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Propositional Epistemic Logics with Quantification Over Agents of Knowledge (An Alternative Approach)

Abstract. In the previous paper with a similar title (see Shtakser in Stud Log 106(2):311–344, 2018), we presented a family of propositional epistemic logics whose languages are extended by two ingredients: (a) by quantification over modal (epistemic) operators or over agents of knowledge and (b) by predicate symbols that take modal (epistemic) operators (or agents) as arguments. We denoted this family by $\mathcal{PEL}_{(QK)}$. The family $\mathcal{PEL}_{(QK)}$ is defined on the basis of a decidable higher-order generalization of the loosely guarded fragment (HO-LGF) of first-order logic. And since HO-LGF is decidable, we obtain the decidability of logics of $\mathcal{PEL}_{(QK)}$. In this paper we construct an alternative family of decidable propositional epistemic logics whose languages include ingredients (a) and (b). Denote this family by $\mathcal{PEL}^{alt}_{(QK)}$. Now we will use another decidable fragment of first-order logic: the two variable fragment of first-order logic with two equivalence relations (FO^2+2E) [the decidability of FO^2+2E was proved in Kieroński and Otto (J Symb Log 77(3):729–765, 2012)]. The families $\mathcal{PEL}^{alt}_{(QK)}$ and $\mathcal{PEL}_{(QK)}$ differ in the expressive power. In particular, we exhibit classes of epistemic sentences considered in works on first-order modal logic demonstrating this difference.

Keywords: Propositional epistemic logics, Quantification over modal operators, Two variable fragment of first-order logic with equivalence relations, Decision problem, Outer and inner scopes.

1. Introduction

There are two basic decidable fragments of first-order logic such that propositional modal logic can be embedded to these fragments: (1) the two variable fragment; (2) the guarded fragment (see, for example, [2, pp. 83–91, 448–460]). We consider more expressive logics (between propositional modal logic and first-order modal, more precisely, term modal logic) whose languages are extended by two ingredients: (a) by quantification over modal operators and (b) by predicate symbols that take modal operators as arguments. And we want these logics to remain decidable. Therefore we need more expressive

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(but decidable) fragments of first-order logic. In [24], we used the loosely guarded fragment (LGF) of first-order logic instead of the simple guarded fragment. And moreover, we generalized LGF to a higher-order decidable loosely guarded fragment (HO-LGF). On the basis of HO-LGF, we constructed a family of decidable propositional epistemic logics whose languages are extended by (a) and (b). This family was denoted by $\mathcal{PEL}_{(QK)}$.

In this paper we will use the extended two variable fragment of first-order logic: the two variable fragment with two equivalence relations (FO^2+2E). It is clear that expressing that E is an equivalence relation requires three variables. Thus if we need equivalence relations in addition to the two variable fragment, we must either (1) add transitivity formulas to the two variable fragment; or (2) consider restricted classes of structures, in which some binary relations are interpreted as equivalence relations. We will use the second option.

The decidability of FO^2+2E was proved in [18, 19]. In [17], the upper complexity bound for satisfiability of FO^2+2E was improved.[1] Using FO^2+2E, we construct an alternative family of decidable propositional epistemic logics whose languages include ingredients (a) and (b). Denote this family by $\mathcal{PEL}^{alt}_{(QK)}$. The families $\mathcal{PEL}^{alt}_{(QK)}$ and $\mathcal{PEL}_{(QK)}$ differ in the expressive power. In particular, we exhibit classes of epistemic sentences considered in works on first-order modal logic demonstrating this difference (see Section 4).

Recently, new interesting expressive and decidable fragments of first-order logic containing basic modal logic has been defined, for example, (1) the unary negation fragment (UNF) of first-order logic and some generalizations of UNF: the guarded negation fragment, the decidable extension of UNF by arbitrary many equivalence relations (see [1, 5, 25]); and (2) monodic fragments of first-order temporal logic (see [13, 14, 16]). We consider extensions of the two variable and guarded fragments as basic decidable fragments for our purpose. But UNF (and its generalizations) and monodic fragments can also be used to construct some families of decidable propositional modal logics whose languages include ingredients (a) and (b). For instance, the paper [21] (forthcoming in Studia Logica) presents an interesting decidable monodic fragment of propositional term modal logic. This fragment contains quantification over modal operators (more precisely, over indexes of modal operators), but now does not contain predicate symbols that take modal operators as arguments.

[1] We will specify properties of FO^2+2E (proved in these papers) in Section 3.

It is worth noting that expressive decidable fragments of first-order logic allow us to define *families* of propositional modal logics with desired properties.

The paper is organized as follows. In Section 2 we begin with a logic $\mathcal{E}_{(QK)}$ of $\mathcal{PEL}_{(QK)}$. Denote the language of this logic by $\mathcal{L}_{(QK)}$. Further, we transform a possible-world structure of $\mathcal{E}_{(QK)}$ to an alternative structure such that (I) the resulting structure is a structure for propositional epistemic logics with quantification over agents of knowledge and (II) this structure is simultaneously a structure for the first-order *correspondence* language (FO$\mathcal{L}_{(QK)}$) for $\mathcal{L}_{(QK)}$ such that FO$\mathcal{L}_{(QK)}$ is a part of FO^2+2E. We prove that the problem of deciding whether a formula in $\mathcal{L}_{(QK)}$ is valid with respect to the class of alternative structures is PSPACE-complete. In Section 3 we give a formal definition of the family $\mathcal{PEL}^{alt}_{(QK)}$ and present an expressive logic of $\mathcal{PEL}^{alt}_{(QK)}$. And finally, in Section 4 we consider epistemic sentences expressible in this logic of $\mathcal{PEL}^{alt}_{(QK)}$. We also compare the expressive power of $\mathcal{PEL}^{alt}_{(QK)}$ with the expressive power of $\mathcal{PEL}_{(QK)}$.

2. Alternative Structures for Propositional Epistemic Logics with Quantification over Agents of Knowledge

In [24, Section 2], we presented a logic $\mathcal{E}_{(QK)}$ of $\mathcal{PEL}_{(QK)}$.[2] The language of $\mathcal{E}_{(QK)}$ was denoted by $\mathcal{L}_{(QK)}$. The alphabet of $\mathcal{L}_{(QK)}$ consists of: a set $\Phi\colon \{p_1, p_2, \ldots\}$ of propositional variables; the propositional connectives $\neg, \wedge$; n modal (epistemic) operators $\{K_1, \ldots, K_n\}$; one modal (epistemic) variable K; the existential quantifier $(\exists)$ over K; a set of unary predicate symbols $\{\mathbf{N}_1, \mathbf{N}_2, \ldots\}$. (The connectives $\vee, \rightarrow$ are defined in terms of $\neg$ and $\wedge$; and the quantifier $\forall$ is defined in terms of $\neg$ and $\exists$ in the usual way.)

Formulas of $\mathcal{L}_{(QK)}$ are recursively defined as follows:

- all propositional variables $p_1, p_2, \ldots$ of Φ are formulas of $\mathcal{L}_{(QK)}$;
- if φ and ψ are formulas of $\mathcal{L}_{(QK)}$, then so are $\neg\varphi$, $\varphi \wedge \psi$, $K_j\varphi$, $(\exists K)\{\mathbf{N}_t(K) \wedge K\varphi\}$, $(\forall K)\{\mathbf{N}_t(K) \rightarrow K\varphi\}$.

DEFINITION 2.1. A possible-world structure for $\mathcal{L}_{(QK)}$ is a tuple $\mathfrak{M} = \langle W, \mathfrak{R}, \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots\}, \{r_1, \ldots, r_n\}, \{N_1, N_2, \ldots\}\rangle$, where

- W is a non-empty set of worlds;
- $\mathfrak{R}$ is a non-empty subset of the powerset of $W \times W$;

[2] The satisfiability problem for $\mathcal{E}_{(QK)}$ is PSPACE-complete.

- $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots\}$ are unary predicates on W;
- $\{r_1, \ldots, r_n\}$ is a finite set of binary predicates on W;
- $\{N_1, N_2, \ldots\}$ are predicates on $\mathfrak{R} \times W$;
- with every propositional variable $p_i \in \mathcal{L}_{(QK)}$ is associated the subset $W_i \in W$ such that $w \in W_i$ iff $\mathfrak{p}_i(w)$ holds.

Let φ be a formula of $\mathcal{L}_{(QK)}$ and let $w \in W$. The *truth-relation* $(\mathfrak{M}, w) \models$ is defined by induction on the construction of φ as follows:

$(\mathfrak{M}, w) \models p_i$ (for $p_i \in \Phi$) iff $\mathfrak{p}_i(w)$ holds in $\mathfrak{M}$;

$(\mathfrak{M}, w) \models \neg\varphi$ iff not $(\mathfrak{M}, w) \models \varphi$;

$(\mathfrak{M}, w) \models \varphi \wedge \psi$ iff $(\mathfrak{M}, w) \models \varphi$ and $(\mathfrak{M}, w) \models \psi$;

$(\mathfrak{M}, w) \models K_j\varphi$ iff $(\mathfrak{M}, w') \models \varphi$ for all w' such that $(w, w') \in r_j$;

$(\mathfrak{M}, w) \models (\exists K)\{\mathbf{N}_t(K) \wedge K\varphi\}$ iff, for some $r \in \mathfrak{R}$ such that $N_t(r, w)$ holds, we have $(\mathfrak{M}, w') \models \varphi$ for all w' with $(w, w') \in r$;

$(\mathfrak{M}, w) \models (\forall K)\{\mathbf{N}_t(K) \rightarrow K\varphi\}$ iff, for all $r \in \mathfrak{R}$ such that $N_t(r, w)$ holds, we have $(\mathfrak{M}, w') \models \varphi$ for all w' with $(w, w') \in r$.

Denote by $\mathfrak{C}$ the class of all structures $\mathfrak{M}$ as above. A formula φ is said to be true in a structure $\mathfrak{M}$, written $\mathfrak{M} \models \varphi$, if $(\mathfrak{M}, w) \models \varphi$ for all worlds $w \in W$; φ is said to be valid with respect to $\mathfrak{C}$ if $\mathfrak{M} \models \varphi$ for all $\mathfrak{M} \in \mathfrak{C}$.

Note that n modal (epistemic) operators $\{K_1, \ldots, K_n\}$ correspond to the set of n binary (accessibility) relations $\{r_1, \ldots, r_n\}$; and the quantifiers $\exists K$ and $\forall K$ range over modal (epistemic) operators corresponding to binary (accessibility) relations of $\mathfrak{R}$.

We can associate with $\mathbf{N}_t$ (in the expression $\mathbf{N}_t(K)$) a group of agents (a name of this group), since every K corresponds to some agent. For example, let us interpret $\mathbf{N}_t$ as 'a group of witnesses to the event t'. Then the formula $(\exists K)\{\mathbf{N}_t(K) \wedge K\varphi\}$ will mean 'there is a witness K to the event t such that K knows φ'.

Let $\mathfrak{M}$ be an arbitrary structure of $\mathfrak{C}$. We are going to transform $\mathfrak{M}$ and obtain an alternative structure $Tr(\mathfrak{M})$ such that (I) $Tr(\mathfrak{M})$ is a structure for propositional epistemic logics with quantification over agents of knowledge and (II) $Tr(\mathfrak{M})$ is simultaneously a structure for the first-order *correspondence* language ($\mathrm{FO}\mathcal{L}_{(QK)}$) for $\mathcal{L}_{(QK)}$ such that $\mathrm{FO}\mathcal{L}_{(QK)}$ is a part of FO^2+2E.

2.1. A Domain of $Tr(\mathfrak{M})$

We transform the structure $\mathfrak{M}$ using the following key idea of [11]: the proposal that knowledge should be regarded as a relation between *pairs* which consist of a world and a view point (agent) in that world (see [11, Sections 4.2–4.5]; see also [12, Sections 2, 4]). This proposal implies that agents consider (world, agent) pairs to be possible, rather than just collection of possible worlds alone.[3] To realize the above mentioned idea, possible world structures defined in [11] contain a set of agents A besides a set of worlds W.

In the structure $\mathfrak{M}$, the set $\mathfrak{R}$ corresponds to A. So we regard $r \in \mathfrak{R}$ (instead of $a \in A$) as an agent. Let us define the following domain Ω of $Tr(\mathfrak{M})$ using the domains W and $\mathfrak{R}$ of $\mathfrak{M}$:

$$(def1) \qquad \Omega = \{\, \langle w, r \rangle \mid w \in W, r \in \mathfrak{R}\}.$$

Different authors consider different advantages of replacing a collection of worlds by a collection of (world, agent) pairs (see, for example, [11, Section 4.5]). In our case, the main advantage is the following. When we quantify over Ω, we simultaneously quantify over worlds and over agents.

In [11], pairs are used in the explicit form $\langle w, a \rangle$ (see [11, Section 4]). In contrast, we will represent a pair of Ω as a single element. Pairs of Ω will be denoted by the symbols ω, ω_1, ω_2. And we will reveal parts of a pair of Ω using the following two equivalence relations EW and ER defined on Ω.

Definition 2.2. EW, ER are the equivalence relations on Ω such that for arbitrary pairs ω_1 and ω_2 of Ω, we have $EW(\omega_1, \omega_2)$ holds iff ω_1 and ω_2 contain the same world, and $ER(\omega_1, \omega_2)$ holds iff ω_1 and ω_2 contain the same agent.

Let us define the set of unary predicates $\{P_1, P_2, \ldots\}$ on the domain Ω of $Tr(\mathfrak{M})$ such that this set corresponds to the set of unary predicates $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots\}$ defined on W.

Definition 2.3. For every $\omega = \langle w, r \rangle$ and for every i, we define: $P_i(\omega_1)$ holds in $Tr(\mathfrak{M})$ for some ω_1 with $EW(\omega, \omega_1)$ iff $\mathfrak{p}_i(w)$ holds in $\mathfrak{M}$.

2.2. Accessibility Relations Defined on Ω

Denote by U the following binary relation defined on Ω.

[3] Note that recently, in works on epistemic logic, there is an increasing tendency to represent knowledge as a relation between (world, agent) pairs (see, for example, [20, p. 81], [7, pp. 139–140], [22, pp. 196–198], [3, p. 1165]).

Definition 2.4. For every $\omega_1 = \langle w_1, r_1\rangle$ and $\omega_2 = \langle w_2, r_2\rangle$, we define: $\langle\omega_1, \omega_2\rangle \in U$ iff $r_1 = r_2$ and $\langle w_1, w_2\rangle \in r_1$.

It is easy to see that for every pairs ω_1 and ω_2 of Ω, if $U(\omega_1, \omega_2)$ holds, then we have $ER(\omega_1, \omega_2)$.

Let ω be an arbitrary pair of Ω. Then the accessibility relation corresponding to the agent in ω is defined as follows:

$$(def2) \qquad R_{(\omega)} = \{\, \langle\omega_1, \omega_2\rangle \mid \omega_1, \omega_2 \in \Omega, \quad ER(\omega, \omega_1), \quad U(\omega_1, \omega_2)\,\}.$$

Rewrite an arbitrary pair of $R_{(\omega)}$ in terms of worlds and agents. Let $\omega = \langle w, r\rangle$, $\omega_1 = \langle w_1, r\rangle$, $\omega_2 = \langle w_2, r\rangle$. Then

$$\langle\omega_1, \omega_2\rangle = \langle\langle w_1, r\rangle, \langle w_2, r\rangle\rangle\,.$$

And this pair can be represented as[4]

$$w_1 \xrightarrow{r} w_2.$$

It is not hard to prove that $R_{(\omega)}$ represents the set of all pairs of the accessibility relation r. Hence the binary relation U can be considered as the union of all relations $r \in \mathfrak{R}$, and each individual relation is singled out using $(def2)$. It is obvious that for every non-empty subset $\mathfrak{R}$ of the powerset $W \times W$, we can define the corresponding binary relation U.

Note that definition $(def2)$ contains three pairs $\omega, \omega_1, \omega_2$. Therefore we need three variables to express this definition in first-order logic, and the corresponding formula does not belong to FO^2+2E. But in possible-world structures of propositional modal logics, for a given world w and for every accessibility relation r, we consider only the worlds accessible from the current world w via the relation r. Thus we regard only pairs of the kind $\langle w, w'\rangle \in r$. In our terms, for an arbitrary current pair ω and an arbitrary accessibility relation $R_{(\omega)}$, we consider only pairs of the kind $\langle\omega, \omega_1\rangle \in R_{(\omega)}$. Clearly, the set of these pairs can be expressed in first-order logic using only two variables.

Besides the set $\mathfrak{R}$, the structure $\mathfrak{M}$ includes the finite set $\{r_1, \ldots, r_n\}$ of accessibility relations on W.[5] Denote by $\{C_1, \ldots, C_n\}$ the following set of unary predicates on Ω corresponding to $\{r_1, \ldots, r_n\}$. Let $r_j \in \{r_1, \ldots, r_n\}$ and let $\omega = \langle w, r_j\rangle$ for some $w \in W$. Then

[4] Notice that a similar expression is used in [3,7]: a ternary relation is introduced between a world w_1, an agent of that world and another world w_2; $w_1 \xrightarrow{a} w_2$ meaning that w_2 is *a-reachable* from w_1 (see, [7, p. 139] and [3, p. 1165]).

[5] Note that the sets $\mathfrak{R}$ and $\{r_1, \ldots, r_n\}$ are defined in $\mathfrak{M}$ independently of each other.

$$(def3) \qquad C_j = \{\, \omega_1 \mid \omega_1 \in \Omega, \quad ER(\omega, \omega_1)\,\}.$$

Thus C_j is an equivalence class induced on Ω by ER. The accessibility relation corresponding to agent r_j can be defined as follows:

$$R_{(\omega)} = \{\, \langle \omega_1, \omega_2 \rangle \mid \omega_1, \omega_2 \in \Omega, \quad C_j(\omega), \quad ER(\omega, \omega_1), \quad U(\omega_1, \omega_2)\,\}.$$

2.3. Names of Agents Defined on Ω and a Full Definition of $Tr(\mathfrak{M})$

In the structure $\mathfrak{M}$, we interpret names of agents $\mathbf{N}_t$ using predicates N_t defined on $\mathfrak{R} \times W$. It is known that in first-order modal logic, we allow the interpretations of predicate symbols to vary from world to world in a given possible-world structure (see, for example, [6, pp. 84–86]). The interpretations of predicate symbols $\mathbf{N}_t$ have the same property, since corresponding predicates N_t take as argument not only an accessibility relation (an agent) r but also a world w. This means that we introduce non-rigid general names $\mathbf{N}_t$ such that some agent K may have a name $\mathbf{N}_t$ in a world w and may not have this name in another world w'.

Similarly, we will interpret predicate symbols $\mathbf{N}_t$ as non-rigid general names in the alternative structure $Tr(\mathfrak{M})$. Denote by $\{N'_1, N'_2, \ldots\}$ the set of unary predicates on Ω corresponding to the set $\{N_1, N_2, \ldots\}$ of $\mathfrak{M}$. Let $\omega = \langle w, r \rangle$. Then the expression $N'_t(\omega)$ will mean that 'agent r has name N'_t in world w'.

We define the interpretation of unary predicates $\{N'_1, N'_2, \ldots\}$ in the structure $Tr(\mathfrak{M})$ as follows: for every $w \in W$ and for every $r \in \mathfrak{R}$, let $\omega = \langle w, r \rangle$; then

$$(def4) \qquad N_t(r, w) \text{ holds in } \mathfrak{M} \text{ iff } N'_t(\omega) \text{ holds in } Tr(\mathfrak{M}).$$

Let us define the resulting structure $Tr(\mathfrak{M})$ and interpret formulas of $\mathcal{L}_{(QK)}$ in this structure.

Definition 2.5. The alternative possible-world structure $Tr(\mathfrak{M})$ for $\mathcal{L}_{(QK)}$ is $\langle\, \Omega, \{P_1, P_2, \ldots\}, EW, ER, U, \{C_1, \ldots, C_n\}, \{N'_1, N'_2, \ldots\}\,\rangle$, where

- Ω is the domain defined by $(def1)$;
- $\{P_1, P_2, \ldots\}$ is the set of unary predicates on Ω that satisfy Definition 2.3;
- EW, ER are the equivalence relations on Ω that satisfy Definition 2.2;
- U is the binary relation on Ω that satisfies Definition 2.4;
- $\{C_1, \ldots, C_n\}$ is the finite set of unary predicates on Ω defined by $(def3)$;
- $\{N'_1, N'_2, \ldots\}$ is the set of unary predicates on Ω defined by $(def4)$.

- with every propositional variable $p_i \in \mathcal{L}_{(QK)}$ is associated the subset $\Omega_i \in \Omega$ such that $\omega \in \Omega_i$ iff, for some ω_1 with $(\omega, \omega_1) \in EW$, $P_i(\omega_1)$ holds.

Let φ be a formula of $\mathcal{L}_{(QK)}$ and let ω be a pair of Ω. The *truth-relation* $(Tr(\mathfrak{M}), \omega) \models$ is defined by induction on the construction of φ as follows:

$(Tr(\mathfrak{M}), \omega) \models p_i$ (for $p_i \in \Phi$) iff $\omega \in \Omega_i$;

$(Tr(\mathfrak{M}), \omega) \models \neg\varphi$ iff not $(Tr(\mathfrak{M}), \omega) \models \varphi$;

$(Tr(\mathfrak{M}), \omega) \models \varphi \wedge \psi$ iff $(Tr(\mathfrak{M}), \omega) \models \varphi$ and $(Tr(\mathfrak{M}), \omega) \models \psi$;

$(Tr(\mathfrak{M}), \omega) \models K_j\varphi$ iff, for some ω_1 with $(\omega, \omega_1) \in EW$, we have $\omega_1 \in C_j$ and, for all ω such that $(\omega_1, \omega) \in U$, we have $(Tr(\mathfrak{M}), \omega) \models \varphi$;

$(Tr(\mathfrak{M}), \omega) \models (\exists K)\{\mathbf{N}_t(K) \wedge K\varphi\}$ iff, for some ω_1 such that $(\omega, \omega_1) \in EW$ and $N'_t(\omega_1)$ holds, we have $(Tr(\mathfrak{M}), \omega) \models \varphi$ for all ω with $(\omega_1, \omega) \in U$;

$(Tr(\mathfrak{M}), \omega) \models (\forall K)\{\mathbf{N}_t(K) \to K\varphi\}$ iff, for all ω_1 such that $(\omega, \omega_1) \in EW$ and $N'_t(\omega_1)$ holds, we have $(Tr(\mathfrak{M}), \omega) \models \varphi$ for all ω with $(\omega_1, \omega) \in U$.

Denote by $Tr(\mathfrak{C})$ the following set of structures: $Tr(\mathfrak{C}) = \{\, Tr(\mathfrak{M}) \mid \mathfrak{M} \in \mathfrak{C} \,\}$. A formula φ is said to be true in a structure $Tr(\mathfrak{M})$, written $Tr(\mathfrak{M}) \models \varphi$, if $(Tr(\mathfrak{M}), \omega) \models \varphi$ for all pairs $\omega \in \Omega$; φ is said to be valid with respect to $Tr(\mathfrak{C})$ if $Tr(\mathfrak{M}) \models \varphi$ for all $Tr(\mathfrak{M}) \in Tr(\mathfrak{C})$.

Note that in the interpretations of $K_j\varphi$, $(\exists K)\{\mathbf{N}_t(K) \wedge K\varphi\}$, and $(\forall K)$ $\{\mathbf{N}_t(K) \to K\varphi\}$, we re-use the symbol ω: first we use ω as the current pair; and secondly we use the same symbol to denote other pairs in the expression 'for all ω with $(\omega, \omega_1) \in U$' (in this case, we can directly see that structures of $Tr(\mathfrak{C})$ are based on the two variable fragment). Similarly, we re-use variables in the standard translation of modal formulas into first-order language (see, for instance, [8, pp. 18–19]). For example,

$$(\forall y)\{R(x, y) \to (\exists x)\{R(y, x) \wedge \varphi^{\dagger}(x)\}\}.$$

In this formula, x is a free variable in the first occurrence and a bound variable in the second occurrence.

Definition 2.6. Let ω be a pair of Ω and let φ be a formula of $\mathcal{L}_{(QK)}$. We have a *bound occurrence* of ω in the interpretation of φ if ω occurs in the scope of 'for some ω' or 'for all ω'. Otherwise we have a *free occurrence*.

Definition 2.7. Let $\mathfrak{Cl}$ be a class of structures and $\mathcal{L}$ be an appropriate language. By $\mathrm{L}(\mathfrak{Cl}, \mathcal{L})$ we will denote the logic that is the set of all sentences of $\mathcal{L}$ valid in $\mathfrak{Cl}$.

2.4. An Embedding of the Logic $\mathbf{L}(Tr(\mathfrak{C}), \mathcal{L}_{(QK)})$ into the Logic $\mathbf{L}(\mathfrak{C}, \mathcal{L}_{(QK)})$

Lemma 2.1. *For every structure $Tr(\mathfrak{M}) \in Tr(\mathfrak{C})$, for every $\varphi \in \mathcal{L}_{(QK)}$, and for every $\omega \in \Omega$ of the form $\omega = \langle w, r \rangle$ (for some r), we have*

$$(Tr(\mathfrak{M}), \omega) \models \varphi \qquad \textit{iff} \qquad (\mathfrak{M}, w) \models \varphi$$

Proof. We will prove this assertion by induction on the length of φ.

Basis step. Suppose φ is a propositional variable p_i and $(Tr(\mathfrak{M}), \omega) \models p_i$ for some $\omega = \langle w, r \rangle$. Then $P_i(\omega_1)$ holds in $Tr(\mathfrak{M})$ for some ω_1 with $(\omega, \omega_1) \in EW$ (by Definition 2.5). Then $\mathfrak{p}_i(w)$ holds in $\mathfrak{M}$ by Definition 2.3. This yields that $(\mathfrak{M}, w) \models p_i$ (by Definition 2.1).

The converse is proved similarly

Inductive step. The proofs of the cases $\neg\varphi, \varphi \wedge \psi$, are trivial. Suppose $(Tr(\mathfrak{M}), \omega) \models K_j\varphi$ for some $\omega = \langle w, r \rangle$. Then for some ω_1 with $(\omega, \omega_1) \in EW$, we obtain $\omega_1 \in C_j$ and, for all ω such that $(\omega_1, \omega) \in U$, we have $(Tr(\mathfrak{M}), \omega) \models \varphi$.

Since $(\omega, \omega_1) \in EW$ and $\omega_1 \in C_j$, we obtain that $\omega_1 = \langle w, r_j \rangle$ by Definition 2.2 and by $(def3)$. As mentioned above, we re-use the symbol ω in the interpretation of $K_j\varphi$. Secondly we use ω to denote an arbitrary pair such that $(\omega_1, \omega) \in U$. Since $(\omega_1, \omega) \in U$, we have $(\omega_1, \omega) \in ER$ (see Definition 2.4). Thus we can represent ω as $\langle w', r_j \rangle$, and $(Tr(\mathfrak{M}), \langle w', r_j \rangle) \models \varphi$ for all $\langle w', r_j \rangle$ with $(\langle w, r_j \rangle, \langle w', r_j \rangle) \in U$.

Then $(\mathfrak{M}, w') \models \varphi$ for all w' with $(w, w') \in r_j$, by the inductive hypothesis. Hence $(\mathfrak{M}, w) \models K_j\varphi$.

The converse is proved similarly.

Let $(Tr(\mathfrak{M}), \omega) \models (\exists K)\{\mathbf{N}_t(K) \wedge K\varphi\}$ for some $\omega = \langle w, r0 \rangle$. Then for some ω_1 such that $(\omega, \omega_1) \in EW$ and $N'_t(\omega_1)$ holds, we have $(Tr(\mathfrak{M}), \omega) \models \varphi$ for all ω with $(\omega_1, \omega) \in U$.

Since $(\omega, \omega_1) \in EW$, we obtain that $\omega_1 = \langle w, r \rangle$ for some r, by Definition 2.2. As mentioned above, we re-use the symbol ω in the interpretation of $(\exists K)\{\mathbf{N}_t(K) \wedge K\varphi\}$. Secondly we use ω to denote an arbitrary pair such that $(\omega_1, \omega) \in U$. Since $(\omega_1, \omega) \in U$, we have $(\omega_1, \omega) \in ER$ (see Definition 2.4). Thus we can represent ω as $\langle w', r \rangle$, and $(Tr(\mathfrak{M}), \langle w', r \rangle) \models \varphi$ for all $\langle w', r \rangle$ with $(\langle w, r \rangle, \langle w', r \rangle) \in U$.

Then $(\mathfrak{M}, w') \models \varphi$ for all w' with $(w, w') \in r$, by the inductive hypothesis. Besides, $N_t(r, w)$ holds in $\mathfrak{M}$, since $N'_t(\omega_1)$ holds in $Tr(\mathfrak{M})$ (see definition $(def4)$). Hence $(\mathfrak{M}, w) \models (\exists K)\{\mathbf{N}_t(K) \wedge K\varphi\}$.

The converse is proved similarly.

The case $(\forall K)\{\mathbf{N}_t(K) \rightarrow K\varphi\}$ is proved in the same way. ■

In [24, Section 2], we proved the following theorem.

Theorem 2.2. *The problem of deciding whether a formula in $\mathcal{L}_{(QK)}$ is valid with respect to $\mathfrak{C}$ is* Pspace*-complete.*

Lemma 2.1 and Theorem 2.2 imply the following theorem.

Theorem 2.3. *The problem of deciding whether a formula in $\mathcal{L}_{(QK)}$ is valid with respect to $Tr(\mathfrak{C})$ is* Pspace*-complete.*

2.5. An Alternative Correspondence Language for $\mathcal{L}_{(QK)}$

In [24], we defined the higher-order *correspondence* language HO$\mathcal{L}_{\mathcal{E}}$ for $\mathcal{L}_{(QK)}$. Using the class of structures $Tr(\mathfrak{C})$, we can construct an alternative first-order *correspondence* language for $\mathcal{L}_{(QK)}$.[6] Denote the alternative correspondence language by FO$\mathcal{L}_{(QK)}$.

When no confusion arises, we will denote a predicate symbol and its value in a structure by the same symbol.

The alphabet of FO$\mathcal{L}_{(QK)}$ consists of: two individual variables x, y; a set of unary predicate symbols $\{P_1, P_2, \ldots\}$; a set of unary predicate symbols $\{N'_1, N'_2 \ldots\}$; a finite set of unary predicate symbols $\{C_1, \ldots, C_n\}$; two binary predicate symbols EW, ER, which are interpreted as equivalence relations; the binary predicate symbol U; the propositional connectives $\neg, \wedge$; and the quantifier $\exists$. (The connectives $\vee, \rightarrow$ are defined in terms of $\neg$ and $\wedge$; and the quantifier $\forall$ is defined in terms of $\neg$ and $\exists$ in the usual way.)

Formulas of FO$\mathcal{L}_{(QK)}$ are recursively defined as follows:[7]

– $P_1(x), P_2(x), \ldots$ are formulas of FO$\mathcal{L}_{(QK)}$;

[6] The notions 'first-order correspondence language' and 'higher-order correspondence language' are defined in [2, pp. 83, 127].

[7] We omit analogous formulas with the free variable y for the sake of simplicity.

– if $\varphi(x)$ and $\psi(x)$ are formulas of $\mathrm{FO}\mathcal{L}_{(QK)}$,
then so are $\neg\varphi(x)$, $\varphi(x) \wedge \psi(x)$,

$$(\exists y)\{EW(x,y) \wedge C_j(y) \wedge (\forall x)\{U(y,x) \rightarrow \varphi(x)\}\},$$
$$(\exists y)\{EW(x,y) \wedge N'_t(y) \wedge (\forall x)\{U(y,x) \rightarrow \varphi(x)\}\},$$
$$(\forall y)\{(EW(x,y) \wedge N'_t(y)) \rightarrow (\forall x)\{U(y,x) \rightarrow \varphi(x)\}\}.$$

It is obvious that $\mathrm{FO}\mathcal{L}_{(QK)}$ is a part of the two variable fragment of first-order logic with two equivalence relations (FO^2+2E).[8]

Clearly, $Tr(\mathfrak{C})$ can be considered as the class of structures for $\mathrm{FO}\mathcal{L}_{(QK)}$. By Definition 2.7, the logic $\mathrm{L}(Tr(\mathfrak{C}), \mathrm{FO}\mathcal{L}_{(QK)})$ is the set of all sentences of $\mathrm{FO}\mathcal{L}_{(QK)}$ valid in $Tr(\mathfrak{C})$.

Predicates of structures of $Tr(\mathfrak{C})$ satisfy certain conditions (see Definition 2.5). This yields that the following formulas must be true in the logic $\mathrm{L}(Tr(\mathfrak{C}), \mathrm{FO}\mathcal{L}_{(QK)})$:

$$(Cond1) \quad (\forall x)(\forall y)\{(C_j(x) \wedge C_j(y)) \rightarrow (ER(x,y) \wedge (\forall y)\{ER(x,y) \rightarrow C_j(y)\})\}, \text{ where } j = 1, \ldots, n.$$

$(Cond1)$ means that the unary predicate symbols $\{C_1, \ldots, C_n\}$ are interpreted as equivalence classes induced on Ω by ER.

$$(Cond2) \quad (\forall x)(\forall y)\{U(x,y) \rightarrow ER(x,y)\}.$$

This formula expresses the consequence of Definition 2.4.

Besides, EW and ER satisfy all the conditions of equivalence relations. It is easy to see that the formulas $(Cond1)$, $(Cond2)$ belong to FO^2+2E.

Let us define the standard translation $\cdot^\dagger$ from $\mathcal{L}_{(QK)}$ into $\mathrm{FO}\mathcal{L}_{(QK)}$.[9]

$$p_i^\dagger = P_i(x), \qquad (\neg\varphi)^\dagger = \neg\varphi^\dagger, \qquad (\varphi \wedge \psi)^\dagger = \varphi^\dagger \wedge \psi^\dagger,$$
$$(K_j\varphi)^\dagger = (\exists y)\{EW(x,y) \wedge C_j(y) \wedge (\forall x)\{U(y,x) \rightarrow \varphi^\dagger(x)\}\},$$

$$((\exists K)\{\mathbf{N}_t(K) \wedge K\varphi\})^\dagger = (\exists y)\{EW(x,y) \wedge N'_t(y) \wedge (\forall x)\{U(y,x) \rightarrow \varphi^\dagger(x)\}\},$$
$$((\forall K)\{\mathbf{N}_t(K) \rightarrow K\varphi\})^\dagger = (\forall y)\{(EW(x,y) \wedge N'_t(y)) \rightarrow (\forall x)\{U(y,x) \rightarrow \varphi^\dagger(x)\}\}.$$

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[8]The equivalence relation ER is used in definitions of predicate symbols $\{C_1, \ldots, C_n\}$ and U in $\mathrm{FO}\mathcal{L}_{(QK)}$. See $(Cond1)$, $(Cond2)$ below.

[9]The standard translation from a modal language into its first-order correspondence language is defined, in particular, in [8, pp. 18–19]) and in [2, pp. 83–91].

Let φ be an arbitrary formula in $\mathcal{L}_{(QK)}$ and let $\varphi^{\dagger}(x)$ be the standard translation of φ into FO$\mathcal{L}_{(QK)}$. By induction on the length of φ, it is easy to prove the following. For every structure $Tr(\mathfrak{M})$ of $Tr(\mathfrak{C})$ and for every pair $\omega \in \Omega$, we have

$$(Tr(\mathfrak{M}), \omega) \models \varphi \qquad \text{iff} \qquad Tr(\mathfrak{M}) \models \varphi^{\dagger}(x)[\omega],$$

where $[\omega]$ means that ω is assigned to the free variable x of $\varphi^{\dagger}(x)$.

Therefore φ is valid with respect to $Tr(\mathfrak{C})$ iff $\varphi^{\dagger}(x)$ is valid with respect to $Tr(\mathfrak{C})$, i.e.,

$$\varphi \in L(Tr(\mathfrak{C}), \mathcal{L}_{(QK)}) \qquad \text{iff} \qquad (\forall x)\{\varphi^{\dagger}(x)\} \in L(Tr(\mathfrak{C}), FO\mathcal{L}_{(QK)}).$$

3. A Formal Definition of $\mathcal{PEL}^{alt}_{(QK)}$ and an Expressive Logic of $\mathcal{PEL}^{alt}_{(QK)}$

DEFINITION 3.1. Let L($\mathfrak{C}l$, $\mathcal{L}_{\mathcal{E}}$) be any propositional epistemic logic with quantification over modal (epistemic) operators and with predicate symbols that take modal (epistemic) operators as arguments. Let FO$\mathcal{L}_{\mathcal{E}}$ be the first-order correspondence language for $\mathcal{L}_{\mathcal{E}}$. Then L($\mathfrak{C}l, \mathcal{L}_{\mathcal{E}}$) $\in \mathcal{PEL}^{alt}_{(QK)}$ if the language FO$\mathcal{L}_{\mathcal{E}}$ is a part of FO^2+2E.

The paper [17] establishes the following properties of FO^2+2E:

- the satisfiability problem (SAT) and the finite satisfiability problem (FINSAT) for FO^2+2E are decidable;
- both SAT and FINSAT for FO^2+2E are 2-NEXPTIME-complete.

Using these properties of FO^2+2E, we obtain the decidability and the 2-NEXPTIME upper complexity bound for satisfiability and finite satisfiability of all logics of the family $\mathcal{PEL}^{alt}_{(QK)}$.

Note that in proofs of [17], the fragment FO^2+2E includes *equality.*

In this section we are going to present an expressive logic of $\mathcal{PEL}^{alt}_{(QK)}$. But first we construct some auxiliary logic. Denote by $\mathcal{L}_0$ the language of this logic. The alphabet of $\mathcal{L}_0$ consists of: a set Φ: $\{p_1, p_2, \ldots\}$ of propositional variables; the propositional connectives $\neg, \wedge$; n modal (epistemic) operators $\{K_1, \ldots, K_n\}$; two modal (epistemic) variables $K1$ and $K2$; the existential quantifier ($\exists$) over $K1$ (and over $K2$); a set of unary predicate symbols $\{\mathbf{N}_1, \mathbf{N}_2, \ldots\}$; a set of unary predicate symbols $\{\mathbf{D}_1, \mathbf{D}_2, \ldots\}$; a set of binary predicate symbols $\{\mathbf{Q}_1, \mathbf{Q}_2, \ldots\}$; and the equality symbol '=' .

(The connectives $\vee, \rightarrow$ are defined in terms of $\neg$ and $\wedge$; and the quantifier $\forall$ is defined in terms of $\neg$ and $\exists$ in the usual way.)

Formulas of $\mathcal{L}_0$ are recursively defined as follows:

- all propositional variables $p_1, p_2, \ldots$ of Φ are formulas of $\mathcal{L}_0$;
- $\mathbf{N}_t(K1)$, $\mathbf{D}_k(K1)$, $\mathbf{Q}_l(K1, K2)$, $K1 = K2$, $K1 = K_j$ are formulas of $\mathcal{L}_0$;
- if φ and ψ are formulas of $\mathcal{L}_0$, then so are $\neg\varphi, \varphi \wedge \psi$, $K1\varphi$, $(\exists K1)\{\varphi\}$;
- let φ be a formula of $\mathcal{L}_0$; if we replace an arbitrary occurrence of $K1$ by $K2$ or (and) replace an arbitrary occurrence of $K2$ by $K1$ in φ, then the resulting formula is a formula of $\mathcal{L}_0$.

Denote by s a function (a *variable-assignment*) from the variables $K1, K2$ to Ω. Let $s(K1)$ be a pair ω of Ω, and let $\omega = \langle w, r\rangle$. Then we interpret $K1$ as 'agent r in world w'.

DEFINITION 3.2. A possible-world structure for $\mathcal{L}_0$ is a tuple

$$M = \langle\, \Omega, \{P_1, P_2, \ldots\}, =, EW, ER, U, \{C_1, \ldots, C_n\}, \{N_1, N_2, \ldots\}, \{D_1, D_2, \ldots\}, \{Q_1, Q_2, \ldots\}\,\rangle, \text{ where}$$

- Ω is a nonempty set;[10]
- $\{P_1, P_2, \ldots\}$ are unary predicates on Ω;
- EW, ER are equivalence relations on Ω such that for all $\omega, \omega_1 \in \Omega$, if $(\omega, \omega_1) \in EW$ and $(\omega, \omega_1) \in ER$, then $\omega = \omega_1$;
- U is a binary relation on Ω that satisfy $(Cond2)$;
- $\{C_1, \ldots, C_n\}$ are unary predicates on Ω that satisfy $(Cond1)$;
- $\{N_1, N_2, \ldots\}$ are unary predicates on Ω;
- $\{D_1, D_2, \ldots\}$ are unary predicates on Ω such that for all $\omega, \omega_1 \in \Omega$, we have if $D_k(\omega)$ and $D_k(\omega_1)$ holds, then $(\omega, \omega_1) \in ER$;
- $\{Q_1, Q_2 \ldots\}$ are binary predicates on Ω;
- with every propositional variable $p_i \in \mathcal{L}_0$ is associated the subset $\Omega_i \in \Omega$ such that $\omega \in \Omega_i$ iff, for some ω_1 with $(\omega, \omega_1) \in EW$, $P_i(\omega_1)$ holds.

Let φ be a formula of $\mathcal{L}_0$ and let ω be a pair of Ω. The *truth-relation* $(M, \omega), s \models$ is defined by induction on the construction of φ as follows:

[10] Formally, Ω is a set of *arbitrary* elements. But we implicitly regard these elements as (world, agent) pairs.

$(M, \omega), s \models p_i$ (for $p_i \in \Phi$) iff $\omega \in \Omega_i$;

$(M, \omega), s \models \mathbf{N}_t(K1)$ iff $N_t(s(K1))$ holds;

$(M, \omega), s \models \mathbf{D}_k(K1)$ iff $D_k(s(K1))$ holds;

$(M, \omega), s \models \mathbf{Q}_l(K1, K2)$ iff $(s(K1), s(K2)) \in Q_l$;

$(M, \omega), s \models (K1 = K2)$ iff $s(K1) = s(K2)$;

$(M, \omega), s \models (K1 = K_j)$ iff $C_j(s(K1))$ holds;

$(M, \omega), s \models \neg\varphi$ iff not $(M, \omega), s \models \varphi$;

$(M, \omega), s \models \varphi \wedge \psi$ iff $(M, \omega), s \models \varphi$ and $(M, \omega), s \models \psi$;

$(M, \omega), s \models K1\varphi$ iff, for all ω such that $(s(K1), \omega) \in U$, we have $(M, \omega), s \models \varphi$;

$(M, \omega), s \models (\exists K1)\{\varphi\}$ iff, for some ω_1 with $(\omega, \omega_1) \in EW$, we have $(M, \omega_1), s' \models \varphi$, where $s'(K1) = \omega_1$ and $s'(K2) = s(K2)$.

The formulas $\mathbf{N}_t(K2)$, $\mathbf{D}_k(K2)$, $\mathbf{Q}_l(K1, K1)$, $\mathbf{Q}_l(K2, K1)$, $\mathbf{Q}_l(K2, K2)$, $(K2 = K_j)$, $K2\varphi$, $(\exists K2)\{\varphi\}$ are interpreted in the same way.

Denote by $\mathcal{C}$ the set of all structures M as above. A formula φ is said to be true in a structure M, written $M \models \varphi$, if $(M, \omega), s \models \varphi$ for all pairs $\omega \in \Omega$ and for every variable-assignment s; φ is said to be valid with respect to $\mathcal{C}$ if $M \models \varphi$ for all $M \in \mathcal{C}$.

Note that it is easy to prove that

- if $(M, \omega), s \models p_i$, then $(M, \omega_1), s \models p_i$ for all ω_1 with $(\omega, \omega_1) \in EW$;
- if $(M, \omega), s \models (\exists K1)\{\varphi\}$, then $(M, \omega_1), s \models (\exists K1)\{\varphi\}$ for all ω_1 with $(\omega, \omega_1) \in EW$. And similarly for $(\exists K2)\{\varphi\}$.

Note also that we re-use the symbol ω in the interpretation of $K1\varphi$: first we use ω as the current pair in the expression $(M, \omega), s \models$; and then we have bound occurrences[11] of ω. Similarly, in the standard translation of modal formulas into first-order language, first we use x as a free variable; and secondly as a bound variable (see comments on Definition 2.5).

As mentioned above (see Section 2.3), we interpret predicate symbols $\mathbf{N}_t$ as non-rigid names in alternative structures. Let $\omega = \langle w, r\rangle$. Then the expression $N_t(\omega)$ means that 'agent r has name N_t in world w'. Clearly, in another world, agent r may not have name N_t. In the same way, we

[11] The notions 'free occurrence of ω' and 'bound occurrence of ω' are defined in Definition 2.6.

interpret binary predicate symbols $\mathbf{Q}_l$ as non-rigid names. Let $\omega = \langle w, r\rangle$ and $\omega_1 = \langle w_1, r_1\rangle$. Then the expression $Q_l(\omega, \omega_1)$ means that 'agent r from world w is Q_l-connected to agent r_1 from world w_1'.

In contrast, a predicate symbol $\mathbf{D}_k$ is a rigid designator, since the corresponding predicate (name) D_k denotes the same agent in every world (see Definition 3.2):

$$(Cond3) \quad (\forall x)(\forall y)\{(D_k(x) \wedge D_k(y)) \rightarrow ER(x, y)\}, \text{ where } k = 1, 2, \ldots.$$

Similarly, $(K1 = K_j)$ means that $K1$ has a rigid value, since the corresponding predicate C_j also fixes the same agent in every world (see $(Cond1)$).

In addition to the predicates U, C_j, and D_k of M, the equivalence relations EW and ER also satisfy a certain condition:

$$(Cond4) \quad (\forall x)(\forall y)\{(EW(x, y) \wedge ER(x, y)) \rightarrow (x = y)\}.$$

Denote by $\mathrm{FO}\mathcal{L}_0$ the following first-order correspondence language for $\mathcal{L}_0$. The alphabet of $\mathrm{FO}\mathcal{L}_0$ consists of: three individual variables x, y_{K1}, y_{K2}; a set of unary predicate symbols $\{P_1, P_2, \ldots\}$; sets of unary predicate symbols $\{C_1, \ldots, C_n\}$; $\{N_1, N_2, \ldots\}$; $\{D_1, D_2, \ldots\}$; a set of binary predicate symbols $\{Q_1, Q_2, \ldots\}$; two binary predicate symbols EW, ER, which are interpreted as equivalence relations; the binary predicate symbol U; the propositional connectives $\neg, \wedge$; the quantifier $\exists$; and the equality symbol '$=$'. (The connectives $\vee, \rightarrow$ are defined in terms of $\neg$ and $\wedge$; and the quantifier $\forall$ is defined in terms of $\neg$ and $\exists$ in the usual way.)

Formulas of $\mathrm{FO}\mathcal{L}_0$ are recursively defined as follows:

- $P_i(x)$, $C_j(y_{K1})$, $N_t(y_{K1})$, $D_k(y_{K1})$, $Q_l(y_{K1}, y_{K2})$, $y_{K1} = y_{K2}$ are formulas of $FO\mathcal{L}_0$;
- if we replace all occurrences of y by y_{K1} in $(Cond1)$, $(Cond2)$, $(Cond3)$, and $(Cond4)$, then the resulting formulas are formulas of $FO\mathcal{L}_0$;
- if φ and ψ are formulas of $\mathrm{FO}\mathcal{L}_0$, then so are $\neg\varphi$, $\varphi \wedge \psi$, $(\forall x)\{U(y_{K1}, x) \rightarrow \varphi\}$, $(\exists y_{K1})\{EW(x, y_{K1}) \wedge \varphi\}$;
- let φ be a formula of $\mathrm{FO}\mathcal{L}_0$; if we replace an arbitrary occurrence of y_{K1} by y_{K2} or (and) replace an arbitrary occurrence of y_{K2} by y_{K1} in φ, then the resulting formula is a formula of $\mathrm{FO}\mathcal{L}_0$.

Let us define the following translation function $\cdot^{\ddagger}$ from $\mathcal{L}_0$ into $\mathrm{FO}\mathcal{L}_0$.

(1) $\quad (p_i)^{\ddagger} = P_i(x)$,

(2) $\quad (\mathbf{N}_t(K1))^{\ddagger} = N_t(y_{K1})$,

(3) $\quad (\mathbf{D}_k(K1))^{\ddagger} = D_k(y_{K1})$,

(4) $(\mathbf{Q}_l(K1, K2))^\ddagger = Q_l(y_{K1}, y_{K2})$,

(5) $(K1 = K2)^\ddagger = (y_{K1} = y_{K2})$,

(6) $(K1 = K_j)^\ddagger = C_j(y_{K1})$,

(7) $(\neg\varphi)^\ddagger = \neg\varphi^\ddagger$,

(8) $(\varphi \wedge \psi)^\ddagger = \varphi^\ddagger \wedge \psi^\ddagger$,

(9) $(K1\varphi)^\ddagger = (\forall x)\{U(y_{K1}, x) \rightarrow \varphi^\ddagger\}$,

(10) $((\exists K1)\{\varphi\})^\ddagger = (\exists y_{K1})\{EW(x, y_{K1}) \wedge \varphi^\ddagger\}$.

The formulas $\mathbf{N}_t(K2)$, $\mathbf{D}_k(K2)$, $\mathbf{Q}_l(K1, K1)$, $\mathbf{Q}_l(K2, K1)$, $\mathbf{Q}_l(K2, K2)$, $(K2 = K_j)$, $K2\varphi$, $(\exists K2)\{\varphi\}$ are translated in the same way.

Observe that in the translation defined above, for every formula $\varphi \in \mathcal{L}_0$, at each stage of the translation of φ, we do not re-use (do not change) variables in the first-order formulas of (1) – (10). Consider, for example, the formula

$$(f) \qquad (\exists K1)\{K1((\exists K2)\{\mathbf{Q}_l(K2, K1)\})\}.$$

The formula $(f)^\ddagger$ is

$$(\exists y_{K1})\{EW(x, y_{K1}) \wedge (\forall x)\{U(y_{K1}, x) \rightarrow (\exists y_{K2})\{EW(x, y_{K2}) \wedge Q_l(y_{K2}, y_{K1})\}\}\}.$$

Given this property of the translation function $\cdot^\ddagger$, for every formula $\varphi \in \mathcal{L}_0$, we obtain the 1-1 correspondence between occurrences of the variable $K1$ in φ and occurrences of y_{K1} in $\varphi^\ddagger$. And similarly for the variables $K2$ and y_{K2}.

Note that the formula $(f)^\ddagger$ has the following epistemic interpretation. In the scope of $(\exists y_{K1})$, let a pair $\langle w, r_1\rangle$ be an assignment to the free variable y_{K1}, where w denotes the current world. In the scope of $(\forall x)$, suppose x is interpreted by a pair $\langle w_1, r_1\rangle$, where w_1 denotes an arbitrary world that r_1 considers possible. Note that the value of x includes r_1, since $U(y_{K1}, x)$ implies $ER(y_{K1}, x)$. In the scope of $(\exists y_{K2})$, let a pair $\langle w_1, r_2\rangle$ be an assignment to the free variable y_{K2}. Then this formula is read as 'there is an agent r_1 in the current world w such that in all the worlds that r_1 considers possible, there is an agent r_2 who is Q_l-connected to r_1'.

Lemma 3.1. *Let s be an arbitrary variable-assignment: $s(K1) = \omega_1$ and $s(K2) = \omega_2$. Denote by $\tilde{s}$ the variable-assignment corresponding to s such that $\tilde{s}(y_{K1}) = \omega_1$ and $\tilde{s}(y_{K2}) = \omega_2$. For every structure $M \in \mathcal{C}$, for every $\varphi \in \mathcal{L}_0$, and for every $\omega \in \Omega$, we have:*

$$(M, \omega), s \models \varphi \qquad \textit{iff} \qquad (M), \tilde{s} \models \varphi^{\ddagger}(x)[\omega]$$

where $[\omega]$ *means that* ω *is assigned to the free variable* x *of the formula* $\varphi^{\ddagger}(x)$*.*

Proof. This lemma is easily proved by induction on the length of φ. ■

Now we define a language $\mathcal{L}^{alt}_{(QK)}$ as follows. The alphabet of $\mathcal{L}^{alt}_{(QK)}$ coincides with the alphabet of $\mathcal{L}_0$. To define the set of formulas of $\mathcal{L}^{alt}_{(QK)}$, we need the following definition.

Definition 3.3. Let $\mathbb{K}$ be a variable that ranges over the set $\{K1, K2\}$. Let φ be an arbitrary formula of $\mathcal{L}_0$ (we assume that φ is expressed in the basic connectives, i.e., $\neg$, $\wedge$, and $\exists$). A quantifier $(\exists\mathbb{K})$ is called *an outermost quantifier* in φ if $(\exists\mathbb{K})$ is not in the scope of another quantifier and is not in the scope of a modal operator. A modal operator $\mathbb{K}$ is called *an outermost modal operator* in φ if $\mathbb{K}$ is not in the scope of another modal operator and is not in the scope of a quantifier.[12]

Definition 3.4. Let φ be an arbitrary formula of $\mathcal{L}_0$ expressed in the basic connectives $\neg$, $\wedge$, and $\exists$. The formula φ belongs to $\mathcal{L}^{alt}_{(QK)}$ iff φ satisfies the following conditions:

(a) φ does not contain $K1$ and $K2$ free;

(b) if $K1$ is a modal operator in φ, then in the scope of $K1$, only the variable $K1$ can occur free; and similarly for $K2$;

(c) if $\varphi = (\exists K1)\{\psi\}$, or $(\exists K1)\{\psi\}$ is a subformula of φ, or $K1\psi$ is a subformula of φ, then ψ does not contain an outermost quantifier of the form $(\exists K1)\{\phi\}$.[13] And similarly for $(\exists K2)\{\psi\}$ and $K2\psi$.

It is known from the literature [2, pp. 87–89] that in the standard translation from modal formulas to first-order formulas, we keep flipping between two variables x and y. That is, in this translation, we use x and y as new bound variables in the following sequence: $x, y, x, y, x, y, \ldots$. Note that the variable x does not occur twice in a row. Similarly for y. Condition (c) means that, in the same way, we keep flipping between two variables $K1$ and $K2$ in the recursive definition of a formula of $\mathcal{L}^{alt}_{(QK)}$.

Since formulas of $\mathcal{L}^{alt}_{(QK)}$ do not contain $K1$ and $K2$ free, we obtain that variable-assignments make no difference. And therefore we can use the usual

[12]The definition of an outermost modal operator will be used in the proof of Lemma 3.3.

[13]That is, if we have an outermost quantifier in the scope of $(\exists K1)$ or in the scope of a modal operator $K1$, then this quantifier is $(\exists K2)$.

truth-relation $(M, \omega) \models$ instead of $(M, \omega), s \models$; and Lemma 3.1 yields the following corollary.

Corollary 3.2. *For every structure $M \in \mathcal{C}$, for every $\varphi \in \mathcal{L}^{alt}_{(QK)}$, and for every $\omega \in \Omega$, we have:*

$$(M, \omega) \models \varphi \qquad \textit{iff} \qquad M \models \varphi^{\ddagger}(x)[\omega]$$

where $[\omega]$ means that ω is assigned to the free variable x of the formula $\varphi^{\ddagger}(x)$.

Definition 3.5. Let M be an arbitrary structure of $\mathcal{C}$. Let φ be an arbitrary formula of $\mathcal{L}^{alt}_{(QK)}$. By $\varphi^{\ddagger\ddagger}$ denote a formula obtained from $\varphi^{\ddagger}$ by re-using variables such that

- $\varphi^{\ddagger\ddagger}$ contains only two individual variables;
- $\varphi^{\ddagger\ddagger}$ and $\varphi^{\ddagger}$ have the same interpretation in M.

The formula $\varphi^{\ddagger\ddagger}$ is called the *two-variable equivalent* of $\varphi^{\ddagger}$.

For example, we again consider the formula

$$(f) \qquad (\exists K1)\{K1((\exists K2)\{\mathbf{Q}_l(K2, K1)\})\}.$$

This formula belongs to $\mathcal{L}^{alt}_{(QK)}$ (see Definition 3.4). The two-variable equivalent of $(f)^{\ddagger}$ is

$$(\exists y)\{EW(x, y) \wedge (\forall x)\{U(y, x) \rightarrow (\exists y)\{EW(x, y) \wedge Q_l(y, x)\}\}\}.$$

The epistemic interpretation of this formula is as follows. In the scope of the first occurrence of $(\exists y)$, let a pair $\langle w, r_1\rangle$ be an assignment to the free variable y, where w denotes the current world. In the scope of $(\forall x)$, suppose x is interpreted by a pair $\langle w_1, r_1\rangle$, where w_1 denotes an arbitrary world that r_1 considers possible. And in the scope of the second occurrence of $(\exists y)$, let a pair $\langle w_1, r_2\rangle$ be an assignment to the free variable y. As a result, $(f)^{\ddagger\ddagger}$ is read as 'there is an agent r_1 in the current world such that in all the worlds that r_1 considers possible, there is an agent r_2 who is Q_l-connected to r_1'. That is, this interpretation coincides with the interpretation of $(f)^{\ddagger}$.

Lemma 3.3. *For an arbitrary formula φ of $\mathcal{L}^{alt}_{(QK)}$, the formula $\varphi^{\ddagger}$ has the two-variable equivalent.*

Proof. Formulas of $\mathcal{L}^{alt}_{(QK)}$ contain only two modal variables $K1$ and $K2$, but we can re-use these variables. Within the framework of this proof, we assume to distinguish between different usages of these variables with the help of superscripts. Let a pair $\langle w, r_1\rangle$ be an assignment to the free variable

$K1$ in the scope of $(\exists K1)$. Then we mark $K1$ as follows: $K1^{(w,r_1)}$. And similarly for $K2$. In the same way, we will mark different usages of variables x, y, y_{K1}, and y_{K2} corresponding to $K1$ and $K2$.

Let φ be an arbitrary formula of $\mathcal{L}^{alt}_{(QK)}$ (we assume that φ is expressed in the basic connectives, i.e., $\neg$, $\wedge$, and $\exists$). The transformation from $\varphi^{\ddagger}$ to $\varphi^{\ddagger\ddagger}$ can be represented as follows:

- take two variables: x and y;
- for every bound variable x, y_{K1}, and y_{K2}, replace this variable by x or y;
- verify that the interpretation of the resulting formula coincides with the interpretation of $\varphi^{\ddagger}$.

It is obvious that this procedure depends only on the quantifiers $(\exists x)$, $(\exists y_{K1})$, and $(\exists y_{K2})$ that occur in the formula $\varphi^{\ddagger}$ and on combinations of these quantifiers. Note that the translation function $\cdot^{\ddagger}$ produces new quantifiers in the formula $\varphi^{\ddagger}$ when we translate subformulas of the kind $(\exists K1)\{\psi\}$, $(\exists K2)\{\psi\}$, $K1\psi$, $K2\psi$ in the formula φ. Thus we will consider only such subformulas.

Since formulas of $\mathcal{L}^{alt}_{(QK)}$ do not contain $K1$ and $K2$ free, we assume that φ begins with $(\exists K1)\{\ldots\}$. Therefore the translation taking φ to the formula $\varphi^{\ddagger}$ begins with $(\exists y_{K1})\{EW(x, y_{K1})\ldots\}$. Let a pair $\langle w, r_1\rangle$ be an assignment to the free variable $K1^{(w,r_1)}$ in the scope of $(\exists K1^{(w,r_1)})$, where w denotes the current world. Then we assign the same pair to the free variable $y^{(w,r_1)}_{K1}$ in the scope of $(\exists y^{(w,r_1)}_{K1})$. Let us replace $y^{(w,r_1)}_{K1}$ by $y^{(w,r_1)}$ without changing the interpretation; and mark x with the superscript $^{(w,r)}$. Thus the formula $\varphi^{\ddagger\ddagger}$ begins with $(\exists y^{(w,r_1)})\{EW(x^{(w,r)}, y^{(w,r_1)})\ldots\}$. In the scope of $(\exists K1^{(w,r_1)})$, we allow:

$\|1\|$ an outermost quantifier $(\exists K2)$;

$\|2\|$ an outermost modal operator $K1^{(w,r_1)}$;

(see Definitions 3.3 and 3.4).[14]

Consider case $\|1\|$. Let φ be $(\exists K1^{(w,r_1)})\{\phi\}$. Then in the interpretation of ϕ, the current pair is $\langle w, r_1\rangle$ (see Definition 3.2). The translation of $(\exists K2)\{\ldots\}$ begins with $(\exists y_{K2})$ $\{EW(x^{(w,r_1)}, y_{K2})\ldots\}$. Let a pair $\langle w, r_2\rangle$ be an assignment to the free variable $K2^{(w,r_2)}$ in the scope of $(\exists K2^{(w,r_2)})$. Then

[14] $K2$ cannot be an outermost modal operator in the scope of $(\exists K1^{(w,r_1)})$, since otherwise φ contains $K2$ free.

we assign the same pair to the free variable $y_{K2}^{(w,r_2)}$ in the scope of $(\exists y_{K2}^{(w,r_2)})$. Let us replace $y_{K2}^{(w,r_2)}$ by $x^{(w,r_2)}$ without changing the interpretation. We use $x^{(w,r_1)}$ as a variable representing the current world. But $y^{(w,r_1)}$ has the same property. Hence we can replace $x^{(w,r_1)}$ by $y^{(w,r_1)}$. As a consequence, we obtain $(\exists x^{(w,r_2)})$ $\{EW(x^{(w,r_2)}, y^{(w,r_1)})\ldots\}$. Note that the variable $y^{(w,r_1)}$ represents not only the current world w, but also the agent r_1. Therefore we can refer to r_1 in the scope of $(\exists x^{(w,r_2)})$ using $y^{(w,r_1)}$. And we do not need the variable $y_{K1}^{(w,r_1)}$ for this purpose.

If there is another outermost quantifier of the form $(\exists K2)$ in the scope of $(\exists K1^{(w,r_1)})$, then we must make the same transformations provided that another pair $\langle w, r_3\rangle$ is assigned to the free variable $K2^{(w,r_3)}$ in the scope of this quantifier $(\exists K2^{(w,r_3)})$.

In the scope of $(\exists K2^{(w,r_2)})$, we allow:

$\|1.1\|$ an outermost quantifier $(\exists K1)$;

$\|1.2\|$ outermost modal operators $K1^{(w,r_1)}$ and $K2^{(w,r_2)}$.

Consider case $\|1.1\|$. We do not need to move to the next levels of this branch, since we keep flipping between two variables $K1$ and $K2$ in the recursive definition of a formula of $\mathcal{L}_{(QK)}^{alt}$. Hence on the next levels the same combinations of $(\exists K1)\{\psi\}$, $(\exists K2)\{\psi\}$, $K1\psi$, $K2\psi$ are repeated.

Let us consider case $\|2\|$. Given that we replace $y_{K1}^{(w,r_1)}$ by $y^{(w,r_1)}$, the translation of $K1^{(w,r_1)}(\ldots)$ begins with $(\forall x)$ $\{U(y^{(w,r_1)}, x)\ldots\}$. Let a pair $\langle w_1, r_1\rangle$ be an assignment to the free variable x in the scope of $(\forall x)$, where w_1 is an arbitrary world that r_1 considers possible. Note that the pair $\langle w_1, r_1\rangle$ contains r_1, since $U(y^{(w,r_1)}, x)$ implies $ER(y^{(w,r_1)}, x)$. Let us mark x with the superscript $^{(w_1,r_1)}$. As a consequence, we obtain $(\forall x^{(w_1,r_1)})$ $\{U(y^{(w,r_1)}, x^{(w_1,r_1)})\ldots\}$. In the scope of the modal operator $K1^{(w,r_1)}$, we allow:

$\|2.1\|$ an outermost quantifier $(\exists K2)$;

$\|2.2\|$ an outermost modal operator $K1^{(w_1,r_1)}$.

Now we consider case $\|1.2\|$.[15] For an outermost modal operator $K1^{(w,r_1)}$, we must make the same transformations as in case $\|2\|$. Given that we replace $y_{K2}^{(w,r_2)}$ by $x^{(w,r_2)}$, the translation of $K2^{(w,r_2)}(\ldots)$ begins with $(\forall y)$ $\{U(x^{(w,r_2)}, y)\ldots\}$. Let a pair $\langle w_2, r_2\rangle$ be an assignment to the free variable y in the scope of $(\forall y)$, where w_2 is an arbitrary world that r_2 considers possible. Let us mark y with the superscript $^{(w_2,r_2)}$. As a consequence, we obtain $(\forall y^{(w_2,r_2)})$ $\{U(x^{(w,r_2)}, y^{(w_2,r_2)})\ldots\}$.

[15] Case $\|1.2\|$ is proved after case $\|2\|$ for technical convenience.

Consider case $\|2.1\|$. The translation of $(\exists K2)\{\ldots\}$ begins with $(\exists y_{K2})$ $\{EW(x^{(w_1,r_1)}, y_{K2})\ldots\}$. Let a pair $\langle w_1, r_4\rangle$ be an assignment to the free variable $K2^{(w_1,r_4)}$ in the scope of $(\exists K2^{(w_1,r_4)})$. Then we assign the same pair to the free variable $y_{K2}^{(w_1,r_4)}$ in the scope of $(\exists y_{K2}^{(w_1,r_4)})$. Let us replace $y_{K2}^{(w_1,r_4)}$ by $y^{(w_1,r_4)}$ without changing the interpretation. As a consequence, we obtain $(\exists y^{(w_1,r_4)})$ $\{EW(x^{(w_1,r_1)}, y^{(w_1,r_4)})\ldots\}$.

And finally we consider case $\|2.2\|$. The translation of $K1^{(w_1,r_1)}(\ldots)$ begins with $(\forall y)$ $\{U(x^{(w_1,r_1)}, y)\ldots\}$. Let a pair $\langle w_3, r_1\rangle$ be an assignment to the free variable y in the scope of $(\forall y)$, where w_3 is an arbitrary world that the agent r_1 consider possible from w_1. Let us mark y with the superscript $^{(w_3,r_1)}$. As a consequence, we obtain $(\forall y^{(w_3,r_1)})$ $\{U(x^{(w_1,r_1)}, y^{(w_3,r_1)})\ldots\}$.

We do not need to move to the next levels of all the branches that we consider above, since on the next levels the same combinations of $(\exists K1)\{\psi\}$, $(\exists K2)\{\psi\}$, $K1\psi$, $K2\psi$ are repeated.

So we have shown that for an arbitrary formula φ of $\mathcal{L}^{alt}_{(QK)}$, we can replace the variables x, y_{K1}, and y_{K2} by the variables x and y in the formula $\varphi^{\ddagger}$ such that the interpretation of the resulting formula coincides with the interpretation of $\varphi^{\ddagger}$. That is, the resulting formula is $\varphi^{\ddagger\ddagger}$. ■

Corollary 3.2 and Lemma 3.3 imply the following theorem.

THEOREM 3.4. *For every structure $M \in \mathcal{C}$, for every $\varphi \in \mathcal{L}^{alt}_{(QK)}$, and for every $\omega \in \Omega$, we have:*

$$(M, \omega) \models \varphi \qquad \textit{iff} \qquad M \models \varphi^{\ddagger\ddagger}(x)[\omega]$$

where $[\omega]$ means that ω is assigned to the free variable x of the formula $\varphi^{\ddagger\ddagger}(x)$.

Since every formula of the kind $\varphi^{\ddagger\ddagger}(x)$ belongs to FO^2+2E, it follows that the logic $\mathrm{L}(\mathcal{C}, \mathcal{L}^{alt}_{(QK)}) \in \mathcal{PEL}^{alt}_{(QK)}$ by Definition 3.1. We obtain the decidability and the 2-NEXPTIME upper complexity bound for satisfiability of $\mathrm{L}(\mathcal{C}, \mathcal{L}^{alt}_{(QK)})$. But the following observation[16] in proofs of [17] allows us to improve this bound. The equivalence relations EW and ER satisfy $(Cond4)$ which says that the intersections of the equivalence classes of EW and ER are of size at most 1. To show the 2-NEXPTIME upper and lower complexity bounds for satisfiability of FO^2+2E, the paper [17, Section 6 and 7] employs intersections of exponential size. If we restrict the class of structures to those with intersections of size at most 1, then the satisfiability problem

[16] This observation was proposed by anonymous referee 1.

for FO^2+2E becomes NEXPTIME-complete. Thus we obtain the NEXPTIME upper complexity bound for satisfiability of $L(\mathcal{C}, \mathcal{L}^{alt}_{(QK)})$.

Denote by $FO\mathcal{L}^{alt}_{(QK)}$ the first-order correspondence language for $\mathcal{L}^{alt}_{(QK)}$. The alphabet of $FO\mathcal{L}^{alt}_{(QK)}$ coincides with the alphabet of $FO\mathcal{L}_0$. Formulas of $FO\mathcal{L}^{alt}_{(QK)}$ are defined as follows:

– $(Cond1), (Cond2), (Cond3), (Cond4)$ are formulas of $FO\mathcal{L}^{alt}_{(QK)}$;
– if φ is an arbitrary formula of $\mathcal{L}^{alt}_{(QK)}$, then $\varphi^{\ddagger\ddagger}$ belongs to $FO\mathcal{L}^{alt}_{(QK)}$.

4. Sentences Expressible in the Logic $L(\mathcal{C}, \mathcal{L}^{alt}_{(QK)})$

The paper [4, p. 29] presents some epistemic sentences expressible in the first-order epistemic language defined in [4]. We consider one of these sentences (the other sentences are formalized in the logic $L(\mathcal{C}, \mathcal{L}^{alt}_{(QK)})$ in the same way).

Someone knows that all Peter's friends know that he likes Mary

$$\begin{aligned}&(\exists K1)\{K1((\exists K2)\{(K2 = K_{Peter}) \wedge (\forall K1)\{\mathbf{Q}_{friend}(K1, K2)\\&\quad \rightarrow K1((\exists K2)\{(K2 = K_{Peter}) \wedge (\exists K1)\{(K1 = K_{Mary})\\&\quad \wedge \mathbf{Q}_{likes}(K2, K1)\}\})\}\})\}.\end{aligned}$$

The corresponding first-order formula is

$$\begin{aligned}&(\exists y)\{EW(x, y) \wedge (\forall x)\{U(y, x) \rightarrow (\exists y)\{EW(x, y) \wedge C_{Peter}(y)\\&\quad \wedge (\forall x)\{(EW(x, y) \wedge Q_{friend}(x, y)) \rightarrow (\forall y)\{U(x, y) \rightarrow (\exists x)\{EW(x, y)\\&\quad \wedge C_{Peter}(x) \wedge (\exists y)\{EW(x, y) \wedge C_{Mary}(y) \wedge Q_{likes}(x, y)\}\}\}\}\}\}\}.\end{aligned}$$

Note that $(K2 = K_{Peter})$ means that $K2$ has the rigid value (agent 'Peter'), since the corresponding predicate C_{Peter} denotes the same agent in every world (see $(Cond1)$). Note also that $\mathbf{Q}_{friend}$ and $\mathbf{Q}_{likes}$ are non-rigid names.

As in the paper [24], let us consider the following classical example:

(Ex) 'Ralph knows that someone is a spy'.

This sentence has the following two readings:

- Ralph knows that there are spies, but it is possible that he does not know any real spy (*de dicto*).

- Let X be some particular person. Ralph knows of X that he is a spy (*de re*).

The *de dicto* reading of (Ex) is expressed by the formula

$$\text{(Ex-dicto)} \qquad (\exists K1)\{\mathbf{N}_{Ralph}(K1) \wedge K1((\exists K2)\{\mathbf{N}_{spy}(K2)\})\}.$$

The corresponding first-order formula is

$$(\exists y)\{EW(x,y) \wedge N_{Ralph}(y) \wedge (\forall x)\{U(y,x) \rightarrow (\exists y)\{EW(x,y) \wedge N_{spy}(y)\}\}\}.$$

The *de re* reading of (Ex) is expressed by the formula

$$\text{(Ex-re)} \qquad (\exists K1)\{\mathbf{N}_{Ralph}(K1) \wedge (\exists K2)\{\mathbf{D}_{man}(K2) \\ \wedge K1((\exists K2)\{\mathbf{D}_{man}(K2) \wedge \mathbf{N}_{spy}(K2)\})\}\}$$

The corresponding first-order formula is

$$(\exists y)\{EW(x,y) \wedge N_{Ralph}(y) \wedge (\exists x)\{EW(x,y) \wedge D_{man}(x) \\ \wedge (\forall x)\{U(y,x) \rightarrow (\exists y)\{EW(x,y) \wedge D_{man}(y) \wedge N_{spy}(y)\}\}\}\}.$$

In this formula, we fix the same person in the current world and in all the worlds that Ralph considers possible using rigid name D_{man} (see ($Cond3$)). And then for this particular person, we use non-rigid name N_{spy}.

In first-order modal logic, the notions of *de re* and *de dicto* readings of epistemic sentences have specific definitions in terms of different *scopes* that can be used in evaluating non-rigid names (see [12, p. 313–315]). We have *outer scope* (corresponding to *de re*) if a non-rigid name is evaluated – its referent determined – just once in the *actual* world. And further, this non-rigid name has the same referent in all the worlds that some agent r considers possible from the actual world. We have *inner scope* (corresponding to *de dicto*) if a non-rigid name can be re-evaluated in each situation that r considers possible.

Clearly, the logic $\mathrm{L}(\mathcal{C}, \mathcal{L}^{alt}_{(QK)})$ does not have the expressive power of first-order modal logic. Therefore we redefine the notions of outer and inner scopes in a simpler form.

Definition 4.1. Let $K1$ be a bound variable in the scope of some modal operator in a formula of $\mathcal{L}^{alt}_{(QK)}$. Suppose also that $K1$ denotes an agent with a non-rigid name (for example, $\mathbf{N}_t(K1)$ or $\mathbf{Q}_l(K1, K2)$). Then the non-rigid name of $K1$ has *outer scope reference* if, in addition, $K1$ has a rigid name, i.e., $(K1 = K_j)$ or $\mathbf{D}_k(K1)$; otherwise we have *inner scope*. And similarly for $K2$.

It is obvious that if $K1$ has a rigid name, then we can refer to the same agent out of the scope of the modal operator, in particular, in the actual world. Therefore using rigid names, we mimic the first-order modal notion of outer scope in evaluating non-rigid names.

Traditionally, *de re* reading of epistemic sentences is formalized using *quantifying-in* over agents, where a free variable is separated from its binding quantifier by some modality. For example, the first-order modal formula that expresses the *de re* reading of the sentence (Ex) is

$$(\exists x)\{K_{Ralph}(SPY(x))\}.$$

This formula differs from the following formula that expresses the *de dicto* reading of (Ex):

$$K_{Ralph}((\exists x)\{SPY(x)\}).$$

It is known that in interpretations of these formulas in a given possible-world structure, we have a rigid valuation for the free variable x in the scope of $(\exists x)$ and a non-rigid interpretation for the predicate symbol SPY. This means that if x is evaluated in the current world, then x denotes the same person in the current world and in all the worlds that Ralph considers possible, while the interpretations of SPY can vary from world to world. Such combination of rigid and non-rigid designators allows us to distinguish between the *de re* and *de dicto* readings of (Ex) and other similar epistemic sentences (see [6, pp. 85–86]).

But we do not need to use exactly these designators. In particular, consider the first-order modal logic defined in [12]. This logic is many-sorted, with two distinguished sorts *agent* and *name*. Denote by $\mathcal{L}$ the language of this logic. The language $\mathcal{L}$ includes variables of sort agent and variables of sort name. The syntax of $\mathcal{L}$ allows quantifying-in only for sort name; quantifying-in for sort agent is prohibited. But in spite of this, we can construct formulas of $\mathcal{L}$ that express *de re* reading of epistemic sentences using variables of sort name as rigid designators. Consider, for example, the following formula of $\mathcal{L}$ (see [12, p. 333]):

$$(\forall x)\{(\exists X)\{Loc(X) \wedge In(me, x, X) \wedge K_{me}((\exists y)\{In(me, y, X) \wedge P(y)\})\}\}.$$

In this formula, x is a variable of sort agent; X is a variable of sort name; $Loc(X)$ means that X is a location-type name; me is the following constant of sort agent: formulas of $\mathcal{L}$ are evaluated at (world, agent) pairs, and if the current pair is (w, a), then me denotes a; $In(me, x, X)$ means that agent a calls agent x by name X. As a result, the whole formula is read ‘every agent x existing in the current world has some location-type name X; a calls x by name X; and a knows that there is an agent y such that a calls y by name X and y has name P'.

Note that in this formula, y is a bound variable in the scope of the modal operator K_{me}; y denotes an agent with non-rigid name P; in addition, y has rigid name X. And we obtain that non-rigid name P has outer scope reference, since X is evaluated just once in the actual world. This is similar to Definition 4.1. It is clear that instead of the variable X of sort name, we can use any other name provided that this name is a rigid designator. So it is easily be checked that rigid name $\mathbf{D}_{man}$ in the formula (Ex-re) plays the same role as the variable X of sort name. Hence our examples of this section and Definition 4.1 are given in line with the approach of [12].

Specific epistemic sentences expressible in the logic $\mathrm{L}(\mathcal{C}, \mathcal{L}^{alt}_{(QK)})$ can be defined as follows. For a given agent $K2$, another agent $K1$ can know non-rigid names of $K2$ if, in addition, $K2$ has a rigid name.

Let us compare the expressive power of $\mathcal{PEL}^{alt}_{(QK)}$ with the expressive power of $\mathcal{PEL}_{(QK)}$. Examples of this section and of [24, Sections 3.3 and 4] allow us to conclude:

- logics of $\mathcal{PEL}^{alt}_{(QK)}$ give a more natural representation of the de re and de dicto modalities;
- logics of $\mathcal{PEL}_{(QK)}$ offer a better formalization in the case where knowledge is relative, e.g., 'an agent $K1$ knows that if he is Q-connected to other agents $K2, \ldots, KN$, then a formula $\varphi(K1, K2, \ldots, KN)$ holds'.[17]

As was mentioned above (see Section 3), the fragment FO^2+2E includes *equality*. Clearly, equality strengthens the expressive power of FO^2+2E, and thus of $\mathcal{PEL}^{alt}_{(QK)}$. Note that we can use equality also for logics of $\mathcal{PEL}_{(QK)}$. We have defined logics of $\mathcal{PEL}_{(QK)}$ using a decidable higher-order generalization of the loosely guarded fragment (HO-LGF). In [24, Section 3], we used the following original definition of LGF without *equality*:

- any atomic formula is in LGF;
- LGF is closed under boolean combinations;
- If (i) G (the 'guard') is a conjunction of atomic formulas; (ii) φ is in LGF; (iii) every free variable of φ is free in G; (iv) $\overline{y}$ is a tuple of free

[17] Such knowledge is essential, in particular, for tasks of dynamic epistemic logic. For example, the paper [20, p. 70] claims that the knowledge required for action is often relative to the agent's perspective – that it is often indexical (or relative) knowledge rather than objective knowledge. For example, if a robot knows the relative position of an object, he can go and pick it up.

variables of G; (v) if x is a free variable of G and y is a variable from $\overline{y}$, then there is a conjunct of G in which x, y both occur;

then $(\exists\overline{y})\{G \wedge \varphi\}$ and $(\forall\overline{y})\{G \rightarrow \varphi\}$ are in LGF (see [15, p. 228]).

The definition of HO-LGF coincides with the definition of LGF provided that the list of all free variables of the formulas G and φ can contain not only individual, but also second-order variables (see [24, Section 3.2]). It is known that LGF with equality is also decidable and, in particular, equalities are allowed as conjuncts of guards (see [9,10], [15, p. 228 (footnotes)]). It is not hard to prove that if LGF with equality is decidable, then HO-LGF with equality is also decidable. Structures of logics of $\mathcal{PEL}_{(QK)}$ (defined in [24]) contain a set of worlds W and a set of accessibility relations $\mathfrak{R}$ (see, for example, Definition 2.1). These structures are simultaneously structures for HO-LGF. Note that it is natural to define equality on W and on $\mathfrak{R}$ separately. In particular, using equality defined on $\mathfrak{R}$, we can extend languages of $\mathcal{PEL}_{(QK)}$ by equalities of the kind $KI = KJ$, where KI and KJ are modal variables.

5. Conclusions

In this paper we have presented the alternative family $\mathcal{PEL}^{alt}_{(QK)}$ of propositional epistemic logics with quantification over agents of knowledge. As was mentioned above (see Introduction), there are two basic decidable fragments of first-order logic such that propositional modal logic can be embedded to these fragments: (1) the two variable fragment; (2) the guarded fragment. The family $\mathcal{PEL}_{(QK)}$ is defined on the basis of the guarded fragment (HO-LGF); and the family $\mathcal{PEL}^{alt}_{(QK)}$ is defined on the basis of the two-variable fragment (FO^2+2E). Thus we obtain examples of such families in all the main directions.

In Section 3 we defined the class $\mathcal{C}$ of structures for logics of $\mathcal{PEL}^{alt}_{(QK)}$ (see Definition 3.2). Let M be an arbitrary structure of $\mathcal{C}$. The domain Ω of M consists of (world, agent) pairs such that a pair $\omega \in \Omega$ is represented as a single element. We reveal parts of ω using two equivalence relations EW and ER defined on Ω. In [23] we have considered another case where

- a domain of some class of structures consists of objects that have constituent parts;
- in a given language, we do not have direct access to these parts and we refer to them indirectly using equivalence relations;

- when we quantify over the domain, we simultaneously quantify over the all constituent parts of objects.

We believe that such an approach can be effective in many areas of logic.

It would be interesting to present other expressive logics of $\mathcal{PEL}^{alt}_{(QK)}$ except $\mathrm{L}(\mathcal{C}, \mathcal{L}^{alt}_{(QK)})$.

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