together, $t^+ \subseteq h_2 \setminus h$, and by the maximality of t and the construction of t^+, we have that t^+ is a maximal chain in $h_2 \setminus h$. The PCP$_{\mathrm{NF}}$ gives us a choice set $\ddot{c}$ s.t. (†) $h \perp_{\ddot{c}} h_2$, and for the $c' \in \ddot{c} \cap h_2$, we have $c' \leqslant t^+$. We observe next that from the fact that t^+ is a maximal chain in $h_2 \setminus h$, it follows that $c' = \inf t^+$. Otherwise, for $i = \inf t^+$ we would have $c' < i \leqslant t^+$. By (†) we have $c' \notin h$, so $\{c'\} \cup t^+ \subseteq h_2 \setminus h$. As this chain extends t^+, it contradicts the maximality of t^+ in $h_2 \setminus h$. Thus, $c' = \inf t^+$, whence $c' = e$. It follows that $e \in h_2 \setminus h$, which contradicts our initial assumption that $e \in h$. So indeed, t contains some $e' \in h$ for which $e' > e$.

Second, we construct t_h by starting with an initial segment of the given t, as follows: Let $t^- =_{\mathrm{df}} \{e^* \in t \mid e^* \leqslant e'\}$; we have $t^- \subseteq h$ and $e \in t^-$. By the Zorn-Kuratowski lemma we can extend t^- with elements of h to form a maximal chain t_h, so that $t_h \in MC_h(e)$. The chains t and t_h share the initial segment t^-. We can now invoke the openness of Z w.r.t. $\mathscr{T}_h$ for e and t_h, which gives us a diamond $D_{e_1^h, e_2^h} \subseteq Z$ for which $e_1^h, e_2^h \in t_h$ and $e_1^h < e < e_2^h$. We set $e_1 =_{\mathrm{df}} e_1^h$ and $e_2 =_{\mathrm{df}} \min\{e', e_2^h\}$. We thus have $e_1 < e < e_2$ with $e_1 = e_1^h \in t$, and also $e_2 \in t$ because $e' \in t$. And as the diamond $D_{e_1, e_2} \subseteq D_{e_1^h, e_2^h}$, we have $D_{e_1, e_2} \subseteq Z$. So we have found the witnessing e-centered diamond with vertices on t, which establishes the openness of Z w.r.t. $\mathscr{T}$. ∎

With the above lemma at hand, we can prove the mentioned theorem:

THEOREM 4. *Let $\langle W, < \rangle$ be a BST$_{\mathrm{NF}}$ structure. If there is an $n \in \mathbb{N}$ such that for every $h \in \mathrm{Hist}(W)$, the topological space $\langle h, \mathscr{T}_h \rangle$ is locally Euclidean with dimension n, then the topological space $\langle W, \mathscr{T} \rangle$ is also locally Euclidean with dimension n.*

PROOF. We need to show that each $e \in W$ has a neighborhood $O_e \in \mathscr{T}$ that is mapped by some homeomorphism φ_e to an open set $R_e \in \mathscr{T}(\mathbb{R}^n)$. Let $e \in W$, and pick some $h \in H_e$. Since h is locally Euclidean with respect to $\mathscr{T}_h$, there is a $\mathscr{T}_h$-open neighborhood $O_e^h \subseteq h$ of e, an open set of $\mathbb{R}^n$, $R_e^h \in \mathscr{T}(\mathbb{R}^n)$, and a homeomorphism φ_e^h such that $\varphi_e^h[O_e^h] = R_e^h$. By Lemma 5, from $O_e^h \in \mathscr{T}_h$ it follows that $O_e^h \in \mathscr{T}$. We let $O_e =_{\mathrm{df}} O_e^h$, $R_e =_{\mathrm{df}} R_e^h$, and we can use $\varphi_e =_{\mathrm{df}} \varphi_e^h$ as our homeomorphism between the open sets $O_e \in \mathscr{T}$ and $R_e \in \mathscr{T}(\mathbb{R}^n)$. ∎

BST$_{\mathrm{NF}}$ thus vindicates the idea that if one starts with locally Euclidean histories (space-times) that allow for the assignment of spatio-temporal coordinates, one does not destroy that feature by analyzing indeterminism

within the framework of branching space-times. This is in marked contrast with the situation in BST_{92}, in which local Euclidicity does not transfer from the individual histories to the global branching structure.[19] So, if one wants to analyze local indeterminism in branching structures that retain local Euclidicity, one has to choose the framework of BST_{NF}.

With respect to the topological condition of Hausdorffness, on the other hand, the two frameworks are on a par: Apart from some trivial cases,[20] both BST_{92} structures and BST_{NF} structures are non-Hausdorff even if their individual histories are Hausdorff.

What is the significance of the failure of transferring topological properties of individual histories to the global structure? Arguably, transfer of local Euclidicity is much more important. Local Euclidicity guarantees the ascription of spatio-temporal coordinates (sets of real numbers) to any spatio-temporal event. The idea that such events have coordinates is deeply entrenched in our concept of space-time. Perhaps there are ways to ascribe coordinates to events that do not require local Euclidicity, but that condition is used in standard physical theories of space-time. Thus, abandoning local Euclidicity will mark a revolutionary break with with the established practice of physics. We thus claim that local Euclidicity should hold both in each history (individual space-time) and in a global BST structure that represents the totality of all possible spatio-temporal events.

The Hausdorff property has a different status. It is standardly postulated to hold in structures representing space-times in General Relativity: theses structures, differential manifolds, are Hausdorff by definition. However, there have been attempts to relax the Hausdorff property, motivated by particular candidates for space-time. It can also be argued that the Hausdorff property is not needed in General Relativity and can be abandoned at a small price [9]. Finally, there is a method of gluing (Hausdorff) differential manifolds into a larger structure, a so-call generalized manifold, that is not Haudorff. It is natural to interpret this result as a modal representation of a family of alternative space-times that overlap on some region [10]. Accordingly, histories should be Hausdorff, but global structures with multiple alternative histories will violate Hausdorffness.

---

[19]Compare our discussion following Definition 11 in Section 3.2.

[20]Main examples are one-dimensional structures of BST_{92} such as pictured in Figure 1. Also, one-history structures are, of course, Hausdorff if their single history is.

5.   From New Foundations to Old Foundations and Back Again

Our target in this section is a set of theorems establishing that we can move freely between BST_{92} and BST_{NF} while preserving the basic indeterministic structure. In Section 5.1 we will show that given a BST_{NF} structure, we can define an accompanying BST_{92} structure via a transformation detailed in Definition 20, in parallel to the derivation of a BST_{NF} structure from a BST_{92} structure above. In Section 5.2, we will then show that the concatenation of these two translations, in any order, is an order isomorphism. In this way, BST_{92} and BST_{NF} can be seen as two alternative representations of the same underlying indeterministic structure. This means that we can represent indeterministic scenarios without having to decide between the different prior choice principles of BST_{92} and BST_{NF}.

5.1.   From New Foundations BST_{NF} to BST_{92}

We have seen how the move from a BST_{92} structure to its full transition structure brings us from BST_{92} to BST_{NF}. In the other direction, there is also a fairly simple translation, viz., combining, or collapsing, all the elements of a choice set to form a single point. The elements of a choice set constitute different history-relative suprema of a chain without an endpoint. The transform, in contrast, contains a chain with a unique endpoint, after which the different outcomes have no first elements.

DEFINITION 20. *(The Λ transformation from BST_{NF} to BST_{92}.)* Let $\langle W, < \rangle$ be a BST_{NF} model. Then we define the companion Λ-transformed ("collapsed") model as follows:[21]

$\Lambda(\langle W, < \rangle) =_{df} \langle W', <' \rangle$,   where

$W' =_{df} \{ \ddot{e} \mid e \in W \}$; $\ddot{e}_1 <' \ddot{e}_2$ iff $e'_1 < e'_2$ for some $e'_1 \in \ddot{e}_1, e'_2 \in \ddot{e}_2$.

It will be useful to extend the Λ-notation to elements and subsets of W, so that $\Lambda(e) =_{df} \ddot{e}$, and $\Lambda(E) =_{df} \{ \ddot{e} \mid e \in E \}$.

FACT 17. (Facts about Definition 20) The following holds:

1. Let $e_1, e_2 \in W$ and let $e_1 < e_2$. Then $\Lambda(e_1) <' \Lambda(e_2)$.

2. Let $t \subseteq W$ be a chain (w.r.t. $<$). Then $\Lambda(t)$ is a chain (w.r.t. $<'$).

3. Let $E \subseteq W$ be directed. Then $\Lambda(E)$ is also directed.

---

[21]Graphically, we have chosen "Λ" as the reverse of "Υ". Note that from here on, we denote the BST_{NF} structures as (unprimed) $\langle W, < \rangle$ and the Λ-transformed structures with primes.

PROOF. (1): holds by the definition of $<'$. (2) and (3) follow immediately. ∎

FACT 18. (Justification of the notation in Definition 20) In BST$_{NF}$: (1) If $e_1 < e_2$ and $e_2' \in \ddot{e}_2$, then $e_1 < e_2'$. So we can write $e_1 <' \ddot{e}_2$. (2) If $e_1 <' \ddot{e}_2$ and $e_1^* \in \ddot{e}_1$, $e_1 \neq e_1^*$, then $e_1^* \not<' \ddot{e}_2$. So given $\ddot{e}_1 <' \ddot{e}_2$, there is a unique $e_1 \in \ddot{e}_1$ for which $e_1 <' \ddot{e}_2$. (3) If $\ddot{e}_1 <' \ddot{e}_2$, then there are no $e_i^* \in \ddot{e}_i$ $(i = 1, 2)$ for which $e_2^* < e_1^*$.

PROOF. (1): By Fact 15.

(2): Let $e_1 <' \ddot{e}_2$ as witnessed by e_2 (i.e., $e_1 < e_2$), and $e_1^* \in \ddot{e}_1$, $e_1 \neq e_1^*$. Assume for reductio $e_1^* <' \ddot{e}_2$, then there has to be some witness $e_2^* \in \ddot{e}_2$ for which $e_1^* < e_2^*$. Then by (1), we also have $e_1^* < e_2$, so that (by downward closure of histories) there is a history containing e_1 and e_1^*, contradicting Fact 9(1).

(3): Let $\ddot{e}_1 <' \ddot{e}_2$ as witnessed by e_1 and e_2 (i.e., $e_1 \in \ddot{e}_1$, $e_2 \in \ddot{e}_2$, and $e_1 < e_2$). Assume for reductio that there are $e_1^* \in \ddot{e}_1$ and $e_2^* \in \ddot{e}_2$ for which $e_2^* < e_1^*$. Then by (1), we have $e_1 < e_2^*$, and by transitivity, $e_1 < e_1^*$. Thereby e_1 and e_1^*, being different elements of $\ddot{e}_1$ (by irreflexivity of $<$), would have to belong to one history, contradicting Fact 9(1). ∎

Similarly to what we established about the properties of the transition structure of a BST$_{92}$ structure, we can characterize the Λ-transform of a BST$_{NF}$ structure. It turns out that, as announced, the Λ-transform leads us back to BST$_{92}$. As above, we split the proof into a number of steps.

FACT 19. Let $\langle W, < \rangle$ be a BST$_{NF}$ model, and let $\langle W', <' \rangle =_{df} \Lambda(\langle W, < \rangle)$. Then $\langle W', <' \rangle$ is (1) non-empty and (2) a strict partial ordering (3) without minima or maxima.

PROOF. (1) By construction, W' is non-empty (given that W was non-empty).

(2) Asymmetry follows from Fact 18(3). For transitivity, let $\ddot{e}_1 <' \ddot{e}_2$ and $\ddot{e}_2 <' \ddot{e}_3$. Then by Fact 18(2), there is a unique $e_2 \in \ddot{e}_2$ for which $e_2 <' \ddot{e}_3$, and a unique $e_1 \in \ddot{e}_1$ for which $e_1 <' \ddot{e}_2$. So by $e_2 \in \ddot{e}_2$ and by transitivity of $<$ we have $e_1 <' \ddot{e}_3$, which proves $\ddot{e}_1 <' \ddot{e}_3$.

(3) Let $\ddot{e} \in W'$. There is some $e_1 \in W$ for which $\ddot{e} = \Lambda(e_1)$. As W has no maximal nor minimal elements, there are $e_2, e_3 \in W$ for which $e_2 < e_1 < e_3$. Then by the definition of the ordering, $\ddot{e}_2 <' \Lambda(e_1) = \ddot{e}$, establishing that $\ddot{e}$ cannot be a minimum, and $\Lambda(e_1) = \ddot{e} <' \ddot{e}_3$, establishing that $\ddot{e}$ cannot be a maximum. ∎

Before we can establish history-relative suprema of upper bounded chains, we need to prove a lemma about the form of histories in W'.

LEMMA 6. *Let $\langle W, < \rangle$ be a BST_{NF} model, and let $\langle W', <' \rangle =_{df} \Lambda(\langle W, < \rangle)$. The histories (maximal directed sets) in $\langle W', <' \rangle$ are exactly the sets $\Lambda(h)$, for $h \in \mathrm{Hist}(W)$. That is, (1) for $h \in \mathrm{Hist}(W)$, the set $\Lambda(h)$ is maximal directed and (2) for any maximal directed set $h' \in \mathrm{Hist}(W')$ there is a unique history $h \in \mathrm{Hist}(W)$ s.t. $h' = \Lambda(h)$.*

PROOF. (1) The set $\Lambda(h) \subseteq W'$ is directed by Fact 17(3), so there is some maximal directed $h' \in \mathrm{Hist}(W')$ for which $\Lambda(h) \subseteq h'$. Note that by Facts 19(3) and 1 (5), h' cannot have a maximum. This allows us to define a function $f : h' \mapsto W$ that establishes the converse of Λ on h', in the following sense: (i) for any $\ddot{e} \in h'$, $\Lambda(f(\ddot{e})) = \ddot{e}$, (ii) for any $\ddot{e}_1, \ddot{e}_2 \in h'$, we have $\ddot{e}_1 <' \ddot{e}_2$ iff $f(\ddot{e}_1) < f(\ddot{e}_2)$, and (iii) for any $e \in h$, $f(\Lambda(e)) = e$. (Note that the primed histories and the primed ordering refer to $\Lambda(\langle W, < \rangle)$, not to the BST_{NF} structure.)

To define f, let $\ddot{e}_1 \in h'$. Let $\ddot{e}_2 \in h'$ s.t. $\ddot{e}_1 <' \ddot{e}_2$; such an element exists as h' has no maxima. By Fact 18(2), there is a unique $v \in W$ for which $\ddot{e}_1 = \Lambda(v)$ and $v <' \ddot{e}_2$. That v is, moreover, independent of the chosen upper bound $\ddot{e}_2 \in h'$: let $v^* \in \ddot{e}_1$ be such that $v^* <' \ddot{e}_3$ for some $\ddot{e}_3 \in h'$ for which $\ddot{e}_1 <' \ddot{e}_3$. Then by directedness of h', there is some common upper bound $\ddot{e}_4$ of $\ddot{e}_2$ and $\ddot{e}_3$, and again invoking Fact 18(2), we have $v^* = v$. So we can set $f(\ddot{e}_1) = v$ as specified. Note that thereby, $f(\ddot{e}_1) \in \ddot{e}_1$. Constraint (i) holds by construction, as $\Lambda(f(\ddot{e})) = \Lambda(v) = \ddot{e}$. For (ii, "$\Rightarrow$"), let $\ddot{e}_1, \ddot{e}_2 \in h'$ satisfy $\ddot{e}_1 <' \ddot{e}_2$. Then $f(\ddot{e}_1) <' \ddot{e}_2$ by construction (as $\ddot{e}_2$ is an upper bound of $\ddot{e}_1$ in h'), and the claim follows by Fact 18(1), noting that $f(\ddot{e}_2) \in \ddot{e}_2$. For (ii, "$\Leftarrow$"), let $\ddot{e}_1, \ddot{e}_2 \in h'$ be such that $f(\ddot{e}_1) < f(\ddot{e}_2)$. By (i) and by the definition of the ordering $<'$, this implies $\ddot{e}_1 <' \ddot{e}_2$. For (iii), let $e_1 \in h$. As h contains no maxima, there is some $e_2 \in h$ for which $e_1 < e_2$. Let $\ddot{e}_i = \Lambda(e_i)$ $(i = 1, 2)$, so that (by the definition of $<'$) we have $\ddot{e}_1 <' \ddot{e}_2$. By the definition of f, $f(\ddot{e}_1) = e^*$ for the unique member $e^* \in \ddot{e}_1$ for which $e^* <' \ddot{e}_2$. Given $e_1 < e_2$, we have $e^* = e_1$, i.e., $f(\Lambda(e_1)) = f(\ddot{e}_1) = e_1$.

Now for maximality of the directed set $\Lambda(h) \subseteq h'$, assume for reductio that $h' = \Lambda(h) \cup A'$ with $\Lambda(h) \cap A' = \emptyset$ and $A' \neq \emptyset$ (i.e., assume that $\Lambda(h)$ is not maximal directed). Let $A =_{df} \{f(\ddot{e}) \mid \ddot{e} \in A'\}$, so that $A \neq \emptyset$ and $A \cap h = \emptyset$. By property (ii) of f, $h \cup A$ is directed, violating the maximality of h.

(2) Let $h' \in \mathrm{Hist}(W')$, and define f as above. Let $E =_{df} \{f(\ddot{e}) \mid \ddot{e} \in h'\}$, so that by (i), $\Lambda(E) = h'$. By (ii), E is directed, so that there is some $h \in \mathrm{Hist}(W)$ with $E \subseteq h$. It follows that $h' = \Lambda(E) \subseteq \Lambda(h)$. By (1), we have that $\Lambda(h) = h''$ for some $h'' \in \mathrm{Hist}(W')$. So we have two histories $h', h'' \in \mathrm{Hist}(W')$ for which $h' \subseteq h''$, whence, by Fact 1(3), $h' = h''$. This

means that we have found some $h \in \text{Hist}(W)$ for which $\Lambda(h) = h'$. For uniqueness of h, let $h_1, h_2 \in \text{Hist}(W)$ be such that $\Lambda(h_1) = \Lambda(h_2) = h'$. Then $h_1 = f(h')$ and $h_2 = f(h')$, establishing $h_1 = h_2$. ∎

LEMMA 7. *Let* $\langle W, < \rangle$ *be a* BST_{NF} *structure. Then its* Λ*-transform,* $\langle W', <' \rangle =_{\text{df}} \Lambda(\langle W, < \rangle)$, *is a common BST structure.*

PROOF. Our task is to show that $\langle W', <' \rangle$ satisfies the postulates of Definition 2.

1. Non-emptiness: By Fact 19(1).

2. Partial odering: By Fact 19(2).

3. No maximal nor minimal elements: By Fact 19(3).

4. $<'$ is dense.
   Let $\ddot{e}_1 <' \ddot{e}_2$, so by Fact 18(2), there is a unique $e_1 \in \ddot{e}_1$ for which $e_1 <' \ddot{e}_2$. Let $e_2 \in \ddot{e}_2$; in particular, $e_1 < e_2$. By density of $<$, we have $e^* \in W$ s.t. $e_1 < e^* < e_2$. By the definition of the $<'$ ordering, this establishes $\ddot{e}_1 = \Lambda(e_1) <' \Lambda(e^*) <' \Lambda(e_2) = \ddot{e}_2$, which proves density of $<'$.

5. Lower bounded chains have infima in $<'$.
   Let $t' \subseteq W'$ be a lower bounded chain, and let $\ddot{b} \in W'$ be a lower bound for t'. The elements of t' are of the form $\ddot{e} = \Lambda(e)$ with $e \in W$. We distinguish two cases. (a) If t' has a least element (which covers the case that t' has only one element), then that least element is the infimum of t' w.r.t. $<'$, by definition. (b) If t' has no least element, pick some $\ddot{e} \in t'$, and let $t^* =_{\text{df}} \{x \in t' \mid x <' \ddot{e}\}$. We have $\inf t^* = \inf t'$ by the definition of the infimum. And by Fact 18(2), for all $\ddot{e}_1 \in t^*$ there are unique $e_1 \in W$ for which $e_1 \in \ddot{e}_1$ and $e_1 < \ddot{e}$, and there is a unique $b^* \in \ddot{b}$ for which $b^* < \ddot{e}$. So there is a unique set $t \subseteq W$ given by

$$t = \{e_1 \in W \mid e_1 < \ddot{e} \wedge \ddot{e}_1 \in t^*\},$$

   which is a chain since t^* is a chain; further t is lower bounded by $b^* \in W$. By the properties of BST_{NF}, t therefore has an infimum $a =_{\text{df}} \inf t$, $a \in W$. We claim that $\ddot{a}$ is the infimum of t' w.r.t. $<'$. As $a < t$, we have $\ddot{a} <' t'$ by the definition of $<'$. Now let $\ddot{c} \leqslant' t'$. Again by Fact 18(2), there is a unique $c \in \ddot{c}$ for which $c < \ddot{e}$. By the fact that a is the infimum of t, we have $c \leqslant a$, which implies $\ddot{c} \leqslant' \ddot{a}$. So $\ddot{a}$ is indeed the greatest lower bound, i.e., the infimum, of t'.

6. Upper bounded chains have history-relative suprema in $<'$.
   Let t' be an upper bounded chain with $\ddot{b}$ an upper bound, and let $\ddot{b} \in h'$ for h' some history in $\langle W', <' \rangle$, so that $t' \subseteq h'$ as well. As h' has a unique

pre-image h under Λ (by Lemma 6), also $\ddot{b}$ and t' have unique pre-images $b \in h$ and $t \subseteq h$. So by the $\mathrm{BST_{NF}}$ axioms, t has an h-relative supremum $s \in h$. By Fact 17(1), we have $t' \leqslant' \ddot{s}$, and for any $\ddot{a} \in h'$ for which $t' \leqslant' \ddot{a}$, we can consider the unique pre-image $a \in h \cap \ddot{a}$, for which $t \leqslant a$. By the fact that s is the h-relative supremum of t, we have $s \leqslant a$, which translates into $\ddot{s} \leqslant' \ddot{a}$, i.e., $\ddot{s}$ is the least upper bound in h', and therefore the h'-relative supremum, of t'.

7. The Weiner postulate.
   Consider two histories $h'_1, h'_2 \in \mathrm{Hist}(W')$, two chains $l', k' \subseteq h'_1 \cap h'_2$, and their history-relative suprema $\ddot{s}_i = sup_{h'_i} l'$ and $\ddot{c}_i = sup_{h'_i} k$ ($i = 1, 2$). Assume that $\ddot{s}_1 \leqslant' \ddot{c}_1$. We denote the unique pre-images of $h'_1, h'_2, l', k', \ddot{s}_1,$ $\ddot{s}_2, \ddot{c}_1$, and $\ddot{c}_2$ under Λ by $h_1, h_2, l, k, s_1, s_2, c_1,$ and c_2, respectively. By the uniqueness of pre-images and properties of $<'$, we have $l, k \subseteq h_1 \cap h_2$, $s_i = sup_{h_i}(l)$, $c_i = sup_{h_i}(k)$ ($i = 1, 2$), and $s_1 \leqslant c_1$. Then by the Weiner Postulate of $\mathrm{BST_{NF}}$, $s_2 \leqslant c_2$. Since $\ddot{s}_2 = \Lambda(s_2)$ and $\ddot{c}_2 = \Lambda(c_2)$, we have $\ddot{s}_2 \leqslant' \ddot{c}_2$.

8. Historical connection. Pick h'_1, h'_2 histories in $\langle W', <' \rangle$, and consider their unique pre-images h_1, h_2 by Λ, which by Lemma 6 are histories in the $\mathrm{BST_{NF}}$ structure $\langle W, < \rangle$. By historical connection for $\mathrm{BST_{NF}}$ structures, we have $h_1 \cap h_2 \neq \emptyset$. Hence $\Lambda(h_1) \cap \Lambda(h_2) \neq \emptyset$, i.e., $h'_1 \cap h'_2 \neq \emptyset$. ∎

LEMMA 8. *The Λ-transform $\langle W, < \rangle =_{\mathrm{df}} \Lambda(\langle W, < \rangle)$ of a BST_{NF} structure $\langle W, < \rangle$ satisfies the BST_{92} prior choice principle.*

PROOF. Let h'_1, h'_2 be histories in $\langle W', <' \rangle$, and let $t' \subseteq h'_1 \setminus h'_2$ be a lower bounded chain in h'_1 that contains no element of h'_2. We have to find a maximal element $c \in h'_1 \cap h'_2$ that lies below t', $c <' t'$, and for which $h'_1 \perp_c h'_2$. The histories h'_1, h'_2 have as unique pre-images the $\langle W, < \rangle$-histories h_1, h_2. As $t' \subseteq h'_1$, the unique pre-image $t \subseteq h_1$. Furthermore, $t \cap h_2 = \emptyset$, for an element $e \in t \cap h_2$ would give rise to $\ddot{e} \in t' \cap h'_2$, violating our assumption about t'. So $t \subseteq h_1 \setminus h_2$. From the $\mathrm{BST_{NF}}$ prior choice principle, we have a choice set $\ddot{s}$ and $s_1 \in h_1 \cap \ddot{s}$ for which $s_1 \leqslant t$, while there is some $s_2 \in \ddot{s} \cap h_2$. Let $c =_{\mathrm{df}} \Lambda(s_1) = \ddot{s}$; we claim that c is the sought-for choice point. (a) By Lemma 4 we have $\ddot{s}_1 = \ddot{s}_2$, and as $s_i \in h_i$, we have $\ddot{s}_i \in h'_i$ ($i = 1, 2$), so that $c = \ddot{s}_1 = \ddot{s}_2 \in h'_1 \cap h'_2$. (b) As c lies in the intersection of h'_1 and h'_2, it cannot be that $\ddot{s}_1 \in t'$. This excludes $s_1 \in t$, so that in fact $s_1 < t$. This in turn implies $c = \ddot{s}_1 <' t'$. (c) For the maximality of c in $h'_1 \cap h'_2$, assume that there is $\ddot{a} \in h'_1 \cap h'_2$ for which $c < \ddot{a}$. Then we have a unique pre-image $a_1 \in h_1 \cap h_2$ for which both $s_1 < a_1$ and $s_2 < a_1$, so that both s_1 and s_2 belong to history h_1. This contradicts Fact 9(1). So $c = \ddot{s}$ is in fact maximal in $h'_1 \cap h'_2$. (d) By the definition of $\perp_c$, we therefore have $h'_1 \perp_c h'_2$. ∎

THEOREM 5. *The Λ-transform $\Lambda(\langle W, < \rangle)$ of a BST_{NF} structure $\langle W, < \rangle$ is a BST_{92} structure.*

PROOF. By Lemma 7 and Lemma 8. ∎

5.2.   Going Full Circle

We have now established that there is a way to get from BST_{92} structures to BST_{NF} structures and from BST_{NF} structures to BST_{92} structures. This raises the question of where we land when we concatenate these transformations. We can show that, as hoped, we land where we started: the resulting structures are order-isomorphic to the ones we started with. Since all important notions of BST are defined in terms of the ordering, this means that in the relevant sense, the Λ transform of the Υ transform of a BST_{92} structure *is* that original BST_{92} structure (Theorem 6), and the Υ transform of the Λ transform of a BST_{NF} structure *is* that original BST_{NF} structure (Theorem 7).

5.2.1.   From BST_{92} to BST_{NF} to BST_{92}

THEOREM 6. *The function $\Lambda \circ \Upsilon$ is an order isomorphism of BST_{92} structures: Let $\langle W_1, <_1 \rangle$ be a BST_{92} structure, let $\langle W_2, <_2 \rangle =_{\text{df}} \Upsilon(\langle W_1, <_1 \rangle)$, and let $\langle W_3, <_3 \rangle =_{\text{df}} \Lambda(\langle W_2, <_2 \rangle)$. Then there is an order isomorphism φ between $\langle W_1, <_1 \rangle$ and $\langle W_3, <_3 \rangle$, i.e., a bijection between W_1 and W_3 that preserves the ordering.*

PROOF. We claim that we can use the mapping φ, defined for $e \in W_1$ to be

$$\varphi(e) =_{\text{df}} \{e \rightarrowtail H \mid H \in \Pi_e\}.$$

We have to show (1) that φ is indeed a mapping from W_1 to W_3, (2) that φ is injective, (3) that φ is surjective, and (4) that φ preserves the ordering.

(1) Mapping: We have to show that for any $e \in W_1$, $\varphi(e) = \{e \rightarrowtail H \mid H \in \Pi_e\} \in W_3$. The set W_2 is the full transition structure of $\langle W_1, <_1 \rangle$, so that for $e \in W_1$ and for any $H \in \Pi_e$, the transition $e \rightarrowtail H \in W_2$. Thus, for $e \in W_1$, the set $\varphi(e) \subseteq W_2$. The set W_3 contains, for any $\tau \in W_2$, the set $\Lambda(\tau) = \ddot{\tau} \in W_3$, and $\ddot{\tau} \subseteq W_2$ as well. Let now $e \in W_1$, and pick some $H^* \in \Pi_e$, which fixes some $\tau = e \rightarrowtail H^* \in W_2$. We claim that

$$\ddot{\tau} = \{e \rightarrowtail H \mid H \in \Pi_e\},$$

which establishes $\ddot{\tau} = \varphi(e)$, so that indeed, $\varphi(e) \in W_3$. The claim is an equality between subsets of W_2, so that we show inclusion both ways.

"$\subseteq$": Let $\tau' = e' \rightarrowtail H' \in \ddot{\tau}$; we have to show that $\tau' \in \{e \rightarrowtail H \mid H \in \Pi_e\}$. We have $\tau \in h$ and $\tau' \in h'$ for $h, h' \in \text{Hist}(W_2)$. By Lemma 1 we know that these histories are of the form

$$h = \{e_1 \rightarrowtail \Pi_{e_1}\langle h_1 \rangle \mid e_1 \in h_1\}; \quad h' = \{e_1' \rightarrowtail \Pi_{e_1'}\langle h_1' \rangle \mid e_1' \in h_1'\}$$

for some $h_1, h_1' \in \text{Hist}(W_1)$. The set $\ddot{\tau}$ is defined as the intersection of all sets of history-relative suprema of any chain $l \subseteq W_2$ ending in, but not containing τ ($l \in \mathscr{C}_\tau$), so that for any $l \in \mathscr{C}_\tau$, we have $\sup_h l = \tau$ and $\sup_{h'} l = \tau'$, since $\tau' \in \ddot{\tau}$. As $\tau = e \rightarrowtail H^* \in h$, we have $e \in h_1$. We now claim that $e \in h_1'$ as well. Assume otherwise, so that $e \in h_1 \setminus h_1'$. By PCP_{92}, there is then some $c <_1 e$ for which $h_1 \perp_c h_1'$. Let $\tau_c =_{\text{df}} c \rightarrowtail \Pi_c\langle h_1 \rangle$, so that $\tau_c \in h$ and $\tau_c <_2 \tau$. There is thus some chain $l \in \mathscr{C}_\tau$ for which $\tau_c \in l$. As $\sup_{h'} l = \tau'$ (by $\tau' \in \ddot{\tau}$, see above), we have $l \subseteq h'$, which implies $\tau_c \in h'$ and $c \in h_1'$. From the form of elements of h' we therefore must have $\Pi_c\langle h_1 \rangle = \Pi_c\langle h_1' \rangle$, contradicting $h_1 \perp_c h_1'$. So indeed $e \in h_1 \cap h_1'$.

Now take some $l \in \mathscr{C}_\tau$; we have $l \subseteq W_2$ and indeed $l \subseteq h \cap h'$. Let l_1 be the set of initials of the elements of l, i.e., $l_1 \subseteq W_1$ and $l = \{e_1 \rightarrowtail \Pi_{e_1}\langle h_1 \rangle \mid e_1 \in l_1\}$. Note that $\sup_{h_1} l_1 = e = \sup_{h_1'} l_1$ from $\sup_h l = \tau$ and $e \in h_1 \cap h_1'$, and as $l \subseteq h \cap h'$, we have $\Pi_{e_1}\langle h_1 \rangle = \Pi_{e_1}\langle h_1' \rangle$ for all $e_1 \in l_1$. We now claim that $\sup_{h'} l = \tau'' =_{\text{df}} e \rightarrowtail \Pi_e\langle h_1' \rangle$. We have $\tau'' \in h'$ because $e \in h_1'$, and $l <_2 \tau''$ because $l_1 <_1 e$, so τ'' is an upper bound of l in h'. Let now $\tau^* = e^* \rightarrowtail \Pi_{e^*}\langle h_1' \rangle \in h'$ be some upper bound of l in h'. Then e^* is an upper bound of l_1 in h_1', and thus $e \leqslant_1 e^*$ as $\sup_{h_1'} l_1 = e$, so that $\tau'' \leqslant_2 \tau^*$, proving that τ'' is the h'-relative supremum of l. So we have shown that $\tau'' = e \rightarrowtail \Pi_e\langle h_1' \rangle = \sup_{h'} l = \tau'$. So indeed, $\tau' \in \{e \rightarrowtail H \mid H \in \Pi_e\}$.

"$\supseteq$": Let $\tau' \in \{e \rightarrowtail H \mid H \in \Pi_e\}$, i.e., $\tau' = e \rightarrowtail H$ for the e in question and for some $H \in \Pi_e$. We have to show that $\tau' \in \ddot{\tau}$. We have $\tau \in h$ and $\tau' \in h'$ for some $h, h' \in \text{Hist}(W_2)$, which are again of the form

$$h = \{e_1 \rightarrowtail \Pi_{e_1}\langle h_1 \rangle \mid e_1 \in h_1\}; \quad h' = \{e_1' \rightarrowtail \Pi_{e_1'}\langle h_1' \rangle \mid e_1' \in h_1'\}$$

for some $h_1, h_1' \in \text{Hist}(W_1)$, so that $\tau' = e \rightarrowtail \Pi_e\langle h_1' \rangle$.

Let $l \in \mathscr{C}_\tau$; we have $\sup_h l = \tau = e \rightarrowtail H^*$ by assumption. We now claim that $\sup_{h'} l = \tau'$, which establishes $\tau' \in \ddot{\tau}$. To prove that τ' is the h'-relative supremum of l, as above, let l_1 be the set of initials of the elements l, so that $l_1 \subseteq W_1$ and $l = \{e_1 \rightarrowtail \Pi_{e_1}\langle h_1 \rangle \mid e_1 \in l_1\}$. Again as above, $l_1 <_1 e$, and thus τ' is an upper bound of l in h'. Let now $\tau^* = e^* \rightarrowtail \Pi_{e^*}\langle h_1' \rangle \in h'$ be some upper bound of l in h'. Then $e^* \in h_1'$ is an upper bound of l_1 in h_1', and thus $e \leqslant_1 e^*$ as $\sup_{h_1'} l_1 = e$ (note that $e \in h_1'$ as $\tau' \in h'$). Therefore $\tau' \leqslant_2 \tau^*$, proving that τ' is the h'-relative supremum of l. As l was an arbitrary chain from $\mathscr{C}_\tau$, we have indeed $\tau' \in \ddot{\tau}$.

(2) Injectivity: Let $e, e' \in W_1$ with $e \neq e'$. Then $\varphi(e) \neq \varphi(e')$. This is clear as the sets $\varphi(e)$ and $\varphi(e')$ have different members.

(3) Surjectivity: Let $a \in W_3$. We have to find some $e \in W_1$ for which $\varphi(e) = a$. As $a \in W_3$, we have $a = \ddot{\tau}$ for some $\tau = e \rightarrowtail H \in W_2$, where $e \in W_1$ and $H \in \Pi_e$. Above under (1) we have established that for $\tau = e \rightarrowtail H \in W_2$, we have $\ddot{\tau} = \{e \rightarrowtail H \mid H \in \Pi_e\}$, i.e., $a = \ddot{\tau} = \varphi(e)$.

(4) Order preservation: We have to show that for $e_1, e_2 \in W_1$, $e_1 <_1 e_2$ iff $\varphi(e_1) <_3 \varphi(e_2)$. (The claim about equality follows from the fact that φ is a bijection.) We know from the definition of φ that $\varphi(e_i) = \{e_i \rightarrowtail H \mid H \in \Pi_{e_i}\}$ $(i = 1, 2)$.

"$\Rightarrow$": Let $e_1, e_2 \in W_1$ with $e_1 <_1 e_2$, and let $h_2 \in H_{e_2}$. Let $\tau_1 =_{\mathrm{df}} e_1 \rightarrowtail \Pi_{e_1}\langle h_2 \rangle$ and $\tau_2 =_{\mathrm{df}} e_2 \rightarrowtail H$ for some $H \in \Pi_{e_2}$; we have $\tau_1, \tau_2 \in W_2$ and $\varphi(e_i) = \ddot{\tau}_i$ $(i = 1, 2)$. By the definition of the transition ordering $<_2$, we have $\tau_1 <_2 \tau_2$, and by the definition of $<_3$ in terms of instances, we thus have $\ddot{\tau}_1 <_3 \ddot{\tau}_2$, i.e., $\varphi(e_1) <_3 \varphi(e_2)$.

"$\Leftarrow$": Let $\varphi(e_1) <_3 \varphi(e_2)$, i.e., there are some $\tau_1 \in \varphi(e_1)$, $\tau_2 \in \varphi(e_2)$ for which $\tau_1 <_2 \tau_2$. These transitions have the form $\tau_i = e_i \rightarrowtail H_i$ for some $e_i \in W_1$ and $H_i \in \Pi_{e_i}$ $(i = 1, 2)$. Thus, in particular, from $\tau_1 <_2 \tau_2$ we have that $e_1 <_1 e_2$. ∎

5.2.2. From $\mathrm{BST_{NF}}$ to $\mathrm{BST_{92}}$ to $\mathrm{BST_{NF}}$ Before we can tackle the main Theorem 7, we need to establish an additional fact.

FACT 20. Let $\langle W_1, <_1 \rangle$ be a $\mathrm{BST_{NF}}$ structure and $\langle W_2, <_2 \rangle =_{\mathrm{df}} \Lambda(\langle W_1, <_1 \rangle)$ the corresponding $\mathrm{BST_{92}}$ structure. Then for any $h_1, h_2 \in \mathrm{Hist}(W_1)$, we have $h_1 \perp^1_{\ddot{e}} h_2$ iff $\Lambda(h_1) \perp^2_{\ddot{e}} \Lambda(h_2)$, where $\perp^i_{\ddot{e}}$ is the relation of splitting for histories in W_i.

PROOF. "$\Rightarrow$": Let $h_1, h_2 \in \mathrm{Hist}(W_1)$, and let $h_1 \perp^1_{\ddot{e}} h_2$. Then there are $e_1, e_2 \in \ddot{e}$ s.t. $e_1 \neq e_2$ and $h_i \cap \ddot{e} = e_i$ $(i = 1, 2)$. Then $\Lambda(e_1) = \Lambda(e_2) = \ddot{e}$, so $\ddot{e} \in \Lambda(h_1) \cap \Lambda(h_2)$. Moreover, $\ddot{e}$ is maximal in $\Lambda(h_1) \cap \Lambda(h_2)$, which establishes $\Lambda(h_1) \perp^2_{\ddot{e}} \Lambda(h_2)$. To prove this, assume for reductio that there is some $\ddot{e}' >_2 \ddot{e}$ in the intersection of $\Lambda(h_1)$ and $\Lambda(h_2)$. This means that there are some $e'_1, e'_2 \in \ddot{e}'$ with $e'_1 \in h_1$, $e'_2 \in h_2$, and $\Lambda(e'_1) = \Lambda(e'_2) = \ddot{e}'$. The ordering $\ddot{e} <_2 \ddot{e}'$ implies that for some $e^* \in \ddot{e}$, $e^* <_2 \ddot{e}_1$, which further implies that $e^* <_1 e'_1$ and $e^* <_1 e'_2$. But then $e^* \in h_1 \cap h_2$, and by Fact 18(2), it must be that $e^* = e_1 = e_2$, which contradicts $h_1 \perp^1_{\ddot{e}} h_2$.

"$\Leftarrow$": Let $\Lambda(h_1) \perp^2_{\ddot{e}} \Lambda(h_2)$, which implies that $\ddot{e} \in \Lambda(h_1) \cap \Lambda(h_2)$. Note that there are e_i s.t. $e_i \in h_i$ and $\Lambda(e_i) = \ddot{e}$ $(i = 1, 2)$. Therefore, $h_i \cap \ddot{e} \neq \emptyset$, so that h_1 and h_2 fulfill the precondition for either $h_1 \equiv^1_{\ddot{e}} h_2$ or $h_1 \perp^1_{\ddot{e}} h_2$ (see Definition 17). For reductio, assume the former, which means that $h_1 \cap \ddot{e} =$

$h_2 \cap \ddot{e}$, i.e., $e_1 = e_2$. As there are no maxima in the intersection of histories in BST_{NF} (Fact 12), there is some $e^* \in h_1 \cap h_2$ for which $e_1 <_1 e^*$. Now for $\ddot{e}^* =_{df} \Lambda(e^*)$ we have $\ddot{e}^* \in \Lambda(h_1) \cap \Lambda(h_2)$, and $\ddot{e} <_2 \ddot{e}^*$. This, however, contradicts the maximality of $\ddot{e}$ implied by $\Lambda(h_1) \perp_{\ddot{e}}^2 \Lambda(h_2)$. So in fact, we have $h_1 \perp_{\ddot{e}}^1 h_2$.

Note that by contraposing the above Fact (and making a simple observation) we have that for any $h_1, h_2 \in \text{Hist}(W_1)$, $h_1 \equiv_{\ddot{e}}^1 h_2$ iff $\Lambda(h_1) \equiv_{\ddot{e}}^2 \Lambda(h_2)$.

∎

THEOREM 7. *The function $\Upsilon \circ \Lambda$ is an order isomorphism of BST_{NF} structures: Let $\langle W_1, <_1 \rangle$ be a BST_{NF} structure, let $\langle W_2, <_2 \rangle =_{df} \Lambda(\langle W_1, <_1 \rangle)$, and let $\langle W_3, <_3 \rangle =_{df} \Upsilon(\langle W_2, <_2 \rangle)$. Then there is an order isomorphism φ between $\langle W_1, <_1 \rangle$ and $\langle W_3, <_3 \rangle$, i.e., a bijection between W_1 and W_3 that preserves the ordering.*

PROOF. We claim that we can use the mapping φ, defined for $e \in W_1$ to be

$$\varphi(e) =_{df} \ddot{e} \rightarrowtail \Pi_{\ddot{e}} \langle \Lambda(h) \rangle \text{ for arbitrary } h \in H_e \subseteq \text{Hist}_1.$$

First we show that $\varphi(e)$ is well-defined. Thus, let $h, h' \in H_e$; we need to show that $\Pi_{\ddot{e}} \langle \Lambda(h) \rangle = \Pi_{\ddot{e}} \langle \Lambda(h') \rangle$. By Lemma 6 (1), $\Lambda(h), \Lambda(h') \in \text{Hist}(W_2)$. Also, by Fact 12, $h \equiv_{\ddot{e}} h'$, and so by Fact 20, $\Pi_{\ddot{e}} \langle \Lambda(h) \rangle = \Pi_{\ddot{e}} \langle \Lambda(h') \rangle$.

We now have to show (1) that φ is indeed a mapping from W_1 to W_3, (2) that φ is injective, (3) that φ is surjective, and (4) that φ preserves the ordering.

(1) Mapping: We have to show that for any $e \in W_1$, $\varphi(e) = \ddot{e} \rightarrowtail \Pi_{\ddot{e}} \langle \Lambda(h) \rangle \in W_3$, where $h \in H_e \subseteq \text{Hist}_1$. The set W_3 is defined via W_2, and the set $W_2 = \Lambda[W_1]$, which means that for every $e \in W_1$, $\ddot{e} \in W_2$. To $\ddot{e}$ is then assigned $\Pi_{\ddot{e}} \langle \Lambda(h) \rangle$, and we need to see if it is an elementary outcome of $\ddot{e}$, i.e., an element of $\Pi_{\ddot{e}}$. Since by Lemma 6 $\Lambda(h)$ is a history in $\langle W_2, <_2 \rangle$ for any $h \in \text{Hist}_1$ and for any $h \in H_e$, $\ddot{e} = \Lambda(e) \in \Lambda(h)$, we get that $\Pi_{\ddot{e}} \langle \Lambda(h) \rangle$ is an elementary outcome of $\ddot{e}$, so indeed $\ddot{e} \rightarrowtail \Pi_{\ddot{e}} \langle \Lambda(h) \rangle \in W_3$.

(2) Injectivity: Let $e, e' \in W_1$ and $e \neq e'$. If $\ddot{e} \neq \ddot{e}'$, then obviously $\varphi(e) \neq \varphi(e')$, as these two transitions then have different initials. If $\ddot{e} = \ddot{e}'$ but $e \neq e'$, then e and e' are incompatible elements of the choice set $\ddot{e}$, and moreover, for any $h, h' \in \text{Hist}_1$, if $e \in h, e' \in h'$, then $h \perp_{\ddot{e}} h'$, and hence by Fact 20, $\Lambda(h) \perp_{\ddot{e}} \Lambda(h')$. Accordingly, $\Pi_{\ddot{e}} \langle \Lambda(h) \rangle \neq \Pi_{\ddot{e}} \langle \Lambda(h') \rangle$, and hence $\varphi(e) \neq \varphi(e')$.

(3) Surjectivity: Let $a \in W_3$. We have to find some $e \in W_1$ for which $\varphi(e) = a$. As $a \in W_3$, we have $a = \ddot{e}' \rightarrowtail H$, where $\ddot{e}' \in W_2$ and $H \in \Pi_{\ddot{e}'}$. Since $\langle W_2, <_2 \rangle$ is the result of Λ-transform applied to $\langle W_1, <_1 \rangle$, there is

(possibly more than one) $e^* \in W_1$ for which $\Lambda(e^*) = \ddot{e}'$. We need to find which of these is the sought-for e. Clearly, there is some $h^* \in \text{Hist}(W_2)$ for which $H = \Pi_{\ddot{e}'}\langle h^* \rangle$. By Lemma 6(2), there is a unique $h \in \text{Hist}(W_1)$ s.t. $h^* = \Lambda(h)$, and hence $H = \Pi_{\ddot{e}'}\langle \Lambda(h) \rangle$. For the sought-for e we thus take the unique $e \in \ddot{e}' \cap h$; clearly $\ddot{e}' = \ddot{e}$. It follows that $\varphi(e) = \ddot{e} \rightarrowtail H$, where $H = \Pi_{\ddot{e}}\langle \Lambda(h) \rangle$.

(4) Order preservation: For $e_1, e_2 \in W_1$, $e_1 <_1 e_2$ iff $\varphi(e_1) <_3 \varphi(e_2)$. (The claim about equality already follows from the fact that φ is a bijection.)

"$\Rightarrow$": Let $e_1, e_2 \in W_1$ with $e_1 <_1 e_2$. We show that $\varphi(e_1) <_3 \varphi(e_2)$. Since for $\varphi(e_1)$ we may pick an arbitrary member of H_{e_1}, we pick $h_2 \in H_{e_2} \subseteq H_{e_1}$, so that $e_1, e_2 \in h_2$. We get, as required, $\ddot{e}_1 <_2 \ddot{e}_2$ and hence, as the elementary outcomes $\varphi(e_i)$ are defined by the same history $\Lambda(h_2)$, we get $H_{\ddot{e}_2} \subseteq \Pi_{\ddot{e}_1}\langle \Lambda(h_2) \rangle$. Hence, $\ddot{e}_1 \rightarrowtail \Pi_{\ddot{e}_1}\langle \Lambda(h_2) \rangle <_3 \ddot{e}_2 \rightarrowtail \Pi_{\ddot{e}_2}\langle \Lambda(h_2) \rangle$.

"$\Leftarrow$": Let $\varphi(e_1) <_3 \varphi(e_2)$, i.e., $(\ddot{e}_1 \rightarrowtail \Pi_{\ddot{e}_1}\langle \Lambda(h_1) \rangle <_3 \ddot{e}_2 \rightarrowtail \Pi_{\ddot{e}_2}\langle \Lambda(h_2) \rangle)$, for $h_i \in H_{e_i}, e_i \in \ddot{e}_i$. Hence for some $e_1' \in \ddot{e}_1$: (i) $e_1' <_2 \ddot{e}_2$, and hence $H_{\ddot{e}_2} \subseteq H_{e_1'}$, so $h_2 \in H_{e_1'}$ (because $H_{e_2} \subseteq H_{\ddot{e}_2}$). Since $h_2 \in H_{e_2} \subseteq H_{e_1}$, and it is impossible that $\{e_1, e_1'\} \subseteq h_2$ (by Fact 9(1)), it must be that $e_1 = e_1'$ and hence $e_1 < e_2$ (by (i)). ∎

6. Conclusion

In this paper we developed a branching space-times theory that constitutes an alternative to the well-studied theory of Belnap [1], BST_{92}. We describe both BST_{92} and our "new foundations" theory, BST_{NF}, as alternative versions of common BST structures (Definition 2). The difference lies in the way in which histories branch locally, as prescribed by the different prior choice principles, PCP_{92} (Definition 6) vs. PCP_{NF} (Definition 18). On the one hand, the difference between BST_{92} and BST_{NF} is substantial, as shown by their different topological properties: in BST_{NF}, locally Euclidean individual histories give rise to a locally Euclidean branching structure, i.e., to a generalized manifold, whereas in BST_{92}, local Euclidicity does not carry over from individual histories to the whole branching structure. On the other hand, we can prove that both frameworks are formally intertranslatable, so that BST_{92} and BST_{NF} can be viewed as different possible representations of a single underlying notion of local indeterminism. In this way, the development of BST_{NF} strengthens the position that branching space-times provides an adequate, formally precise analysis of local indeterminism.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] BELNAP, N., Branching space-time. *Synthese* 92(3):385–434, 1992. See also the postprint, Belnap (2003).

[2] BELNAP, N., Concrete transitions. In G. Meggle, (ed.), *Actions, Norms, Values: Discussions with Georg Henrik von Wright*, Berlin: de Gruyter, 1999, pp. 227–236.

[3] BELNAP, N., Branching space-time. *Pittsburgh PhilSci-Archive.* 2003. http://philsci-archive.pitt.edu/1003/.

[4] BELNAP, N., A theory of causation: *Causae causantes* (originating causes) as inus conditions in branching space-times. *British Journal for the Philosophy of Science* 56:221–253, 2005.

[5] BELNAP, N., M. PERLOFF, and M. XU, *Facing the future: Agents and choices in our indeterminist world.* Oxford University Press, Oxford. 2001.

[6] EARMAN, J., Pruning some branches from branching space-times. In D. Dieks, (ed.), *The Ontology of Spacetime II*, Amsterdam: Elsevier, 2008, pp. 187–206.

[7] JACOBSEN, D., P. ØHRSTRØM, and P. HASLE, In celebration of *Past, Present and Future.* In P. Hasle, P. Blackburn, and P. Øhrstrøm, (eds.), *Logic and Philosophy of Time: Themes from Prior*, Aalborg University Press, Aalborg, 2017, pp. 9–27.

[8] LEWIS, D. K., *On the plurality of worlds.* Blackwell, Oxford. 1986.

[9] LUC, J., Generalised manifolds as basic objects of General Relativity. *Foundations of Physics.* Forthcoming; 2019. https://doi.org/10.1007/s10701-019-00292-w.

[10] LUC, J., and T. PLACEK, Interpreting non-Hausdorff (generalized) manifolds in General Relativity. *Philosophy of Science* 87(1):21–42, 2020.

[11] MCCABE, G., The topology of branching universes. *Foundations of Physics Letters* 18(7):665–676, 2005.

[12] McCall, S., Choice trees. In J. Dunn, and A. Gupta, (eds.), *Truth or consequences. Essays in honor of Nuel Belnap*, Kluwer, Dordrecht, 1990, pp. 231–244.

[13] Müller, T., Probability theory and causation: a branching space-times analysis. *British Journal for the Philosophy of Science* 56(3):487–520, 2005.

[14] Müller, T., Towards a theory of limited indeterminism in branching space-times. *Journal of Philosophical Logic* 39:395–423, 2010.

[15] Müller, T., A generalized manifold topology for branching space-times. *Philosophy of Science* 80:1089–1100, 2013.

[16] Placek, T., Quantum state holism: a case for holistic causation. *Studies in History and Philosophy of Modern Physics* 35(4):671–692, 2004.

[17] Placek, T., Bell-type correlations in branching space-times. In T. Czarnecki, K. Kijania-Placek, O. Poller, and J. Woleński, (eds.), *The Analytic Way. Proceedings of the European Congres on Analytic Philosophy, Kraków, August 2008*. London: College Publications. 2010.

[18] Placek, T., N. Belnap, and K. Kishida, On topological issues of indeterminism. *Erkenntnis* 79:403–436, 2014.

[19] Rumberg, A., Transition semantics for branching time. *Journal of Logic, Language and Information* 25:77–108, 2016.

[20] Rumberg, A., *Transitions: toward a Semantics for Real Possibility*. PhD thesis, University of Utrecht. 2016.

[21] Rumberg, A., and A. Zanardo, First-order definability of transition structures. *Journal of Logic, Language and Information* 28:459–488, 2019.

[22] Strobach, N., *Alternativen in der Raumzeit. Eine Studie zur philosophischen Anwendung multidimensionaler Aussagenlogiken*, volume 16 of *Logische Philosophie*. Logos, Berlin. 2007.

[23] von Wright, G. H., *Norm and Action. A Logical Inquiry*. London: Routledge. 1963.

[24] Wald, R. M., *General Relativity*. Chicago University Press, Chicago, IL. 1984.

[25] Weiner, M., and N. Belnap, How causal probabilities might fit into our objectively indeterministic world. *Synthese* 149:1–36, 2006.

N. Belnap
Department of Philosophy
University of Pittsburgh
401 Shady Avenue, Apt. A-806
Pittsburgh PA15206
USA

belnap@pitt.edu

T. Müller
Department of Philosophy
University of Konstanz
P.O. Box 1778457 Konstanz
Germany
`Thomas.Mueller@uni-konstanz.de`

T. Placek
Department of Philosophy
Jagiellonian University Kraków
Grodzka 52
31-044 Kraków
Poland
`uzplacek@cyf-kr.edu.pl`