Aldo Figallo Orellano
Inés Pascual

On Monadic Operators on Modal Pseudocomplemented De Morgan Algebras and Tetravalent Modal Algebras

Abstract. In our paper, monadic modal pseudocomplemented De Morgan algebras (or mmpM) are considered following Halmos' studies on monadic Boolean algebras. Hence, their topological representation theory (Halmos–Priestley's duality) is used successfully. Lattice congruences of an mmpM is characterized and the variety of mmpMs is proven semisimple via topological representation. Furthermore and among other things, the poset of principal congruences is investigated and proven to be a Boolean algebra; therefore, every principal congruence is a Boolean congruence. All these conclusions contrast sharply with known results for monadic De Morgan algebras. Finally, we show that the above results for mmpM are verified for monadic tetravalent modal algebras.

Keywords: Tetravalent modal algebras, Monadic operators, Pseudocomplemented De Morgan algebras.

1. Introduction and Preliminaries

In 1978, Monteiro introduced tetravalent modal algebras (or TMA for short) as algebras $\langle L, \wedge, \vee, \sim, \nabla, 0, 1\rangle$ of type $(2, 2, 1, 1, 0, 0)$ such that $\langle L, \wedge, \vee, \sim, 0, 1\rangle$ are De Morgan algebras which satisfy the following conditions:

(i) $\nabla x \vee \sim x = 1$,

(ii) $\nabla x \wedge \sim x = \sim x \wedge x$.

These algebras arise as a generalization of three–valued Łukasiewicz algebras by omitting the identity $\nabla(x \wedge y) = \nabla x \wedge \nabla y$. The variety of TMAs is generated by the well-known four-element De Morgan algebra expanded with a simple modal operator ∇ (i.e., $\nabla 1 = 1$ and $\nabla x = 0$ for $x \neq 1$. Besides, $\sim 0 = 1$ and $\sim x = x$ for $x \neq 0, 1$). These algebras were studied by I. Loureiro (see [11,12]) in her Ph. D. studies under Monteiro's supervision (for more historical information see [10]).

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TMAs have been studied by different authors. For example, Font and Rius studied this class of algebras with the methods of *abstract algebraic logic* ([10]). More recently, Celani proved that the variety of TMAs has the Amalgamation and the Superamalgamation Property [4]. Besides, M. Coniglio and M. Figallo studied them under the perspective of *paraconsistent logics* [6].

On the other hand, in [8], Figallo considered the subvariety of pseudocomplemented De Morgan algebras which verifies:

(tm) $x \vee \sim x \leq x \vee x^*$.

This author called them modal pseudocomplemented De Morgan algebras (or mpM-algebras). Recall that a pseudocomplemented De Morgan algebra A is a De Morgan algebra with a unary operator * such that every $a \in A$ the element a^* is the pseudocomplement of a; i.e. $a \wedge x = 0$ if and only if $x \leq a^*$. Later, he showed that all mpM-algebra is a TMA by defining $\nabla x = \sim (\sim x \wedge x^*)$. In [9], the authors have proven that the subdirectly irreducible mpM-algebras are three as TMAs, in fact Hasse diagrams are the same in each case, but 3-chain-mpM-algebra is not a subalgebra of four-elements.

On the other hand, it is worth mentioning that mpM-algebras constitute a proper subvariety of the variety $\mathcal{V}_0$ studied by Sankappanavar in [15]. Furthermore, the variety of mpM-algebras is a proper subvariety of the variety of classical modal De Morgan algebras introduced by S. Celani in [4].

In order to simplify the reading, we will summarize the main notions and results needed throughout this work.

If X is a partially ordered set and $Y \subseteq X$, we will denote by $\uparrow Y$ ($\downarrow Y$) the set of all $x \in X$ such that $y \leq x$ ($x \leq y$) for some $y \in Y$, and we will say that Y is increasing (decreasing) if $Y = \uparrow Y$ ($Y = \downarrow Y$). In particular, we will write $\uparrow y$ ($\downarrow y$) instead of $\uparrow \{y\}$ ($\downarrow \{y\}$). Furthermore, we will denote by $max\, Y$ the set of maximum elements of Y.

In [14], Priestley described a topological duality for pseudocomplemented distributive lattices (for short p-algebras). For this purpose, the category whose objects are d-spaces and whose morphisms are d-functions was considered. More precisely, a d-space is a Priestley space X which satisfies the following condition: $\downarrow U$ is an open subset of X for all $U \in D(X)$, where $D(X)$ denotes the family of increasing, closed and open subsets of X. Furthermore, a d-function f from a d-space X_1 into another one X_2 is an increasing and continuous function (i.e. a Priestley function) such that $f(maxX_1 \cap \uparrow x) = maxX_2 \cap \uparrow f(x)$ for each $x \in X_1$. Besides, it is proved

(P1) If A is a p-algebra, then the Priestley space $X(A)$ of all prime filters of A is a d-space. Moreover, $\sigma_A : A \longrightarrow D(X(A))$ defined by $\sigma_A(a) = \{P \in X(A) : a \in P\}$ is a d-isomorphism.

(P2) If X is a d-space, then $\langle D(X), \cup, \cap, ^*, \emptyset, X\rangle$ is a p-algebra where $U^* = X\backslash \downarrow U$ for each $U \in D(X)$ and $\varepsilon_X : X \longrightarrow X(D(X))$ defined by $\varepsilon_X(x) = \{U \in D(X) : x \in U\}$ is a homeomorphism and an order isomorphism.

Then, the category of d-spaces and d-functions is naturally equivalent to the dual of the category of p-algebras and their corresponding homomorphisms, where the isomorphisms σ_L and ε_X are the corresponding natural equivalences.

On the other hand, she proved that

(P3) the lattice of all closed subsets Y of $X(A)$ with $maxX(A) \cap \uparrow Y \subseteq Y$ is isomorphic to the dual lattice of all congruences on A.

In 1977, Cornish and Fowler ([7]) restricted Priestley duality for bounded distributive lattices to De Morgan algebras by considering the De Morgan spaces (or m-spaces) as pairs (X, g), where X is a Priestley space and $g : X \longrightarrow X$ is decreasing and continuous function satisfying $g^2 = id_X$. They also defined the m-functions f from an m-space (X_1, g_1) into another one, (X_2, g_2) as Priestley functions which satisfy the additional condition $f \circ g_1 = g_2 \circ f$.

In order to restrict Priestley duality to the case of De Morgan algebras, these authors defined the unary operation $\sim$ on $D(X)$ by

(P4) $\sim U = X\backslash g(U)$ for each $U \in D(X)$,

and the homeomorphism $g_A : X(A) \longrightarrow X(A)$ by

(P5) $g_A(P) = A\backslash\{\sim x : x \in P\}$.

Then, the category of m-spaces and m-functions is naturally equivalent to the dual of the category of De Morgan algebras and their corresponding homomorphisms. In addition, these authors showed that:

(P6) the lattice of all involutive closed subsets of $X(A)$ is isomorphic to the dual of the lattice of all congruences on the De Morgan algebra A, where $Y \subseteq X(A)$ is involutive if $g_A(Y) = Y$.

In our work, we consider monadic modal pseudocomplemented De Morgan algebras ($\mathcal{M}$-algebra) as an algebraic model of monadic predicate four valued logic, following Halmos' studies on monadic Boolean algebras. With the addition of an existential quantifier $\exists$ the variety of $\mathcal{M}$-algebra is no longer finitely generated as occurs in monadic boolean algebras.

The main objective of this paper is to study the lattice of congruences of given $\mathcal{M}$-algebra. Also, the properties of generating algebras were investigated and the poset of principal congruences was proven to be a Boolean algebra. Our results are not verified in monadic De Morgan algebra despite the fact that an $\mathcal{M}$-algebra is special monadic De Morgan algebras, this is because the variety of monadic De Morgan algebras are not semisimple and the intersection of two principal congruences for a given algebra is not always principal congruence (see [1,13]).

2. $\mathcal{M}$-Algebras

Definition 1. A $\mathcal{M}$-algebra is an algebra $(A, \wedge, \vee, \sim, ^*, \exists, 0, 1)$, where the reduct $(A, \wedge, \vee, \sim, ^*, 0, 1)$ is an mpM-algebra and the unary operator $\exists$ (existential quantifier) verifies the following conditions:

(m1) $x \wedge \exists x = x$,

(m2) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$,

(m3) $\exists \sim \exists x = \sim \exists x$,

(m4) $\exists((\sim x)^* \wedge x) = (\sim \exists x)^* \wedge \exists x$,

(m5) $\exists \sim x^* = \sim (\exists x)^*$.

We will denote the variety of $\mathcal{M}$-algebras by $\mathbf{M}$ and as usual, we denote an algebra of the variety by $(A, \exists)$ or simply A.

Next, we indicate some properties of $\mathcal{M}$-algebras.

Proposition 2. *For every $\mathcal{M}$-algebra the following properties hold:*

(m6) $x \leq \exists x$, (m7) $\exists 1 = 1, \exists 0 = 0$, (m8) $\exists\exists x = \exists x$,

(m9) $x \leq y$ *implies* $\exists x \leq \exists y$,

(m10) $\sim x \vee \nabla \exists x = 1$, *where* $\nabla x = \sim \triangle \sim x$ *and* $\triangle x = (\sim x)^* \wedge x$,

(m11) $\exists x \vee \nabla \sim x = 1$

(m12) $\exists A = \{x \in A : \exists x = x\}$ *is an mpM-subalgebra of* A,

(m13) $\exists(x \vee y) = \exists x \vee \exists y$, (m14) $x \leq \nabla x$, (m15) $\nabla(x \vee y) = \nabla x \vee \nabla y$,

(m16) $\triangle \nabla x = \nabla x$, (m17) $\nabla \triangle x = \triangle x$, (m18) $\exists \nabla x = \nabla \exists x$.

Proof. We only prove (m12) and (m18), the others can be seen in [16].

(m12): From (m2) and (m3), it is easy to see that $\exists A$ is a subalgebra of De Morgan algebra A. Besides, let $a \in \exists A$, then $a^* = (\exists a)^*$. According to (m5), we have $\sim a^* =\sim (\exists a)^* = \exists \sim a^*$ and so, $\sim a^* \in \exists A$. Therefore, $a^* \in \exists A$.

(m18): We will show that $\exists \nabla x \leq \nabla \exists x$. Firstly, we note that (1) $\exists \nabla \exists x = \nabla \exists x$. Indeed, as $\exists \nabla \exists x = \exists \sim \triangle \sim \exists x$, from (m3) and (m4) we infer $\exists \nabla \exists x = \exists \sim \triangle \exists \sim \exists x = \exists \sim \exists \triangle \sim \exists x =\sim \exists \triangle \sim \exists x =\sim \triangle \exists \sim \exists x =\sim \triangle \sim \exists x = \nabla \exists x$. On the other hand, according to (m6) and (m15), we conclude that $\nabla x \leq \nabla \exists x$. Therefore, by (m9) we can write $\exists \nabla x \leq \exists \nabla \exists x$. Hence, from (1) we have $\exists \nabla x \leq \nabla \exists x$.

On the oher hand, from $x \leq \nabla x$ and (m9), we have $\exists x \leq \exists \nabla x$ and therefore, by (m15) we infer (2) $\nabla \exists x \leq \nabla \exists \nabla x$. Furthermore, as (3) $\nabla \exists \nabla x = \exists \nabla x$ holds and according to (m16), (m4) and (m17), we write $\nabla \exists \nabla x = \nabla \exists \triangle \nabla x = \nabla \triangle \exists \nabla x = \triangle \exists \nabla x = \exists \triangle \nabla x = \exists \nabla x$. Therefore, from (2) and (3) the proof is complete. ■

From Proposition 2, we can see that $(A, \wedge, \vee, \sim, \triangle, \exists, 0, 1)$ is a monadic tetravalent modal algebra according to [16]. As a consequenceof this, from (m1), (m2) and (m3), we have the reduct $(A, \wedge, \vee, \sim, \exists, 0, 1)$ is a monadic De Morgan algebra. Also, if $\mathcal{M}$-algebra $(A, \exists)$ verifies $\nabla(x \wedge y) = \nabla x \wedge \nabla y$, then $(A, \exists)$ is a monadic 3-valued Łukasiewicz-Moisil algebra (see for example [3]). The latter assertions were our motivations to define an $\mathcal{M}$-algebra.

Definition 3. Let $A \in \mathbf{M}$ and $x \in A$. We say that x is a regular element if $\sim x = x^*$ holds, and we denote the set of regular elements of A by $Reg(A)$.

Lemma 4. *If $A \in \mathbf{M}$, then the following properties hold:*

(m19) $\triangle A = \nabla A = Reg(A)$ *where* $\triangle A = \{\triangle x : x \in A\}$ *and* $\nabla A = \{\nabla x : x \in A\}$,

(m20) $Reg(A) \subseteq B(A)$, *where* $B(A) = \{x \in A : x \vee x^* = 1\}$ *is the set of boolean elements of* A,

(m21) x *is a fixed point of* A *iff* $\nabla x = 1$ *and* $\triangle x = 0$, *where* x *is a fixed point if* $\sim x = x$,

(m22) *if* $\exists A \simeq T_2$, *then* $\sim x$ *is a boolean complement of* x *for all* $x \in A$,

(m23) $(\triangle A, \exists)$ *is a monadic Boolean algebra,*

(m24) $\exists \triangle A$ *is a Boolean algebra,*

(m25) *if* $\exists A \simeq T_3$ *and* z *is a fixed point of* A, *then* z *is a unique fixed point of* A *and it is not a boolean element,* (m26) *if* $\exists A \simeq T_4$, *then there exist two fixed points of* A *such that each one is a boolean complement of the other, where*

$T_2 = \{0,1\}$ *where* $0 < 1$, $\sim 0 = 0^* = 1$, $\sim 1 = 1^* = 0$,
$T_3 = \{0,a,1\}$, *where* $0 < a < 1$, $\sim a = a$, $a^* = 0$, $\sim 0 = 0^* = 1$, $\sim 1 = 1^* = 0$,
$T_4 = \{0,a,b,1\}$ *where* $a \not\leq b$, $b \not\leq a$ *and* $0 < a,b < 1$, $\sim b = a^* = b$, $\sim a = b^* = a$, $\sim 0 = 0^* = 1$, $\sim 1 = 1^* = 0$.

Proof. The proof of (m19), (m20), (m21) and (m22) are easy to get and left to the reader.

(m23): It is easy to see that $\triangle A$ is a Boolean algebra, and from (m4) we infer that $\exists x \in \triangle A$, for each $x \in \triangle A$. Besides, from (m7), (m1) and (m2) we have complete the proof.

(m24): It is a direct consequence of (m23).

(m25): Let $z \in A$ be a fixed point, then $\exists z \in \{0,c,1\}$. Let us see that $\exists z = c$. Indeed, suppose that $\exists z = 0$, then $0 = \nabla\exists z$. On the other hand, from the hypothesis, (m21) and (m18) we conclude $\nabla\exists z = 1$, which is a contradiction. Analogously, we can prove that $\exists z \neq 1$. Furthermore, by (m6) we have $z \leq \exists z = c$, and as $c = \sim c \leq \sim z = z$ holds, we obtain $z = c$.

Suppose now that z is a boolean element of A. Then, since z is a fixed point, we have $z \neq 0$ and $z \neq 1$, and so, $z' \notin \exists A$, where z' is the complement element of z. Therefore, from $\exists z = z$, (m7) and (m2) we infer that $0 = \exists(z \wedge z') = \exists z \wedge \exists z' = z \wedge \exists z'$. Besides, according to (m7) and (m13) we have $1 = \exists(z \vee z') = z \vee \exists z'$. Hence, $\exists z' = z'$ which is a contradiction.

(m26): Let z be a fixed point of A. Following an analogous reasoning to (m25), we prove that $\exists z \neq 0$ and $\exists z \neq 1$. Hence, $\exists z = a$ or $\exists z = b$. Suppose that $\exists z = a$, then $z \leq a$ and therefore, $a = \sim a \leq \sim z = z$. From the latter, we have $z = a$. It is clear that if $\exists z = b$, we have the proof of $z = b$ analogously to the last case. ■

3. Relationship Between Special Subalgebras and Quantifiers

In this section, we will indicate how to obtain all quantifiers that can be defined over an mpM-algebra A by special family of subalgebras.

Let us note that (m9), (m6), (m8) and (m13) are both conditions to define a quantifier on bounded distributive lattices (for short, Q-lattices) (see [5]) and an additive closure operator (see [2, pag. 47]). Therefore, the quantifier $\exists$ is determined by its range $\exists A$.

Proposition 5. *Let* $(A, \exists) \in \mathbf{M}$, *then* $\exists A$ *verifies the following conditions:*

(i) *for each* $x \in A$, $\exists x$ *is the smallest element of* $[x) \cap \exists A$, *where* $[x) = \{y \in A : x \leq y\}$,

(ii) *for each* $x, y \in \exists A$, *if* $x \Rightarrow y$ *exists in* A, *then* $x \Rightarrow y \in \exists A$, *where* $x \Rightarrow y$ *is the pseudocomplemented element of* x *relative to* y,

(iii) $\exists x \in \triangle A$, *for each* $x \in \triangle A$

(iv) $[\triangle x) \cap (\exists x] \cap \exists A \cap \triangle A$ *has a unique element,*

(v) $\sim (\exists x)^* \in [\exists \sim x^*)$.

Proof. According to [5, Proposition 1.2], the conditions (i) and (ii) hold. Besides, from (m4) and (m5) we have that (iii) and (v) are verified. Then, we only have to prove (iv).

(iv): Let us put $B = [(\sim x)^* \wedge x) \cap (\exists x] \cap \exists A \cap \triangle A$. From (m4), we infer that $\exists \triangle x \in B$. Hence, if $z \in B$, then $\exists z = z = \triangle z$ and $\triangle x \leq z \leq \exists x$. From the latter, we conclude $\exists \triangle x \leq z = \exists z$. On the other hand, $z = \triangle z \leq \triangle \exists x$ and according to (m4), we have $z \leq \exists \triangle x$. Therefore, $z = \exists \triangle x$. ■

Proposition 6. *Let* A *be an mpM-algebra and let* M *be a subalgebra of* A. *Assume that the following conditions hold:*

(i) *the set* $[x) \cap M$ *has the smallest element, for each* $x \in M$,

(ii) *if* $x, y \in M$ *and exists* $x \Rightarrow y$ *in* A, *then* $x \Rightarrow y \in M$,

(iii) *for each* $x \in \triangle A$, *then* $\exists_M x \in \triangle A$, *where* $\exists_M x$ *is the smallest element of* $[x) \cap M$,

(iv) $[\triangle x) \cap (\exists_M x] \cap \triangle A \cap M$ *has an unique element,*

(v) $\sim (\exists_M x)^* \in [\exists_M \sim x^*)$.

Then $(A, \exists_M)$ *is an* $\mathcal{M}$*-algebra and* $\exists_M A = M$.

Proof. According to [5, Proposition 1.3], we have that $\exists_M$ verifies (m1), (m2) and $\exists_M A = M$. We shall see that (m3), (m4) and (m5) are verified. Indeed,

(m3): Since $\exists_M x \in M$ and M is a subalgebra of A, then $\sim \exists_M x \in M$. Hence, $\sim \exists_M x \in [\sim \exists_M x) \cap M$ and from the latter, we have $\exists_M \sim \exists_M x \leq \sim \exists_M x$. The other inequality is obtained analogously.

(m4): From (iii) and the fact that M is a subalgebra of A, we infer that $\exists_M \triangle x \in [(\sim x)^* \wedge x) \cap (\exists_M x] \cap \triangle A \cap M$. Furthermore, $\triangle \exists_M x \in [(\sim x)^* \wedge x) \cap (\exists_M x] \cap \triangle A \cap M$. From this last assertion and (iv), we conclude the proof.

(m5): Let $x \in A$. Since $\exists_M$ is an operator which verifies (m1), we have $x \leq \exists_M x$ and so, $\sim x^* \leq \sim (\exists_M x)^*$. Hence, $\sim (\exists_M x)^* \in [\sim x^*) \cap M$. Then, by definition of $\exists_M$, we infer that $\exists_M \sim x^* \leq\sim (\exists_M x)^*$ and from (v), we conclude $\exists_M \sim x^* =\sim (\exists_M x)^*$, which completes the proof. ■

4. Topological Representation for $\mathcal{M}$-Algebras

Next, we are going to show a topological duality for $\mathcal{M}$-algebras. For this task, we will extend the dualities for Q-lattices [5] and for mpM-algebras [9] to our case. Firstly, we will make a synthesis of the duality obtained in [9] and to simplify reading, we summarize the fundamental concepts.

Recall from [9] that (X, g) is a mp_M-space if it is a De Morgan space and a d-space, which satisfies the following condition:

(pm1) $x \leq y$ implies $x = y$ or $g(x) = y$.

An mp_M-function from an mp_M-space to another one is both a De Morgan function and an d-function. Furthermore, we have

(DmpI) If (X, g) is an mp_M-space, then $\langle D(X), \cup, \cap, \sim, ^*, \emptyset, X\rangle$ is an mp_M-algebra where $\sim U = X\backslash g(U)$ and $U^* = X\backslash(U]$, for each $U \in D(X)$. Besides, the function $\varepsilon_X : X \longrightarrow X(D(X))$ is a homeomorphism and order isomorphism.

(DmpII) If A is an mp_M-algebra, then $(X(A), g_A)$ is an mp_M-space. Moreover, the function $\sigma_A : A \longrightarrow D(X(A))$ is an mpM-isomorphism.

Therefore, using the usual procedures, we conclude that the category of mpM-spaces and mpM-functions is naturally equivalent to the dual of the category of mpM-algebras and their corresponding homomorphisms.

Remark 7. ([9]) By virtue of (pm1) we infer that any mpM-space is the cardinal sum of chains, each of them with two elements at most. Then, any totally ordered mpM-space has two elements at most.

Theorem 8. *Let $(A, \exists)$ be an $\mathcal{M}$-algebra and let $S, T \in X(A)$ such that $S \cap \exists A \subseteq T$. Then, there exists $Q \in X(A)$ such that*

(i) *$S \subseteq Q$,*

(ii) *$Q \cap \exists A = T \cap \exists A$.*

Proof. Firstly, let us consider the filter F generated by $\{S \cup (T \cap \exists A)\}$ as $F = F(S \cup (T \cap \exists A))$ and the ideal J generated by $\{\exists A\backslash T\}$ as $J = I(\exists A\backslash T)$, then it is clear that $F \cap J = \emptyset$. Indeed, suppose $x \in F \cap J$ then, there are (1) $k \in \exists A\backslash T$, $s \in S$ and $\exists t \in T \cap \exists A$ such that $s \wedge \exists t \leq x \leq k$. Hence, $\exists(s \wedge \exists t) \leq \exists k = k$ and form (m2), we infer that (2) $\exists s \wedge \exists t \leq k$. Since

$\exists s \in S \cap \exists A$ and by the hypothesis, we have $\exists s \in T$ and so, by (2), we write $k \in T$, which contradicts (1). Furthermore, according to Birkhoff-Stones Theorem we know that there exists $Q \in X(A)$ such that (3) $F \subseteq Q$ and $Q \cap J = \emptyset$. By (3), we have that $S \subseteq Q$ and $T \cap \exists A \subseteq Q$. Therefore, $T \cap \exists A \subseteq Q \cap \exists A$. On the other hand, let us consider $x \in Q \cap \exists A$ and suppose that $x \notin T$ then, we obtain $x \in \exists A \backslash T \subseteq J$ and so, $Q \cap J \neq \emptyset$ which is a contradiction. ■

Theorem 9. *Let $(A, \exists)$ be an $\mathcal{M}$-algebra and let $R_\exists$ an equivalence relation on $X(A)$ defined by $R_\exists = \{(P,Q) \in X(A) \times X(A) : P \cap \exists A = Q \cap \exists A\}$. Then, the following properties hold:*

(i) *the equivalence classes for $R_\exists$ are closed in $X(A)$.*

(ii) *$(P,Q) \in R_\exists$ implies $(g(P), g(Q)) \in R_\exists$.*

(iii) *If for each $(P,Q) \in R_\exists$ there exists $R \in X(A)$ such that $g(R) \subseteq P$, then there are $S,T \in X(A)$, such that $(g(S),T) \in R_\exists$ and $T \subseteq Q$.*

(iv) *If $S,T,Q \in X(A)$ such that $(g(S),T) \in R_\exists$ and $T \subseteq Q$, then there exist $P,R \in X(A)$ such that $(P,Q) \in R_\exists$ and $g(R) \subseteq Q$.*

Proof. (i) and (ii): It follows immediately from [5] and [13], respectively.

(iii): Let $P,Q \in X(A)$ such that (1) $P \cap \exists A = Q \cap \exists A$ and suppose that there exists $R \in X(A)$ such that $g(R) \subseteq P$. Hence, $g(P) \cap \exists A \subseteq R$ and by (1) and (ii), we have that $g(Q) \cap \exists A \subseteq R$. According to Theorem 8, there exists $W \in X(A)$ such that (2) $g(Q) \subseteq W$ and (3) $W \cap \exists A = R \cap \exists A$. Therefore, let us put $g(W) = T$ and $R = S$ and so, by (2), we infer that $T \subseteq Q$. Besides, from the latter and from (3) and (ii), we obtain $g(S) \cap \exists A = T \cap \exists A$.

(iv): Let $S,T,Q \in X(A)$ such that $g(S) \cap \exists A = T \cap \exists A$ and $T \subseteq Q$. Let us put $R = g(T)$ then, it is clear that $g(R) \subseteq Q$ and therefore, $g(R) \cap \exists A \subseteq Q$. By Theorem 8, there is $P \in X(A)$ such that $P \cap \exists A = Q \cap \exists A$, which completes the proof. ■

Recall that in [5], Cignoli defined q-space as a pair (X, R) where X is a Priestley space, R is a equivalence relation such that $R(U) \in D(X)$ for every $U \in D(X)$, and the equivalence classes for R are closed in X. Besides, if (X_1, R_1) and (X_2, R_2) are q-spaces, then $f : X_1 \to X_2$ is an q-function iff $f^{-1}(R_2(U)) = R_1(f^{-1}(U))$ for every $U \in D(X_2)$.

Definition 10. A monadic space (or $\mathcal{M}$-space) is a triple (X, g, R), where (X, g) is an mp_M-space and (X, R) is a q-space that satisfies the following conditions:

(ms1) if $(x, y) \in R$, then $(g(x), g(y)) \in R$, (ms2) $R(U) \cap R(g(U)) \subseteq R(U \cap g(U))$ for every $U \in D(X)$,

(ms3) if for each $U \in D(X)$ there exist $(x, y) \in R$ and $u_0 \in U$ such that $g(u_0) \leq y$, then there is $t_0 \in X$ such that $(t_0, g(u_0)) \in R$ and $t_0 \leq x$,

(ms4) if for each $U \in D(X)$ there are $s_0 \in U$ and $x, t \in X$ such that $t \leq x$ and $(t, g(s_0)) \in R$, then there exist $u_0 \in U$ and $y \in X$ such that $g(u_0) \leq y$ and $(x, y) \in R$.

If (X_1, g_1, R_1) and (X_2, g_2, R_2) are $\mathcal{M}$-spaces, then $f : X_1 \to X_2$ is an $\mathcal{M}$-function iff f is an mp_M-function and a q-function, simultaneously.

Remark 11. Let (X, g) be a De Morgan space and let $W \in D(X)$. Then, $g(\downarrow W) = \uparrow g(W)$. Indeed, suppose that $x \in g(\downarrow W)$ then, there is $w_0 \in W$ such that $g(x) \leq w_0$. Hence, we have $g(w_0) \leq x$ and so, $x \in \uparrow g(W)$. Analogously, we have the other inclusion.

Lemma 12. *If (X, g, R) is an $\mathcal{M}$-space, then it verifies the following properties:*

(i) $g(R(x)) = R(g(x))$ *for every* $x \in X$

(ii) $g(R(Y)) = R(g(Y))$ *for every* $Y \subseteq X$*, where it is clear that* $R(T) = \bigcup_{x \in T} R(x)$ *for all* $T \subseteq X$*,*

Proof. (i): It follows immediately from (ms1).

(ii): It follows immediately from (i) and the fact that $g = g^{-1}$. ■

Definition 13. Let (X, g, R) be an $\mathcal{MS}$-space and let Y be an subset of X is said to be an involutive if $g(Y) = Y$.

Corollary 14. *Let (X, g, R) be an $\mathcal{M}$–space. If Y is an involutive subset of X, so is $R(Y)$ too.*

Proof. It is a direct consequence of Lemma 12. ■

Proposition 15. *If (X, g, R) is an $\mathcal{M}$-space, then* $\mathsf{M}(X) = (D(X), \cup, \cap, \sim, ^*, \exists_R, \emptyset, X)$ *is an $\mathcal{M}$-algebra where for each* $U \in D(X)$*,* $U^* = X \setminus \downarrow U$*,* $\sim U = X \setminus g(U)$ *and* $\exists_R(U) = R(U)$*.*

Proof. By the hypothesis, (DmpI) and from [9], we only have to prove that (m3), (m4) and (m5) hold:

(m3): Let $x \in \exists_R \sim \exists_R U$. Then, there is $y \in \sim \exists_R U$ such that $(x, y) \in R$. From the latter and (ms1), we have (1) $(g(x), g(y)) \in R$ and (2) $g(y) \notin \exists_R U$. Hence, $g(x) \notin \exists_R U$. Indeed, suppose that there exists $z \in U$ such that $(g(x), z) \in R$, then from (1) we infer that $(g(y), z) \in R$ from which it follows $g(y) \in \exists_R U$, which contradicts (2). Therefore, $x \in \sim \exists_R U$. The other inclusion follows similarly.

(m4): It is a direct consequence of (ms2) and Lemma 12.

(m5): According to Remark 11 and Lemma 12, we have that (1) $\sim (\exists_R U)^* = X \backslash g((\exists_R U)^*) = X \backslash g(X \backslash \downarrow \exists_R U) = g(\downarrow \exists_R U) = \uparrow g(\exists_R U) = \uparrow g(\bigcup_{s \in U} R(s)) = \uparrow \bigcup_{s \in U} R(g(s))$. Therefore, (2) $\exists_R \sim U^* = \bigcup_{s \in \sim U^*} R(s) = \bigcup_{s \in \uparrow g(U)} R(s)$.

On the other hand, let $x \in \sim (\exists_R U)^*$, then as a consequence of (1) there is $t_0 \in \bigcup_{s \in U} R(g(s))$ such that $t_0 \leq x$. Hence, from (ms4) there exist $u_0 \in U$ and $y \in X$ such that $g(u_0) \leq y$ and $(x, y) \in R$. From the latter, we have that $y \in \uparrow g(U)$, from which it follows by (2) that $x \in \exists_R \sim U^*$. Vice versa, let $x \in \exists_R \sim U^*$. Then, from (2) we infer that there exists $s_0 \in \uparrow g(U)$ such that $(x, s_0) \in R$ and besides, there is $u_0 \in U$ such that $g(u_0) \leq s_0$. From the last assertions and (ms3), we obtain that there exists $t_1 \in X$ such that $(t_1, g(u_0)) \in R$ and $t_1 \leq x$. Therefore, $t_1 \in \bigcup_{s \in U} R(g(s))$ and so, $x \in \uparrow \bigcup_{s \in U} R(g(s)) = \sim (\exists_R U)^*$, which completes the proof. ■

Proposition 16. *If $(A, \exists)$ is an $\mathcal{M}$-algebra, then $\mathsf{m}(A) = (X(A), g_A, R_\exists)$ is the associated $\mathcal{M}$-space where for each $P \in X(A)$, $g_A(P) = A \backslash \sim P$ and $R_\exists = \{(P, Q) \in X(A) \times X(A) : P \cap \exists A = Q \cap \exists A\}$. Furthermore, the function σ_A is an $\mathcal{M}$-isomorphism.*

Proof. From the hypothesis, we have that $(X(A), \exists_R)$ is a q-space and $(X(A), g_A)$ is an mp_M-space. Then, (ms1) is verified.

(ms2): As a consequence of $\sigma_A(a) = U$ for some $U \in D(X(A))$ and $a \in A$, we only have to prove that $R_\exists(\sigma_A(a)) \cap R_\exists(g_A(\sigma_A(a))) \subseteq R_\exists(\sigma_A(a) \cap g_A(\sigma_A(a))$ holds. Which is an immediately consequence of Lemma 12 and $\sigma_A(\exists \triangle a) = \sigma_A(\triangle \exists a)$.

The conditions (ms3) and (ms4) follow immediately from Theorem 9. ■

From Propositions 15 and 16 and using the usual procedures, we conclude the following theorem.

Theorem 17. *The category of the $\mathcal{M}$-space and of the $\mathcal{M}$-function is naturally equivalent to the dual of the category of the $\mathcal{M}$-algebras and their corresponding homomorphisms.*

5. Congruences

Bearing in mind the above results, our next task is to characterize the lattice $Con(A)$ of $\mathcal{M}$-congruences on $\mathcal{M}$-algebra A.

Definition 18. Let (X, g, R) be an $\mathcal{M}S$-space and let Y be an subset of X. We will say that Y is R-saturated if $Y = R(Y)$.

Theorem 19. *Let $(A, \exists)$ be an $\mathcal{M}$-algebra. Then, the lattice $\mathcal{C}_{IR_S}(X(A))$ of all closed, involutive and $R_\exists$-saturated subsets of $X(A)$ is isomorphic to the dual lattice $Con(A)$, and the isomorphism is the function $\Theta_{CIR_S} : \mathcal{C}_{IR_S}(X(A)) \longrightarrow Con(A)$ defined by $\Theta_{CIR_S}(Y) = \{(a, b) \in A \times A : \sigma_A(a) \cap Y = \sigma_A(b) \cap Y\}$.*

Proof. Suppose $Y \in X(A)$ then, from [9] we know that $\Theta_{IR_S}(Y)$ is a mpM-congruence. So, by virtue of the results established [5, Lemma 3.1], we have that the congruence is compatible with $\exists$.

Conversely, suppose now that $\theta \in Con(A)$ and $Y = \{P \in X(A) : |1|_\theta \subseteq P\}$. Then, we will prove that $Y \in \mathcal{C}_{IR_S}(\mathsf{m}(A))$ and $\theta = \Theta(Y)$. Indeed,

(i) Y is closed: if $Q \notin Y$, then by the hypothesis we infer that $|1|_\theta \not\subseteq Q$ and therefore, $a \in |1|_\theta \backslash Q$. As consequence from the latter, we have that $Q \in X(A) \backslash \sigma_A(a)$. Furthermore, it is simple to verify that $(X(A) \backslash \sigma_A(a)) \cap Y = \emptyset$. Hence, we obtain that $X(A) \backslash \sigma_A(a)$ is an open set that contains Q and $X(A) \backslash \sigma_A(a) \subseteq X(A) \backslash Y$. Therefore, Y is closed.

(ii) Y is involutive: let $P \in Y$ and let $x \in |1|_\theta$. Since $\triangle x \in |1|_\theta$ and then, we can conclude that $\sim x \notin P$. Hence, $|1|_\theta \subseteq g_A(P)$ and so, $g_A(P) \in Y$. The other inclusion follows similarly.

(iii) Y is $R_\exists$-saturated: We only prove that $R(Y) \subseteq Y$ because the other inclusion is immediate. Let us suppose (1) $P \in Y$, then $R(P) \subseteq Y$. Indeed, let (2) $Q \in R(P)$ and let $t \in |1|_\theta$. So, $\sim \exists \sim t \in |1|_\theta$ and then, from (m3), (1) and (2) we have that $\sim \exists \sim t \in P \cap \exists A = Q \cap \exists A$. And, since $\sim \exists \sim t \leq t$ then, we can write $t \in Q$. Therefore, $|1|_\theta \subseteq Q$ and as consequence we infer that $Q \in Y$. Finally, according to Priestley representation for bounded distributive lattices and from Theorem 2.4.6 of [9], we conclude that $\theta = \Theta(Y)$, then the proof is now complete. ■

The following version of Theorem 19 will facilitate to determine the lattice of principal congruences of an given $\mathcal{M}$-algebra. Firstly, let us remark the following:

(i) $Y \subseteq X(A)$ is a closed (open) and involutive set if only if $X(A) \backslash Y$ is an open (closed) and involutive one,

(ii) $\sigma_A(a) \cap Y = \sigma_A(b) \cap Y$ if only if $\sigma_A(a) \triangle \sigma_A(b) \subseteq X(A) \backslash Y$, where $W \triangle Z = (W \backslash Z) \cup (Z \backslash W)$.

Next, we will give another characterization of the lattice of congruences via open sets.

THEOREM 20. *Let A be an $\mathcal{M}$-algebra. Then, the lattice $\mathcal{O}_{IR_S}(X(A))$ of all open, involutive and $R_\exists$-saturated set of $X(A)$ is an isomorphism to lattice $Con(A)$; and this isomorphism is defined as follows: $\Theta_{OIR_S} : \mathcal{O}_{IR_S}(X(A)) \longrightarrow Con(A)$ where $\Theta_{OIR_S}(G) = \{(a,b) \in A \times A : \sigma_A(b) \triangle \sigma_A(a) \subseteq G\}$.*

6. The Semi-simplicity of the Variety of $\mathcal{M}$-Algebras

Our next task is to prove that subdirectly irreducible algebras are simple in the variety of $\mathcal{M}$-algebras, that is to say, the variety is a semisimple one. For this specific purpose, we recall some properties from [9].

LEMMA 21. [9] *If (X, g) is an mp_M-space, then $min\, X \cup max\, X = X$.*

LEMMA 22. [9] *Let (X, g) be an mp_M-space, and let Y be a non-empty and involutive set of X, then Y is increasing and decreasing.*

As direct consequence from Lemma 22, we have the following:

COROLLARY 23. *Let (X, g) be an mp_M-space. If Y is a non-empty and involutive subset of X, then $min\, Y \cup max\, Y \subseteq Y$.*

PROPOSITION 24. *Let (X, g, R) be an $\mathcal{M}S$-space, and let Y be an non-empty and involutive subset of X. Then, the followings conditions are equivalent:*

(i) *$min\, R(y) \subseteq Y$, for each $y \in Y$,*

(ii) *Y is R-saturated.*

PROOF. (i)$\Rightarrow$ (ii): Since R is a reflexive relation, we have that $Y \subseteq R(Y)$. On the other hand, let us suppose $z \in R(Y)$, then there is $y \in Y$ such that $z \in R(y)$. As $R(y)$ is a closed set, then $min\, R(y) \neq \emptyset$. Therefore, there exists $m \in min\, R(y)$ such that $m \leq z$ and by (i), we conclude that $m \in Y$. So, from Lemma 22, we can infer that $z \in Y$ and then, $R(Y) \subseteq Y$.

(ii)$\Rightarrow$ (i): It is an immediate consequence of (ii). ∎

PROPOSITION 25. *Let (X, g, R) be an $\mathcal{M}S$-space. Then X is the only closed and involutive set which contains $min\, X$.*

Proof. Suppose now $Y \subseteq X$, $Y \neq \emptyset$ a closed and involutive set such that (1) $min\, X \subseteq Y$. Let us consider $x \in X$ then, from Lemma 21 we obtain (2) $x \in min\, X$ or (3) $x \in max\, X$. If (2) holds, we have $X \subseteq Y$, and if (3) holds, we infer that there exists $t \in min\, X$ such that $t < x$, then $t \in Y$. From the latter, Lemma 22 and since Y is involutive, we can conclude that Y is increasing and therefore, $x \in Y$. ■

Theorem 26. *Let (X, g, R) be an $\mathcal{MS}$-space such that $D(X)$ is an subdirectly irreducible $\mathcal{M}$-algebra. If Y is a non-empty, closed and involutive subset of X, then the following conditions are equivalent:*

(i) *Y is R-saturated,*

(ii) *$min\, X \subseteq Y$.*

Proof. (i) ⇒ (ii): Let $Y \in \mathcal{C}_{IR_S}(X)$ and suppose that $Y \neq \emptyset$. If $min\, X \not\subseteq Y$, then $Y \neq X$ and therefore $Y \in \mathcal{C}_{IR_S}(X) \backslash \{\emptyset, X\}$. From Theorem 19 and since $D(X)$ is a subdirectly irreducible algebra, we have that there exists $M_0 \in \mathcal{C}_{IR_S}(X) \backslash \{\emptyset, X\}$ such that (1) $S \subseteq M_0$ for each $S \in \mathcal{C}_{IR_S}(X) \backslash \{X\}$. As $M_0 \neq X$ and from Proposition 25, we infer that there is $m \in min\, X$ such that $m \notin M_0$. Besides, as M_0 is involutive then, $g(m) \notin M_0$. Now, let us put $W = R(m) \cup g(R(m))$. Then, W is a closed and involutive subset of X. Furthermore, from Lemma 12 we have that $R(W) = W$. From the last assertions, we can infer that $R(m) \cap M_0 = \emptyset$ and $g(R(m)) \cap M_0 = R(g(m)) \cap M_0 = \emptyset$. Consequently, $W \cap M_0 = \emptyset$ and so, $W \in \mathcal{C}_{IR_S}(X) \backslash \{\emptyset, X\}$ and $W \not\subseteq M_0$, which contradicts (1). Therefore, $min\, X \subseteq Y$.

(ii) ⇒ (i): It is a direct consequence from Proposition 25. ■

Now, we are in a position to prove that subdireclty irreducible algebras are each a simple algebra. Indeed,

Corollary 27. *The variety of $\mathcal{M}$-algebras is semisimple.*

Proof. Let us suppose $Y \in \mathcal{C}_{IR_S}(X)$ such that $Y \neq \emptyset$ then, from Theorem 26 we have that $min\, X \subseteq Y$ and as, Y increasing we infer $Y = X$. Therefore, $\mathcal{C}_{IR_S}(X) = \{\emptyset, X\}$ and so, $D(X)$ is simple. The proof is now complete. ■

7. Principal and Boolean Congruences

In this section we will investigate the poset of principal congruences and prove to be a Boolean algebra.

Let us remark if $a, b \in A$ and suppose $\theta(a, b)$ is a principal congruence generated by (a, b), then since $\theta(a, b) = \theta(a \wedge b, a \vee b)$, we can suppose without loss of generality that $a \leq b$.

Remark 28. Let us suppose that A is an $\mathcal{M}$-algebra, $a, b \in A$ and G is an open, involutive and $R_\exists$-saturated subset of $X(A)$. According to Theorem 20, we have that $(a, b) \in \Theta_{OIR_S}(G)$ if only if $\sigma_A(b) \triangle \sigma_A(a) \subseteq G$. If we suppose that $a \leqslant b$, then we can infer that $(a, b) \in \Theta(G)$ iff $\sigma_A(b) \backslash \sigma_A(a) \subseteq G$.

Proposition 29. *Let A be an $\mathcal{M}$-algebra and let $(X(A), g_A, R_\exists)$ be the $\mathcal{M}$-space associated with A. Let G be an open and involutive subset of $X(A)$. If $a, b \in A$ are such that $a \leqslant b$, then the following conditions are equivalent:*

(i) $\Theta_{OIR_S}(G) = \theta(a, b)$, *where Θ_{OIR_S} is defined in the Theorem 20,*

(ii) *G is the smallest subset of $\mathcal{O}_{IR_S}(X(A))$, in the sense of set inclusion, which contains* $\sigma_A(b) \backslash \sigma_A(a)$.

Proof. (i) $\Rightarrow$ (ii): According to hypothesis and Remark 28, we have that $\sigma_A(b) \backslash \sigma_A(a) \subseteq G$. On the other hand, if $H \in \mathcal{O}_{IR_S}(X(A))$ is such that $\sigma_A(b) \backslash \sigma_A(a) \subseteq H$, then we can infer $(a, b) \in \Theta_{OIR_S}(H)$. Therefore, from (i) we obtain that $\Theta_{OIR_S}(G) \subseteq \Theta_{OIR_S}(H)$ and from Theorem 20, we have $G \subseteq H$.

(ii) $\Rightarrow$ (i): From the hypothesis and Remark 28, we can write $(a, b) \in \Theta_{OIR_S}(G)$. Besides, if $\varphi \in Con(A)$ is such that $(a, b) \in \varphi$, then from Theorem 20 there exists $H \in \mathcal{O}_{IR_S}(X(A))$ such that $\Theta_{OIR_S}(H) = \varphi$. From the latter, we have that $\sigma_A(b) \backslash \sigma_A(a) \subseteq H$. Hence, according to (ii) and from Theorem 20, we conclude that $\Theta_{OIR_S}(G) \subseteq \varphi$ and so, $\Theta_{OIR_S}(G) = \theta(a, b)$. ■

Recall that the poset P is *convex* iff for each $S \subseteq P$ and for every $a, b \in S$, then $\{x \in P : a \leq x \leq b\} \subseteq S$ ([2, p. 41]).

Lemma 30. *Let X be an mpM-space and let $R \subseteq X$. Then the following conditions are equivalent:*

(i) *R is a closed, open (henceforth clopen) and convex subset of X,*

(ii) *there exist $W, V \in D(X)$ such that $W \subseteq V$ and $R = V \backslash W$.*

Proof. (i)$\Rightarrow$ (ii): From the hypothesis and [9, Lemma 3.1.1], we have that $\uparrow R \backslash R$ is closed and increasing. On the other hand, from [9, Lemma 3.1.2] there exists $W \in D(X)$ such that $\uparrow R \backslash R \subseteq W$ and $R \cap W = \emptyset$. Let $V = R \cup W$ and V is increasing. Indeed, let $x \in V$ and $y \in X$ such that $x \leq y$. If $x \in W$, then $y \in W \subseteq V$ and if $x \in R$, we have that $y \in R$ or $y \notin R$. In the first case, it is clear that $y \in V$ and in the second case, we infer that $y \in (\uparrow R) \backslash R$, therefore $y \in W \subseteq V$. From the latter, $V, W \in D(X)$.

(ii)$\Rightarrow$ (i): It is an immediate consequence of hypothesis. ■

Next, we we will describe with more precision the open, involutive and R-saturated subset of the space that determine a principal congruence.

Proposition 31. *Let $(A, \exists)$ be an $\mathcal{M}$-algebra and let $(X(A), g_A, R_\exists)$ be the $\mathcal{M}$-space associated. Let G be an open and involutive subset of $X(A)$. If $a, b \in A$ is such that $a \leq b$, then the following conditions are equivalent:*

(i) $\Theta_{OIR_S}(G) = \theta(a, b)$,

(ii) $G = R_\exists((\sigma_A(b) \backslash \sigma_A(a)) \cup g_A(\sigma_A(b) \backslash \sigma_A(a)))$,

(iii) *there is a clopen subset T, of $X(A)$, such that $G = R_\exists(T \cup g_A(T))$.*

Proof. (i) $\Rightarrow$ (ii): It is clear that $\sigma_A(b) \backslash \sigma_A(a) \subseteq G$, and since G is an involutive subset of $X(A)$, we conclude that $(\sigma_A(b) \backslash \sigma_A(a)) \cup g_A(\sigma_A(b) \backslash \sigma_A(a)) \subseteq G$. Furthermore, as G is $R_\exists$-saturated, we have that $R_\exists((\sigma_A(b) \backslash \sigma_A(a)) \cup g_A(\sigma_A(b) \backslash \sigma_A(a))) \subseteq G$.

On the other hand, as $(\sigma_A(b) \backslash \sigma_A(a)) \cup g_A(\sigma_A(b) \backslash \sigma_A(a))$ is a clopen and involutive subset of $X(A)$, then from Lemma 22, we can infer that it is increasing of $X(A)$. Hence, $R_\exists((\sigma_A(b) \backslash \sigma_A(a)) \cup g_A(\sigma_A(b) \backslash \sigma_A(a))) \in \mathcal{O}_{IR_S}(X(A))$. As $\sigma_A(b) \backslash \sigma_A(a) \subseteq R_\exists(\sigma_A(b) \backslash \sigma_A(a)) \cup g_A(\sigma_A(b) \backslash \sigma_A(a)))$, then according to Proposition 29 we have that $G \subseteq R_\exists((\sigma_A(b) \backslash \sigma_A(a)) \cup g_A(\sigma_A(b) \backslash \sigma_A(a)))$.

(ii) $\Rightarrow$ (iii): If we put $T = \sigma_A(b) \backslash \sigma_A(a)$, then the proof is easy to get and left to the reader.

(iii) $\Rightarrow$ (ii): Now, we have that T is a clopen and convex subset of $X(A)$. Besides, from Lemma 30 there exist $U, V \in D(X(A))$ such that $U \subseteq V$ and $T = V \backslash U$. From the latter, we have that there exist a, $b \in A$ such that $a \leq b$, $U = \sigma_A(a)$ and $V = \sigma_A(b)$. Therefore, $T = \sigma_A(b) \backslash \sigma_A(a)$ which completes the proof.

(ii) $\Rightarrow$ (i): It is a direct consequence of Proposition 29. ■

Theorem 32. *Let A be an $\mathcal{M}$-algebra and $(X(A), g_A, R_\exists)$ the $\mathcal{M}$-space associated. Then, there exists an isomorphism from $\mathcal{CO}_{IR_S}(X(A))$ (the clopen, involutive and $R_\exists$-saturated subsets of $X(A)$) into $Con_P(A)$ (the poset of principal $\mathcal{M}$-congruences of A).*

Proof. Let $G \in \mathcal{CO}_{Is}X(A)$. Then, it is easy to see that $G = R_\exists(G \cup g_A(G))$. Hence, according to Proposition 31, we have $\Theta_{OIR_S}(G) \in Con_P(A)$.

Vice versa, if $\rho \in Con_P(A)$, then from Theorem 20 we have there is $G \in \mathcal{O}_{IR_S}(X(A))$ such that $\rho = \Theta_{OIR_S}(G)$. Then, from Proposition 31 we infer that there exists $T \subseteq X(A)$ clopen such that $G = R_\exists(T \cup g_A(T))$ and so, $G \in \mathcal{CO}_{IR_S}(X(A))$. ■

Let us note that in general two principal congruences of a given De Morgan algebra has non-principal intersection ([1]). In our case, from Theorem 32, we have $Con_P(A)$ is an isomorphic order to a special lattice. Moreover,

Corollary 33. *If A is an $\mathcal{M}$-algebra, then $Con_P(A)$ is a boolean algebra.*

Proof. It follows as a consequence of Theorem 32 and Lemma 12. ■

According to Corollary 33, every principal $\mathcal{M}$-congruence is a boolean $\mathcal{M}$-congruence, now we will show the reciprocal condition.

Proposition 34. *Let A be an $\mathcal{M}$-algebra and $\varphi \in Con(A)$. Then, φ is a boolean $\mathcal{M}$-congruence iff φ is a principal $\mathcal{M}$-congruence.*

Proof. From Theorem 20, we have that $\varphi = \Theta_{OIR_S}(G)$. On the other hand, there is $\rho \in Con(A)$ such that $\rho \vee \Theta_{OIR_S}(G) = A \times A$ and $\rho \wedge \Theta_{OIR_S}(G) = Id_A$. Furthermore, according to Theorem 20, we infer that there exists $H \in \mathcal{O}_{IR_S}(X(A))$ such that $\rho = \Theta_{OIR_S}(H)$. From the latter, we can conclude that $G \cup H = X(A)$ and $G \cap H = \emptyset$, and so $G = X(A) \backslash H$ is a closed subset of $X(A)$. Therefore, $G \in \mathcal{CO}_{Is}X(A))$ and from Theorem 20, we obtain that $\varphi \in Con_P(A)$. The reciprocal condition is a direct consequence from Corolary 33. ■

When A is a finite $\mathcal{M}$-algebra, we have an important consequence:

Corollary 35. *If A is a finite $\mathcal{M}$-algebra, then $Con(A) = Con_P(A) = Con_B(A)$, where $Con_B(A)$ is the lattice of boolean $\mathcal{M}$-congruences of A.*

Next, we are to indicate some facts about mp_M-spaces:

(N1) Let (X, g) be an mp_M-space and let $\{C_i\}_{i \in I}$ be the set of all maximal chains of X. Then, the following conditions hold for every $U \in D(X)$:

(i) $\triangle U = U \cap g(U) = \bigcup_{C_i \subseteq U \cap g(U)} C_i$,

(ii) $\nabla U = U \cup g(U) = \bigcup_{\substack{C_i \cap U \neq \emptyset \\ or \\ C_i \cap g(U) \neq \emptyset}} C_i$.

(N2) Let (X, g) be an mp_M-space and let $U \in D(X)$. Then, we have the following:

(i) $U \in \nabla(D(X))$ iff $U = \nabla U$,
(ii) $U \in \triangle(D(X))$ iff $U = \triangle U$,
(iii) $\nabla(D(X)) = \triangle(D(X))$.

(N3) Let (X, g) be an mp_M-space and $U \in D(X)$. Then, the following is verified:

(i) $U \in \triangle(D(X))$ iff U is a closed, open and involutive subset,
(ii) $U \in B(D(X))$ iff U is a clopen and $U = \bigcup_{x \in U} C_x$, where C_x is a maximal chain which contains to x,
(iii) $\nabla(D(X)) = \triangle(D(X)) \subseteq B(D(X))$.

Proposition 36. *Let (X, g, R) be an $\mathcal{M}$-space and $U \in D(X)$. Then, $U \in \exists_R(\nabla(D(X)))$ if only if U is a clopen, involutive and R-saturated of X.*

Proof. If $U \in \exists_R(\nabla(D(X)))$, then from (N3) and Lemma 22 we have that U is a clopen, involutive and R-saturated of X. The reciprocal condition is a direct consequence of the (N2) and (N3). ■

Theorem 37. *Let A be an $\mathcal{M}$-algebra. Then, the lattices $\exists\nabla(A)$ and Con_P (A) are isomorphic.*

Proof. It is a direct consequence of Proposition 36 and Theorem 32. ■

Corollary 38. *Let A be a finite $\mathcal{M}$-algebra and let $(X(A), g_A, R_\exists)$ be the $\mathcal{M}$-space associated. If $X(\exists(A))$ is the cardinal sum of n chains with two elements, m involutive chains with one element, and $2l$ non-involutive chains (n, m, l positive integers). Then, $|Con_P(A)| = |Con(A)| = 2^{n+m+l}$.*

Proof. It is a direct consequence of Theorem 37 and the fact that $|Con_P$ $(A)| = |\nabla\exists(A)| = |Con_{mp_M}(\exists A)| = 2^{n+m+l}$. ■

8. Another Characterization of Principal $\mathcal{M}$-congruences

Our next task is to obtain a new characterization of principal $\mathcal{M}$-congruences via certain subset of given algebra.

It is well-know that, given a bounded distributive lattice L, we have the congruence determined by a proper filter F of L in the follow way: $S(F) = \{(a, b) \in L \times L :$ *there exist* $f \in F$ *and* $a \wedge f = b \wedge f\}$. Then, it is verified that $Y_F = \{P \in X(L) : F \subseteq P\}$ is a clopen and increasing subset of the Priestley space of L and $\Theta(Y_F) = S(F)$. Furthermore, if $a \in L$ and $F = \uparrow a$, then $Y_{\uparrow a} = \{P \in X(L) : \uparrow a \subseteq P\} = \sigma_L(a)$. Therefore, (#) $\Theta(\sigma_L(a)) = S(\uparrow a)$.

Proposition 39. *Let A be an $\mathcal{M}$-algebra. Then, the following conditions are equivalent:*

(i) *φ is a principal $\mathcal{M}$-congruence of A,*

(ii) *there is $a \in \exists\nabla(A)$ such that $\varphi = S(\uparrow a)$.*

Proof. (i) $\Rightarrow$ (ii): From Theorem 32 there exists G, a clopen, involutive and $R_\exists$-saturated of $X(A)$, such that $\Theta_{OIR_S}(G) = \varphi$. On the other hand, from Lemma 22 and Corolary 36, we infer that $G \in \exists_{R_\exists}\nabla D(X(A))$. From Theorem 37, it is verified that $\exists\nabla(A)$ and $\exists_{R_\exists}\nabla D(X(A))$ are isomorphic. Hence, there is $a \in \exists\nabla(A)$ such that $G = \sigma_A(a)$. From the latter, we can conclude that $\varphi = \Theta_{OIR_S}(\sigma_A(a))$. Bearing in mind that the application Θ_{OIR_S} is a restriction of Θ over $\mathcal{CO}_{IR_S}(X(A))$ and from #, we have that $\varphi = S(\uparrow a)$.

(ii) $\Rightarrow$ (i): Let $\varphi = S(\uparrow a)$ and as $a \in \exists\nabla(A)$, then from (m18) we have that $\sigma_A(a) \in \nabla(\exists_{R_\exists} D(X(A)))$. From (N3), we infer that $\sigma_A(a) \in \mathcal{CO}_{IR_S}(X(A))$. On the other hand, from # it is verified that $S(\uparrow a) = \Theta(\sigma_A(a)) = \Theta_{OIR_S}(\sigma_A(a))$. From the latter and Theorem 32, we conclude that φ is a principal $\mathcal{M}$-congruence. ■

Proposition 40. *Let A be an mpM-algebra and let $(X(A), g_A)$ be the mp_M-space associated. If $a, b \in A$ such that $a \leq b$, then:*

$$\begin{aligned}\theta(a,b) &= S(\uparrow((\nabla a \vee \sim \nabla b) \wedge (\triangle a \vee \sim \triangle b)))\\ &= \Theta_{COI}(\sigma_A((\nabla a \wedge \sim \nabla b) \vee (\triangle a \wedge \sim \triangle b))).\end{aligned}$$

Proof. According to [9, Theorem 3.4.6, Theorem 2.4.6], we can infer that $\theta(a,b) = S(|1|_{\theta(a,b)}) = S(\uparrow d)$ with $d = (\nabla a \vee \sim \nabla b) \wedge (\triangle a \vee \sim \triangle b)$. On the other hand, from (#) and [9, Corolario 3.3.9], and bearing in mind that $\triangle d = d$, we can conclude that $S(\uparrow d) = \Theta_{COI}(\sigma_A(d))$. ■

Proposition 41. *Let A be an $\mathcal{M}$-algebra and let $(X(A), g_A, R_\exists)$ be the $\mathcal{M}$-space associated. If $a, b \in A$ such that $a \leq b$, then:*

$$\begin{aligned}\theta(a,b) &= S(\uparrow(\exists((\nabla a \vee \sim \nabla b) \wedge (\triangle a \vee \sim \triangle b))))\\ &= \Theta_{COIR_S}(\exists_{R_\exists}(\sigma_A((\nabla a \wedge \sim \nabla b) \vee (\triangle a \wedge \sim \triangle b)))).\end{aligned}$$

Proof. It follows from Theorem 32, and Propositions 40 and 39. ■

9. Final Conclusions

It is not hard to see that the congruence properties of the variety of mpM-algebras are verified for variety of tetravalent modal algebras. For instance, for every mpM-algebra A, the lattice of congruences of A is isomorphic to the lattice of $\triangle$-filters; i.e., filters that verify $\triangle F = F$. The same occurs for every tetravalent modal algebra. Moreover, the proof of properties for congruences in an mpM-algebra are the same for a tetravalent modal algebras, which can be transferred to the monadic case. Therefore, all above results are verified in monadic tetravalent modal algebras.

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A. Figallo Orellano
Departamento de Matemática
Universidad Nacional del Sur
Buenos Aires
Argentina
aldofigallo@gmail.com

and

Centro de Lógica, Epistemologia e História da Ciência
Universidade Estadual de Campinas
Campinas
Brasil

I. Pascual
Instituto de Ciencias Básicas
Universidad Nacional de San Juan
San Juan
Argentina
inespascual756@hotmail.com