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Reusing Topological Nexttime Logic

Abstract. In this paper, a particular extension of the constitutive bi-modal logic for single-agent subset spaces will be provided. That system, which originally was designed for revealing the intrinsic relationship between knowledge and topology, has been developed in several directions in recent years, not least towards a comprehensive knowledge-theoretic formalism. This line is followed here to the extent that subset spaces are supplied with a finite number of functions which shall represent certain knowledge-enabling actions. Due to the corresponding functional modalities, another basic system for subset spaces, topological nexttime logic, comes into play. The resulting merge of logics can, for example, be applied to comparing the different effects of those actions in respect of knowledge. Subsequently, the completeness and the decidabilty of the basic combined system and of a certain extension thereof will be proved.

Keywords: Reasoning about knowledge, Subset space semantics, Epistemic actions, Topological nexttime logic.

1. Introduction and Overview

The starting point for this paper is *reasoning about knowledge.* This important foundational issue has been given a solid logical basis right from the beginning of the research into theoretical aspects of artificial intelligence, as can be seen, e.g., from the classic textbooks [6, 18]. According to this, a binary *accessibility relation* R_A connecting *possible worlds* or *conceivable states of the world,* is associated with any agent A. The *knowledge of* A is then defined through the set of all *correspondingly valid formulas,* where this kind of validity is understood with regard to every state the agent considers possible at the actual one. This widespread and well-established view of knowledge has been complemented by Moss and Parikh's bi-modal *logic of subset spaces,* LSS, of which the basic idea is reported straightaway; cf. [5, 19], or [1, Chap. 6].

The *knowledge state* of the agent in question, i.e., the set of all those states that cannot be distinguished by what the agent topically knows,

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can be viewed as a certain *neighborhood* U of the actual state x of the world. Formulas are now interpreted with respect to the resulting pairs x, U called *neighborhood situations*. Thus, both the set of all states and the set of all knowledge states constitute the relevant semantic domains as particular subset structures. The two modalities involved, K and $\Box$, quantify over all elements of U and 'downward' over all neighborhoods contained in U, respectively. This means that K captures the notion of knowledge as usual (see [6] again), and $\Box$ reflects a kind of *effort to acquire knowledge,* since gaining knowledge goes hand in hand with a shrinkage of the knowledge state. In fact, knowledge acquisition is reminiscent of a *topological procedure* this way (inasmuch as, ideally, an effective descent within a system of sets is caused). Thus, it was natural to ask for the appropriate logic of 'real' topological spaces, which was determined by Georgatos shortly afterwards [7]. The subsequent research into subset and topological spaces, respectively, is quoted in the handbook [1], whereas more recent developments include, among others, the papers [2,21,22].[1]

As of now, we shall pay attention to the knowledge-theoretic side of LSS exclusively by proceeding to our object of modeling here, *actions*. To remain in accordance with the general setting for subset spaces, the actions the agent is able to perform shall be of such a nature that no decrease of knowledge can result. They will semantically be captured by certain functions, called *knowledge-enabling functions,* which are assumed to operate on knowledge states. This is the place where a multi-operator version of *topological nexttime logic* (TNL) enters the field, with shifting the temporal context from [9] to a dynamic-epistemical one.

Taking functions for realizing the agent's epistemic actions is quite in conformity with the traditional practice; cf. [6, Section 5.1]. This view is retained in a rather abstract sense in this paper, contrasting the very recent and more advanced approaches from [3,20] focussing on *public announcements* as the distinct type of action.

Technically speaking, we add a finite number n of functional modalities to LSS and fix both the well-known and the new interrelations between the resulting $n + 2$ operators. Then, we prove the soundness and completeness as well as the decidability of the arising logic with respect to the intended semantics. The standard proofs for the individual systems involved are completely different in each case so that it is not immediately clear how

[1] Moreover, some additional research into the intrinsic bi-topological nature of LSS has been done lately; see [13].

to proceed here. However, it turns out that the methods for LSS can suitably be extended, with some new technical peculiarities appearing.

The rest of the paper is organized as follows. In the next section, the language and the logic of subset spaces for single agents is recapitulated. In Section 3, the idea of *enabling knowledge through actions* is treated formally, thereby revisiting the system TNL as needed. In Section 4, the completeness of the ensuing logic is proved. In Section 5, the corresponding decidability problem is dealt with. In Section 6, one of the possible applications of the new system is discussed in quite some detail. Moreover, we give an everyday example and reason about the relationship between the general effort operator $\Box$ and the new action modalities there. Finally, two brief comments on related issues are given.

All relevant facts from modal logic not explicitly introduced in this paper can be found in the standard textbook [4]. Acquaintance with the proofs in Section 2 of the paper [5] is of advantage for Sections 4 and 5 below.

2. The Language and the Logic of Subset Spaces Revisited

The purpose of this section is twofold: to clarify the starting point of our investigation on a technical level and to prepare some concepts and results to be introduced and proved, respectively, later on.

To begin with, we define the syntax of $\mathcal{L}$. Let $\mathsf{Prop} = \{p, q, \ldots\}$ be a denumerably infinite set of symbols called *proposition variables* (which shall represent the basic facts about the states of the world). Then, the set SF of all *subset formulas* over Prop is defined by the rule

$$\alpha \ ::= \ \top \mid p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \mathsf{K}\alpha \mid \Box\alpha.$$

The missing boolean connectives are treated as abbreviations, as needed. The operators which are dual to K and $\Box$ are abbreviated by L and $\Diamond$, respectively. In view of our remarks in the previous section, K is called the *knowledge operator* and $\Box$ the *effort operator;* moreover, $\mathsf{K}\alpha$ and $\Box\alpha$ are formalizations of the colloquial statements 'the agent knows α' and, respectively, 'systemic effort yields α'. The latter phrase needs some explanation. We think of effort as an abstract procedure, with the device performing it being unspecified in the first instance; therefore, the attribute 'systemic' is used here. Furthermore, the formulation has been deliberately kept short and catchy, but shall clearly imply that α is valid after the application of such kind of effort.

Second, we fix the semantics of $\mathcal{L}$. For a start, we single out the relevant domains. We let $\mathcal{P}(X)$ designate the powerset of a given set X.

DEFINITION 2.1. (*Semantic Domains*)

1. Let $X \neq \emptyset$ be a set (of *states*) and $\mathcal{O} \subseteq \mathcal{P}(X)$ a set of subsets of X. Then, the pair $\mathcal{S} = (X, \mathcal{O})$ is called a *subset frame*.
2. Let $\mathcal{S} = (X, \mathcal{O})$ be a subset frame. Then, the set
$$\mathcal{N}_{\mathcal{S}} := \{(x, U) \mid x \in U \text{ and } U \in \mathcal{O}\}$$
is called the *set of neighborhood situations* of $\mathcal{S}$.
3. Let $\mathcal{S} = (X, \mathcal{O})$ be a subset frame. Under an $\mathcal{S}$-*valuation,* we understand a mapping $V : \mathsf{Prop} \to \mathcal{P}(X)$.
4. Let $\mathcal{S} = (X, \mathcal{O})$ be a subset frame and V an $\mathcal{S}$-valuation. Then, $\mathcal{M} := (X, \mathcal{O}, V)$ is called a *subset space (based on* $\mathcal{S}$).

Note that neighborhood situations denominate the semantic atoms of the bi-modal language $\mathcal{L}$. The first component of such a situation indicates the actual state of the world, while the second reflects the uncertainty of the agent in question about it. Furthermore, Definition 2.1.3 shows that values of proposition variables depend on states only. This is in accordance with the common practice in epistemic logic; see [6] once more. Finally, the term 'subset space' (instead of 'subset model') is borrowed from the foundational papers [5,19].

For a given subset space $\mathcal{M}$, we now define the relation of *satisfaction,* $\models_{\mathcal{M}}$, between neighborhood situations of the underlying frame and formulas from SF. Based on that, we define the notion of *validity* of formulas *in subset spaces*. In the following, neighborhood situations are often written without parentheses.

DEFINITION 2.2. (*Satisfaction and Validity*) Let $\mathcal{S} = (X, \mathcal{O})$ be any subset frame.

1. Let $\mathcal{M} = (X, \mathcal{O}, V)$ be a subset space based on $\mathcal{S}$, and let $x, U \in \mathcal{N}_{\mathcal{S}}$ be a neighborhood situation of $\mathcal{S}$. Then
$$\begin{array}{lll} x, U \models_{\mathcal{M}} \top & & \text{is always true} \\ x, U \models_{\mathcal{M}} p & :\Longleftrightarrow & x \in V(p) \\ x, U \models_{\mathcal{M}} \neg\alpha & :\Longleftrightarrow & x, U \not\models_{\mathcal{M}} \alpha \\ x, U \models_{\mathcal{M}} \alpha \wedge \beta & :\Longleftrightarrow & x, U \models_{\mathcal{M}} \alpha \text{ and } x, U \models_{\mathcal{M}} \beta \\ x, U \models_{\mathcal{M}} \mathsf{K}\alpha & :\Longleftrightarrow & \forall\, y \in U : y, U \models_{\mathcal{M}} \alpha \\ x, U \models_{\mathcal{M}} \Box\alpha & :\Longleftrightarrow & \forall\, U' \in \mathcal{O} : [x \in U' \subseteq U \Rightarrow x, U' \models_{\mathcal{M}} \alpha], \end{array}$$
where $p \in \mathsf{Prop}$ and $\alpha, \beta \in \mathsf{SF}$. In case $x, U \models_{\mathcal{M}} \alpha$ is true we say that α *holds in* $\mathcal{M}$ *at* the neighborhood situation x, U.

2. Let $\mathcal{M} = (X, \mathcal{O}, V)$ be a subset space based on $\mathcal{S}$. A subset formula α is called *valid in* $\mathcal{M}$ iff it holds in $\mathcal{M}$ at every neighborhood situation of $\mathcal{S}$.

Note that the idea of both knowledge and effort, as described in the introduction, is made precise by the first item of this definition. In particular, knowledge is defined as 'validity at all states the agent considers possible' in the context of subset spaces, too. In addition, effort is given by 'validity at all knowledge states obtained through shrinking the actual one'.

Subset frames and subset spaces can be considered from a different perspective, as is known since [5] and reviewed in the following, for the reader's convenience. Let a subset frame $\mathcal{S} = (X, \mathcal{O})$ and a subset space $\mathcal{M} = (X, \mathcal{O}, V)$ based on it be given. Take $X_{\mathcal{S}} := \mathcal{N}_{\mathcal{S}}$ as a set of worlds, and define two accessibility relations $R^{\mathsf{K}}_{\mathcal{S}}$ and $R^{\Box}_{\mathcal{S}}$ on $X_{\mathcal{S}}$ by

$$(x, U)\, R^{\mathsf{K}}_{\mathcal{S}}\, (x', U') :\Longleftrightarrow U = U' \text{ and}$$
$$(x, U)\, R^{\Box}_{\mathcal{S}}\, (x', U') :\Longleftrightarrow (x = x' \text{ and } U' \subseteq U),$$

for all $(x, U), (x', U') \in X_{\mathcal{S}}$. Furthermore, let a valuation be defined by $V_{\mathcal{M}}(p) := \{(x, U) \in X_{\mathcal{S}} \mid x \in V(p)\}$, for all $p \in \mathsf{Prop}$. Then, bi-modal Kripke structures $S_{\mathcal{S}} := \big(X_{\mathcal{S}}, \{R^{\mathsf{K}}_{\mathcal{S}}, R^{\Box}_{\mathcal{S}}\}\big)$ and $M_{\mathcal{M}} := \big(X_{\mathcal{S}}, \{R^{\mathsf{K}}_{\mathcal{S}}, R^{\Box}_{\mathcal{S}}\}, V_{\mathcal{M}}\big)$ result in such a way that $M_{\mathcal{M}}$ is equivalent to $\mathcal{M}$ in the following sense.

PROPOSITION 2.3. *For all* $\alpha \in \mathsf{SF}$ *and* $(x, U) \in X_{\mathcal{S}}$*, we have that* $x, U \models_{\mathcal{M}} \alpha$ *iff* $M_{\mathcal{M}}, (x, U) \models \alpha$.

Here (and later on as well), the non-indexed symbol '$\models$' denotes the usual satisfaction relation of modal logic.

The proposition can easily be proved by structural induction on α. We call $S_{\mathcal{S}}$ and $M_{\mathcal{M}}$ the Kripke structures *induced* by the subset structures $\mathcal{S}$ and $\mathcal{M}$, respectively.

We now turn to the *logic* of subset spaces, LSS. The subsequent axiomatization from [19] was proved to be sound and complete in detail in [5, Sections 1.2 and 2.2], respectively, with the proof rules involved being the standard ones, i.e., modus ponens and necessitation with respect to each modality.

1. All instances of propositional tautologies
2. $\mathsf{K}(\alpha \to \beta) \to (\mathsf{K}\alpha \to \mathsf{K}\beta)$
3. $\mathsf{K}\alpha \to (\alpha \wedge \mathsf{KK}\alpha)$
4. $\mathsf{L}\alpha \to \mathsf{KL}\alpha$
5. $(p \to \Box p) \wedge (\Diamond p \to p)$
6. $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$

7. $\Box\alpha \rightarrow (\alpha \wedge \Box\Box\alpha)$
8. $\mathsf{K}\Box\alpha \rightarrow \Box\mathsf{K}\alpha$,

where $p \in \mathsf{Prop}$ and $\alpha, \beta \in \mathsf{SF}$.

The last schema is by far the most interesting one, as it displays the interrelation between knowledge and effort. The members of this schema have been called the *Cross Axioms* since [19]. Note that only proposition variables are involved in the fifth schema. Thus, the logic LSS is not closed under substitution. On the other hand, this schema is in accordance with the remark on Definition 2.1.3 above, which means that the system mirrors the fact that the knowledge of the agent leaves unchanged the state of the world.

As the next step, let us take a brief look at the effect of the axioms from the above list within the framework of common modal logic. To this end, we consider bi-modal Kripke models $M = (W, R, R', V)$ satisfying the following four properties:

- the accessibility relation R of M belonging to the knowledge operator K is an equivalence relation,
- the accessibility relation R' of M belonging to the effort operator $\Box$ is reflexive and transitive,
- the composite relation $R' \circ R$ is contained in $R \circ R'$ (this is usually called the *cross property*), and
- the valuation V of M is constant along every R'-path, for all proposition variables.

Such a model M is called a *cross axiom model* (and the frame underlying M a *cross axiom frame*). Now, it can be verified without difficulty that LSS is sound with respect to the class of all cross axiom models. And it is also easy to see that every induced Kripke model is a cross axiom model (and every induced Kripke frame a cross axiom frame). Thus, the completeness of LSS for cross axiom models follows from that for subset spaces (which is Theorem 2.4 in [5]) by means of Proposition 2.3. This inferred (soundness and) completeness result can be used for proving the decidability of LSS; see [5, Section 2.3]. We shall proceed in a similar way below, in Section 5.

3. Subset Spaces with Knowledge-Enabling Functions

The formalism from the previous section will now be extended to the case of $n \geq 1$ actions additionally, as described in the introduction. We again start with the logical language, which comprises n new operators $\mathsf{A}_1, \ldots, \mathsf{A}_n$ as of now. Thus, the set $n\mathsf{SF}$ of all *n-subset formulas* over Prop, is defined by the rule

$$\alpha \ ::= \ \top \mid p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \mathsf{K}\alpha \mid \Box\alpha \mid \mathsf{A}_1\alpha \mid \cdots \mid \mathsf{A}_n\alpha.$$

Note that $\mathsf{SF} \subseteq n\mathsf{SF}$. For $i = 1, \ldots, n$, the modality A_i is called the *knowledge-enabling operator associated with action i,* and $\mathsf{A}_i\alpha$ shall formalize the phrase 'action i yields α'.[2] The syntactic conventions from Section 2 apply correspondingly here. The dual to A_i is denoted by C_i.

Concerning the semantics of the extended language, the modifications implementing the ideas from the introduction follow right now.

Definition 3.1. (*n-Action Subset Structures*)

1. Let $n \in \mathbb{N}$ be as above. Furthermore, let $\mathcal{S} = (X, \mathcal{O})$ be a subset frame. For all 'actions' $i \in \{1, \ldots, n\}$, let $f_i : \mathcal{O} \to \mathcal{O}$ be a partial function satisfying $f_i(U) \subseteq U$ whenever $U \in \mathcal{O}$. (In this case, we say that f_i is *contracting.*) Then, the triple

$$\mathcal{S} = (X, \mathcal{O}, \{f_i\}_{1 \leq i \leq n})$$

   is called an *n-action subset frame* (or an *n-a-subset frame* for short), and, for every $i \in \{1, \ldots, n\}$, the mapping f_i is called the *knowledge-enabling function for action i.*

2. The notions of *neighborhood situation, $\mathcal{S}$-valuation* and *n-action subset space* (*n-a-subset space*) are completely analogous to those introduced in Definition 2.1.

With regard to satisfaction and validity, we need not repeat Definition 2.2 at this place, but may confine ourselves to the clauses for the new operators.

Definition 3.2. (*Satisfaction*) Let $\mathcal{S} = (X, \mathcal{O}, \{f_i\}_{1 \leq i \leq n})$ be an n-a-subset frame, $\mathcal{M}$ an n-a-subset space based on $\mathcal{S}$, and $x, U \in \mathcal{N}_{\mathcal{S}}$ a neighborhood situation of $\mathcal{S}$. Then, for every $i \in \{1, \ldots, n\}$ and $\alpha \in n\mathsf{SF}$, we let

$$x, U \models_{\mathcal{M}} \mathsf{A}_i\alpha :\Longleftrightarrow \text{ if } f_i(U) \text{ exists and } x \in f_i(U), \text{ then } x, f_i(U) \models_{\mathcal{M}} \alpha.$$

It should be mentioned that a particular *multi-agent reading* of n-a-subset spaces was indicated in the previous paper [12], in which the set $\{1, \ldots, n\}$ represented the different agents. At that time, we did not know the paper [21], which makes that reading dispensable to some extent nowadays. In any case, the present interpretation appears to be more natural.

[2] Cf. the explanation concerning the reading of $\Box\alpha$ at the beginning of Section 2.

The final semantic issue to be mentioned is that of induced Kripke structures in the broader context. Letting $\mathcal{S} = (X, \mathcal{O}, \{f_i\}_{1 \leq i \leq n})$ be any n-a-subset frame, the following definition suggests itself.

$$(x, U)\, R_{\mathcal{S}}^{\mathsf{A}_i}\, (x', U') :\Longleftrightarrow\ (x = x' \text{ and } U' = f_i(U)),$$

where $i \in \{1, \dots, n\}$, $x, x' \in X$, and $U, U' \in \mathcal{O}$. With that, the corresponding analogue of Proposition 2.3 is obviously valid.

The *logic of n-action subset spaces,* ALSS_n, is given by the following list of axioms and the standard proof rules. (That is to say, the necessitation rules for each of the A_i's are joined with the proof rules for LSS.)

1. All instances of propositional tautologies
2. $\mathsf{K}(\alpha \to \beta) \to (\mathsf{K}\alpha \to \mathsf{K}\beta)$
3. $\mathsf{K}\alpha \to (\alpha \wedge \mathsf{KK}\alpha)$
4. $\mathsf{L}\alpha \to \mathsf{KL}\alpha$
5. $(p \to \Box p) \wedge (\Diamond p \to p)$
6. $\Box (\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$
7. $\Box\alpha \to (\alpha \wedge \Box\Box\alpha)$
8. $\mathsf{K}\Box\alpha \to \Box\mathsf{K}\alpha$
9. $\mathsf{A}_i(\alpha \to \beta) \to (\mathsf{A}_i\alpha \to \mathsf{A}_i\beta)$
10. $\mathsf{C}_i\alpha \to \mathsf{A}_i\alpha$
11. $\mathsf{KA}_i\alpha \to \mathsf{A}_i\mathsf{K}\alpha$
12. $\mathsf{KA}_i(\alpha \to \mathsf{L}\beta) \vee \mathsf{KA}_i(\beta \to \mathsf{L}\alpha)$
13. $\Box\alpha \to \mathsf{A}_i\alpha$,

where $1 \leq i \leq n$, $p \in \mathsf{Prop}$, and $\alpha, \beta \in n\mathsf{SF}$.

Evidently, the first eight schemata of this list coincide with the LSS-axioms presented in Section 2. Hence we only comment on the others. Axioms 9 – 12 are essentially the relevant TNL-axioms from [9] with subscripts i for every $i \in \{1, \dots, n\}$. Axiom 10 captures the partial functionality of the accessibility relation R associated with A_i in ordinary modal logic (i.e., $\forall s \forall t \forall u (s\, R\, t \text{ and } s\, R\, u \Rightarrow t = u)$); see, e.g., [8, Section 1]. In the present context, it comes along with the fact that we have assigned partial functions to the actions. The schema 11 is formally similar to the eighth one, thus comprising the Cross Axioms for K and A_i. The second-last schema mirrors the fact that the enabling functions, when defined, use knowledge states as input. This schema is related to the one for *weak connectivity* of any accessibility relation R (i.e., $\forall s \forall t \forall u (s\, R\, t$ and $s\, R\, u \Rightarrow (t\, R\, u$ or $t = u$ or $u\, R\, t)))$; see [8, Section 1] again. With regard to the relational semantics, the final axiom says that the accessibility relation for A_i is contained in that for $\Box$. This schema is as well responsible, together with Axiom 10, for the fact that the counterpart of Axiom 5 for A_i is ALSS_n-derivable and hence not necessary.

The logic of n-a-subset spaces might be regarded as not very interesting at first glance, since the simple Axiom 13 represents the only connection between the systems LSS and TNL. However, the outcome of this paper is at least that the system TNL can fully be embedded into the basic system LSS, with interpreting the formerly temporal operators as action modalities quite naturally. This yields a complementary view on the established models of *knowledge and action* in exactly the same way as the original LSS-system has complemented the classical view on *knowledge*. Beyond that, the proofs in Sections 4 and 5 show that the upcoming technical modifications are worth noting. All this is true for the extension of ALSS_n discussed in Section 6 to an even greater extent.

Finally in this section, we prove that the logic ALSS_n is *sound* with respect to the class of all n-action subset spaces.

PROPOSITION 3.3. *Let $\mathcal{M} = (X, \mathcal{O}, \{f_i\}_{1 \leq i \leq n}, V)$ be an n-a-subset space. Then, every axiom from the above list is valid in $\mathcal{M}$; moreover, every ALSS_n-rule preserves the validity of formulas.*

PROOF. We note that most axioms and rules were shown valid, respectively validity preserving, in previous papers as [5] or [9]. We therefore may here confine ourselves to proving that Axiom 11 is valid. Actually, we verify the dual of that schema, i.e., $\mathsf{C}_i \mathsf{L}\alpha \to \mathsf{L}\mathsf{C}_i\alpha$. So suppose that $x, U \models_{\mathcal{M}} \mathsf{C}_i\mathsf{L}\alpha$, for any neighborhood situation x, U of the frame underlying $\mathcal{M}$. Then there exists $V \in \mathcal{O}$ such that $V = f_i(U)$, $x \in V$, and $x, V \models_{\mathcal{M}} \mathsf{L}\alpha$. Furthermore, there is some $y \in V$ such that $y, V \models_{\mathcal{M}} \alpha$. Since f_i is contracting, we know that $y \in U$. This means that $y, U \models_{\mathcal{M}} \mathsf{C}_i\alpha$, whence we finally obtain $x, U \models_{\mathcal{M}} \mathsf{L}\mathsf{C}_i\alpha$. ■

4. Completeness

In this section, we present the special features required for proving the semantic completeness of ALSS_n on the class of all n-action subset spaces. The overall structure of that proof consists of an infinite step-by-step model construction, as it is often the case with subset space logics. Note that one or another proof of such a kind can be found in the literature; see, e.g., [5] for a fully completed proof regarding LSS and [11] for a fairly detailed proof outline suitable for a particular multi-agent variation. Thus, it is really sufficient to confine ourselves to the case-specific issues here.

First, let us fix some notations concerning the canonical model of ALSS_n. Let $\mathcal{C}$ be the set of all maximal ALSS_n-consistent sets of formulas. Furthermore, let $\stackrel{\mathsf{K}}{\longrightarrow}$, $\stackrel{\Box}{\longrightarrow}$, and $\stackrel{\mathsf{A}_i}{\longrightarrow}$ be the accessibility relations induced on $\mathcal{C}$ by the modalities K, $\Box$, and A_i, respectively, where $i \in \{1, \ldots, n\}$. And finally, let $\alpha \in n\mathsf{SF}$ be a formula which is *not* contained in ALSS_n. Then, the formula $\neg\alpha \in n\mathsf{SF}$ is ALSS_n-consistent, hence contained in some maximal consistent set $\Gamma \in \mathcal{C}$. This indicates that the required model for $\neg\alpha$ could be found with the aid of the canonical one.

The desired model is constructed by recursion in such a way that better and better intermediary structures are obtained, which means that more and more 'existential' formulas $\mathsf{L}\beta$, $\Diamond\beta$, and $\mathsf{C}_i\beta$, are realized. In order to ensure that the resulting limit structure behaves as desired, several requirements on those 'approximations' have to be met at every stage. Describing the corresponding details and verifying the necessary properties makes up the technical core of the proof, which will be done by following the procedure in [5] as far as possible; the latter will enable us to utilize some of the achievements there.

We, in more detail, proceed as follows. First, we fix the essential characteristics of the approximating models, called *pre-models* for convenience. To this end, we formulate a first group, 'G 1', of five requirements, since we shall be dealing with quintuples in the case of these pre-models. Then, we specify a further group, 'G 2', of five requirements saying how pre-models extend each other. Additionally, we list a third group, 'G 3', of three requirements describing what the construction must respect with regard to the formulas of type $\mathsf{L}\beta$, $\Diamond\beta$, or $\mathsf{C}_i\beta$. The actual construction and the verification of the requirements, however, is put last. Instead, we first derive the target structure from the given conditions and state two auxiliary results (Propositions 4.1 and 4.2) which are useful for the subsequent *Truth Lemma* 4.3. This lemma will easily yield the desired completeness of ALSS_n (Theorem 4.4). The elaboration of the construction and the verification of the requirements from the three groups are carried out after that.

For a start, we fix the basic domains. The possible worlds we use will successively be taken from a denumerably infinite set of points, Y, chosen in advance. Also, another denumerably infinite set, Q, is chosen such that

$Y \cap Q = \emptyset$. This set shall gradually contribute to a partially ordered set representing the subset space structure of the limit model we strive for. Finally, we fix particular 'starting elements' $x_0, x_0' \in Y$, $\bot \in Q$, and $\Gamma \in \mathcal{C}$ containing the formula $\neg\alpha$ from above. Then, the sequence of pre-models $(X_m, P_m, h_m, \{g_i^m\}_{1 \le i \le n}, t_m)$ will be defined inductively in such a way that, for all $m \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$,

- G1-1. X_m is a finite subset of Y containing $\{x_0, x_0'\}$,
- G1-2. P_m is a finite subset of Q containing $\bot$ and carrying a partial order $\le_m$ with *least element* $\bot$,
- G1-3. $h_m : P_m \to \mathcal{P}(X_m)$ is a function such that, for all $\pi, \rho \in P_m$,
  - (a) $h_m(\pi)$ contains at least two elements,[3] and
  - (b) $\pi \le_m \rho \iff h_m(\pi) \supseteq h_m(\rho)$,
- G1-4. $g_i^m : P_m \to P_m$ is a partial function such that, for all $\pi \in P_m$, if $g_i^m(\pi)$ exists, then $\pi \le_m g_i^m(\pi)$,
- G1-5. $t_m : X_m \times P_m \to \mathcal{C}$ is a partial function such that, for all $x, y \in X_m$ and $\pi, \rho \in P_m$,
  - (a) $t_m(x, \pi)$ is defined iff $x \in h_m(\pi)$; in this case it holds that
    - i. if $y \in h_m(\pi)$, then $t_m(x, \pi) \xrightarrow{\mathsf{K}} t_m(y, \pi)$
    - ii. if $\pi \le_m \rho$ and $x \in h_m(\rho)$, then $t_m(x, \pi) \xrightarrow{\Box} t_m(x, \rho)$
    - iii. if $g_i^m(\pi)$ exists and $x \in h_m(g_i^m(\pi))$, then

$$t_m(x, \pi) \xrightarrow{\mathsf{A}_i} t_m(x, g_i^m(\pi)), \text{ and}$$

  - (b) $t_m(x_0, \bot) = \Gamma$.

Note that approximating partial functions $f_i^m : \mathcal{P}(X_m) \to \mathcal{P}(X_m)$ could easily be derived from the functions g_i^m, for $i = 1, \ldots, n$. However, this is needed for the final structure only.

The next five conditions reveal to what extent the limit model is approximated by the pre-models $(X_m, P_m, h_m, \{g_i^m\}_{1 \le i \le n}, t_m)$. Actually, it will be ensured that, for all $m \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$,

- G2-1. $X_m \subseteq X_{m+1}$,
- G2-2. $(P_{m+1}, \le_{m+1})$ is an *end extension* of $(P_m, \le_m)$, i.e., a superstructure of $(P_m, \le_m)$ such that no element $\pi \in P_{m+1} \setminus P_m$ is smaller than any element of P_m,
- G2-3. $h_{m+1}(\pi) \cap X_m = h_m(\pi)$ for all $\pi \in P_m$,

[3] This is a technical requirement which, however, we do not need to care about in this paper; see [5, p. 88] for the role it plays.

G2-4. $g_i^{m+1} |_{P_m} = g_i^m$,
G2-5. $t_{m+1} |_{X_m \times P_m} = t_m$.

Note that nontrivial changes regarding item G 2-2 will appear in Section 6. Finally, the construction complies with the following overall requirements dealing with existential formulas: for all $m \in \mathbb{N}$, $x \in X_m$, $\pi \in P_m$, $n \in \{1, \ldots, n\}$, and $\beta \in n\mathsf{SF}$,

G3-1. if $\mathsf{L}\beta \in t_m(x, \pi)$, then there are $m < k \in \mathbb{N}$ and $y \in h_k(\pi)$ such that $\beta \in t_k(y, \pi)$,
G3-2. if $\Diamond\beta \in t_m(x, \pi)$, then there are $m < k \in \mathbb{N}$ and $\pi \leq_k \rho \in P_k$ such that $\beta \in t_k(x, \rho)$,
G3-3. if $\mathsf{C}_i\beta \in t_m(x, \pi)$, then there is an $m < k \in \mathbb{N}$ such that
  (a) $g_i^k(\pi)$ exists and
  (b) $\beta \in t_k\left(x, g_i^k(\pi)\right)$.

So far, we have outlined our construction plan and formulated the intermediate goals. Now is the time to state our first interim result. To this end, suppose that all that above has successfully been carried out. Then, we define a quintuple called *limit structure* which is quite close to the desired model already. We let

- $X := \bigcup_{m \in \mathbb{N}} X_m$,
- $P := \bigcup_{m \in \mathbb{N}} P_m$, carrying the partial order $\leq := \bigcup_{m \in \mathbb{N}} \leq_m$,
- $h : P \to \mathcal{P}(X)$ be defined by $h(\pi) := \bigcup_{k > m} h_k(\pi)$, where m is the smallest natural number k such that $\pi \in P_k$, for all $\pi \in P$,
- $g_i : P \to P$ be defined by $g_i(\pi) := g_i^m(\pi)$, where m is any natural number such that $g_i^m(\pi)$ is defined, for all $\pi \in P$, and
- $t : X \times P \to \mathcal{C}$ be defined by $t(x, \pi) := t_m(x, \pi)$, where m is any natural number such that $t_m(x, \pi)$ is defined, for all $x \in X$ and $\pi \in P$.

Note that the definitions of the mappings g_i and t are correct because of item G 2-4 and, respectively, item G 2-5 above.

Most items of the following proposition were proved in the paper [5]; see Prop. 2.6 there.

PROPOSITION 4.1. *Let conditions* G 1-1 – G 1-5, G 2-1 – G 2-5, and G 3-1 – G 3-3 *be satisfied. Then,*

1. *X is a set containing the designated element x_0,*

2. $(P, \leq)$ *is a partially ordered set having* $\bot$ *as the least element,*
3. $h : P \to \mathcal{P}(X)$ *is an order-reversing injective function having only non-empty sets in the image and satisfying* $h(\bot) = X$*,*
4. *for all* $i \in \{1, \ldots, n\}$ *and* $\pi \in P$*, if the value* $g_i(\pi)$ *is defined, then* $\pi \leq g_i(\pi)$*, and*
5. *for all* $x \in X$ *and* $\pi \in P$*, the value* $t(x, \pi)$ *is defined iff* $x \in h(\pi)$*; furthermore, for all* $i \in \{1, \ldots, n\}$*,* $y \in X$*,* $\rho \in P$*, and* $\beta \in n\mathsf{SF}$*, we then obtain*
   (a) *if* $y \in h(\pi)$*, then* $t(x, \pi) \xrightarrow{\mathsf{K}} t(y, \pi)$*,*
   (b) *if* $\mathsf{L}\beta \in t(x, \pi)$*, then there is some* $y \in h(\pi)$ *such that* $\beta \in t(y, \pi)$*,*
   (c) *if* $\pi \leq \rho$ *and* $x \in h(\rho)$*, then* $t(x, \pi) \xrightarrow{\Box} t(x, \rho)$*,*
   (d) *if* $\Diamond\beta \in t(x, \pi)$*, then there is some* $\pi \leq \rho$ *such that* $\beta \in t(x, \rho)$*,*
   (e) *if* $g_i(\pi)$ *is defined and* $x \in h(g_i(\pi))$*, then it is the case that*
   $$t(x, \pi) \xrightarrow{\mathsf{A}_i} t(x, g_i(\pi)),$$
   (f) *if* $\mathsf{C}_i\beta \in t(x, \pi)$*, then* $g_i(\pi)$ *is defined and* $\beta \in t\,(x, g_i(\pi))$*,*
   (g) $t(x_0, \bot) = \Gamma$ *(containing* $\neg\alpha$*; see above).*

Proof. We need to prove the assertions 4, 5 (e), and 5 (f) only. But these are easy consequences of the definitions stated right before the proposition and the previous requirements G 1-4, G 1-5 (a) iii, and G 3-3 (b). ■

We now take X as the carrier set of the desired n-a-subset space, and we let $\mathcal{O} := \mathrm{Im}(h)$, the image set of the mapping h. Furthermore, for every $1 \leq i \leq n$, the knowledge-enabling function f_i is defined as follows:

$$f_i(h(\pi)) := \begin{cases} h(g_i(\pi)) & \text{if } g_i(\pi)\text{exists} \\ \text{undefined} & \text{otherwise,} \end{cases}$$

for all $\pi \in P$. With that, we obtain

Proposition 4.2. *The tuple* $\mathcal{S} := (X, \mathcal{O}, \{f_i\}_{1 \leq i \leq n})$ *is an* n*-a-subset frame.*

Proof. First, we must show that each of the functions f_i is well-defined, where $i \in \{1, \ldots, n\}$. But this is clear from the fact that h is injective, due to Proposition 4.1.3. Second, the items 3 and 4 of Proposition 4.1 imply that f_i is contracting. This proves the proposition. ■

Letting $V(p) := \{x \in X \mid p \in t(x, \bot)\}$, for all $p \in \mathsf{Prop}$, gives us an n-a-subset space $\mathcal{M}$ based on the frame just defined. It turns out that $\mathcal{M}$ satisfies the relevant *Truth Lemma;* cf. [4, (Truth) Lemma 4.21].

Lemma 4.3. *Let $\mathcal{M} := (X, \mathcal{O}, \{f_i\}_{1\leq i\leq n}, V)$, and let h and t be the functions defined right before Proposition 4.1. Then, for all $\beta \in n\mathsf{SF}$ and $x, h(\pi) \in \mathcal{N}_{\mathcal{S}}$,*

$$x, h(\pi) \models_{\mathcal{M}} \beta \iff \beta \in t(x, \pi).$$

Proof. Referring for other cases to [5], we only consider the case $\beta = \mathsf{C}_i\gamma$, for $1 \leq i \leq n$ (which is equivalent to the case $\beta = \mathsf{A}_i\gamma$). First, let $x, h(\pi) \models_{\mathcal{M}} \mathsf{C}_i\gamma$ be satisfied. This means that $f_i(h(\pi))$ is defined, $x \in f_i(h(\pi))$, and $x, f_i(h(\pi)) \models_{\mathcal{M}} \gamma$, according to Definition 3.2. The induction hypothesis now implies that $\gamma \in t(x, g_i(\pi))$, since $f_i(h(\pi)) = h(g_i(\pi))$ by definition. As, in particular, $g_i(\pi)$ as well as $t(x, g_i(\pi))$ are defined, $t(x, \pi) \xrightarrow{\mathsf{A}_i} t(x, g_i(\pi))$ ensues from both the first statement of Proposition 4.1.5 and item 4.1.5 (e). This gives us $\mathsf{C}_i\gamma \in t(x, \pi)$, as desired.

Second, let $\mathsf{C}_i\gamma$ be contained in $t(x, \pi)$. Then, $g_i(\pi)$ is defined and $\gamma \in t\,(x, g_i(\pi))$, owing to Proposition 4.1.5 (f). Thus, $x, h(g_i(\pi))) \models_{\mathcal{M}} \gamma$ because of the induction hypothesis. From the existence of $f_i(h(\pi))$ (which follows from $h(g_i(\pi)) = f_i(h(\pi)))$, and the fact that $x \in f_i(h(\pi))$ (due to Proposition 4.1.5, as above), we finally conclude that $x, h(\pi) \models_{\mathcal{M}} \mathsf{C}_i\gamma$; see Definition 3.2 once more. ∎

From Lemma 4.3, the completeness of ALSS_n with respect to the n-a-subset space semantics follows immediately.

Theorem 4.4. (Completeness I) *Let $\alpha \in n\mathsf{SF}$ be any formula which is valid in all n-a-subset spaces. Then α belongs to the logic ALSS_n.*

To complete the proof of this theorem, it remains to specify, for all $m \in \mathbb{N}$, the pre-models $(X_m, P_m, h_m, \{g_i^m\}_{1\leq i\leq n}, t_m)$ in a way that all the requirements stated above are finally met. As a matter of fact, the tuples we define will contain some additional parameters.

First of all, we let $\mathbb{N}_{>0}$ denote the set of all positive natural numbers, and we fix a mapping $\nu : \mathbb{N}_{>0} \to \{\mathsf{C}_1, \ldots, \mathsf{C}_n, \mathsf{L}, \Diamond\} \times \mathbb{N} \times \mathbb{N}$ satisfying, for all (∇, l, k) in the co-domain of ν,

- there is some $j > l$ such that $\nu(j) = (\nabla, l, k)$, and
- for any $j \in \mathbb{N}_{>0}$, if $\nu(j) = (\nabla, l, k)$, then $l < j$.

Note that such a mapping ν actually exists.

The further ingredients of the tuples to be defined are enumerations ν_1^m, ν_2^m, …, ν_{n+2}^m of, respectively, the sets

$$T_\nabla^m := \{(x, \pi, \beta) \in X_m \times P_m \times n\mathsf{SF} \mid \nabla\beta \in t_m(x, \pi)\},$$

where ∇ runs through the set $\{\mathsf{C}_1, \ldots, \mathsf{C}_n, \mathsf{L}, \Diamond\}$ (in that order).

We now consider the basic case $m = 0$. We let $X_0 := \{x_0, x_0'\}$, $P_0 := \{\bot\}$, $h_0(\bot) := X_0$, g_i^0 be the nowhere defined function, for every $i \in \{1, \ldots, n\}$, and $t(x_0, \bot) = t(x_0', \bot) := \Gamma$. Then the requirements G 1-1–G 1-5 are easily seen to be satisfied. (The other conditions are irrelevant to this case.) Moreover, we fix enumerations ν_1^0, ν_2^0, $\ldots$, ν_{n+2}^0 according to the above description.

The proceeding in step $m+1$ of the construction depends on the value of $\nu(m+1)$. We may confine ourselves to the case $\nu(m+1) = (\mathsf{C}_i, l, k)$, where $i \in \{1, \ldots, n\}$ and $l, k \in \mathbb{N}$, except for a little detail concerning the L-case.[4] So let $\nu_i^l(k) = (x, \pi, \beta)$, whence $\mathsf{C}_i\beta \in t_l(x, \pi)$. (Note that $l \leq m$ so that ν_i^l can actually be accessed.) We now distinguish three cases.

Case 1. First, assume that $g_i^m(\pi)$ is undefined. Then, in principle the same proceeding as in [5, p. 89 f] in the $\Diamond$-case leads to success. This means that we choose both a new point $y \in Y$ and a fresh $\sigma \in Q$, and we let $X_{m+1} := X_m \cup \{y\}$ and $P_{m+1} := P_m \cup \{\sigma\}$. The partial order is extended to P_{m+1} by letting $\pi \leq_{m+1} \sigma$ and σ be not comparable with any other element of P_m that is not contained in a $\leq_m$-path through π. The function h_{m+1} is defined as follows. We let $h_{m+1}(\tau) := h_m(\tau) \cup \{y\}$ for all $\tau \in P_m$ satisfying $\tau \leq_m \pi$, and $h_{m+1}(\sigma) := \{x, y\}$; for all other arguments, h_{m+1} equals h_m by definition. Furthermore, the extension of the function g_i^m is defined by $g_i^{m+1}(\pi) := \sigma$ and $g_i^{m+1}(\tau) := g_i^m(\tau)$ for all $\tau \neq \pi$. Finally, the mapping t_m is adjusted as follows. We conclude from Axiom 10 and the fact that $\mathsf{C}_i\beta \in t_m(x, \pi)$, see condition G 2-5, that there is a *unique* maximal consistent set $\Delta \in \mathcal{C}$ satisfying $t_m(x, \pi) \xrightarrow{\mathsf{A}_i} \Delta$ and $\beta \in \Delta$. Thus, we let $t_{m+1}(x, \sigma) := \Delta$. After extending t_m to t_{m+1} on $X_m \times P_m$ trivially, we at last define $t_{m+1}(y, \rho) := t_{m+1}(x, \rho)$ for all $\rho \in P_{m+1}$ satisfying $\rho \leq_{m+1} \sigma$. This completes the definition of t_{m+1} and thus that of $\left(X_{m+1}, P_{m+1}, h_{m+1}, \{g_i^{m+1}\}_{1 \leq i \leq n}, t_{m+1}\right)$ in the case under consideration.

Case 2. Second, let $g_i^m(\pi)$ be defined and $x \in h(g_i(\pi))$. Then, nothing will be changed, i. e., the structure $(X_m, P_m, h_m, \{g_i^m\}_{1 \leq i \leq n}, t_m)$ is extended to $\left(X_{m+1}, P_{m+1}, h_{m+1}, \{g_i^{m+1}\}_{1 \leq i \leq n}, t_{m+1}\right)$ trivially.

Case 3. Finally, let $g_i^m(\pi)$ be defined and $x \notin h\left(g_i(\pi)\right)$. In this case, we let $X_{m+1} := X_m$, $P_{m+1} := P_m$, and $g_i^{m+1} := g_i^m$. The function h_{m+1} is now given in the following way. We let $h_{m+1}\left(g_i^m(\pi)\right) := h_m\left(g_i^m(\pi)\right) \cup \{x\}$; for all

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[4]According to the enumerations used respectively, it might happen that we cannot solely work with the usual cross property as in [5, p. 89] in this case, but have to apply the one corresponding to Axiom 11 additionally.

other arguments $\rho \in P_{m+1}$, we let $h_{m+1}(\rho) := h_m(\rho)$. The mapping t_{m+1} is determined through $t_{m+1}(x, g_i^m(\pi)) := \Delta$, as in *Case 1*. Thus, the tuple $\left(X_{m+1}, P_{m+1}, h_{m+1}, \{g_i^{m+1}\}_{1\leq i\leq n}, t_{m+1}\right)$ is completely defined here as well.

Additionally, the enumerations ν_1^{m+1}, ν_2^{m+1}, ..., ν_{n+2}^{m+1} are suitably chosen in each of these cases.

We must now check that the properties stated above in the first group G 1 of requirements are satisfied for $m + 1$ and that the validity of those stated in the second group G 2 is transferred from m to $m + 1$. Doing so, several items prove to be evident from the construction just described so that we may confine ourselves to the less obvious ones.

Concerning G 1, we need only care about the conditions G 1-5 (a) i, G 1-5 (a) ii, and G 1-5 (a) iii. In *Case 1* of the construction, G 1-5 (a) i is valid due to the definition of t_{m+1} and the reflexivity of the relation $\xrightarrow{\mathsf{K}}$. For item G 1-5 (a) ii, only the subcase $\rho = \sigma$ must be considered. The validity of $t_{m+1}(x, \pi) \xrightarrow{\Box} t_{m+1}(x, \sigma)$ is obtained from $t_m(x, \pi) \xrightarrow{\mathsf{A}_i} t_{m+1}(x, \sigma)$ with the aid of Axiom 13 and the definition of t_{m+1}. Finally, G 1-5 (a) iii is clear from the construction in this case.

Obviously, nothing has to be proved in *Case 2* above. Thus, we turn to *Case 3*. After what has been said about *Case 1,* only the requirement G 1-5 (a) i is critical here. To verify this condition, we choose any $y \in h(g_i(\pi))$ (which exists since $g_i^m(\pi)$ is defined) and let $\Theta := t_{m+1}(y, g_i^m(\pi))$. We must now show that $\Delta \xrightarrow{\mathsf{K}} \Theta$. We know that $t_{m+1}(x, \pi) \xrightarrow{\mathsf{K}} t_{m+1}(y, \pi)$ and $t_{m+1}(y, \pi) \xrightarrow{\mathsf{A}_i} t_{m+1}(y, g_i^m(\pi)) = \Theta$ from the induction hypothesis and the definition of t_{m+1}. Furthermore, $t_{m+1}(x, \pi) \xrightarrow{\mathsf{A}_i} t_{m+1}(x, g_i^m(\pi)) = \Delta$ is valid due to the construction in *Case 3*. Thus, the desired relation $\Delta \xrightarrow{\mathsf{K}} \Theta$ is forced by Axiom 12.[5] With that, the verification of the properties stated in G 1 is finished.

It is easy to convince oneself that $(P_{m+1}, \leq_{m+1})$ is an end extension of $(P_m, \leq_m)$. This is all what should perhaps be said on the verification of the group G 2 of requirements.

Finally, we must establish the group G 3 of 'global' properties. We may confine ourselves to the C_i-case once more. Suppose that, for some $m \in \mathbb{N}$, $x \in X_m$, and $\pi \in P_m$, we have that $\mathsf{C}_i\beta \in t_m(x, \pi)$. Let k be a natural number such that $\nu_i^m(k) = (x, \pi, \beta)$. Furthermore, let $N \in \mathbb{N}_{>0}$ be such

[5] A detailed proof, using Axiom 12, of the fact that $\xrightarrow{\mathsf{K}}$-relations are preserved in this sense by applying $\xrightarrow{\mathsf{A}_i}$, is contained in [9, Proof of Prop. 3.5].

that $\nu(N) = (\mathsf{C}_i, m, k)$. Then $N > m$, and $\mathsf{C}_i\beta$ is processed in step N of the construction in such a way that the conditions G 3-3 (a) and G 3-3 (b) become true.

All in all, the completeness theorem 4.4 is thus proved.

5. Decidability

The standard method for proving the decidability of a given modal logic is *filtration*. By that method, inspection of the relevant models is restricted to those not exceeding a specified size, in this way making a decision procedure possible. However, just as subset spaces are not compatible with canonical models in a direct manner, they are incompatible with filtration. A detour is therefore required, which takes us back into the relational semantics. In the following, we shall single out a certain class of multi-modal Kripke structures for which ALSS_n is as well sound and complete, and which is closed under filtration in a suitable manner. This will give us the desired decidability result.

That class of models subsumes those *induced* by n-a-subset spaces (see Sections 2 and 3 above), in particular; moreover, the ALSS_n-axioms are mirrored in terms of correponding properties of the accessibility relations. This will be applied in the proof of Proposition 5.2 below, which marks the first step towards the decidability of ALSS_n.

As we are in ordinary modal logic in this section as of now, we shall turn to the respective notation of models. Subsequently, K is supposed to correspond to R, $\Box$ to R', and A_i to S_i, for $i = 1, \ldots, n$.

Definition 5.1. (*n-A-Model*) Given a natural number $n \geq 1$, let $M := (W, R, R', S_1, \ldots, S_n, V)$ be a multi-modal Kripke model (i.e., W is a non-empty set, $R, R', S_1, \ldots, S_n \subseteq W \times W$ are binary relations, and $V : \mathsf{Prop} \to \mathcal{P}(W)$ is a valuation). Then M is called an *n-a-model,* iff the following conditions are satisfied.

1. R is an equivalence relation,
2. R' is reflexive and transitive,
3. S_i is a partially functional relation contained in R', for every $1 \leq i \leq n$,
4. each of the pairs (R, R'), (R, S_1), …, (R, S_n) satisfies the cross property, i.e., $R' \circ R \subseteq R \circ R'$ and, for all $i \in \{1, \ldots, n\}$, $S_i \circ R \subseteq R \circ S_i$,
5. for every $1 \leq i \leq n$, the relation S_i induces a partial function, with domain and range being contained in the set of all R-equivalence classes

(i.e., the property $\forall s \forall t \forall u \forall v (s\, R\, t$ and $s\, S_i\, u$ and $t\, S_i\, v \Rightarrow u\, R\, v)$ is valid), and

6. for all proposition variables, the valuation V of M is constant along every R'-path.

The following proposition ties in with the remark at the end of Section 2.

PROPOSITION 5.2. *The logic* ALSS_n *is sound and complete with respect to the class of all* n*-a-models.*

PROOF. Soundness is proved quite straightforwardly so that we only comment on Axiom 12. Suppose that this schema is not valid in some n-a-model M at some point s. Then there are formulas $\alpha, \beta \in n\mathsf{SF}$ such that $M, s \models \mathsf{LC}_i(\alpha \wedge \mathsf{K}\neg\beta) \wedge \mathsf{LC}_i(\beta \wedge \mathsf{K}\neg\alpha)$. This means that there is an R-equivalence class throughout which, in particular, $\neg\alpha$ is valid, namely the 'S_i-induced successor' of the class of s. However, α is valid at some point of this class as well, due to the property stated in item 5 of the above definition. This is a contradiction. Consequently, Axiom 12 is sound for n-a-models.

Completeness follows from Theorem 4.4 and the fact that every Kripke model induced by an n-a-subset space (as set out in Section 3) is an n-a-model (which can be seen easily, too). ■

Thus, it suffices to establish the *finite model property,* cf. [4, Def. 3.22], for ALSS_n with respect to some recursively enumerable set of n-a-models in order to obtain the desired decidability result; cf. [4, Th. 6.13].[6] For that purpose, take any $\alpha \in n\mathsf{SF}$ such that $\alpha \notin \mathsf{ALSS}_n$. Then, the ALSS_n-consistent formula $\neg\alpha \in n\mathsf{SF}$ is contained in some maximal consistent set, say Γ. We must refute α (respectively, realize $\neg\alpha$) in some *finite* n-a-model. This will be done by *filtering* the canonical model of ALSS_n appropriately. That is to say, a suitable 'filter set' Σ of formulas will be defined, splitting the canonical model into equivalence classes in the following way: two maximal consistent sets of formulas shall belong to the same class, iff they contain the same formulas from Σ.

For any $\beta \in n\mathsf{SF}$, let $\mathrm{sf}(\beta)$ denote the set of all subformulas of β. The set Σ is now defined in steps. We start off with the set $\Sigma_0 := \mathrm{sf}(\neg\alpha) \cup \{\mathsf{A}_i\neg\beta \mid \mathsf{A}_i\beta \in \mathrm{sf}(\neg\alpha)\}$. Then, we let $\Sigma_1 := \Sigma_0 \cup \{\neg\beta \mid \beta \in \Sigma_0\}$. In the next step, we take the closure of Σ_1 under finite conjunctions of pairwise distinct elements of Σ_1. After that, we close under single applications of the operator L. And finally, we join the sets of subformulas of all the elements of the set obtained

[6] Note that the normality of the logic under discussion is required in [4]. However, the kind of non-normality of ALSS_n proves to be irrelevant in this respect.

last.[7] The resulting set of formulas is the desired Σ. Note that Σ is built like the filter set used for LSS in [5], with the A_i-case being different here. Further note that Σ is, in fact, a *finite* set closed under subformulas.

Let W be the result of filtering the set of all maximal ALSS_n-consistent sets, $\mathcal{C}$, through Σ. Moreover, take the *smallest* filtration of the corresponding accessibility relations in each of the $n+2$ cases; see [4, p. 79]. Finally, let V be the valuation on W which, for all proposition variables occurring in Σ, is induced by the filtration of $\mathcal{C}$, and is defined by assigning $\emptyset$ to all other proposition variables.[8] Let $M = (W, R, R', S_1, \ldots, S_n, V)$ be the resulting model. Then, the following considerations are crucial.

Let $i \in \{1, \ldots, n\}$ and $w \in W$ be any point. Since w emerged from a filtration, it could be the case that w has more than one S_i-successor. However, if S_i were a *functional* relation, then, for every formula $\mathsf{A}_i\beta \in \Sigma$ being false (or true) in M at w, we would infer that β is false (respectively, true) in M at *all* S_i-successors of w, according to [8, Lemma 9.9], the so-called *Fun-Lemma*. Thus, choosing any of them (and forgetting the others) would not change the truth value of any formula of that kind at w. It would follow that S_i could be 'thinned out' in a way that a functional relation results and the truth value behaviour of all A_i-prefixed formulas from Σ is preserved, which would enable the transferring of the truth value of $\neg\alpha$ from the canonical model of ALSS_n down to the modified filtration.

A suitable version of the *Fun-Lemma* is valid in the case of partial functionality, too; see [10, Lemma 3.7]. We here prove a result of equal value. For any maximal consistent set $\Delta \in \mathcal{C}$, let $\overline{\Delta} \in W$ denote the class of Δ obtained by filtration through Σ.

LEMMA 5.3. *Let $i \in \{1, \ldots, n\}$, $\mathsf{A}_i\beta \in \Sigma$, and $\Delta \in \mathcal{C}$. Then,*

$$\text{if } M, \overline{\Delta} \models \neg\mathsf{A}_i\beta, \text{ then } M, \overline{\Theta} \models \neg\beta \text{ for all } \overline{\Theta} \in W \text{ such that } \overline{\Delta}\, S_i\, \overline{\Theta}.$$

PROOF. We freely use the *Filtration Theorem* [4, Th. 2.39]. Let $M, \overline{\Delta} \models \neg\mathsf{A}_i\beta$. Then, according to this, $\neg\mathsf{A}_i\beta \in \Delta$ since $\neg\mathsf{A}_i\beta \in \Sigma$. Let $\Theta \in \mathcal{C}$ be any maximal consistent set such that $\overline{\Delta}\, S_i\, \overline{\Theta}$. Due to the choice of the minimal filtration, there are $\Delta' \in \overline{\Delta}$ and $\Theta' \in \overline{\Theta}$ satisfying $\Delta' \xrightarrow{\mathsf{A}_i} \Theta'$. The formula $\neg\mathsf{A}_i\beta$, as an element of Σ, is contained in Δ' as well. It follows that there exists some $\Xi \in \mathcal{C}$ such that $\neg\beta \in \Xi$. We now obtain $\Xi = \Theta'$, thus $\neg\beta \in \Theta'$, from the fact that $\xrightarrow{\mathsf{A}_i}$ is a partially functional relation. As we have $\neg\beta \in \Sigma$,

[7] This final step is necessary because L was introduced as an abbreviation.

[8] Thus, the definition for the proposition variables is different from the usual one; cf. [4, Def. 2.36]. However, this does not affect the validity of the *Filtration Theorem* [4, Th. 2.39].

too, $\neg\beta \in \Theta$ is valid. Consequently, $M, \overline{\Theta} \models \neg\beta$, which is what we wanted to show. ∎

As a next step it is shown that the process of 'thinning out' described above can even be applied to the relation induced by S_i on the set of all R-equivalence classes, transforming the latter relation into a partial function thus as well (where $i \in \{1, \ldots, n\}$). To this end, we prove the following proposition.

Proposition 5.4. *Let $i \in \{1, \ldots, n\}$ and $\Delta, \Theta, \Xi, \Phi \in \mathcal{C}$ be given. Suppose that $\overline{\Delta}\, S_i\, \overline{\Xi}$, $\overline{\Theta}\, R\, \overline{\Delta}$ and $\overline{\Theta}\, S_i\, \overline{\Phi}$. Additionally, let $M, \overline{\Delta} \models \mathsf{A}_i\beta$, for some formula $\mathsf{A}_i\beta \in \Sigma$. Then, there exists $\Psi \in \mathcal{C}$ such that $\overline{\Delta}\, S_i\, \overline{\Psi}$ and $\overline{\Phi}\, R\, \overline{\Psi}$.*

Proof. Due to the choice of the minimal filtration, there are $\Delta', \Delta'' \in \overline{\Delta}$, $\Theta', \Theta'' \in \overline{\Theta}$, $\Xi' \in \overline{\Xi}$, and $\Phi' \in \overline{\Phi}$ such that $\Delta' \xrightarrow{\mathsf{A}_i} \Xi'$, $\Theta' \xrightarrow{\mathsf{K}} \Delta''$ and $\Theta'' \xrightarrow{\mathsf{A}_i} \Phi'$. The special form of the filter set Σ ensures that we may assume that $\Theta' = \Theta''$, as in the basic LSS-case.[9] From $M, \overline{\Delta} \models \mathsf{A}_i\beta$ we conclude that $M, \overline{\Xi} \models \beta$, hence $\beta \in \Xi'$. Consequently, $\mathsf{C}_i\beta \in \Delta'$. It suffices to show that $\mathsf{C}_i\beta \in \Delta''$ because there would then exist some $\Psi \in \mathcal{C}$ satisfying $\Delta'' \xrightarrow{\mathsf{A}_i} \Psi$ and it would follow from Axiom 12 that $\Phi' \xrightarrow{\mathsf{K}} \Psi$. Thus, the proposition would have been proved by going down to the filtration. We distinguish two cases.

Case 1: $\mathsf{A}_i\beta \in \mathrm{sf}(\neg\alpha)$. Then we obtain $\mathsf{C}_i\beta \in \Sigma$, due to the definition of the filter set Σ in case of an A_i-prefixed subformula of $\neg\alpha$. Of course, $\mathsf{C}_i\beta \in \Delta''$ is valid in this case.

Case 2: $\mathsf{A}_i\beta \notin \mathrm{sf}(\neg\alpha)$. Then, again by the definiton of Σ, there is some $\gamma \in \mathrm{sf}(\neg\alpha)$ such that $\beta = \neg\gamma$ and $\mathsf{A}_i\gamma \in \mathrm{sf}(\neg\alpha)$. Now suppose towards a contradiction that $\mathsf{C}_i\beta \notin \Delta''$. From this we conclude that $\neg\mathsf{C}_i\beta \in \Delta''$, thus $\mathsf{A}_i\gamma \in \Delta''$ is valid, too. It follows that $M, \overline{\Delta} \models \mathsf{A}_i\gamma$, which is why $M, \overline{\Xi} \models \gamma$. But the latter contradicts $M, \overline{\Xi} \models \beta$, which ensues from $M, \overline{\Delta} \models \mathsf{A}_i\beta$. Therefore, $\mathsf{C}_i\beta \in \Delta''$, as desired. In this way, the proof of the proposition is completed. ∎

The above proof shows in addition that the existence of $\Psi \in \mathcal{C}$ is independent of the concrete formula $\mathsf{A}_i\beta \in \Sigma$ satisfying $M, \overline{\Delta} \models \mathsf{A}_i\beta$. Instead, it only depends on the fact that such a formula actually exists.

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[9]In fact, it is forced by the L-closure property of Σ that not only the clause defining the filtration, $(\exists\Theta' \in \overline{\Theta})(\exists\Delta'' \in \overline{\Delta})(\Theta' \xrightarrow{\mathsf{K}} \Delta'')$, but even $(\forall\Theta' \in \overline{\Theta})(\exists\Delta'' \in \overline{\Delta})(\Theta' \xrightarrow{\mathsf{K}} \Delta'')$ is valid here.

The subsequent corollary represents the intended 'strengthening' of Lemma 5.3.

COROLLARY 5.5. *Let* $\Delta, \Theta, \Xi, \Phi \in \mathcal{C}$ *satisfy the same presuppositions as for the previous proposition. Furthermore, let* $i \in \{1, \ldots, n\}$ *and* $\mathsf{A}_i\beta \in \Sigma$*. Then* $M, \overline{\Xi} \models \beta$*, iff there exists a maximal consistent set* $\Psi \in \mathcal{C}$ *as claimed there, satisfying* $\overline{\Psi} \models \beta$*.*

PROOF. First, let $M, \overline{\Xi} \models \beta$. Then, $M, \overline{\Delta} \models \mathsf{C}_i\beta$. From this we conclude that $M, \overline{\Delta} \models \mathsf{A}_i\beta$ by means of Lemma 5.3. Now, Proposition 5.4 yields the existence of some $\Psi \in \mathcal{C}$ such that $\overline{\Delta}\, S_i\, \overline{\Psi}$. It follows that $M, \overline{\Psi} \models \beta$. This proves the left-to-right direction. The converse ensues in the same manner. ∎

Summarizing our preparatory results, we shall now take the decisive step towards the desired decidability theorem.

LEMMA 5.6. *The structure* M *can be changed into a finite* n*-a-model* M' *falsifying* α *at some point, by manipulating the relations* R *and* $S_1, \ldots, S_n$ *suitably. Furthermore, the size of* M *(respectively,* M'*) is bounded by a computable function of the length of* α*.*

PROOF. The finiteness of W follows from that of Σ, and the number of elements of W is obviously bounded above by a computable function of $l = \text{length}(\alpha)$.

The alteration of M is now done in the way indicated above. First, consider any $i \in \{1, \ldots, n\}$ and $w \in W$. Let $[w]$ denote the R-equivalence class of w and suppose that there exist $w' \in [w]$ and $v \in W$ such that $w'\, S_i\, v$. Then, select such a v and 'forget' all S_i-connections from $[w]$ into $[v']$, where v' represents an S_i-induced successor of $[w]$ different from $[v]$. Second, select a *unique* S_i-successor in $[v]$ of every $w' \in [w]$ having at least two. Afterwards, treat any other R-equivalence class of M correspondingly. Let $M' = (W, R, R', S'_1, \ldots, S'_n, V)$ be the resulting model.

We must now show that the six conditions from Definition 5.1 are satisfied. However, a large section of this proof is covered by the verifications in [5, Section 2.3] already. We need only still care about the items 3, 4, and 5.

The just specified modification of S_i obviously results in a partially functional relation S'_i, for $i = 1, \ldots, n$. Thus, the first part of item 3 is true. The relation induced by S'_i on the set of all R-equivalence classes is likewise a partial function by construction, whence item 5 is satisfied. For completing the verification of item 3, let $i \in \{1, \ldots, n\}$ and $w, w' \in W$ be such that $w\, S'_i\, w'$. Then $\Delta \xrightarrow{\mathsf{A}_i} \Theta$ for some representatives $\Delta, \Theta \in \mathcal{C}$ of w and w', respectively. According to Axiom 10, $\Delta \overset{\Box}{\longrightarrow} \Theta$ follows from that. But this implies that $w\, R'\, w'$. Therefore, we obtain $S_i' \subseteq R'$.

As to item 4, the first assertion is the cross property in case of LSS, which as well was established in the paper [5]; see the proof of Lemma 2.10 there. Thus, it remains to prove that the cross property $S_i' \circ R \subseteq R \circ S_i'$ is valid for every $i \in \{1, \ldots, n\}$. However, an easy inspection of that proof shows that we can proceed in exactly the same way here.

Finally, we must show that $M, w \models \beta$ iff $M', w \models \beta$, for all $w \in W$ and $\beta \in \Sigma$. This is done by induction on β, with the A_i-case being the only critical one ($i = 1, \ldots, n$). However, Lemma 5.3 and Corollary 5.5 obviously ensure that the induction is successful in this case, too. It follows that α is falsified in M' at the point $\overline{\Gamma}$. This completes the proof of the lemma. ■

Note that using the smallest filtrations seems to be necessary for transferring the functionality of the action-specific relations from the canonical model down to the modified model M'; as opposed to this, one can use a different filtration of $\overset{\Box}{\longrightarrow}$ in case of LSS, simplifying proofs in this way; see [1, p. 322 f].

The finite model property for ALSS_n we are heading for follows from the previous lemma in a standard manner. Due to the decidability criterion quoted above, we therefore obtain our next theorem as the main outcome of this section.

Theorem 5.7. (Decidability I) *The logic* ALSS_n *is a decidable set of formulas.*

6. Possible Extensions and Final Comments

In view of potential applications, the basic logic for n-a-subset spaces appears to be hardly exciting, since really significant or more challenging relationships between the LSS- and the TNL-modalities are missing. Thus, it is natural to ask for useful extensions. Regarding this, the discussion on comparing the different knowledge statuses of the agents involved in a multi-agent system, see Section 1 of to the paper [17], has led us to ask for a comparison between the effects of the distinct actions in respect of knowledge. In particular, we would like to model scenarios where a distinguished action $j \in \{1, \ldots, n\}$ provides the agent with more knowledge than all the others. As to a corresponding illustration, consider the following variation of the policeman example from [5]. Suppose that a policeman is controlling the speed of passing cars. Then, his knowledge of the speeds of these cars is limited by several factors, for example, his attentiveness, the volume of traffic, or the precision of the measuring device he is using. But the policeman will always be in a position to improve his actual knowledge, e.g., by interpolating multiply measured data or using a more precise instrument. This is reflected by the effort operator $\Box$ in the original system of Moss and Parikh. The system presented in Section 3 can, however, distinguish between the different actions to be taken. Now, imagine that the policeman feels compelled to apply the state-of-the-art measurement tool instead of the standard one he normally has to use. Then this action definitely yields more knowledge than retaining the stipulated method.

So let us add the following schema to the list of all ALSS_n-axioms.

14. $\mathsf{A}_i \Box \alpha \rightarrow \mathsf{A}_j \alpha$,

for all $\alpha \in n\mathsf{SF}$ and every $i = 1, \ldots, n$.

In case all the modalities involved in (14) were equal, we would have the axioms capturing the *weak density* of the corresponding accessibility relation R (i.e., $\forall s \forall t (s\,R\,t \Rightarrow \exists u (s\,R\,u \text{ and } u\,R\,t))$); see [8, Section 1]. In the present case, a related *lying-in-between property* is imposed on the relation associated with the operator A_j. This property is now formulated for the canonical model of the resulting logic, ALSS_n^+.

LEMMA 6.1. *Let $n, j \in \mathbb{N}$ be fixed as above. Suppose that $\Delta, \Theta \in \mathcal{C}$ are maximal ALSS_n^+-consistent sets of formulas satisfying $\Delta \xrightarrow{\mathsf{A}_j} \Theta$. Then, for all $i \in \{1, \ldots, n\}$, there is some $\Xi_i \in \mathcal{C}$ such that $\Delta \xrightarrow{\mathsf{A}_i} \Xi_i \xrightarrow{\Box} \Theta$.*

PROOF. We can argue in a similar way as in the case of weak density in ordinary modal logic, cf. [8, p. 26]. Thus, we may be brief here. As is known, it suffices to prove that the set $\Sigma_1 \cup \Sigma_2$ of formulas is consistent, where

$$\Sigma_1 := \{\alpha \in n\mathsf{SF} \mid \mathsf{A}_i \alpha \in \Delta\} \text{ and } \Sigma_2 := \{\neg\Box\beta \in n\mathsf{SF} \mid \beta \notin \Theta\}.$$

Assuming towards a contradiction that this is not the case, then there are finitely many elements $\alpha_1, \ldots, \alpha_n \in \Sigma_1$ and $\neg\Box\beta_1, \ldots, \neg\Box\beta_m \in \Sigma_2$ such that the formula $\alpha_1 \wedge \cdots \wedge \alpha_n \rightarrow \Box\beta_1 \vee \cdots \vee \Box\beta_m$ is an ALSS_n^+-theorem. Letting $\beta := \beta_1 \vee \cdots \vee \beta_m$, the formula $\alpha_1 \wedge \cdots \wedge \alpha_n \rightarrow \Box\beta$ is an ALSS_n^+-theorem as well, due to some propositional and modal reasoning. Applying a little more modal proof theory now yields $\mathsf{A}_i\alpha_1 \wedge \cdots \wedge \mathsf{A}_i\alpha_n \rightarrow \mathsf{A}_i\Box\beta \in \mathsf{ALSS}_n^+$, which implies that $\mathsf{A}_i\Box\beta \in \Delta$. According to Axiom 14, we obtain $\mathsf{A}_j\beta \in \Delta$, hence $\beta \in \Theta$, a contradiction. This proves the lemma.[10] ■

[10]Note that one cannot simply apply a Sahlqvist argument here since we are dealing with a non-normal logic.

Evidently, the schema (14) exemplifies the interaction of an LSS-operator with some TNL-modalities in a non-trivial way. However, it becomes apparent at this place that we must be careful in reading the operator $\Box$ in the new context.[11] In a sense, it here measures the 'distance' between the actions i and j 'qualitatively', but in a way not further specified. Thus, $\Box\alpha$ should now be regarded as a formalization of the clause 'unspecified effort yields α',[12] contrasting the further specified effects of the operators A_i $(i = 1, \ldots, n)$.

Regarding the semantics in n-a-subset spaces $\mathcal{S} = (X, \mathcal{O}, \{f_i\}_{1 \leq i \leq n})$, the specific role of the **d**istinguished **a**ction j finds expression in the following property (DA).

(DA) For all $i \in \{1, \ldots, n\}$ and $U \in \mathcal{O}$: if $f_j(U)$ exists, then $f_i(U)$ exists as well and $f_j(U) \subseteq f_i(U)$.

We now prove the completeness of the logic ALSS_n^+ with respect to the semantics extended in this way. For that, it suffices to state the significant changes from the proof for the system ALSS_n. First, the requirement G 2-2 there must be reformulated as follows.

- $(P_{m+1}, \leq_{m+1})$ is an *almost end extension* of $(P_m, \leq_m)$, i.e., a superstructure of $(P_m, \leq_m)$ such that, if $\pi \in P_{m+1} \setminus P_m$, then either π is not strictly smaller than some element of P_m, or there are uniquely determined $\rho, \sigma \in P_m$ and $i \in \{1, \ldots, n\}$ such that σ is an immediate $\leq_m$-successor of ρ, $\sigma = g_j^m(\rho)$, $\pi \leq_{m+1} \sigma$, and $\pi = g_i^{m+1}(\rho)$ (this means, in particular, that $g_j^m(\rho)$ and $g_i^{m+1}(\rho)$ are defined).

Note that this is the first time that end extensions are insufficient for completeness, since some of the new elements must suitably be edged in. This constitutes a technical novelty of this paper.

Second, item 4 from Proposition 4.1 now reads

- for all $i \in \{1, \ldots, n\}$ and $\pi \in P$, if the value $g_i(\pi)$ is defined, then the value $g_j(\pi)$ as well is defined and satisfies $\pi \leq g_i(\pi) \leq g_j(\pi)$.

Third, we approach the essential construction step of the completeness proof. Let $\nu(m+1) = (\mathsf{C}_i, l, k)$, as in Section 4. We distinguish several cases.

First, let $i = j$. If $g_j^m(\pi)$ is undefined, then we proceed as we did in the same case in Section 4 so that the function g_j^{m+1} becomes definitive for the

[11] Basically, this applies to Section 3 already.

[12] Cf. the remark right after the syntax definition at the beginning of Section 2 once again.

argument π, in particular. On the other hand, if $g_j^m(\pi)$ has already been defined, then nothing has to be changed in this step.

Now, suppose that $i \neq j$. We consider two subcases separately.

1. The value $g_j^m(\pi)$ is undefined. In this case, we again do nothing. The construction will ensure that the situation $\mathsf{C}_i\beta \in t_l(x,\pi)$, for some $l \in \mathbb{N}$, will occur again and yet again so that we may assume that.
2. $\rho = g_j^m(\pi)$ has already been defined. If $g_i^m(\pi)$ as well has been defined in advance, then again nothing has to be changed. Thus suppose that $g_i^m(\pi)$ is undefined. In this case, as earlier, we choose new elements $y \in Y$ and $\sigma \in Q$, and we let $X_{m+1} := X_m \cup \{y\}$ and $P_{m+1} := P_m \cup \{\sigma\}$. However, the partial order is extended to P_{m+1} by letting $\pi \leq_{m+1} \sigma \leq_{m+1} \rho$ and σ be not comparable with any other element of P_m that is not contained in a $\leq_m$-path through π and ρ. The function h_{m+1} is defined as follows. We let $h_{m+1}(\tau) := h_m(\tau) \cup \{y\}$ for all $\tau \in P_m$ satisfying $\tau \leq_m \pi$, and $h_{m+1}(\sigma) := h_m(\rho) \cup \{y\}$; for all other arguments, h_{m+1} equals h_m by definition. Furthermore, the extension of the function g_i^m is defined by $g_i^{m+1}(\pi) := \sigma$ and $g_i^{m+1}(\tau) := g_i^m(\tau)$ for all other $\tau \in P_{m+1}$.
   Finally, the mapping t_m has to be adjusted. This is a little more difficile than defining the other components. Since $\rho = g_j^m(\pi)$ is defined, we know from the construction in the case $i = j$ and the case G 1-5 (a) iii of the induction hypothesis that $t_m(x,\pi) \xrightarrow{\mathsf{A}_j} t_m(x,\rho)$ is then satisfied. From Lemma 6.1 we therefore obtain the existence of a maximal ALSS_n^+-consistent set Θ_i such that $t_m(x,\pi) \xrightarrow{\mathsf{A}_i} \Theta_i \xrightarrow{\Box} t_m(x,\rho)$. Now, we let $t_{m+1}(x,\sigma) := \Theta_i$. The same proceeding applies to all other pairs (z,σ) with $z \in h_m(\rho)$ correspondingly. It remains to extend t_m trivially to t_{m+1} for the rest of $X_m \times P_m$ and to define $t_{m+1}(y,\tau) := t_{m+1}(x,\tau)$ for all $\tau \in P_m$ satisfying $\tau \leq_{m+1} \sigma$. Thus, the definition of $\big(X_{m+1}, P_{m+1}, h_{m+1}, \{g_i^{m+1}\}_{1\leq i \leq n}, t_{m+1}\big)$ is completed so that the enumerations ν_1^{m+1}, ν_2^{m+1}, …, ν_{n+2}^{m+1} can suitably be chosen.

The fourth and final point is the verification of the requirements stated in Section 4. This is similar to the basic case treated there in many respects so that we can be brief here. We only consider G 1-5 (a) i for σ in the last case of the construction. So let $z \in h_{m+1}(\sigma)$. If $z = x$ or $z = y$, then the relation $t_{m+1}(x,\sigma) \xrightarrow{\mathsf{K}} t_{m+1}(z,\sigma)$ follows from the reflexivity of $\xrightarrow{\mathsf{K}}$. In case $x \neq z \neq y$, we obtain $t_{m+1}(z,\pi) \xrightarrow{\mathsf{K}} t_{m+1}(x,\pi) \xrightarrow{\mathsf{A}_i} t_{m+1}(x,\sigma)$ from the definition of t_{m+1}, the induction hypothesis in this case, and the symmetry of $\xrightarrow{\mathsf{K}}$. Furthermore, according to the construction there exists a unique

$\Delta' \in \mathcal{C}$ such that $t_{m+1}(z,\pi) \xrightarrow{\mathsf{A}_i} \Delta'$ and $t_{m+1}(z,\sigma) := \Delta'$, since we here have $z \in h_m(\rho)$. We now obtain $t_{m+1}(x,\sigma) \xrightarrow{\mathsf{K}} t_{m+1}(z,\sigma)$ from Axiom 12, as desired. This is what we wanted to say about the necessary verifications.

In this way, we have proved the semantic completeness of the logic ALSS_n^+.

Theorem 6.2. (Completeness II) *The logic* ALSS_n^+ *is sound and complete for the class of all n-a-subset spaces satisfying the property* (DA).

The decidability of ALSS_n^+ too can be proved in a way similar to the case of ALSS_n. For that purpose, Definition 5.1 is to be extended by the clause

- $S_j \subseteq S_i \circ R'$, for every $i \in \{1, \ldots, n\}$.

Obviously, this property corresponds to Axiom 14 with respect to the Kripke semantics.

However, we must be careful in proving the counterpart of Lemma 5.6 as far as the definition of S_i' for $i \neq j$ is concerned. It will become clear in a minute which S_i-successor of any point of W will actually be the right one to choose in this case. But first of all the above property will be established on M'. To this end, take any $w \in W$ and assume that a unique S_j'-successor w' of w has already been selected. Due to the fact that we have taken the smallest filtrations, there are $\Delta, \Theta \in \mathcal{C}$ such that w is the filtrate of Δ, w' that of Θ, and $\Delta \xrightarrow{\mathsf{A}_j} \Theta$ is valid. We conclude from Lemma 6.1 that there exists some $\Xi_i \in \mathcal{C}$ such that $\Delta \xrightarrow{\mathsf{A}_i} \Xi_i \xrightarrow{\Box} \Theta$. Letting $v \in W$ be the filtrate of Ξ_i, the relation $w\, S_i\, v\, R' w'$ now follows from the general features of any filtration. This proves the above property for the structure modified so far on the one hand, on the other hand, the point v is then chosen as the unique S_i'-successor of w. Thus, the modified Lemma 5.6 is proved.

We now obtain the desired decidability result as above.

Theorem 6.3. (Decidability II) *The logic* ALSS_n^+ *is decidable.*

It could be interesting to examine further connections between the modalities considered in this paper. This would increasingly substantiate the combination of LSS and TNL in respect of the modeling of epistemic actions.

The *computational complexity* of the satisfiability problem for the logics ALSS_n and ALSS_n^+ has not been examined up to now. This is mainly because only partial results were known even for the much more basic system LSS until quite recently; see [2,16]. However, the complexity of the latter logic has just been determined; see [14,15].

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