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A Study in Grzegorczyk Point-Free Topology Part I: Separation and Grzegorczyk Structures

Abstract. This is the first, out of two papers, devoted to Andrzej Grzegorczyk's point-free system of topology from Grzegorczyk (Synthese 12(2–3):228–235, 1960. https://doi.org/10.1007/BF00485101). His system was one of the very first fully fledged axiomatizations of topology based on the notions of *region*, *parthood* and *separation* (the dual notion of *connection*). Its peculiar and interesting feature is the definition of *point*, whose intention is to grasp our geometrical intuitions of points as systems of shrinking regions of space. In this part we analyze (quasi-)separation structures and Grzegorczyk structures, and establish their properties which will be useful in the sequel. We prove that in the class of Urysohn spaces with countable chain condition, to every topologically interpreted representative of a point in the sense of Grzegorczyk's corresponds exactly one point of a space. We also demonstrate that Tychonoff first-countable spaces give rise to complete Grzegorczyk structures. The results established below will be used in the second part devoted to points and topological spaces.

Keywords: Grzegorczyk structures, Point-free topology, Region-based topology, Foundations of topology, Mereology, Mereological fields, Mereological structures.

1. Introduction

Andrzej Grzegorczyk's paper "Axiomatizability of geometry without points" [10] is devoted to construction of points and topological spaces thereof.[1] The presentation is based on a theory of *mereological fields*, whose primitive notions are *spatial body* and *containment of one body in another* (see Section 2). These two are enriched with the binary relation of *being separated* characterized by additional postulates.

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[1] Although in the title of the paper the word 'geometry' is used, the paper itself does not have any reference whatsoever to the typical geometrical concepts, such us the ternary *betweenness* relation or the quaternary relation of *congruence*. Thus what the paper deals with is rather "axiomatizability of *topology* without points".

About the theory developed in [10, p. 231] Grzegorczyk writes:

> The specific axioms of theory T say that:
>
> (i) the basic elements which are named spatial bodies [...] constitute a mereological field (a boolean algebra without zero-element) with a primitive predicate $x \subset y$ (the body x is contained in the body y);
>
> (ii) $x \,)($ y is the relation of being separated: [further are stated the specific axioms A1–A4 for $)($].

From the content of [10] (especially from the isomorphism theorem on p. 235) we infer that for the author mereological fields are mereological structures in the sense of Tarski's, i.e., relational structures which can be obtained from complete Boolean algebras by removing the zero-element.[2]

Our strategy is different, since at the outset we assume rather weak mereological theory and point to its strengthenings in case they are needed to prove any results. For us, *mereological fields* are partially ordered sets $\langle R, \sqsubseteq \rangle$, where R is a set of *regions* and $\sqsubseteq$ is a *parthood relation* such that for all regions x and y:[3]

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[2]The aforementioned isomorphism theorem aside, interpretation of mereological fields based on the quoted passage above poses some difficulties. Grzegorczyk himself points to two papers: Leonard and Goodman [14] and Grzegorczyk [9], which contain theories of different classes of structures. Leonard and Goodman [14] write about mereological structures *obtained* from complete Boolean lattices by removing the zero-element, which are the same (although axiomatized in a different way) structures as used by Tarski [22]. On the other hand, in [9] broader class of structures is studied, some of them do not correspond to *complete* Boolean lattices, others do not correspond to Boolean lattices at all. The author's intention is to study structures which *do not have to* have the unit element, since from his theory the axiom of existence of the unity is absent, yet there is nothing which excludes it from the domain. Among models of his theory there are such which have the unity (none of them, on the other hand, has the zero-element). Similarly, Grzegorczyk mereologies from [9] do not have to be (yet may be) complete, i.e., the counterpart of our ($\exists$sum) (see page 10) does not figure among the axioms from the paper. It is not explicitly assumed in [10] either, however it is needed to prove the isomorphism theorem (which will be included in the second installment of this paper). The theorem in question can be found in [10] with its proof appealing to completeness of the underlying structures, so it is reasonable to admit that ($\exists$sum) is implicitly assumed in [10].

[3]In the sequel we adopt widespread terminology tradition and our *regions* replace *spatial bodies* from [10]. In case $x \sqsubseteq y$ we say that x *is part of* y. In [9, p. 91] '*ingr*' is a syntactical counterpart of our '$\sqsubseteq$' and the author says: "The proposition "A *ingr* B" can be read "A is contained in B" or, after Leśniewski, "A is an ingredient of B" [i.e., either A is a proper part of B or A is identical with B]".

(i) if $x \not\sqsubseteq y$ then there exists the difference of x and y, i.e., the largest region $x - y$ which is part of x and is incompatible with y (x and y do not have a common part);

(ii) x and y have the least upper bound which is their mereological sum $x \sqcup y$.

Note that, by (i), for all x and y from R we obtain:

(iii) if x and y have at least one common part, then they have the greatest lower bound $x \sqcap y$.

These conditions correspond directly to the theory from [9], composed of the axioms M1–M6, as denoted in the paper. The above-mentioned strengthenings we have in mind concern existence of mereological sums of infinite sets and existence of the unity. For example, the proofs of propositions 5.9 and 6.6, and of Corollary 6.7 witness applications of a bounded version of sum existence axiom from page 28 (w∃sum) for infinite sets (finite sets always have sums due to accepted axioms), and some of the representation theorems in the second part of the paper will require its stronger version (∃sum), which we introduce on page 10.

We proceed along the similar line of thought with proofs concerning separations structures, i.e., we prove as much as we can with only three axioms put upon separation relation, since these are enough to demonstrate interesting results.

The main achievement of the first part is analysis of the notion of *representative of a point* (intuitively: the collection of regions shrinking to a unique location in space), with the proofs of facts, that under some standard topological interpretation of *regions*, *parthood* and *separation* (see (df][)):

• in the class of Urysohn spaces with countable chain condition, to every representative of point of a space S corresponds a unique point of S,

• the class of regular open sets of any first-countable Tychonoff space satisfies all axioms of Grzegorczyk's from [10]; this, in a nutshell, means that any first-countable Tychonoff space has enough point representatives in the sense that the place of contact of regions of the space is represented by at least one pre-point in the sense of Grzegorczyk's (in consequence it is also represented by a point, but this will be the object of our study in the second part).

Terminology and properties of topological spaces we make use of in the paper are presented in the "Appendix" on page 39.

2.   Mereological Fields and Mereological Structures

2.1.   Basic Properties of Parthood

We assume that $R \neq \emptyset$ is a set of regions and $\sqsubseteq \subseteq R \times R$ is *a parthood* relation which is reflexive, antisymmetric and transitive. These postulates correspond directly to the axioms M1–M3 from [9, p. 91].

By means of $\sqsubseteq$ we introduce three auxiliary relations: *of being proper part*, *of overlapping* and *of being exterior to*, respectively:

$$x \sqsubset y \ :\Longleftrightarrow\ x \sqsubseteq y \wedge x \neq y\,, \qquad\qquad (\mathrm{df}\sqsubset)$$

$$x \bigcirc y \ :\Longleftrightarrow\ \exists_{z \in R}(z \sqsubseteq x \wedge z \sqsubseteq y), \qquad\qquad (\mathrm{df}\bigcirc)$$

$$x \wr y \ :\Longleftrightarrow\ \neg x \bigcirc y. \qquad\qquad (\mathrm{df}\wr)$$

In the case $x \sqsubset y$ (resp. $x \bigcirc y$; $x \wr y$) we say that x *is proper part of* y (resp. x *overlaps* y; x *is exterior to* y). By definitions, $\bigcirc$ and $\wr$ are symmetric; so if $x \bigcirc y$ (resp. $x \wr y$), then we also say that x and y *overlap* (resp. *are exterior to each other*). Moreover, $\bigcirc$ is reflexive and $\wr$ is irreflexive, and, of course, $\sqsubset$ is irreflexive, transitive, and asymmetric.

The next axiom:

$$\forall_{x,y \in R}\big(x \not\sqsubseteq y \Longrightarrow \exists_{z \in R}(z \sqsubseteq x \wedge z \wr y \wedge \forall_{u \in R}(u \sqsubseteq x \wedge u \wr y \Rightarrow u \sqsubseteq z))\big), \ (\exists -)$$

says that if x is not part of y then there is part z of x which not only is exterior to y but is also the largest among all parts of x which are exterior to y. By means of simple logical transformations and $(\mathrm{df}\wr)$ one may show that $(\exists -)$ is equivalent to the axiom M4 from [9, p. 91].

So for all regions x and y such that $x \not\sqsubseteq y$, the axiom $(\exists -)$ postulates existence of unique region which can be treated as *the difference* of x and y (or the relative complement of y with respect to x), and will be denoted by '$x - y$'. Moreover, the region $x - y$ is equal to the least upper bound of the set $\{u \in R \mid u \sqsubseteq x \wedge u \wr y\}$. Thus, for any $x, y \in R$ such that $x \not\sqsubseteq y$ we put:[4]

$$
\begin{aligned}
x - y &:= (\iota\, z)\big(z \sqsubseteq x \wedge z \wr y \wedge \forall_{u \in R}(u \sqsubseteq x \wedge u \wr y \Rightarrow u \sqsubseteq z)\big) \\
&= \sup_{\sqsubseteq}\{z \in R \mid z \sqsubseteq x \wedge z \wr y\}
\end{aligned}
\qquad (\mathrm{df}-)
$$

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[4]The Greek letter 'ι' stands for the standard description operator. The expression $(\iota\, x)\, \varphi(x)$ is read "the only object x which satisfies the condition $\varphi(x)$". To use 'ι' we first have to ensure both existence and uniqueness of the object that satisfies φ, i.e., we have: $\exists^1_{x \in S}\, \varphi(x)$.

In an obvious way, from $(\exists-)$ we obtain that the partial order $\sqsubseteq$ is separative:

$$\forall_{x,y \in R}\big(x \not\sqsubseteq y \Longrightarrow \exists_{z \in R}(z \sqsubseteq x \land z \wr y)\big). \qquad (\text{sep}_\sqsubseteq)$$

The formula $(\text{sep}_\sqsubseteq)$ traditionally bears the name of *Strong Supplementation Principle*. From $(\text{sep}_\sqsubseteq)$ we obtain the so-called *Weak Supplementation Principle*:

$$\forall_{x,y \in R}\big(x \sqsubset y \Longrightarrow \exists_{z \in R}(z \sqsubset y \land z \wr x)\big). \qquad (\text{WSP})$$

If R has at least two members, then the set R has at least two members which are exterior to each other[5]:

$$|R| > 1 \Longrightarrow \exists_{x,y \in R}\, x \wr y.$$

From this we obtain that non-trivial structures do not have zero-element, i.e.:

$$|R| > 1 \Longrightarrow \neg\, \exists_{x \in R} \forall_{y \in R}\, x \sqsubseteq y.$$

Our last basic axiom has the following form:

$$\forall_{x,y \in R} \exists_{z \in R}\big(x \sqsubseteq z \land y \sqsubseteq z \land \forall_{u \in R}(x \sqsubseteq u \land y \sqsubseteq u \Rightarrow z \sqsubseteq u)\big). \qquad (\exists\sup_2)$$

$(\exists\sup_2)$ is a direct counterpart of Grzegorczyk's M5 from [9] (where it was formulated only by means of primitive notions) and it says that any regions x and y have the least upper bound.

Due to the absence of zero there cannot exist unrestricted infimum operation. Yet still—thanks to the following lemma—we can define a partial operation of *mereological product*.

LEMMA 2.1. ([17]) *The following condition*:

$$\forall_{x,y \in R}\big(x \mathbin{\bigcirc} y \Longrightarrow \exists_{z \in R}(z \sqsubseteq x \land z \sqsubseteq y \land \forall_{u \in R}(u \sqsubseteq x \land u \sqsubseteq y \Rightarrow u \sqsubseteq z))\big)$$
$$(\exists\inf_2)$$

is a consequence of antisymmetry, transitivity and $(\exists-)$.

PROOF. Indeed, if $x \sqsubseteq y$ then we put $z := x$. If $x \mathbin{\bigcirc} y$ and $x \not\sqsubseteq y$, then we put $z := x - (x - y)$, since $x \not\sqsubseteq x - y$. We have $x - (x - y) \sqsubset x$ and $x - (x - y) \sqsubseteq y$. Moreover, assume towards contradiction that for some u: $u \sqsubseteq x$, $u \sqsubseteq y$, and $u \not\sqsubseteq x - (x - y)$. Hence, by $(\exists-)$, for some v we have: $v \sqsubseteq u$ and $v \wr x - (x - y)$. So $v \sqsubseteq x$ and $v \sqsubseteq y$; and also $v \mathbin{\bigcirc} y$. Hence $v \sqsubseteq x - y$; so $v \wr y$. ∎

$(\exists\inf_2)$ says that any overlapping regions x and y have the greatest lower bound. This condition (again, formulated in the primitive terms only) is

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[5]$|A|$ is the cardinal number of a set A.

found in [9] as the axiom M6, and due to Lemma 2.1, it is redundant. Therefore, naming the structures satisfying M1–M6 *Grzegorczyk mereologies* we see that the class **MF** defined by antisymmetry, transitivity, $(\exists -)$, and $(\exists \sup_2)$ is nothing but the class of all Grzegorczyk mereologies.

The classical Leśniewski mereology is based on the notion of mereological sum (not that of supremum). We now show that the theory of the class **MF** is closely related to the aforementioned notion.

2.2. Mereological Sums

We define a relation $\mathsf{sum} \subseteq R \times \mathcal{P}(R)$ by means of the following formula:[6]

$$z \mathsf{\ sum\ } X \ :\Longleftrightarrow\ \forall_{x \in X}\, x \sqsubseteq z \wedge \forall_{y \in R}\big(y \sqsubseteq z \Longrightarrow \exists_{x \in X}\, x \circ y\big) \qquad (\mathsf{df\ sum})$$

and say that *z is a mereological sum of* all members of X in case $z \mathsf{\ sum\ } X$. By reflexivity we have that $\mathsf{sum} \subseteq R \times \mathcal{P}^+(R)$, i.e.:

$$\neg\, \exists_{z \in R}\, z \mathsf{\ sum\ } \emptyset.$$

It is known that (cf. e.g. [15–17]) antisymmetry, transitivity and $(\mathsf{sep}_\sqsubseteq)$ guarantee the uniqueness of mereological sum:

$$\forall_{X \in \mathcal{P}(R)} \forall_{y,z \in R}(y \mathsf{\ sum\ } X \wedge z \mathsf{\ sum\ } X \Longrightarrow y = z) \qquad (\mathsf{u_{sum}})$$

and we obtain:

$$\forall_{X \in \mathcal{P}(R)} \forall_{z \in R}\big(z \mathsf{\ sum\ } X \ \Longleftrightarrow\ \forall_{y \in R}(y \circ z \ \Leftrightarrow\ \exists_{x \in X}\, x \circ y)\big).$$

From the same conditions it follows that (see [8,15,17]):

$$\forall_{X \in \mathcal{P}(R)} \forall_{z \in R}(z \mathsf{\ sum\ } X \ \Longleftrightarrow\ X \neq \emptyset\ \wedge\ z = \sup{}_\sqsubseteq X),$$
$$|R| > 1 \ \Longleftrightarrow\ \mathsf{sum} = \sup{}_\sqsubseteq. \qquad (2.1)$$

Thus, for any mereological field $\langle R, \sqsubseteq \rangle$ we have that: the field is non-trivial iff the relation sum is equal to the relation *of being the least upper bound*.

In light of (2.1), on the base of the remaining axioms, our last axiom $(\exists \sup_2)$ is equivalent to:

$$\forall_{x,y \in R} \exists_{z \in R}\, z \mathsf{\ sum\ } \{x,y\}. \qquad (\exists \mathsf{sum}_2)$$

So, via $(\exists \sup_2)$, we postulate existence of mereological sums for arbitrary pairs of regions. Moreover, by $(\mathsf{df} -)$, for all regions x and y such that $x \not\sqsubseteq y$

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[6]For any set S let $\mathcal{P}(S)$ (resp. $\mathcal{P}_{\mathrm{fin}}(S)$) be the family of all (resp. of all finite) subsets of S. We put $\mathcal{P}^+(S) := \mathcal{P}(S) \backslash \{\emptyset\}$ and $\mathcal{P}^+_{\mathrm{fin}}(S) := \mathcal{P}_{\mathrm{fin}}(S) \backslash \{\emptyset\}$. In general, for any family $\mathcal{F}$ of sets we put $\mathcal{F}^+ := \mathcal{F} \backslash \{\emptyset\}$, i.e., $\mathcal{F}^+$ is the family all non-empty sets from $\mathcal{F}$.

we obtain:

$$x - y \text{ sum } \{z \in R \mid z \sqsubseteq x \wedge z \wr y\}.$$

2.3. Binary Operations of Sum and Product

By $(\exists \sup_2)$, $(\exists \mathsf{sum}_2)$, and (2.1) for all $x, y \in R$ we can put:

$$x \sqcup y := \sup_{\sqsubseteq} \{x, y\} = (\iota z) \, z \text{ sum } \{x, y\}. \tag{df $\sqcup$}$$

For all regions x, y and z we have (cf. e.g. [17]):

$$x \sqcup y = y \sqcup x \qquad\qquad x \sqsubseteq x \sqcup y \tag{2.2}$$

$$z \bigcirc x \sqcup y \iff z \bigcirc x \vee z \bigcirc y \qquad z \wr x \sqcup y \iff z \wr x \wedge z \wr y. \tag{2.3}$$

For all overlapping regions x and y, *the infimum* of the set $\{x, y\}$ will be denoted by '$x \sqcap y$' and will be called the *mereological product* of x and y. Formally, for all $x, y \in R$ such that $x \bigcirc y$ we put:

$$x \sqcap y := \inf_{\sqsubseteq} \{x, y\} = \sup_{\sqsubseteq} \{z \in R \mid z \sqsubseteq x \wedge z \sqsubseteq y\}$$
$$= (\iota z) \, z \text{ sum } \{z \in R \mid z \sqsubseteq x \wedge z \sqsubseteq y\}. \tag{df $\sqcap$}$$

For all overlapping regions x and y we have (cf. e.g. [17]):

$$x \sqcap y = x \sqcap x, \qquad x \sqcap y \sqsubseteq x, \tag{2.4}$$

$$z \sqsubseteq x \sqcap y \iff z \sqsubseteq x \wedge z \sqsubseteq y, \tag{2.5}$$

$$x \sqsubseteq y \iff x \sqcap y = x. \tag{2.6}$$

2.4. The Unity

We call *the unity* (sometimes *the space*) the maximum region in a mereological field $\mathfrak{M}$, if such a maximum exists (and in such case it will be denoted by '$\mathbf{1}$'). For example:

MODEL 2.1 ([**17, p. 118**]). For any set S the structure $\langle \mathcal{P}^+(S), \subseteq \rangle$ is a mereological field with the unity S. But if S is infinite, then $\langle \mathcal{P}_{\text{fin}}^+(S), \subseteq \rangle$ is a mereological field without unity. It is also interesting that $\langle \mathcal{P}_{\text{fin}}^+(S) \cup \{S\}, \subseteq \rangle \notin \mathsf{MF}$ (cf. Remark 2.1).

Convention. If K is a class of structures and $\varphi_1, \ldots, \varphi_n$ are conditions formulated in their language, then:

$$\mathsf{K} + \varphi_1 + \cdots + \varphi_n$$

is a subclass of K which consists of all structures from K which additionally satisfy all $\varphi_1, \ldots, \varphi_n$. Moreover, if among structures from K there are such

that have unity, K1 is a subclass of K restricted to its elements with the unity.   ⊣

According to the above convention, if (∃1) is the condition which postulates existence of unity, **MF1** := **MF** + (∃1) and thanks to Model 2.1 we have: **MF1** ⊊ **MF**.

It is worth observing that structures from **MF****MF1** are not only infinite, but also have a property postulated by Whitehead [25]:

PROPOSITION 2.2. *If* $\mathfrak{M} \in$ **MF****MF1** *then* $\mathfrak{M}$ *has no maximal element with respect to* $\sqsubseteq$, *i.e.,:* $\forall_{x \in R} \exists_{y \in R}\ x \sqsubset y$.

PROOF. For any $x \in R$ there is $z \in R$ such $z \not\sqsubseteq x$. So $x \neq x \sqcup z$ and $x \sqsubset x \sqcup z$, by (2.2).   ∎

LEMMA 2.3. *Let* $\langle R, \sqsubseteq \rangle \in$ **MF** *and for any* $x \in R$ *we put* $R \restriction x := \{y \in R \mid y \sqsubseteq x\}$. *Then the structure* $\langle R \restriction x, \sqsubseteq|_{R \restriction x} \rangle$ *is a mereological field with the unity* x.

PROOF. By (df$\sqcup$) the subset $R \restriction x$ is closed under $\sqcup$. By (2.5), $R \restriction x$ is closed under $\sqcap$ for any two overlapping members of $R \restriction x$. Moreover, if $y \sqsubseteq x$, $z \sqsubseteq x$, and $y \not\sqsubseteq z$, then $y - z \sqsubseteq y \sqsubseteq x$.   ∎

2.5.   Mereological Complement

Let $\langle R, \sqsubseteq \rangle$ be any mereological field with the unity 1. Then for any $x \in R$ we have:

$$x \neq 1 \iff x \sqsubset 1 \iff 1 \not\sqsubseteq x.$$

Hence, by (WSP), since all members of R overlap 1, we have:

$$\forall_{x \in R}\big(x \neq 1 \Longrightarrow \exists_{y \in R}(y \neq 1 \,\wedge\, y \wr x)\big), \tag{2.7}$$

$$\forall_{x,y \in R}\big(x \wr y \Longrightarrow x \neq 1 \neq y\big). \tag{2.8}$$

For any $x \in R$ such that $x \neq 1$ we can define:

$$-x := 1 - x.$$

The object $-x$ will be called the *mereological complement of* x. The operation of complement has the following properties (cf. e.g. [17]):

$$\forall_{x \in R \setminus \{1\}}\ x \wr -x,$$
$$\forall_{x \in R \setminus \{1\}}\ -x \neq 1,$$
$$\forall_{x \in R \setminus \{1\}}\ x = --x,$$
$$\forall_{x \in R \setminus \{1\}}\ x \sqcup -x = 1,$$

$$\forall_{x,y\in R\setminus\{1\}}(-x = -y \iff x = y),$$
$$\forall_{x,y\in R\setminus\{1\}}(x \sqsubseteq y \iff -y \sqsubseteq -x),$$
$$\forall_{x,y\in R\setminus\{1\}}(x \sqsubset y \iff -y \sqsubset -x),$$
$$\forall_{x,y\in R}(x \wr y \iff y \neq 1 \wedge x \sqsubseteq -y),$$
$$\forall_{x,y\in R}(x \not\sqsubseteq y \iff y \neq 1 \wedge x \mathbin{\bigcirc} -y),$$
$$\forall_{x\in R}\forall_{y\in R\setminus\{1\}}(x \mathbin{\bigcirc} -y \implies x - y = x \sqcap -y).$$

2.6. Mereological Fields and Boolean Lattices

There is a strong kinship between mereological fields with unity and Boolean lattices (i.e. lattices that are bounded, complemented and distributive), expressed in the following theorems.

THEOREM 2.4 ([17]). *Let* $\langle B, \leq, {\scriptstyle 0}, {\scriptstyle 1} \rangle$ *be a non-trivial Boolean lattice (i.e.,* ${\scriptstyle 0} \neq {\scriptstyle 1}$*). We put* $R := B\setminus\{{\scriptstyle 0}\}$ *and* $\sqsubseteq := \leq|_R := \leq \cap (R \times R)$*. Then* $\langle R, \sqsubseteq \rangle$ *is a mereological field with the unity* ${\scriptstyle 1}$*.*

THEOREM 2.5 ([17]). *Let* $\langle R, \sqsubseteq \rangle$ *be a mereological field with the unity* 1 *and* 0 *be an arbitrary object such that* $0 \notin R$*. We put* $R^0 := R \cup \{0\}$ *and* $\sqsubseteq^0 := \sqsubseteq \cup (\{0\} \times R^0)$*, i.e., for any* $x, y \in R^0 \colon x \sqsubseteq^0 y :\iff x \sqsubseteq y \vee x = 0$*. Then*:

(i) $\langle R^0, \sqsubseteq^0, 0, 1 \rangle$ *is a non-trivial Boolean lattice.*

(ii) *Let* $\langle R^0, +, \cdot, -, 0, 1 \rangle$ *be the non-trivial Boolean algebra obtained from the Boolean lattice* $\langle R^0, \sqsubseteq^0, 0, 1 \rangle$*. Then for all* $x, y \in R^0$ *we have*:

$$x + y = \begin{cases} x \sqcup y & \text{if } x, y \in R \\ x & \text{if } y = 0 \\ y & \text{if } x = 0 \end{cases} \qquad -x = \begin{cases} -x & \text{if } x \in R\setminus\{1\} \\ 0 & \text{if } x = 1 \\ 1 & \text{if } x = 0 \end{cases}$$

$$x \cdot y = \begin{cases} x \sqcap y & \text{if } x, y \in R \text{ and } x \mathbin{\bigcirc} y \\ 0 & \text{otherwise.} \end{cases}$$

In consequence, if in the set R^0 we define three operations $+$, $\cdot$, and $-$, using equations from (ii), then from the mereological field $\langle R, \sqsubseteq, 1 \rangle$ we obtain a non-trivial Boolean algebra $\langle R^0, +, \cdot, -, 0, 1 \rangle$, the same that we obtain from the Boolean lattice $\langle R^0, \sqsubseteq^0, 0, 1 \rangle$.

In light of Theorems 2.4 and 2.5 we have the following theorem.

THEOREM 2.6 ([17]). *For any non-empty set* R *and for any binary relation* $\sqsubseteq$ *in* R *the following conditions are equivalent*:

(a) $\langle R, \sqsubseteq \rangle$ *is a mereological field with the unity* 1.

(b) *For any* $0 \notin R$, $R^0 := R \cup \{0\}$ *and* $\sqsubseteq^0 := \sqsubseteq \cup (\{0\} \times R^0)$ *the structure* $\langle R^0, \sqsubseteq^0, 0, 1 \rangle$ *is a non-trivial Boolean lattice.*

(c) *For some non-trivial Boolean lattice* $\langle B, \leq, 0, 1 \rangle$ *we have* $R = B \setminus \{0\}$, $\sqsubseteq = \leq|_R$, *and* $1 = 1$.

(d) *For some non-trivial Boolean algebra* $\langle A, +, \cdot, -, 0, 1 \rangle$ *we have* $R = A \setminus \{0\}$, $1 = 1$, *and* $\sqsubseteq = \leq|_R$, *where* $\leq$ *is defined by:* $x \leq y \; :\Longleftrightarrow x + y = y$.

REMARK 2.1 ([17]). We assume that $\langle R, \sqsubseteq \rangle \in \mathbf{MF} \setminus \mathbf{MF1}$.

(i) Let 1 be an arbitrary object such that $1 \notin R$. We put $R^1 := R \cup \{1\}$ and $\sqsubseteq^1 := \sqsubseteq \cup (R \times \{1\})$, i.e., for any $x, y \in R^1$: $x \sqsubseteq^1 y \; :\Longleftrightarrow \; x \sqsubseteq y \lor y = 1$. Then $\langle R^1, \sqsubseteq^1 \rangle$ is not a mereological field (cf. Model 2.1).

Indeed, assume towards contradiction that $\langle R^1, \sqsubseteq^1 \rangle$ is a mereological field. Then 1 is the unity of $\langle R^1, \sqsubseteq^1 \rangle$, $\wr^1 = \wr$, $\sqcup^1|_R = \sqcup$, and for any $x \in R$: $1 -^1 x \in R$ and $x \wr 1 -^1 x$. So we get a contradiction: $1 = x \sqcup (1 -^1 x) \neq 1$ (since for any $x, y \in R$, $x \sqcup y \in R$).

(ii) Notice that $\langle R, \sqsubseteq \rangle$ cannot be created from any mereological field $\langle R', \sqsubseteq' \rangle$ with the unity 1 by deleting this unity.

Indeed, assume towards contradiction that for $\langle R', \sqsubseteq' \rangle$ we have $R := R' \setminus \{1\}$ and $\sqsubseteq := \sqsubseteq'|_R$. Then $\sqcup = \sqcup'|_R$ and $1 -' x \in R$, for any $x \in R$. Hence $x \sqcup (1 -' x) \in R$ and $1 = x \sqcup (1 -' x) \notin R$. $\dashv$

2.7.   Mereological Structures

A *mereological structure* is any separative poset $\langle R, \sqsubseteq \rangle$ that satisfies the following condition:
$$\forall_{X \in \mathcal{P}^+(R)} \exists_{z \in R} \; z \text{ sum } X. \tag{$\exists$sum}$$
Since ($\mathrm{u_{sum}}$) holds in all separative posets, so we also have:
$$\forall_{X \in \mathcal{P}^+(R)} \exists^1_{z \in R} \; z \text{ sum } X. \tag{$\exists^1$sum}$$

Let $\langle R, \sqsubseteq \rangle \in \mathbf{MS}$. Since every nonempty subset of the domain R has the unique sum, we can introduce a unary operation on $\mathcal{P}^+(R)$ of *being the mereological sum of all members of a given non-empty set*:
$$\bigsqcup X := (\iota\, z)\, z \text{ sum } X. \tag{df$\bigsqcup$}$$

All mereological structures satisfy $(\mathrm{df}-)$ and (2.1) (cf. e.g. [8,15]). So for any $X \in \mathcal{P}^+(R)$ we have $\bigsqcup X = \sup_{\sqsubseteq} X$. Moreover:

THEOREM 2.7 ([15,17]). $\mathbf{MS} \subseteq \mathbf{MF1}$. *In more detail*: *if* $\langle R, \sqsubseteq \rangle \in \mathbf{MS}$ *then* $\langle R, \sqsubseteq \rangle \in \mathbf{MF1}$, *where* $1 = \bigsqcup R$ *and for all* $x, y \in R$:

(a) $x \sqcup y = \bigsqcup\{x, y\}$,

(b) *if $x \circ y$ then $x \sqcap y = \bigsqcup\{z \in R \mid z \sqsubseteq x \ \wedge \ z \sqsubseteq y\}$,*

(c) *if $x \not\sqsubseteq y$ then $x - y = \bigsqcup\{z \in R \mid z \sqsubseteq x \ \wedge \ z \wr y\}$,*

(d) *if $x \neq 1$ then $-x = \bigsqcup\{z \in R \mid z \wr x\}$.*

Again, as in case of mereological fields, there is a strong dependence between mereological structures and complete Boolean lattices, expressed in theorems which we obtain, respectively, from Theorems 2.4–2.6 replacing the term 'mereological field with the unity' by the term 'mereological structure' and adding to the term 'Boolean lattice (algebra)' the word 'complete'.[7]

THEOREM 2.8 ([15–18, 23]). *For any non-empty set R and any binary relation $\sqsubseteq$ in R, the following conditions are equivalent*:

(a) *$\langle R, \sqsubseteq \rangle$ is a mereological structure, were 1 is its unity.*

(b) *For some (equivalently: any) $0 \notin R$, for $R^0 := R \cup \{0\}$ and for $\sqsubseteq^0 := \sqsubseteq \cup (\{0\} \times R^0)$ the structure $\langle R^0, \sqsubseteq^0, 0, 1 \rangle$ is a non-trivial complete Boolean lattice.*

(c) *For some non-trivial complete Boolean lattice $\langle B, \leq, 0, 1 \rangle$ we have $R = B \backslash \{0\}$, $\sqsubseteq = \leq|_R$, and $1 = 1$.*

(d) *For some non-trivial complete Boolean algebra $\langle A, +, \cdot, -, 0, 1 \rangle$ we have $R = A \backslash \{0\}$, $1 = 1$, and $\sqsubseteq = \leq|_R$, where $\leq$ is defined by: $x \leq y \ :\Longleftrightarrow x + y = y$.*

Thus, mereological structures may be called *complete mereological fields*.

REMARK 2.2. Biacino and Gerla [2] interpret the term 'mereological field' as "the structure obtained from a complete Boolean algebra **B** by deleting the zero-element, i.e., $\mathcal{R} = \mathbf{B} - \{0\}$" (p. 432). Therefore, their mereological fields are mereological structures, i.e., our complete mereological fields (cf. Theorem 2.8). ⊣

LEMMA 2.9. *Let $= \langle S, \mathscr{O} \rangle$ be any topological space and let $r\mathscr{O}$ be the family of all regular open sets of $\mathcal{T}$.[8] Then the pair $\langle r\mathscr{O}, \subseteq \rangle$ is a complete Boolean lattice with the zero-element $\emptyset$, the unity S, and such that for all $U, V \in r\mathscr{O}$: $U \subseteq V$ iff $\mathrm{Cl}\, U \subseteq \mathrm{Cl}\, V$ (see e.g. [13]). So, in the light of Theorem 2.8, the pair $\langle r\mathscr{O}^+, \subseteq \rangle$ is a mereological structure with $\sqsubseteq := \subseteq$, where for all $U, V \in r\mathscr{O}^+$ we have: $U \circ V$ iff $U \cap V \neq \emptyset$; $U \sqcup V = \mathrm{Int}\,\mathrm{Cl}(U \cup V)$;*

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[7]Concerning these theorems see footnote 1 in ([23, pp. 333–334]).

[8]For all relevant data concerning topological spaces see e.g. [5] and Appendix A.

$U \sqcap V = U \cap V$, *if* $U \mathbin{\bigcirc} V$; $-U = \mathrm{Int}(S \backslash U)$, *if* $U \neq S$; *and* $\bigsqcup \mathcal{U} = \mathrm{Int}\,\mathrm{Cl}\bigcup \mathcal{U}$, *for any* $\mathcal{U} \in \mathcal{P}^+(\mathrm{r}\mathscr{O}^+)$.

By Theorems 2.6 and 2.8, we have **MS** $\subsetneq$ **MF1**. However:

PROPOSITION 2.10 ([15,17]). *All finite mereological fields are mereological structures.*

2.8. Atoms in Mereological Fields

Let $\mathfrak{M} = \langle R, \sqsubseteq \rangle \in$ **MF**. Due to absence of zero-element we have a "natural" notion of *atom*, according to which an atom is a member of R that has no proper parts. Let $\mathrm{At}_{\mathfrak{M}}$ be the set of all atoms of $\mathfrak{M}$. We have that:
$a \in \mathrm{At}_{\mathfrak{M}} \iff \forall_{x \in R}(a \sqsubseteq x \lor a \mathrel{\wr} x).$

We say that $\mathfrak{M}$ is *atomic* iff for any $x \in R$ there exists $a \in \mathrm{At}_{\mathfrak{M}}$ such that $a \sqsubseteq x$.

LEMMA 2.11. $\mathfrak{M}$ *is atomic iff for every* $x \in R$, x sum $\{a \in \mathrm{At}_{\mathfrak{M}} \mid a \sqsubseteq x\}$.

We say that $\mathfrak{M}$ is *atomistic* iff for every $x \in R$, $x = \sup_{\sqsubseteq}\{a \in \mathrm{At}_{\mathfrak{M}} \mid a \sqsubseteq x\}$. If $\mathfrak{M} = \langle R, \sqsubseteq \rangle$ is trivial, then $R = \{1\} = \mathrm{At}_{\mathfrak{M}}$ and 1 sum $\{1\}$ and $1 = \sup_{\sqsubseteq}\{1\}$. So, by the above lemma and (2.1), we obtain: $\mathfrak{M}$ is atomistic iff $\mathfrak{M}$ is atomic.

If $\mathfrak{M}$ has the unity 1 then: $\mathfrak{M}$ is atomic iff the non-trivial Boolean lattice $\langle R^0, \sqsubseteq^0, 0, 1 \rangle$ is atomic. Existence of atoms is independent from all axioms listed above.

A subset of R is an *antichain* iff its any two distinct elements are exterior to each other. We say that a structure $\langle R, \sqsubseteq \rangle$ has *the countable chain condition* (abbrv.: c.c.c.) iff every antichain of its regions is countable.

LEMMA 2.12. *If* $\mathfrak{M}$ *is infinite, then*:

1. *Either* $\mathfrak{M}$ *is atomic and has infinitely many atoms, or for some* $x \in R$ *the set* $R \upharpoonright x$ *is infinite.*

2. $\mathfrak{M}$ *has some infinite antichain.*

PROOF. Suppose that R is infinite.

Ad 1. If for any $x \in R$ the set $R \upharpoonright x$ is finite, then $\mathfrak{M}$ is atomic and $\mathrm{At}_{\mathfrak{M}}$ is infinite, in light of Lemma 2.11.

Ad 2. If $\mathfrak{M}$ is atomic and has infinitely many atoms then $\mathrm{At}_{\mathfrak{M}}$ is an infinite antichain. Otherwise, by the previous point, for some $x \in R$ the set $R \upharpoonright x$ is infinite. Then, by Lemma 2.3 and Theorem 2.5, respectively, $\langle R \upharpoonright x, \sqsubseteq|_{R \upharpoonright x}, x \rangle$ is a mereological field with the unity x and $\langle (R \upharpoonright x)^0, \sqsubseteq|^0_{R \upharpoonright x}, 0, x \rangle$ is a Boolean lattice to which we can apply Proposition 3.4 from Koppelberg [13] and obtain an infinite antichain in $\mathfrak{M}$. ■

We say that $x \in R$ is *atomless* iff there is no atom a such that $a \sqsubseteq x$. $\mathfrak{M}$ is *atomless* iff all its elements are atomless (iff $\mathrm{At}_{\mathfrak{M}} = \emptyset$).

2.9. Filters and Ultrafilters in Mereological Fields

A non-empty subset F of R is a *filter* in $\mathfrak{M} \in$ **MF** iff F fulfills the following two conditions:

- if $x, y \in F$, then both $x \bigcirc y$ and $x \sqcap y \in F$;
- if $x \in F$ and $x \sqsubseteq y$, then $y \in F$.

If $\mathfrak{M}$ has the unity $\mathbf{1}$ then, obviously, $\mathbf{1}$ belongs to all filters in $\mathfrak{M}$.

We say that a filter F is an *ultrafilter* in $\mathfrak{M}$ iff F is a maximal filter in $\mathfrak{M}$ with respect to set theoretical inclusion. Let $\mathrm{Ult}(\mathfrak{M})$ be the family of all ultrafilters of $\mathfrak{M}$. The *Stone map of* $\mathfrak{M}$ is the function $\mathrm{s}: R \to \mathcal{P}(\mathrm{Ult}(\mathfrak{M}))$ defined by $\mathrm{s}(x) := \{F \in \mathrm{Ult}(\mathfrak{M}) \mid x \in F\}$. Standardly we obtain the following fact (as for Boolean lattices; cf. Theorems 2.4–2.6):[9]

PROPOSITION 2.13. *Let* $\mathfrak{M} \in$ **MF1** *and let* F *be any filter in* $\mathfrak{M}$. *Then the following conditions are equivalent*:

(a) $F \in \mathrm{Ult}(\mathfrak{M})$;

(b) *for any* $x \in R \backslash \{\mathbf{1}\}$, *either* $x \in F$ *or* $-x \in F$;

(c) *for all* $x, y \in R$, *if* $x \sqcup y \in F$ *then either* $x \in F$ *or* $y \in F$;

(d) F *is an ultrafilter in the non-trivial Boolean lattice* $\langle R^0, \sqsubseteq^0, 0, \mathbf{1} \rangle$.

We say that a non-empty subset X of R has *finite intersection property* (abbrv.: *f.i.p.*) iff for all $x_1, \ldots, x_n \in X$ $(n > 0)$ there exists the product $x_1 \sqcap \cdots \sqcap x_n$. If X has f.i.p., then X generates the filter $\mathrm{F}_X := \{y \in R \mid \exists_{x_1,\ldots,x_n \in X} \ x_1 \sqcap \cdots \sqcap x_n \sqsubseteq y\}$ in $\mathfrak{M}$. If $X = \{x\}$ then $\mathrm{F}_x := \{y \in R \mid x \sqsubseteq y\}$ is called a *principal filter generated by* x. Moreover, $\mathrm{F}_a \in \mathrm{Ult}(\mathfrak{M})$, for any $a \in \mathrm{At}_{\mathfrak{M}}$. We also have:

LEMMA 2.14. *If* $\mathfrak{M}$ *is finite then the set* $\mathrm{Ult}(\mathfrak{M})$ *is equal to the set of all principal filters generated by atoms in* $\mathfrak{M}$, *i.e.,* $\mathrm{Ult}(\mathfrak{M}) = \{\mathrm{F}_a \mid a \in \mathrm{At}_{\mathfrak{M}}\}$.

We now give general conditions for ultrafilters in structures from **MF**:[10]

PROPOSITION 2.15. *For any filter* F *in* $\mathfrak{M} \in$ **MF** *the following conditions are equivalent*:

(a) $F \in \mathrm{Ult}(\mathfrak{M})$;

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[9]For the class **MS** see e.g. [7,15].

[10]For **MF1** Proposition 2.15 follows from Proposition 2.13 and the definition of a filter.

(b) *for any $x \in R$, either $x \in F$ or there is $y \in F$ such that $y \wr x$;*

(c) *for all $x, y \in R$ such that $x \sqcup y \in F$, either $x \in F$ or $y \in F$.*

Proof. (a) $\Rightarrow$ (b) Let $F \in \mathrm{Ult}(\mathfrak{M})$, $x \in R$ and assume that for any $y \in F$ we have $y \bigcirc x$. Thus $F \cup \{x\}$ has f.i.p. and generates the filter $\mathrm{F}_{F \cup \{x\}}$ such that $x \in \mathrm{F}_{F \cup \{x\}}$ and $F \subseteq \mathrm{F}_{F \cup \{x\}}$. So $x \in F = \mathrm{F}_{F \cup \{x\}}$, by maximality of F.

(b) $\Rightarrow$ (a) Assume that $F \notin \mathrm{Ult}(\mathfrak{M})$. Then there is a filter G in $\mathfrak{M}$ such that $F \subsetneq G$. Hence for some $x \in G$ we have $x \notin F$. Moreover, for any $y \in F \subseteq G$ we have $y \bigcirc x$.

(b) $\Rightarrow$ (c) Suppose that $x \sqcup y \in F$. Assume towards contradiction that $x, y \notin F$. Then there are $z_x, z_y \in F$ such that $z_x \wr x$ and $z_y \wr y$. Hence $z_x \bigcirc z_y$ and $z_x \sqcap z_y \in F$. But $z_x \sqcap z_y \wr x$ and $z_x \sqcap z_y \wr y$, so $z_x \sqcap z_y \wr x \sqcup y$ by (2.3), a contradiction.

(c) $\Rightarrow$ (b) Suppose $x \in R$ and $x \notin F$. We take any z from F. Suppose that $z \bigcirc x$. Then, we have $z = (z - x) \sqcup (z \sqcap x)$, because $z \not\sqsubseteq x$. Since $z \sqcap x \sqsubseteq x$, so $z \sqcap x \notin F$. Therefore, by (c), $z - x \in F$. Of course, $z - x \wr x$.   ∎

3.   Quasi-separation Structures

3.1.   Definition and Basic Properties

Let R be any non-empty set and $\sqsubseteq$ and $)($ be binary relation in R. A triple $\langle R, \sqsubseteq,)(\rangle$ is a *quasi-separation structure* iff it satisfies the following conditions:

$$\langle R, \sqsubseteq \rangle \in \mathbf{MF}, \tag{MF}$$

$$\forall_{x,y \in R}\bigl(x \,)(\, y \implies x \not\sqsubseteq y\bigr), \tag{S1}$$

$$\forall_{x,y \in R}\bigl(x \,)(\, y \implies y \,)(\, x\bigr), \tag{S2}$$

$$\forall_{x,y \in R}\bigl(x \sqsubseteq y \implies \forall_{z \in R}(z \,)(\, y \Rightarrow z \,)(\, x)\bigr). \tag{S3}$$

So $\sqsubseteq$ is a *parthood relation*, $)($ will be called a *relation of being separated* and in the case $x \,)(\, y$ we say that x *is separated from* y or that x and y *are separated*, since the relation $)($ is symmetric, by (S2).

The condition (S1) says that the relations $\sqsubseteq$ and $)($ are disjoint. Thus, from (S1) we obtain that the relation $)($ is irreflexive, i.e.:

$$\forall_{x \in R} \, \neg x \,)(\, x. \tag{irr$_{)(}$}$$

Moreover, the relation $)($ is included in the relation $\wr$, i.e.:

$$\forall_{x,y \in R}\bigl(x \,)(\, y \implies x \wr y\bigr). \tag{$\mathrm{I}^{\wr}_{)(}$}$$
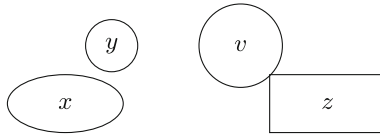
Figure 1. In both pairs regions are external to each other. Regions x and y are separated, but regions v and z are not—they are externally tangent to each other. The relation $\wr$ does not differentiate between these two situations

Indeed, assume towards contradiction that $x \mathbin{)(} y$ and $x \bigcirc y$, i.e., for some z both $z \sqsubseteq x$ and $z \sqsubseteq y$. Then, by (S2) and (S3): $z \mathbin{)(} y$ and $z \mathbin{)(} x$. So $z \not\sqsubseteq y$, by (S1).

Finally, by (2.2) and (S3), we obtain:

$$\forall_{x,y,z \in R}\big(z \mathbin{)(} x \sqcup y \Longrightarrow z \mathbin{)(} x \wedge z \mathbin{)(} y\big). \tag{3.1}$$

The class of all quasi-separation structures is defined as:

$$\mathbf{qSep} := \mathbf{MF} + (\text{S1}) + (\text{S2}) + (\text{S3}). \tag{df\,$\mathbf{qSep}$}$$

The reason behind introducing $)($ is that the relation $\wr$ does not differentiate between two kinds of situations that may hold between regions. The first kind involves regions that are separated, the second one such that are externally tangent to each other (see Figure 1). Clearly, it must be the case that $)(\subseteq \wr$; cf. ($\mathrm{I}_{)(}^{\wr}$). The notion of *external tangency* can be thus expressed by the following difference: $\wr \backslash)($. So the motivation for introducing $)($ could be justified as follows: find a binary relation in R, which will share the essential properties of $\wr$ and will differentiate between regions that are external but are not tangent to each other and those that are both external and tangent (of course in the case there exist such regions in some structure $\langle R, \sqsubseteq,)(\rangle$).

Note that ($\mathrm{I}_{)(}^{\wr}$) "allows for two extreme cases": one in which $)(= \emptyset$, and the other in which $)(= \wr$. Namely:

PROPOSITION 3.1. *Let $\langle R, \sqsubseteq \rangle \in \mathbf{MF}$ and either $)(:= \emptyset$ or $)(:= \wr$. Then in both cases the conditions* (S1)*–*(S3) *are satisfied.*

A quasi-separation structure $\langle R, \sqsubseteq,)(\rangle$ has the unity iff $\langle R, \sqsubseteq \rangle \in \mathbf{MF1}$. In general, we say that a quasi-separation structure $\langle R, \sqsubseteq,)(\rangle$ is *complete* iff $\langle R, \sqsubseteq \rangle \in \mathbf{MS}$. Let $\mathbf{qSep_c}$ be the class of all structures from $\mathbf{qSep}$ which are complete. Since $\mathbf{MS} \subsetneq \mathbf{MF1} \subsetneq \mathbf{MF}$, the inclusions $\mathbf{qSep_c} \subsetneq \mathbf{qSep1} \subsetneq \mathbf{qSep}$ hold by Proposition 3.1.

Let $\langle R, \sqsubseteq,)(\rangle$ belongs to **qSep1**. Since all regions overlap the unity 1, then by $(\mathrm{I}_{)(}^{?})$ we have:

$$\forall_{x \in R} \; \neg x)(1. \tag{3.2}$$

Moreover, if $x \wr y$, then $x \neq 1 \neq y$ and $x \sqsubseteq -y$, by (2.8) and the properties of the operation of complement from pp. 8–9. By (S3) therefore we have:[11]

$$\forall_{x \in R} \forall_{y \in R \setminus \{1\}} \bigl(x \wr y \wedge y)(-y \Longrightarrow y)(x \bigr).$$

3.2. The Relation of *Connection of Regions*

Let $\mathfrak{R} = \langle R, \sqsubseteq,)(\rangle$ be any quasi-separation structure. We introduce the following auxiliary binary relation C in R:

$$x \mathsf{C} y \; :\Longleftrightarrow \; \neg x)(y, \tag{df C}$$

which is called a *relation of being connected*; in the case $x \mathsf{C} y$ we say that regions *x and y are connected*. Of course, by (df C) and, respectively, (S1)–(S3), $(\mathrm{irr}_{)(})$, $(\mathrm{I}_{)(}^{?})$, (3.1) the following conditions hold:

$$\forall_{x,y \in R} \bigl(x \sqsubseteq y \Longrightarrow x \mathsf{C} y \bigr), \tag{C1}$$

$$\forall_{x,y \in R} \bigl(x \mathsf{C} y \Longrightarrow y \mathsf{C} x \bigr), \tag{C2}$$

$$\forall_{x,y \in R} \bigl(x \sqsubseteq y \Longrightarrow \forall_{z \in R} (z \mathsf{C} x \Rightarrow z \mathsf{C} y) \bigr), \tag{C3}$$

$$\forall_{x \in R} \; x \mathsf{C} x, \tag{r_{c}}$$

$$\forall_{x,y \in R} \bigl(x \circ y \Longrightarrow x \mathsf{C} y \bigr), \tag{$\mathrm{I}_{\circ}^{\mathsf{c}}$}$$

$$\forall_{x,y,z \in R} \bigl(z \mathsf{C} x \vee z \mathsf{C} y \Longrightarrow z \mathsf{C} x \sqcup y \bigr). \tag{3.3}$$

The conditions $(\mathrm{r}_{\mathsf{c}})$, (C2), (C1), and $(\mathrm{I}_{\circ}^{\mathsf{c}})$ say, respectively, that the relation C is reflexive, symmetrical, and it includes the relations $\sqsubseteq$ and $\circ$.

For any quasi-separation structure $\langle R, \sqsubseteq,)(\rangle$ with the unity 1, by (3.2), we obtain:

$$\forall_{x \in R} \; x \mathsf{C} 1. \tag{3.4}$$

Every relational structure $\langle R, \sqsubseteq, \mathsf{C} \rangle$ satisfying (MF) and (C1)–(C3) is called a *quasi-connection structure*.

PROPOSITION 3.2. *Any quasi-separation structure* $\langle R, \sqsubseteq,)(\rangle$ *is definitionally equivalent to the quasi-connection structure* $\langle R, \sqsubseteq, \mathsf{C} \rangle$ *via* (df C) *and the following formula:*

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[11] If there exists region u such that u and $-u$ are separated, then the space 1 is not coherent, since $1 = u \sqcup -u$ and $u)(-u$. Generally, we say that a region x is *coherent* iff x is not the sum of any separated regions, i.e., $\neg \exists_{y,z \in R}(x = y \sqcup z \wedge y)(z)$. The term and its definition come from [20]; see also [7]. For some complete G-structures there are regions that are separated from their complements (see e.g. Proposition 6.8).
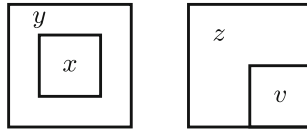
Figure 2. Geometrical interpretation of non-tangential inclusion: x is non-tangentially included in y, while z touches the complement of v

$$\forall_{x,y\in R}\big(x \mathbin{)\!(} y \iff \neg\, x \mathbin{\mathsf{C}} y\big). \tag{df)(}$$

3.3. Non-tangential Inclusion of Regions

From the point of view of developing topology on the basis of quasi-separation structures we need the binary relation $\ll$ of *non-tangential inclusion* between regions (i.e., if $x \ll y$ then we say that x *is non-tangentially included in* y or that x *is a non-tangential part of* y). Intuitively, we want to express the situation in which a region x is part of a region y and is separated from its complement (see Figure 2).[12] The following formula concerns such a situation in any quasi-separation structure $\mathfrak{R} = \langle R, \sqsubseteq, \mathbin{)\!(} \rangle$:

$$x \ll y \ :\iff\ \forall_{z\in R}(z \mathbin{\wr} y \ \Rightarrow\ z \mathbin{)\!(} x), \tag{df $\ll$}$$

i.e., x is non-tangentially included in y iff all regions exterior to y are separated from x. Of course, from $(\mathrm{df}\ll)$, $(\mathrm{df}\wr)$, and $(\mathrm{df}\,\mathsf{C})$ we have:

$$\forall_{x,y\in R}\big(x \ll y \iff \forall_{z\in R}(z \mathbin{\mathsf{C}} x \ \Rightarrow\ z \circ y)\big), \tag{df$'\ll$}$$

i.e., x is non-tangentially included in y iff all regions connected to x also overlap y.

The relation $\ll$ is included in the relation $\sqsubseteq$, i.e., we have:

$$\forall_{x,y\in R}\big(x \ll y \Longrightarrow x \sqsubseteq y\big). \tag{$\mathrm{I}^{\sqsubseteq}_{\ll}$}$$

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[12] The notion of *non-tangential inclusion* expresses inclusion of one region in the interior of another. This is an intuitive interpretation, since the notion of *being the interior of a region* concerns abstract objects, which are represented by non-empty subsets of $\mathcal{P}^+(R)$, i.e., elements of $\mathcal{P}^+(\mathcal{P}^+(R))$. Namely, the interior of a given region is to be a non-empty set of points which will be represented as sets of regions. "Natural" topological interpretation of the relation in question is the following: x is non-tangentially included in y iff the closure of the interior of x is a subset of the interior of y. To speak about the closure of the set we of course need some topology, and this will be introduced via construction of points which will be described in the second part. Then we will be able to show that the aforementioned property is indeed preserved in Grzegorczyk structures.

Indeed, assume that $x \ll y$ and $x \not\sqsubseteq y$. Then $x - y \wr y$ and $x - y \sqsubseteq x$, by (MF). Hence we obtain a contradiction: $x - y \,)(\, x$ and $x - y \,\mathsf{C}\, x$, by (df $\ll$) and (C1), respectively.

Moreover, the relation $\ll$ has the following properties:

$$\forall_{x,y\in R}\big(\forall_{u\in R}\, u \sqsubseteq y \Longrightarrow x \ll y\big), \tag{3.5}$$

$$\forall_{x,y\in R}\big(x \ll y \Longrightarrow \forall_{u\in R}\, u \sqsubseteq y \ \vee\ \exists_{z\in R}(z \wr y \ \wedge\ z \,)(\, x)\big). \tag{3.6}$$

Indeed, for (3.5): If $\forall_{u\in R}\, u \sqsubseteq y$, i.e., if y is the unity, then there is no $z \in R$ such that $z \wr y$. For (3.6): Suppose that $x \ll y$ and there is $u \in R$ such that $u \not\sqsubseteq y$. Then for $z := u - y$ we have $z \wr y$, by (MF). Hence $z \,)(\, x$, by (df $\ll$).

Now notice that from (I$_\ll^\sqsubseteq$) and parthood antisymmetry we obtain:

$$\forall_{x,y\in R}(x \ll y \wedge y \ll x \Longrightarrow x = y). \tag{antis$_\ll$}$$

Moreover, we have the following two conditions:

$$\forall_{x,y,z\in R}\big(x \ll y \wedge y \sqsubseteq z \Longrightarrow x \ll z\big), \tag{3.7}$$

$$\forall_{x,y,z\in R}\big(x \sqsubseteq y \wedge y \ll z \Longrightarrow x \ll z\big). \tag{3.8}$$

Indeed, for (3.7): Let (a) $x \ll y$ and (b) $y \sqsubseteq z$, and (c) $u \wr z$. Then $u \wr y$, by (b), (c), and (MF). Hence $u \,)(\, x$, by (a) and (df $\ll$). Therefore $x \ll z$, by (df $\ll$). For (3.8): Let (a) $x \sqsubseteq y$, (b) $y \ll z$, and (c) $u \wr z$. Then $u \,)(\, y$, by (b), (c), and (df $\ll$). Hence $u \,)(\, x$, by (a) and (S3). Therefore $x \ll z$, by (df $\ll$).

Thus, by (I$_\ll^\sqsubseteq$) and one of (3.7) and (3.8), we have:

$$\forall_{x,y,z\in R}\big(x \ll y \wedge y \ll z \Longrightarrow x \ll z\big), \tag{t$_\ll$}$$

Now we prove:

PROPOSITION 3.3. *For any quasi-separation structure* $\mathfrak{R} = \langle R, \sqsubseteq,)(\,\rangle$:

1. $\ll$ *is reflexive iff* $)(\, = \wr$ *iff* $\wr$ *is included in* $)(\,$ *iff* $\sqsubseteq$ *is included in* $\ll$ *iff* $\sqsubseteq\, = \ll$.

2. *If* $\mathfrak{R}$ *has the unity* 1 *then* $R \times \{1\} \subseteq \ll$.

3. *If* $)(\, = \emptyset$ *and* $\ll\, \neq \emptyset$, *then* $\mathfrak{R}$ *has the unity* 1 *and* $\ll\, = R \times \{1\}$.

4. *If* $\mathfrak{R}$ *has the unity* 1 *and* $\ll\, \subseteq R \times \{1\}$, *then* $)(\, = \emptyset$.

PROOF. *Ad* 1. By (df $\ll$) and (I$_{)(}^\wr$): $\ll$ is reflexive iff $\wr\, \subseteq\,)(\,$ iff $)(\, = \wr$. But, by (I$_\ll^\sqsubseteq$): $\sqsubseteq\, = \ll$ iff $\sqsubseteq\, \subseteq\, \ll$. Moreover, if $\wr\, \subseteq\,)(\,$ then $\sqsubseteq\, \subseteq\, \ll$. Indeed, let $x \sqsubseteq y$. Suppose that $z \wr y$. Then $z \wr x$, by (MF). So also $z \,)(\, x$. Thus, $x \ll y$, by (df $\ll$).

Finally, suppose that $\sqsubseteq\, \subseteq\, \ll$. Then, since $x \sqsubseteq x$, so $x \ll x$. Hence, by (df $\ll$), we have $\forall_{z\in R}(z \wr x \Rightarrow z \,)(\, x))$, i.e., $\wr\, \subseteq\,)(\,$.

Ad 2. If $\mathfrak{R}$ has the unity 1, then $R \times 1 \subseteq \ll$, by (3.5).

Ad 3. If $)(= \emptyset$ and $\mathfrak{R}$ does not have the unity, then for any $y \in R$ there is $z \in R$ such that $z \wr y$. So $\ll = \emptyset$, by (df $\ll$).

Thus, if $)(= \emptyset$ and $\ll \neq \emptyset$, then $\mathfrak{R}$ has the unity 1. If $x \ll y$ and $y \neq 1$, then for some $z \in R$ we have $z \wr y$. So, by (df $\ll$), we obtain a contradiction: $z)(x$. Thus, $\ll \subseteq R \times 1$. Hence $\ll = R \times 1$, by 2.

Ad 4. Suppose that $\mathfrak{R}$ has the unity 1 and $\ll \subseteq R \times \{1\}$. Assume towards contradiction that $)(\neq \emptyset$, i.e., for some $x, y \in R$ we have $x)(y$. Hence $y \neq 1 \neq -y$, by (3.2). If $z \wr -y$, then $z \sqsubseteq y$. So $z)(x$, by (S2) and (S3). Thus, $x \ll -y$, by (df $\ll$). And this entails a contradiction: $-y = 1$.[13] ∎

If $\langle R, \sqsubseteq,)(\rangle$ has the unity 1, then (3.5) and (3.6) have the following forms:
$$\forall_{x \in R} \; x \ll 1, \tag{3.5'}$$
$$\forall_{x,y \in R}\big(x \ll y \Longrightarrow y = 1 \; \vee \; \exists_{z \in R}(z \wr y \; \wedge \; z)(x)\big). \tag{3.6'}$$

We also have the following characterization of the relation $\ll$:
$$\forall_{x,y \in R}\big(x \ll y \iff y = 1 \; \vee \; (y \neq 1 \; \wedge \; x)(-y)\big). \tag{3.9}$$

Indeed, for "$\Rightarrow$" suppose that $x \ll y$ and $y \neq 1$. Then $-y \wr y$. So $x)(-y$, by (df $\ll$) and (S2). "$\Leftarrow$" If $y = 1$, then we use (3.5'). Now suppose that $y \neq 1$, $x)(-y$, and $z \wr y$. Then $z \sqsubseteq -y$. So $z)(x$, by (S3). Thus, $x \ll y$.

Finally, by Proposition 3.3 and (3.9), we obtain:

PROPOSITION 3.4. *For any quasi-separation structure* $\langle R, \sqsubseteq,)(\rangle$ *with the unity* 1:

1. $\sqsubseteq$ *is included in* $\ll$ *iff* $\forall_{y \in R \setminus \{1\}} \; y)(-y$.
2. $)(= \emptyset$ *iff* $\ll \subseteq R \times \{1\}$ *iff* $\ll = R \times \{1\}$.

PROOF. *Ad* 1. Let $\sqsubseteq \; \subseteq \; \ll$. Then $\ll$ is reflexive, by Proposition 3.3. If $y \neq 1$ then $y)(-y$, by (df $\ll$), since $y \wr -y$. Conversely, let for any $y \neq 1$: $y)(-y$. Suppose that $x \sqsubseteq y$. Then $x)(-y$, by (S2) and (S3). Hence $x \ll y$, by (3.9).

Ad 2. Let $)(= \emptyset$ and $x \ll y$. Then $y = 1$. Otherwise, by (3.5'), it would be $x)(-y$, which is contrary to $)(= \emptyset$. Conversely, we use Proposition 3.3 (or (3.9) and (3.5')). ∎

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[13]Notice that from (MF), (S1)–(S3) we cannot infer: if $\ll = \emptyset$, then $)(= \emptyset$. Cf. Proposition 3.4(2).

4.   Separation Structures

A quasi-separation structure $\langle R, \sqsubseteq, \rangle\langle \rangle$ will be called a *separation structure* iff it satisfies both implications converse to (S3) and (3.1), i.e.:

$$\forall_{x,y,z \in R}\big(z \,\rangle\langle\, x \wedge z \,\rangle\langle\, y \Longrightarrow z \,\rangle\langle\, x \sqcup y\big), \tag{S4}$$

$$\forall_{x,y \in R}\big(\forall_{z \in R}(z \,\rangle\langle\, y \Rightarrow z \,\rangle\langle\, x) \Longrightarrow x \sqsubseteq y\big). \tag{S5}$$

We define:

$$\mathbf{Sep} := \mathbf{qSep} + (S4) + (S5).$$

A separation structure $\langle R, \sqsubseteq, \rangle\langle \rangle$ is complete (resp. has the unity) iff $\langle R, \sqsubseteq \rangle \in$ **MS** (resp. $\langle R, \sqsubseteq \rangle \in$ **MF1**). Let $\mathbf{Sep_c}$ be the class of complete separation structures. We have $\mathbf{Sep_c} \subsetneq \mathbf{Sep1} \subsetneq \mathbf{Sep}$.

In all structures from **Sep**, by (S3) and (S5), the relation $\sqsubseteq$ is definable by $\rangle\langle$, i.e.:

$$\forall_{x,y \in R}\big(x \sqsubseteq y \iff \forall_{z \in R}(z \,\rangle\langle\, y \Rightarrow z \,\rangle\langle\, x)\big). \tag{4.1}$$

Note that $\mathbf{Sep} \subsetneq \mathbf{qSep}$, $\mathbf{Sep1} \subsetneq \mathbf{qSep1}$, and $\mathbf{Sep_c} \subsetneq \mathbf{qSep_c}$. Indeed, there is a finite quasi-separation structure that fulfills neither (S4) nor (S5).

MODEL 4.1. Let $\langle R, \sqsubseteq \rangle$ be any mereological structure whose domain contains exactly seven elements, that is $\langle R, \sqsubseteq \rangle$ is obtained from the eight element atomic Boolean lattice by deleting zero. $\mathrm{At}_{\mathfrak{R}}$ has three members a, b, c. For all $x, y \in R$ we put: $x \,\rangle\langle\, y \;:\iff\; x, y \in \mathrm{At}_{\mathfrak{R}} \;\wedge\; x \neq y$. We have $\emptyset \neq \rangle\langle \subsetneq \mathrm{At}_{\mathfrak{R}} \times \mathrm{At}_{\mathfrak{R}} \subsetneq \wr$ and it is easy to check that the conditions (S1)–(S3) are satisfied. But this model does not satisfy (S4), since $c \,\rangle\langle\, a$ and $c \,\rangle\langle\, b$, but $c \,\mathsf{C}\, a \sqcup b$. Moreover, (S5) is not satisfied either. Indeed, since $a \sqcup b \notin \mathrm{At}_{\mathfrak{R}}$, $\forall_{z \in R}(z \,\rangle\langle\, a \sqcup b \Rightarrow z \,\rangle\langle\, 1)$ is trivially true. But $1 \not\sqsubseteq a \sqcup b$. $\dashv$

The conditions (S4) and (S5) are absent from Grzegorczyk's axiomatization, but they are consequences of (MF), (S1)–(S3) and his own axiom (G), that we call *Grzegorczyk axiom* (see Theorem 6.4). For this reason, we establish some properties of separation structures which will be useful later in examination of Grzegorczyk structures.

By (df C), (S4), and (S5), respectively, we obtain the implications converse to (S3) and (3.3), i.e.:

$$\forall_{x,y,z \in R}\big(z \,\mathsf{C}\, x \sqcup y \Longrightarrow z \,\mathsf{C}\, x \vee z \,\mathsf{C}\, y\big), \tag{C4}$$

$$\forall_{x,y \in R}\big(\forall_{z \in R}(z \,\mathsf{C}\, x \Rightarrow z \,\mathsf{C}\, y) \Longrightarrow x \sqsubseteq y\big). \tag{C5}$$

Thus, by (C3) and (C5) (or by (df C) and (4.1)), $\sqsubseteq$ is also definable by C, i.e.:

$$\forall_{x,y\in R}\big(x \sqsubseteq y \iff \forall_{z\in R}(z \mathsf{C} x \Rightarrow z \mathsf{C} y)\big). \tag{4.2}$$

All quasi-connection structures that satisfy (C4) and (C5) are called *connection structures*. Of course, not all quasi-connection structures are connection structures, and this can be seen via Model 4.1, in which: $x \mathsf{C} y \iff x \notin \mathrm{At}_{\mathfrak{R}} \vee y \notin \mathrm{At}_{\mathfrak{R}} \vee x = y$. This model fulfills neither (C4) nor (C5) (cf. Proposition 3.2).

Finally, by application of (3.1), (3.3), (S4), and (C4), we get:

$$\forall_{x,y,z\in R}\big(z \mathbin{)\!(} x \sqcup y \iff z \mathbin{)\!(} x \wedge z \mathbin{)\!(} y\big),$$
$$\forall_{x,y,z\in R}\big(z \mathsf{C} x \sqcup y \iff z \mathsf{C} x \vee z \mathsf{C} y\big).$$

REMARK 4.1. In the literature there is no standard definition of a separation (resp. connection) structure. The axioms chosen by us may be considered as "natural" properties of separation (resp. contact) derived from basic geometrical intuitions concerning the space. Moreover, the axioms are either postulates or theorems of theories which are known as some standard approaches to the problem in the literature (see e.g. [1, 2, 7, 20, 24]). ⊣

Proposition 3.1 says that for any mereological field $\langle R, \sqsubseteq \rangle$ the triple $\langle R, \sqsubseteq, \emptyset \rangle$ is a quasi-separation structure, where $\mathbin{)\!(} := \emptyset$. The situation is different in case of non-trivial separation structures.

PROPOSITION 4.1. *If a non-trivial quasi-separation structure $\langle R, \sqsubseteq, \mathbin{)\!(} \rangle$ satisfies* (S5)*, then $\emptyset \neq \mathbin{)\!(} \subseteq \wr$ and $\circ \subseteq \mathsf{C} \neq R \times R$.*

PROOF. If $\mathbin{)\!(} = \emptyset$, then for all $x, y \in R$ the condition $\forall_{z\in R}(z \mathbin{)\!(} y \Leftrightarrow z \mathbin{)\!(} x)$ is trivially true. So $x = y$, by (S5), i.e., the structure is trivial. By $(\mathrm{I}_{\mathbin{)\!(}}^{\wr})$ and $(\mathrm{I}_{\circ}^{\mathsf{C}})$, respectively, we have suitable inclusions. ∎

It is not difficult to notice that the relation $\wr$ shares all the properties expressed in the axioms (MF), (S1)–(S5).

PROPOSITION 4.2. $\langle R, \sqsubseteq \rangle \in \mathbf{MF}$ *iff* $\langle R, \sqsubseteq, \wr \rangle \in \mathbf{Sep}$ *(with $\mathbin{)\!(} := \wr$).*

More interesting quasi-separation (resp. separation) structures are obtained from topological spaces by means of the following well-known method, which will be useful in various constructions further in the paper.

Let $\mathcal{T} = \langle S, \mathscr{O} \rangle$ be a topological space. Then $\langle \mathrm{r}\mathscr{O}^{+}, \subseteq \rangle \in \mathbf{MS}$, by Lemma 2.9. In $\mathrm{r}\mathscr{O}^{+}$ we define the separation relation $][$ by:

$$U \mathbin{][} V :\iff \mathrm{Cl}\, U \cap \mathrm{Cl}\, V = \emptyset. \tag{df][}$$

PROPOSITION 4.3. *For any topological space* $\mathcal{T} = \langle S, \mathcal{O} \rangle$:

1. [6, p. 87] $\langle \mathrm{r}\mathcal{O}^+, \subseteq,][\,\rangle$ *belongs to* **qSep$_c$** *and satisfies* (S4).[14]

2. [6, p. 92] *The relation* $\ll$ *in* $\mathrm{r}\mathcal{O}^+$, *defined by* $(\mathrm{df} \ll)$, *meets*:[15]

$$U \ll V \iff \mathrm{Cl}\,U \subseteq V.$$

3. [4, Proposition 3.7] *If* $\mathcal{T}$ *is semiregular, then*: $\langle \mathrm{r}\mathcal{O}^+, \subseteq,][\,\rangle$ *satisfies* (S5) *iff* $\mathcal{T}$ *is weakly regular.*

4. [2, Proposition 5.2] *If* $\mathcal{T}$ *is normal, then the relation* $\ll$ *in* $\mathrm{r}\mathcal{O}^+$ *is dense, i.e., satisfies the so-called interpolation axiom*:

$$\forall_{U,V \in \mathrm{r}\mathcal{O}^+}\big(U \ll V \implies \exists_{W \in \mathrm{r}\mathcal{O}^+}(U \ll W \,\wedge\, W \ll V)\big),$$
$$\forall_{U,V \in \mathrm{r}\mathcal{O}^+}\big(\mathrm{Cl}\,U \subseteq V \implies \exists_{W \in \mathrm{r}\mathcal{O}^+}(\mathrm{Cl}\,U \subseteq W \,\wedge\, \mathrm{Cl}\,W \subseteq V)\big). \tag{IA}$$

PROOF. *Ad* 1. Clearly, $\langle \mathrm{r}\mathcal{O}^+, \subseteq,][\,\rangle$ satisfies (MF) and (S1)–(S3). For (S4): For all $A, B, C \in \mathcal{P}(S)$: $A \cap \mathrm{Cl}(B \cup C) = A \cap (\mathrm{Cl}\,B \cup \mathrm{Cl}\,C) = (A \cap \mathrm{Cl}\,B) \cup (A \cap \mathrm{Cl}\,C)$. So for all $U, V, W \in \mathrm{r}\mathcal{O}^+$: $\mathrm{Cl}\,W \cap \mathrm{Cl}(U \sqcup V) = \mathrm{Cl}\,W \cap \mathrm{Cl}\,\mathrm{Int}\,\mathrm{Cl}(U \cup V) = \mathrm{Cl}\,W \cap \mathrm{Cl}(U \cup V) = (\mathrm{Cl}\,W \cap \mathrm{Cl}\,U) \cup (\mathrm{Cl}\,W \cap \mathrm{Cl}\,V)$. Hence: $W\,][\, (U \sqcup V)$ iff $W\,][\, U$ and $W\,][\, V$.

Ad 2. By 1. and (3.9) for all $U, V \in \mathrm{r}\mathcal{O}$ we obtain: $U \ll V$ iff $V = S$ or both $V \neq S$ and $U\,][\, -V$ iff $V = S$ or $\mathrm{Cl}\,U \cap \mathrm{Cl}(-U) = \emptyset$ iff $V = S$ or $\mathrm{Cl}\,U \subseteq S \setminus \mathrm{Cl}(-V)$ iff $V = S$ or $\mathrm{Cl}\,U \subseteq \mathrm{Int}(S \setminus \mathrm{Cl}(-V))$, but $-\mathrm{Cl}-V := \mathrm{Int}(S \setminus \mathrm{Cl}(\mathrm{Int}(S \setminus U)) = \mathrm{Int}(S \setminus (S \setminus \mathrm{Int}(S \setminus -U))) = \mathrm{Int}\,\mathrm{Int}(S \setminus \mathrm{Int}(S \setminus U)) = \mathrm{Int}(S \setminus \mathrm{Int}(S \setminus U)) = \mathrm{Int}(S \setminus \mathrm{Int}(S \setminus U)) = \mathrm{Int}\,\mathrm{Cl}\,U = U$. Thus, we obtain: $U \ll V$ iff $V = S$ or $\mathrm{Cl}\,U \subseteq V$ iff $\mathrm{Cl}\,U \subseteq V$. ∎

REMARK 4.2. Given a topological space $\mathcal{T} = \langle S, \mathcal{O} \rangle$ what we are mainly interested in are its non-empty regular open sets. So instead of $\mathcal{T}$ we can take its semi-regularization $\mathcal{T}_{\mathrm{sr}} := \langle S, \mathcal{O}_{\mathrm{sr}} \rangle$, where the topology $\mathcal{O}_{\mathrm{sr}}$ is generated on S by the basis consisting of all sets from $\mathrm{r}\mathcal{O}$. If $\mathcal{T}$ is semiregular, then $\mathcal{O}_{\mathrm{sr}} = \mathcal{O}$ and $\mathcal{T}_{\mathrm{sr}} = \mathcal{T}$. The space $\mathcal{T}_{\mathrm{sr}} = \langle S, \mathcal{O}_{\mathrm{sr}} \rangle$ itself is of course semiregular and $\mathrm{r}\mathcal{O}_{\mathrm{sr}} = \mathrm{r}\mathcal{O}$ (see Lemma A.6). ⊣

The complete structure $\langle \mathrm{r}\mathcal{O}^+, \subseteq,][\,\rangle$ will be called the *quasi-separation structure associated with* $\mathcal{T}$ and we write: $\mathsf{qsep}\mathcal{T}$. We have $\mathsf{qsep}\mathcal{T} = \mathsf{qsep}\mathcal{T}_{\mathrm{sr}}$, for any topological space $\mathcal{T}$ (see Remark 4.2).

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[14]Düntsch and Winter [4] prove this for regular closed sets, for which (S4) reduces to: $C\,][\, D \iff C \cap D = \emptyset$. As it can be seen in the proof of ours, the transition to regular open sets is not immediate and requires some effort.

[15]Biacino and Gerla [2] state this fact without proof (as Theorem 3.3) unnecessarily requiring that $\mathcal{T}$ be a Hausdorff space.
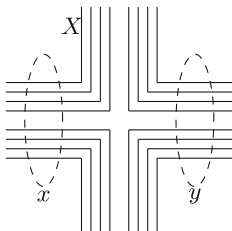
Figure 3. The set X of cross-like regions is not a pre-point

Moreover, by Proposition 4.3, if $\mathcal{T}$ is weakly regular, then $\mathsf{qsep}\mathcal{T}$ belongs to $\mathbf{Sep_c}$ and so it will be called the *separation structure associated with* $\mathcal{T}$; so we write: $\mathsf{sep}\mathcal{T}$. Notice that in this case we have $\mathsf{sep}\mathcal{T} = \mathsf{sep}\mathcal{T}_{\mathrm{sr}}$.

5. Representatives of Points and Points in Quasi-separation Structures

Let $\mathfrak{R} = \langle R, \sqsubseteq, \rangle\langle\;\rangle$ be a quasi-separation structure. Every non-empty subset X of R which satisfies the following three conditions:

$$\forall_{u,v \in X}(u = v \lor u \ll v \lor v \ll u), \tag{r1}$$

$$\forall_{u \in X} \exists_{v \in X}\; v \ll u, \tag{r2}$$

$$\forall_{x,y \in R}\big(\forall_{u \in X}(u \mathbin{\bigcirc} x \land u \mathbin{\bigcirc} y) \implies x \mathbin{\mathsf{C}} y\big), \tag{r3}$$

will be called a *representative of a point* in $\mathfrak{R}$, or a *pre-point* of $\mathfrak{R}$, for short. Let $\mathbf{Q}_{\mathfrak{R}}$ be the family of all pre-points of $\mathfrak{R}$.

Let us analyze a couple of examples in order to grasp the geometrical meaning hidden in the definition of pre-points. For their formal description we may assume that our underlying structure is the Cartesian space $\mathbb{R}^2$, regions are its regular open non-empty subsets and non-tangential inclusion is interpreted as in Proposition 4.3.

In Figure 3 a descending set X of cross-like regions is not a pre-point, since x and y overlap all regions in X, but are not connected with each other. So X does not meet the condition (r3). What X represents is rather a pair of perpendicular lines than a point.

Assuming completeness, in Figure 4 a descending set S of "unbounded" regions is not a pre-point, since both $x := \bigsqcup_{i \in \omega} x_i$ and $y := \bigsqcup_{i \in \omega} y_i$ (where ω is the set of all natural numbers) overlap all regions in S, but are not connected with each other. In consequence, S does not meet the condition (r3). Intuitively, the intention of the third condition is to eliminate points in
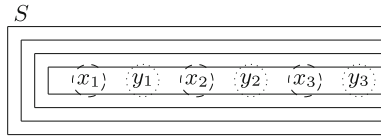
Figure 4. The descending set S of "unbounded" regions is not a pre-point
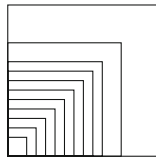

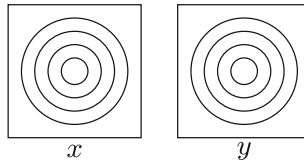Figure 5. The set of rectangular regions is not a pre-point

Figure 6. The set of pairs of circles with the same diameter is not a pre-point

infinity, and as we will show later, this works in the case of quasi-separation structures associated with a certain class of topological spaces.

If we treat the whole sheet of paper as the space, then the set of rectangular regions in Figure 5 is not a pre-point, since the relation ordering the rectangles is not non-tangential inclusion (only parthood). So this set does not meet the condition (r1).

In Figure 6 we consider pairs of circles with the same diameter as regions (which are not coherent). The set depicted above is not a pre-point. The regions x and y overlap all regions in this set but are separated. So the set does not meet the condition (r3). In this case the intention is to eliminate those sets of regions that represent more than one location in space.

Since the role of pre-points is to represent points, what are the points themselves? These are, in a given quasi-separation structure, all filters generated by pre-points.[16]

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[16]We characterize the notion of *point* in quasi-separation structures similarly to Biacino and Gerla [2]. Grzegorczyk himself introduced the definition solely for G-structures (see the definition D_1 in [10, p. 232]).
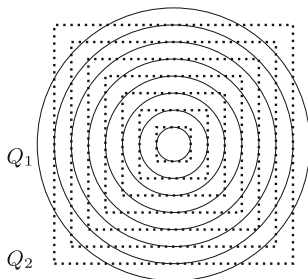
Figure 7. Q_1 and Q_2 represent the same point

Let $\mathfrak{R} = \langle R, \sqsubseteq)(\rangle$ be any quasi-separation structure. In the light (r1), ($\mathrm{I}_{\ll}^{\sqsubseteq}$), and (2.6) all pre-points from $\mathbf{Q}_{\mathfrak{R}}$ have the finite intersection property (see p. 13). So we obtain:

PROPOSITION 5.1. *Every pre-point in $\mathfrak{R}$ generates some filter in mereological field $\langle R, \sqsubseteq \rangle$. For any $Q \in \mathbf{Q}_{\mathfrak{R}}$ the filter generated by Q has the form* $\mathrm{F}_Q := \{ x \in R \mid \exists_{u_1,\dots,u_n \in Q} \ u_1 \sqcap \cdots \sqcap u_n \sqsubseteq x \} = \{ x \in R \mid \exists_{u \in Q} \ u \sqsubseteq x \}.$

By a *point* in $\mathfrak{R}$ we will mean any filter in the mereological field $\langle R, \sqsubseteq \rangle$ generated by some pre-point in $\mathfrak{R}$. Let $\mathbf{Pt}_{\mathfrak{R}}$ be the set of all points in $\mathfrak{R}$. Thus, for any $X \in \mathcal{P}(R)$:

$$X \in \mathbf{Pt}_{\mathfrak{R}} \ :\Longleftrightarrow \ \exists_{Q \in \mathbf{Q}_{\mathfrak{R}}} X = \mathrm{F}_Q. \qquad (\mathrm{df}\,\mathbf{Pt}_{\mathfrak{R}})$$

We will denote elements of $\mathbf{Pt}_{\mathfrak{R}}$ by means of small Gothic letters.

The situation depicted in Figure 7 justifies the definition of point. If we agreed to treat pre-points as points then we would have the situation in which two pre-points representing the same location in space were different points. In other words, we would (usually) have more than one point in the same location, in extreme cases even uncountably many of them.

More formally, there is a class of topological spaces (e.g. first-countable Tychonoff spaces having the countable chain condition; which includes Euclidean spaces) such that for any space $\mathcal{T} = \langle S, \mathscr{O} \rangle$ from this class and for any point $p \in S$ there are distinct pre-points $\mathscr{Q}_1$ and $\mathscr{Q}_2$ in $\mathtt{qsep}\mathcal{T}$ (where $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are non-empty subsets of $\mathrm{r}\mathscr{O}^+$) which correspond to p in the following sense: $\bigcap \mathscr{Q}_1 = \{p\} = \bigcap \mathscr{Q}_2$. But in such case we obtain: $\mathrm{F}_{\mathscr{Q}_1} = \mathrm{F}_{\mathscr{Q}_1}$, i.e., pre-points $\mathscr{Q}_1$ and $\mathscr{Q}_2$ generate the same point (cf. Proposition 5.2 and Theorem 5.13).

Already for quasi-separation structures we get the following interesting facts:

PROPOSITION 5.2 (**cf.** [**10**]). *Let* $\mathfrak{R} = \langle R, \sqsubseteq,)(\rangle \in$ **qSep**. *Then*:

1. *For all* $Q_1, Q_2 \in \mathbf{Q}_\mathfrak{R}$: *if for all* $x \in Q_1$ *and* $y \in Q_2$ *we have* $x \circ y$, *then* Q_1 *and* Q_2 *generate the same point, i.e.,* $\mathrm{F}_{Q_1} = \mathrm{F}_{Q_1}$.

2. *For all* $\mathfrak{p}, \mathfrak{q} \in \mathbf{Pt}_\mathfrak{R}$: *if for all* $x \in \mathfrak{p}$ *and* $y \in \mathfrak{q}$ *we have* $x \circ y$, *then* $\mathfrak{p} = \mathfrak{q}$.

PROOF. *Ad* 1. Assume towards contradiction that for all $x \in Q_1$ and $y \in Q_2$ we have $x \circ y$, but $\mathrm{F}_{Q_1} \neq \mathrm{F}_{Q_1}$. Then either for some $x_0 \in \mathrm{F}_{Q_1}$ we have $x_0 \notin \mathrm{F}_{Q_2}$, or for some $y_0 \in \mathrm{F}_{Q_2}$ we have $y_0 \notin \mathrm{F}_{Q_1}$.

In the first case for any $y \in \mathrm{F}_{Q_2}$, $y \not\sqsubseteq x_0$. Note that for some $u \in Q_1$ we have $u \sqsubseteq x_0$. So, by (r2), for some $x_1 \in Q_1$, $x_1 \ll u$. Hence $x_1 \ll x_0$, by (3.7). Moreover, by assumption, for any $z \in Q_2$ we have $z \circ x_1$. Let y be an arbitrary member of F_{Q_2}. Then for any $z \in Q_2$, $z \circ y - x_0$ (indeed, either $y - x_0 \sqsubseteq y \sqsubseteq z$ or both $y - x_0 \sqsubseteq y$, $z \sqsubseteq y$, and $z \not\sqsubseteq x_0$, so $z - x_0 \sqsubseteq y - x_0$). Hence, by (r3), $x_1 \mathsf{C} y - x_0$. Thus, we obtain a contradiction. Namely, since $x_0 \wr y - x_0$, $x_1)(y - x_0$, by (df $\ll$).

The second case is proved in an analogous way.

Ad 2. Directly by 1. and definition of $\mathbf{Pt}_\mathfrak{R}$.   ∎

Thanks to Proposition 5.2(2), by definition of filters, we also have:

COROLLARY 5.3. *Let* $\mathfrak{R} = \langle R, \sqsubseteq,)(\rangle \in$ **qSep**. *For all* $\mathfrak{p}, \mathfrak{q} \in \mathbf{Pt}_\mathfrak{R}$: *if* $\mathfrak{p} \subseteq \mathfrak{q}$ *then* $\mathfrak{p} = \mathfrak{q}$.

In this way we obtain the next:

COROLLARY 5.4. *For all* $Q \in \mathbf{Q}_\mathfrak{R}$ *and* $\mathfrak{p} \in \mathbf{Pt}_\mathfrak{R}$: *if* $Q \subseteq \mathfrak{p}$ *then* $\mathfrak{p} = \mathrm{F}_Q$.

PROOF. Let $Q \in \mathbf{Q}_\mathfrak{R}$ and $\mathfrak{p} \in \mathbf{Pt}_\mathfrak{R}$, i.e., $\mathfrak{p} = \mathrm{F}_{Q_\mathfrak{p}} := \{ y \in R \mid \exists_{z \in Q_\mathfrak{p}} \, z \sqsubseteq y \}$, for some $Q_\mathfrak{p} \in \mathbf{Q}_\mathfrak{R}$. Assume $Q \subseteq \mathfrak{p}$. Let $x \in \mathrm{F}_Q$, i.e., there is $y_0 \in Q$ such that $y_0 \sqsubseteq x$. Since $Q \subseteq \mathfrak{p} = \mathrm{F}_{Q_\mathfrak{p}}$, so there is $z_0 \in Q_\mathfrak{p}$ such that $z_0 \sqsubseteq y_0$. Hence $z_0 \sqsubseteq x$. So $x \in \mathrm{F}_{Q_\mathfrak{p}}$. Thus, we obtain $\mathrm{F}_Q \subseteq \mathrm{F}_{Q_\mathfrak{p}}$. Hence, by Corollary 5.3, $\mathrm{F}_Q = \mathrm{F}_{Q_\mathfrak{p}}$.   ∎

Generally, the theory of quasi-separation structures is too weak to prove that $\mathbf{Q}_\mathfrak{R} \neq \emptyset$. For example, the seven-element structure from Model 4.1 has no pre-points. However, in light of Lemma 5.5 below, in order to show separation structures without pre-points we must resort to atomless structures. Existence of such structures is a consequence of Proposition 5.9 (see also Proposition 5.10).

For G-structures existence of pre-points is guaranteed axiomatically and entails that $\mathbf{Pt}_\mathfrak{R} \neq \emptyset$. In Theorem 6.10 we prove that for any first-countable Tychonoff space $\mathcal{T}$ the separation structure $\mathsf{sep}\mathcal{T}$ is a G-structure. Points of G-structures will be the object of our study in the second installment to

this paper, for now we move on to formal description of representatives of points and their properties.

Atoms (resp. atomless elements) in a quasi-separation structure $\langle R, \sqsubseteq,)(\rangle$ are exactly atoms (resp. atomless elements) in the mereological field $\langle R, \sqsubseteq \rangle$. Thus, $\langle R, \sqsubseteq,)(\rangle$ is *atomic* (resp. *atomless*) iff $\langle R, \sqsubseteq \rangle$ is atomic (resp. atomless).

The following lemmas will be used in the sequel. Firstly, in the light of Proposition 4.2, for any mereological field $\langle R, \sqsubseteq \rangle$, we have the separation structure $\langle R, \sqsubseteq, \wr \rangle$ with $)(:= \wr$ and $\mathsf{C} = \bigcirc$ and $\ll = \sqsubseteq$ (see Proposition 3.3).

LEMMA 5.5. *For any separation structure* $\mathfrak{R} = \langle R, \sqsubseteq, \wr \rangle$: *if* $a \in \mathrm{At}_{\mathfrak{R}}$ *then* $\{a\} \in \mathbf{Q}_{\mathfrak{R}}$.

PROOF. For any atom a the singleton $\{a\}$ trivially satisfies (r1) and (r2). For (r3): If $a \bigcirc x$ and $a \bigcirc y$, then $a \sqsubseteq x$ and $a \sqsubseteq y$. Hence $x \bigcirc y$, i.e., $x \mathsf{C} y$. ∎

Let $\mathfrak{R} = \langle R, \sqsubseteq,)(\rangle \in \mathbf{qSep}$. For given $X, Y \in \mathcal{P}(R)$ we say that X is *coinitial with* Y iff $\forall_{y \in Y} \exists_{x \in X} \, x \sqsubseteq y$.

LEMMA 5.6. *Let* $\mathfrak{R} \in \mathbf{qSep}$ *and* $Q \in \mathbf{Q}_{\mathfrak{R}}$. *All subsets of* Q *which are coinitial with* Q *also belong to* $\mathbf{Q}_{\mathfrak{R}}$. *In consequence, for any* $x \in Q$ *we have* $Q \upharpoonright x \in \mathbf{Q}_{\mathfrak{R}}$.

PROOF. Suppose that $X \subseteq Q$ and X is coinitial with $Q \in \mathbf{Q}_{\mathfrak{R}}$. Then (r1) is immediate, since Q satisfies (r1). For (r2) take $x \in X$. Q satisfies (r2) and $x \in Q$, so there is $y \in Q$ such that $y \ll x$. Since X is coinitial with Q, there is $z \in X$ such that $z \sqsubseteq y$. But then $z \ll x$, by (3.8). For (r3) assume that regions y and z are given such that $\forall_{u \in X}(u \bigcirc y \land u \bigcirc z)$. Let $v \in Q$. Again, we use the assumption that X is coinitial with Q and take some $x_0 \in X$ for which $x_0 \sqsubseteq v$. Moreover, $x_0 \bigcirc y$ and $x_0 \bigcirc z$. So, by (MF), we get that $v \bigcirc y$ and $v \bigcirc z$. Thus, $x \mathsf{C} y$, by (r3) for Q, since v was arbitrary. ∎

For the sake of the presentation, as we are interested in descending chains (with respect to $\sqsubseteq$) being the representatives of points, we define the set X of regions to be *well-ordered* iff X is linearly ordered by $\sqsubseteq$ and such that its every non-empty subset has the largest element with respect to $\sqsubseteq$ (thus it may be said that X is *dually well-ordered* with respect to parthood).

For a limit non-zero ordinal λ, $\langle x_\alpha \mid \alpha < \lambda \rangle$ is a transfinite sequence of regions indexed by elements of λ. For a given ordinal α, if there is an ordinal β such that $\alpha = 2 \cdot \beta$ (where the dot is the ordinal multiplication), then α is *even* ordinal number, otherwise it is *odd*. For a given limit ordinal λ, $\mathbb{E}_\lambda$ is the set of all even ordinals below λ and $\mathbb{O}_\lambda$ is the set of all odd ordinals below λ. In the sequel we will use the standard set-theoretical result:

LEMMA 5.7 ([11] **Exercise on p. 68 and Counting Theorem on p. 80**). *Every linearly ordered set $\langle L, \leq \rangle$ has a coinitial well-ordered subset $\langle W, \leq' \rangle$ with $\leq' := \leq \cap (W \times W)$. Since every well-ordered set is order-isomorphic to an ordinal α, the set $\langle W, \leq' \rangle$ may be arranged into a sequence $\langle w_\beta \mid \beta < \alpha \rangle$ such that for all $\beta_1, \beta_2 < \delta$, if $\beta_1 < \beta_2$, then $w_{\beta_1} \leq' w_{\beta_2}$ and $w_{\beta_1} \neq w_{\beta_2}$.*

Thanks to this we can prove:

LEMMA 5.8. *For any $Q \in \mathbf{Q}_\mathfrak{R}$ there is $Q' \in \mathbf{Q}_\mathfrak{R}$ such that $Q' \subseteq Q$, Q' is coinitial with Q, and Q' is well-ordered by $\sqsubseteq$. Moreover, for any $x \in Q$ there is $y \in Q'$ such that $y \ll x$.*

PROOF. By (r1), since $\sqsubseteq \subseteq \ll$, Q is linearly ordered by $\sqsubseteq$. Therefore, by Lemma 5.7, there is $Q' \subseteq Q$ which is well-ordered by $\sqsubseteq$ and coinitial with Q. Hence, $Q' \in \mathbf{Q}_\mathfrak{R}$, by Lemma 5.6. Now let $x \in Q$. Since Q' is coinitial with Q, there is $z \in Q'$ such that $z \sqsubseteq x$. By (r2) for Q', there is $y \in Q'$ such that $y \ll z$. Hence $y \ll x$, by (3.7).   ∎

PROPOSITION 5.9. *For each separation structure $\mathfrak{R} = \langle R, \sqsubseteq, \wr \rangle$ satisfying the following weakened version of ($\exists$sum):[17]*

$$\forall_{X \in \mathcal{P}^+(R)} (\, \exists_{y \in R} \forall_{x \in X} \; x \sqsubseteq y \implies \exists_{z \in R} \; z \text{ sum } X \,), \qquad (\text{w}\exists\text{sum})$$

no atomless region from R belongs to $\bigcup \mathbf{Q}_\mathfrak{R}$.

PROOF. Since $)(= \wr$, we have that $\mathsf{C} = \circ$, $\ll = \sqsubseteq$, and $\ll$ is reflexive.

Suppose that $\mathfrak{R}$ satisfies (w$\exists$sum) and assume towards contradiction that $x \in R$ is an atomless region which belongs to some pre-point Q in $\mathfrak{R}$. We consider two cases.

First, assume that (†) Q has the minimal element y with respect to $\sqsubset$. Then $y \notin \text{At}_\mathfrak{R}$, $y \sqsubseteq x$, and for some u we have $u \notin Q$ and $u \sqsubset y$. Hence $y - u \sqsubset y$ and $y - u \wr u$, so $y \circ u$ and $y \circ y - u$. By (†), for any $z \in Q$ we have $z \circ u$ and $z \circ y - u$, which contradicts (r3), since $u)(y - u$.

Second, assume that Q does not have the minimal element with respect to $\sqsubset$ and consider the set $Q \upharpoonright x$ which belongs to $\mathbf{Q}_\mathfrak{R}$ by Lemma 5.6. Moreover, by Lemma 5.8, there is a transfinite sequence $\langle y_\alpha \mid \alpha < \lambda \rangle$ of regions from $Q \upharpoonright x$ such that $y_0 = x$, $y_{\alpha+1} \sqsubset y_\alpha$, and the set $Q' := \{y_\alpha \mid \alpha < \lambda\}$ is coinitial with $Q \upharpoonright x$ and belongs to $\mathbf{Q}_\mathfrak{R}$. For all $\alpha < \lambda$ there exists $z_\alpha := y_\alpha - y_{\alpha+1}$

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[17](w$\exists$sum) says that every non-empty set which is bounded from above has its mereological sum (this condition corresponds to (w₁$\exists$sum) from [8,17]). It is obvious that ($\exists$sum) is equivalent to the pair ($\exists$1) and (w$\exists$sum). Hence every mereological field with unity satisfying (w$\exists$sum) is a mereological structure. Yet the axioms of the class **MF1** do not entail (w$\exists$sum) (see [17]).

and $\langle z_\alpha \mid \alpha < \lambda \rangle$ is an antichain. We divide it into $\langle z_\alpha \mid \alpha \in \mathbb{E}_\lambda \rangle$ and $\langle z_\beta \mid \beta \in \mathbb{O}_\lambda \rangle$. Both sequences are bounded by x. Thus, by (w∃sum), there are z_1 and z_2 such that $z_1 := \bigsqcup \{z_\alpha \mid \alpha \in \mathbb{E}_\lambda\}$ and $z_2 := \bigsqcup \{(z_\beta \mid \beta \in \mathbb{O}_\lambda\}$. By construction, $z_1 \wr z_2$, i.e., $z_1)(z_2$. Of course, for any $y_\alpha \in Q'$ both $y_\alpha \circ z_1$ and $y_\alpha \circ z_2$. So, by (r3) for Q', we obtain a contradiction: $z_1 \mathsf{C} z_2$. ∎

Directly by the above proposition we have:

PROPOSITION 5.10. *For each atomless separation structure $\mathfrak{R} = \langle R, \sqsubseteq, \wr \rangle$ satisfying the condition (w∃sum) we have $\mathbf{Q}_{\mathfrak{R}} = \emptyset$.*

For any topological space $\mathcal{T} = \langle S, \mathcal{O} \rangle$, $\mathsf{qsep}\mathcal{T}$ is the quasi-separation structure associated with $\mathcal{T}$. In $\mathsf{qsep}\mathcal{T}$ representatives of points are non-empty subfamilies of $\mathrm{r}\mathcal{O}^+$, which satisfy (r1)–(r3). Note that, by (df][) and Proposition 4.3, for any family $\mathscr{X} \in \mathcal{P}^+(\mathcal{O}^+)$ the conditions (r1)–(r3) have the following form:

$$\forall_{U,V \in \mathscr{X}} \big(U = V \vee \mathrm{Cl}\, U \subseteq V \vee \mathrm{Cl}\, V \subseteq U \big), \tag{R1}$$

$$\forall_{U \in \mathscr{X}} \exists_{V \in \mathscr{X}} \ \mathrm{Cl}\, V \subseteq U, \tag{R2}$$

$$\forall_{A,B \in \mathrm{r}\mathcal{O}} (\forall_{U \in \mathscr{X}} \ U \cap A \neq \emptyset \neq U \cap B \Rightarrow \mathrm{Cl}\, A \cap \mathrm{Cl}\, B \neq \emptyset). \tag{R3}$$

These conditions can also be used for any non-empty family $\mathscr{X}$ of non-empty subsets of the set S.

LEMMA 5.11. *Let $\mathcal{T} = \langle S, \mathcal{O} \rangle$ be a T_1-space and $p \in S$. Suppose that there is a base $\mathscr{B}^p$ at p satisfying (R1). Then $\mathscr{B}^p$ also satisfies (R2) and (R3). Moreover, if $\mathscr{B}^p \subseteq \mathrm{r}\mathcal{O}^+$, then $\mathscr{B}^p$ is a pre-point of $\mathsf{qsep}\mathcal{T}$.*

PROOF. For (R2): Let $B \in \mathscr{B}^p$. If $B = \{p\}$ then $\mathrm{Cl}\, B = B$, since $\mathcal{T}$ is a T_1-space. So suppose that for some $p, q \in B$ and $U_p \in \mathcal{O}^p$ we have $p \neq q$ and $q \notin U_p$. Hence $B \cap U_p \in \mathcal{O}^p$ and $B \cap U_p \subsetneq B$. Moreover, for some $B_p \in \mathscr{B}^p$ we have $B_p \subseteq B \cap U_p$, since $\mathscr{B}^p$ is a base at p. Therefore $B_p \subsetneq B$ and $B \nsubseteq B_p$. So $\mathrm{Cl}\, B \nsubseteq B_p$, since $B \subseteq \mathrm{Cl}\, B$. Thus, $\mathrm{Cl}\, B_p \subseteq B$, by (R1).

For (R3): Let U and V be members of $\mathrm{r}\mathcal{O}^+$ such that for any $B \in \mathscr{B}^p$ we have $B \circ U$ and $B \circ V$, i.e., $B \cap U \neq \emptyset \neq B \cap V$. Then $p \in \mathrm{Cl}\, U$ and $p \in \mathrm{Cl}\, V$, since $\mathscr{B}^p$ is a base at p. So $\mathrm{Cl}\, U \cap \mathrm{Cl}\, V \neq \emptyset$.

Finally, if $\mathscr{B}^p \subseteq \mathrm{r}\mathcal{O}^+$, then $\mathscr{B}^p$ is a pre-point in $\mathsf{sep}\mathcal{T}$, since $\mathrm{r}\mathscr{B}^p \subseteq \mathrm{r}\mathcal{O}^+$. ∎

PROPOSITION 5.12. *Let $\mathcal{T} = \langle S, \mathcal{O} \rangle$ be a T_1-space such that for any point p there is a base $\mathscr{B}^p$ at p satisfying (R1). Then:*

1. *$\mathcal{T}$ is a Hausdorff space, so $\mathcal{T}$ is weakly regular.*

2. *If $\mathcal{T}$ is second-countable, then $\mathcal{T}$ is perfectly normal.*

3. $\mathtt{qsep}\mathcal{T}$ belongs to **Sep$_c$** (*and therefore we refer to this structure as* $\mathtt{sep}\mathcal{T}$).

4. *For any* $p \in S$ *the family* $\mathrm{r}\mathscr{B}^p := \{\mathrm{Int\,Cl}\,B \mid B \in \mathscr{B}^p\}$ *is a base at* p *and a pre-point in* $\mathtt{sep}\mathcal{T}$.

PROOF. *Ad* 1. By Lemmas A.2 and A.1. *Ad* 2. By Lemma A.3. *Ad* 3. By 1 and Proposition 4.3(3) (see Remark 4.2). *Ad* 4. By Lemma A.2(2), for any $p \in S$, the family $\mathrm{r}\mathscr{B}^p$ is a base at p which satisfies (R1). The rest by Lemma 5.11. ∎

Generally it does not have to be the case that to every point of a given topological space $\mathcal{T}$ corresponds some pre-point $\mathscr{Q}$ of $\mathtt{qsep}\mathcal{T}$ which uniquely determines this point. To be more precise, the following is not always true: for any point p of $\mathcal{T}$ there is a pre-point $\mathscr{Q}$ in $\mathtt{qsep}\mathcal{T}$ such that $\bigcap \mathscr{Q} = \{p\}$. In the extreme case, we may take any set S with at least two points with the anti-discrete topology $\{\emptyset, S\}$. Then $\mathtt{sep}\mathcal{T}$ is based on the degenerate mereological field $\langle \{S\}, \subseteq \rangle$, where $\ll = \{\langle S, S \rangle\} = \mathsf{C}$, and for the only pre-point $\{S\}$ in $\mathtt{sep}\mathcal{T}$ we have $\bigcap \{S\} = S$. However, in the next theorem we demonstrate for which class of topological spaces we may establish such correspondence.

THEOREM 5.13. *Let* $\mathcal{T} = \langle S, \mathscr{O} \rangle$ *be a first-countable Tychonoff space. Then*:

1. *For any point* $p \in S$ *there is a base* $\mathscr{Q}$ *at* p *such that* $\mathscr{Q} \subseteq \mathrm{r}\mathscr{O}^+$, $\mathscr{Q}$ *is a pre-point in* $\mathtt{sep}\mathcal{T}$, *and* $\bigcap \mathscr{Q} = \{p\}$. *So to any* $p \in S$ *corresponds a pre-point* $\mathscr{Q}$ *in* $\mathtt{sep}\mathcal{T}$ *such that* $\bigcap \mathscr{Q} = \{p\}$.

2. *If in addition* $\mathcal{T}$ *is second-countable, then* $\mathcal{T}$ *is perfectly normal and has c.c.c., so in consequence* $\mathtt{sep}\mathcal{T}$ *satisfies* (IA) *and has c.c.c. as a structure.*

PROOF. *Ad* 1. Let $p \in S$. By Lemma A.4 for some continuous function $f \colon S \to [0,1]$ we have that $f(p) = 0$ and the family $\mathscr{B}^p := \{f^{-1}[[0, \frac{1}{2^n})] \mid n \in \omega^+\}$[18] is a base at p such that $\bigcap \mathscr{B}^p = \{p\}$. For any $n \in \omega$ we put $U_n := f^{-1}[[0, \frac{1}{2^n})]$ and we show (∗): $\mathrm{Cl}\,U_{n+1} \subseteq U_n$.

Indeed, assume that $q \notin U_n$. Then $f(q) \notin [0, \frac{1}{2^n})$, i.e., $f(q) \in [\frac{1}{2^n}, 1]$. Take $t \in (\frac{1}{2^{n+1}}, \frac{1}{2^n})$. We have: $f(q) \in (t, 1]$, the set $(t, 1]$ is open in natural topology on $[0,1]$, and $[0, \frac{1}{2^{n+1}}) \cap (t, 1] = \emptyset$. Hence $U_{n+1} \cap f^{-1}[(t,1]] = \emptyset$. Note that $f^{-1}[(t,1]]$ belongs to $\mathscr{O}^q$. Thus, $q \notin \mathrm{Cl}\,U_{n+1}$.

By (∗) the base $\mathscr{B}^p$ satisfies (R1). Hence, by Proposition 5.12, the family $\mathrm{r}\mathscr{B}^p := \{\mathrm{Int\,Cl}\,B \mid B \in \mathscr{B}^p\}$ is a base at p and a pre-point in $\mathtt{sep}\mathcal{T}$. Since $\mathcal{T}$ is T$_1$-space, we obtain that $\bigcap \mathrm{r}\mathscr{B}^p = \{p\}$.

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[18] We put $\omega^+ := \omega \setminus \{0\}$, i.e., ω^+ is the set of all positive integers.

Ad 2. We use 1, Lemma A.3, and Proposition 4.3(4), plus Lemma A.5 for c.c.c. for $\mathrm{sep}\mathcal{T}$. ∎

Theorem 5.13 does not resolve the problem of uniqueness of points determined by intersections of pre-points, i.e., we would like to know as well for which class of spaces topologically interpreted pre-points are unambiguous in "pin-pointing unique locations in space", in the sense that for every pre-point there is exactly one point of the space which there is within every regular open set from the pre-point. In Theorem 5.17 we prove that all separation structures based on Urysohn spaces with c.c.c. have this property. In consequence we have that in the class of separation structures for first-countable Tychonoff spaces with c.c.c.[19] every pre-point determines a unique point of the space, and to every point p corresponds a pre-point $\mathscr{Q}$ such that $\bigcap \mathscr{Q} = \{p\}$. This, in particular, shows that Grzegorczyk's definitions of pre-points and points are "correct" in the sense, that the aforementioned properties hold in the class of separations structures for Euclidean spaces (see Corollary 5.18), which is of course the subclass of the former class of structures.

For all ordinals κ and λ, a function $f\colon \kappa \to \lambda$ is *monotone* iff for all $\alpha, \beta \in \kappa$, if $\alpha \leqslant \beta$ than $f(\alpha) \leqslant f(\beta)$. Moreover, for a given limit ordinal λ, *the cofinality of* λ (in symbols: $\mathrm{cf}(\lambda)$) is the smallest ordinal κ such that there is a monotone function $f\colon \kappa \to \lambda$ with $f[\kappa]$ unbounded in λ, i.e., for any $\alpha \in \lambda$ there is $\beta \in \kappa$ such that $\alpha < f(\beta)$.

In the proof of Lemma 5.15 we use the standard fact from set theory:

LEMMA 5.14. *If $\lambda > 0$ is a countable limit ordinal, then $\mathrm{cf}(\lambda) = \omega$.*

LEMMA 5.15. *Let $\mathcal{T} = \langle S, \mathscr{O} \rangle$ be a topological space having c.c.c. and $\mathscr{U}$ be an infinite chain of open sets satisfying* (R1). *Then there is a monotone ω-sequence of elements of $\mathscr{U}$ which is coinitial with $\mathscr{U}$ and such that $\mathrm{Cl}\,U_{n+1} \subsetneq U_n$, for any $n \in \omega$.*

PROOF. In $\mathscr{U}$ we define: $U \leq V$ iff either $U = V$ or $\mathrm{Cl}\,U \subseteq V$. Then $\langle \mathscr{U}, \leq \rangle$ is a linearly ordered set in light of (R1). By Lemma 5.7, $\langle \mathscr{U}, \leq \rangle$ has a coinitial well-ordered subset $\langle \mathscr{V}, \leq' \rangle$, with $\leq' := \leq \cap (\mathscr{V} \times \mathscr{V})$, and we can arrange it into a sequence $\langle U_\alpha \mid \alpha < \lambda \rangle$, where for any $\alpha < \lambda$ we have: $U_{\alpha+1} \subseteq \mathrm{Cl}\,U_{\alpha+1} \subseteq U_\alpha$ and $U_{\alpha+1} \neq U_\alpha$. As a subsequence of $\mathscr{U}$ it satisfies (R1), so $\mathrm{Cl}\,U_{\alpha+2} \subseteq U_{\alpha+1} \subsetneq U_\alpha$.

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[19]We need both first countability and c.c.c. The long line space (see [21] ex. 45) is a first-countable connected Tychonoff space which does not have c.c.c., while the uncountable Cartesian product of the unit interval $\prod_{i \in [0,1]} [0,1]_i$ is a connected Tychonoff space with c.c.c., but not first countable (see [21] ex. 105).

Let $V_\alpha := U_\alpha \setminus \mathrm{Cl}\, U_{\alpha+2}$. The sequence $\langle V_\alpha \mid \alpha < \lambda \rangle$ is an antichain of non-empty open sets of $\mathcal{T}$, and so it is countable. Hence λ is countable and $\mathrm{cf}(\lambda) = \omega$, by Lemma 5.14. Thus, there is a monotone function $f \colon \omega \to \lambda$ such that $f[\omega]$ is unbounded in λ, and in consequence for any $\alpha \in \lambda$ there is $n \in \omega$ such that $\alpha < f(n)$. Hence $U_\alpha \subseteq U_{f(n)}$. This shows that $\langle U_{f(n)} \mid n < \omega \rangle$ is coinitial with $\langle U_\alpha \mid \alpha < \lambda \rangle$, and so with $\mathscr{U}$ as well.

Now we take every second element from $\langle U_{f(n)} \mid n < \omega \rangle$, i.e., we put $A_0 := U_{f(0)}$ and $A_{n+1} := U_{f(2n)}$. This guarantees that $\mathrm{Cl}\, A_{n+1} \subsetneq A_n$ and $\langle A_n \mid n \in \omega \rangle$ is a monotone ω-sequence which coinitial with $\mathscr{U}$. ∎

The condition (R3) can be expanded onto the whole family $\mathscr{O}^+$:

$$\forall_{A,B \in \mathscr{O}}(\forall_{U \in \mathscr{X}}\, U \cap A \neq \emptyset \neq U \cap B \Rightarrow \mathrm{Cl}\, A \cap \mathrm{Cl}\, B \neq \emptyset). \qquad (\mathrm{R3}°)$$

LEMMA 5.16. *Let $\mathcal{T} = \langle S, \mathscr{O} \rangle$ be a Urysohn space having c.c.c. and $\mathscr{U}$ be a non-empty subfamily of $\mathscr{O}^+$ satisfying* (R1) *and* (R3°). *Then $|\bigcap \mathscr{U}| = 1$.*

PROOF. If $|S| = 1$, then $\mathscr{U} = \{S\}$ and $|\bigcap \mathscr{U}| = 1$.[20] So suppose that $|S| > 1$. Then we have $\mathscr{U} \neq \{S\}$. Indeed, assume the opposite. For some $p \neq q \in S$ and for some $A \in \mathscr{O}^p$ and $B \in \mathscr{O}^q$ we have $\mathrm{Cl}\, A \cap \mathrm{Cl}\, B = \emptyset$, since $\mathcal{T}$ is a Urysohn space. But $S \cap A \neq \emptyset \neq S \cap B$ and hence, by (R3°), we obtain a contradiction: $\mathrm{Cl}\, A \cap \mathrm{Cl}\, B \neq \emptyset$.

Note that $|\bigcap \mathscr{U}| \leqslant 1$. Indeed, assume $p \neq q \in \bigcap \mathscr{U}$. Then for some $A \in \mathscr{O}^p$ and $B \in \mathscr{O}^q$ we have $\mathrm{Cl}\, A \cap \mathrm{Cl}\, B = \emptyset$. Moreover, for any $U \in \mathscr{U}$ we have $U \cap A \neq \emptyset \neq U \cap B$. So, by (R3°) again, $\mathrm{Cl}\, A \cap \mathrm{Cl}\, B \neq \emptyset$.

Now we show that $(*)$: $\mathscr{U}$ is a chain of open sets. By (R1), for all $U, V \in \mathscr{U}$ either $U = V$, or $\mathrm{Cl}\, U \subseteq V$, or $\mathrm{Cl}\, V \subseteq U$. So for any $U, V \in \mathscr{U}$ such that $U \neq V$, either $U \subseteq \mathrm{Cl}\, U \subseteq V$ or $V \subseteq \mathrm{Cl}\, V \subseteq U$, and so either $U \subsetneq V$ or $V \subsetneq U$.

Finally we prove that $\bigcap \mathscr{U} \neq \emptyset$. Assume $\bigcap \mathscr{U}$ is empty. Then $(**)$: $\mathscr{U}$ must be infinite, since otherwise, by $(*)$, it would have to contain the minimal set V, and then $\bigcap \mathscr{U} = V \neq \emptyset$. So, by $(*)$, $(**)$, and c.c.c., from Lemma 5.15 follows existence of a monotone sequence $\langle U_n \mid n < \omega \rangle$ of sets from $\mathscr{U}$ which is coinitial with $\mathscr{U}$ and such that $\mathrm{Cl}\, U_{n+1} \subsetneq U_n$, for any $n \in \omega$. For all $n \in \omega$ we put $V_n := U_n \setminus \mathrm{Cl}\, U_{n+1} \in \mathscr{O}^+$, $A := \bigcup_{n \in \omega} V_{4n}$, and $B := \bigcup_{n \in \omega} V_{4n+2}$. By construction, for any $U \in \mathscr{U}$ we have $U \cap A \neq \emptyset \neq U \cap B$. So $\mathrm{Cl}\, A \cap \mathrm{Cl}\, B \neq \emptyset$, by (R3°). Let $p_0 \in \mathrm{Cl}\, A \cap \mathrm{Cl}\, B$.

Notice that $(\dagger)$: $\mathrm{Cl}\, A = \bigcup_{n \in \omega} \mathrm{Cl}\, V_{4n}$ and $\mathrm{Cl}\, B = \bigcup_{n \in \omega} \mathrm{Cl}\, V_{4n+2}$. Indeed, first, $\bigcup_{n \in \omega} \mathrm{Cl}\, V_{4n} \subseteq \mathrm{Cl}\, A$. Second, for any $n \in \omega$ we have: $A \subseteq V_0 \cup \cdots \cup V_{4n} \cup U_{4n+4}$. So $\mathrm{Cl}\, A \subseteq \mathrm{Cl}\, V_0 \cup \cdots \cup \mathrm{Cl}\, V_{4n} \cup \mathrm{Cl}\, U_{4n+4}$. Assume towards

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[20]We reconstruct, *mutatis mutandis*, the proof of Dorais [3] for Euclidean spaces.

contradiction that $p \in \mathrm{Cl}\, A \setminus \bigcup_{n \in \omega} \mathrm{Cl}\, V_{4n}$. Then $p \in \bigcap_{n>0} \mathrm{Cl}\, U_n$. Yet the sequence $\langle \mathrm{Cl}\, U_n \mid 0 < n < \omega \rangle$ is coinitial with the sequence $\langle U_n \mid n < \omega \rangle$ and therefore also with $\mathscr{U}$. Hence we obtain a contradiction: $p \in \bigcap \mathscr{U}$. In a similar way we prove that $\mathrm{Cl}\, B = \bigcup_{n \in \omega} \mathrm{Cl}\, V_{4n+2}$.

By (†) there are $m, k \in \omega$ such that the difference between them is at least two, say $k \geqslant m + 2$, and $p_0 \in \mathrm{Cl}\, V_m \cap \mathrm{Cl}\, V_k$. Hence, by definition of V_k, we have $\mathrm{Cl}\, V_k \subseteq U_{m+1}$. So also $p_0 \in U_{m+1}$. Since $p_0 \in \mathrm{Cl}\, V_m$, $\emptyset \neq U_{m+1} \cap V_m = U_{m+1} \cap (U_m \setminus \mathrm{Cl}\, U_{m+1}) = \emptyset$. A contradiction. ∎

Thanks to Lemma 5.16 we obtain:

THEOREM 5.17. *If $\mathcal{T} = \langle S, \mathscr{O} \rangle$ is a Urysohn space having c.c.c., then for any pre-point $\mathscr{Q}$ in $\mathrm{qsep}\mathcal{T}$ we have $|\bigcap \mathscr{Q}| = 1$.*

PROOF. Just note that for any non-empty subfamily $\mathscr{X}$ of $\mathcal{P}^+(S)$: if $\mathscr{X}$ satisfies (R3), then $\mathscr{X}$ satisfies (R3°). Indeed, for any $V \in \mathscr{O}$ we have (a) $V \subseteq \mathrm{Int}\, \mathrm{Cl}\, V$, (b) $\mathrm{Int}\, \mathrm{Cl}\, V \in \mathrm{r}\mathscr{O}$, and (c) $\mathrm{Cl}\, V = \mathrm{Cl}\, \mathrm{Int}\, \mathrm{Cl}\, V$. We take $A, B \in \mathscr{O}$ such that for all $U \in \mathscr{X}$: $U \cap A \neq \emptyset \neq U \cap B$. By (a), also $U \cap \mathrm{Int}\, \mathrm{Cl}\, A \neq \emptyset \neq U \cap \mathrm{Int}\, \mathrm{Cl}\, B$. Hence, by (b), (c) and (R3), we have $\mathrm{Cl}\, A \cap \mathrm{Cl}\, B \neq \emptyset$. ∎

Let $n > 0$ and $\mathscr{E}(\mathbb{R}^n)$ be the standard topology on the Cartesian product $\mathbb{R}^n$ of the set of real numbers $\mathbb{R}$. By *the topological Euclidean n-space* we mean the space $\mathcal{E}^n := \langle \mathbb{R}^n, \mathscr{E}(\mathbb{R}^n) \rangle$. Let $\mathrm{r}\mathscr{E}(\mathbb{R}^n)$ be the family of all regular open sets of $\mathcal{E}^n$ and let $\mathrm{r}\mathscr{E}_+(\mathbb{R}^n) := \mathrm{r}\mathscr{E}(\mathbb{R}^n) \setminus \{\emptyset\}$.

COROLLARY 5.18. 1. *For any pre-point $\mathscr{Q}$ in $\mathrm{sep}\mathcal{E}^n$ we have $|\bigcap \mathscr{Q}| = 1$.*

2. *For any $p \in \mathbb{R}^n$ the family $\mathcal{B}^p$ of all open balls with center at p is a pre-point in $\mathrm{sep}\mathcal{E}^n$ (of course, $\bigcap \mathcal{B}^p = \{p\}$).*

6. Grzegorczyk Structures

After Grzegorczyk [10] we enrich quasi-separation structures with an axiom which postulates existence of particular pre-points (cf. the axiom A4 in [10], and also e.g. [7]).

6.1. Definition and Basic Properties

Let $\mathfrak{R} = \langle R, \sqsubseteq, \rangle\langle \rangle$ be a quasi-separation structure. The Grzegorczyk axiom says that for all connected $x, y \in R$ there exists $Q \in \mathbf{Q}_\mathfrak{R}$ such that:

(g1) either $x \wr y$ or there is $z \in Q$ such that $z \sqsubseteq x \sqcap y$,

(g2) for any $z \in Q$ we have $z \mathbin{\bigcirc} x$ and $z \mathbin{\bigcirc} y$.

Formally:

$$\forall_{x,y \in R}\Big(x \mathbin{\mathsf{C}} y \Longrightarrow \exists_{Q \in \mathbf{Q}_\mathfrak{R}} \big((x \mathbin{\wr} y \vee \exists_{z \in Q}\, z \sqsubseteq x \sqcap y) \wedge \forall_{z \in Q}(z \circ x \wedge z \circ y) \big) \Big). \tag{G}$$

By a *G-structure* we will mean any quasi-separation structure $\langle R, \sqsubseteq, \mathbin{)(}\rangle$ satisfying (G). Let **G** be the class of all G-structures, i.e.:

$$\mathbf{G} := \mathbf{qSep} + (\text{G}).$$

Moreover, let $\mathbf{G_c}$ be the class of complete G-structures. Then we have $\mathbf{G_c} \subsetneqq \mathbf{G1} \subsetneqq \mathbf{G}$.

REMARK 6.1. In [10] in place of (g1) we have:

- there is $z \in Q$ such that either $x \not\sqsubseteq y$ or $z \sqsubseteq x$.

In the second part we will prove that replacing both forms with one another we obtain equivalent versions of the axiom (G).   ⊣

Let us notice after Biacino and Gerla [2, p. 435] that thanks to (r3), the implication '⇒' in (G) can be replaced by equivalence '⇔'.

Now we show that:

PROPOSITION 6.1. *The axiom* (G) *can be replaced with the following simpler conditions*:

$$\forall_{x,y \in R}\big(x \circ y \Longrightarrow \exists_{Q \in \mathbf{Q}_\mathfrak{R}} \exists_{z \in Q}\, z \sqsubseteq x \sqcap y \big), \tag{$\text{G}_\circ$}$$

$$\forall_{x,y \in R}\big(x \mathbin{\mathsf{C}} y \,\wedge\, x \mathbin{\wr} y \Longrightarrow \exists_{Q \in \mathbf{Q}_\mathfrak{R}} \forall_{z \in Q}(z \circ x \wedge z \circ y) \big). \tag{$\text{G}_\wr$}$$

PROOF. For "$(\text{G}_\circ) \wedge (\text{G}_\wr) \Rightarrow (\text{G})$": Suppose that $x \mathbin{\mathsf{C}} y$. If $x \mathbin{\wr} y$ then we use $(\text{G}_\wr)$. If $x \circ y$ then, by $(\text{G}_\circ)$, for some $Q_0 \in \mathbf{Q}_\mathfrak{R}$ and $z_0 \in Q_0$ we have $z_0 \sqsubseteq x \sqcap y$. But, by (r1) and $(\text{I}^{\text{c}}_\ll)$, for any $z \in Q_0$ either $z = z_0$, or $z \sqsubseteq z_0$, or $z_0 \sqsubseteq z$. So $z \circ x \sqcap y$, and also both $z \circ x$ and $z \circ y$. "$(\text{G}) \Rightarrow (\text{G}_\circ)$" and "$(\text{G}) \Rightarrow (\text{G}_\wr)$" are obvious in light of $(\text{I}^{\text{c}}_\circ)$.[21]   ∎

The relations $\circ$ and C are reflexive, so in light of (G), $(\text{I}^{\text{c}}_\circ)$, and (MF) we obtain:

$$\mathbf{Q}_\mathfrak{R} \neq \emptyset \quad \text{and} \quad \mathbf{Pt}_\mathfrak{R} \neq \emptyset. \tag{6.1}$$

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[21]Other axiomatizations of G-structures will be presented in the second part of the paper.

Moreover we have:

$$\forall_{x \in R} \exists_{Q \in \mathbf{Q}_{\mathfrak{R}}} \left(\exists_{z \in Q} \, z \sqsubseteq x \ \wedge \ \forall_{z \in Q} \, z \bigcirc x \right), \tag{6.2}$$

$$\forall_{x,y \in R} \left(x \bigcirc y \Longrightarrow \exists_{Q \in \mathbf{Q}_{\mathfrak{R}}} (\exists_{z \in Q} \, z \sqsubseteq x \sqcap y \ \wedge \ \forall_{z \in Q} (z \bigcirc x \ \wedge \ z \bigcirc y))), \tag{6.3}$$

$$\forall_{x,y \in R} \left(x \bigcirc y \Longrightarrow \exists_{Q \in \mathbf{Q}_{\mathfrak{R}}} (\exists_{z \in Q} \, z \ll x \sqcap y \ \wedge \ \forall_{z \in Q} (z \bigcirc x \ \wedge \ z \bigcirc y))). \tag{6.4}$$

Indeed, for (6.2): because $x \bigcirc x$, $x \mathsf{C} x$, and $x \sqcap x = x$. For (6.3): if $x \bigcirc y$, then also $x \mathsf{C} y$, by ($\mathrm{I}_{\bigcirc}^{\mathsf{c}}$). So we apply (G). For (6.4): by (6.3), (r2), and (3.7).

In all G-structures the following two facts hold.

PROPOSITION 6.2. *Every region has at least one non-tangential part.*

PROOF. By (6.2), for every $x \in R$ there is a $Q \in \mathbf{Q}_{\mathfrak{R}}$ with $z \in Q$ such that $z \sqsubseteq x$. Q has the property (r2), so for some $u \in Q$ we have $u \ll z$, and thus $u \ll x$, by (3.7). ∎

PROPOSITION 6.3. *Every region is the mereological sum of its non-tangential parts.*

PROOF. Fix $x \in R$ and put $S := \{z \in R \mid z \ll x\}$. First, in light of ($\mathrm{I}_{\ll}^{\sqsubseteq}$), for any $z \in S$ we have $z \sqsubseteq x$. Second, let $y \sqsubseteq x$. Then, by Proposition 6.2, for some u we have $u \ll y$. So $u \ll x$, by (3.7). Hence $u \in S$ and $u \bigcirc y$, by ($\mathrm{I}_{\ll}^{\sqsubseteq}$). Thus, $x \, \mathsf{sum} \, S$. ∎

6.2. Grzegorczyk Structures versus Separation Structures

Theorem 6.4 below is similar to results that can be found in [2, pp. 435–436]. The differences lie in two facts: in [2] the counterparts of the theorem were proven after the notion of *point* had been introduced, and different spaces were taken into account.

THEOREM 6.4. *All G-structures are separation structures, i.e.,* $\mathbf{G} \subseteq \mathbf{Sep}$, $\mathbf{G1} \subseteq \mathbf{Sep1}$, *and* $\mathbf{G_c} \subseteq \mathbf{Sep_c}$.

PROOF. Let $\mathfrak{R} = \langle R, \sqsubseteq, \rangle \langle \rangle$ be any G-structure.

Ad (S5): Suppose that $x \not\sqsubseteq y$. Then in R there exists $x - y$. By reflexivity of $\bigcirc$ and (6.4) there are $Q \in \mathbf{Q}_{\mathfrak{R}}$ and $z \in Q$ such that $z \ll x - y$. Hence, by (df $\ll$) and (S2), we have $z \rangle \langle y$, since $y \wr x - y$. On the other hand $z \sqsubseteq x$. Hence $z \mathsf{C} x$, by (C1).

Ad (S4): Suppose that $z \mathsf{C} x \sqcup y$. Then, by (G), there exists $Q \in \mathbf{Q}_{\mathfrak{R}}$ such that $z \wr x \sqcup y \ \vee \ \exists_{u \in Q} \, u \sqsubseteq z \sqcap (x \sqcup y)$ and $\forall_{u \in Q} (u \bigcirc z \ \wedge \ u \bigcirc x \sqcup y)$.

Consider two sets: $Q_x := \{u \in Q \mid u \bigcirc x\}$ and $Q_y := \{u \in Q \mid u \bigcirc y\}$. From (2.3) we have $Q_x \cup Q_y = \{u \in Q \mid u \bigcirc x \sqcup y\} = Q$. We show that either $Q_x = Q$ or $Q_y = Q$.

Indeed, suppose that $Q \nsubseteq Q_x$. Then for some $u_0 \in Q$ we have $u_0 \wr x$. Hence $u_0 \bigcirc y$, by (2.3), since $u_0 \bigcirc x \sqcup y$. So $u_0 \in Q_y$. Let now u be arbitrary member of Q. In the case when $u = u_0$, $u \in Q_y$. Otherwise, by (r1), either $u_0 \ll u$ or $u \ll u_0$. So, by ($\mathrm{I}_\ll^\sqsubseteq$), either (a) $u_0 \sqsubseteq u$ or (b) $u \sqsubseteq u_0$. In (a) we have $u \bigcirc y$, since $u_0 \sqsubseteq u$ and $u_0 \bigcirc y$. So $u \in Q_y$. In (b) we have $u \wr x$, since $u \sqsubseteq u_0$ and $u_0 \wr x$. Hence $u \bigcirc y$, since $u \bigcirc x \sqcup y$. So again $u \in Q_y$, and $Q = Q_y$.

Now let u be an arbitrary member of Q. Suppose that $Q_x = Q$. Then $u \bigcirc x$, $u \bigcirc z$ and $z \complement x$, by (r3). Similarly, if $Q_y = Q$ then $z \complement y$. Thus, either $z \complement x$ or $z \complement y$, as required. ∎

We will show now that the class of separation structures is broader than that of G-structures; i.e., $\mathbf{G} \subsetneq \mathbf{Sep}$. By Proposition 5.10 and (6.1) we obtain:

PROPOSITION 6.5. *If an atomless separation structure with $\kern0.1em)\kern-0.3em(\kern0.1em := \wr$ satisfies* (w∃sum)*, then it is not a G-structure.*

Thus, since the structure $\langle \mathrm{r}\mathscr{E}_+(\mathbb{R}^n), \subseteq, \wr \rangle$ is an atomless complete separation structure, it must be an element of $\mathbf{Sep}$ which is not in $\mathbf{G}$.

More generally, by Proposition 5.9, we have:

PROPOSITION 6.6. *If a separation structure with $\kern0.1em)\kern-0.3em(\kern0.1em := \wr$ satisfies* (w∃sum) *and has at least one atomless element, then it is not a G-structure.*

PROOF. Assume towards contradiction that $\mathfrak{R} = \langle R, \sqsubseteq, \wr \rangle$ belongs to $\mathbf{G}$ and $x \in R$ is an atomless region. Then, by (6.2), for some $Q \in \mathbf{Q}_\mathfrak{R}$ there is $z \in Q$ such that $z \sqsubseteq x$. Yet then z must be atomless, which contradicts Proposition 5.9. ∎

In consequence, from Theorem 6.4 and Propositions 4.1 and 6.6 we have:

COROLLARY 6.7. *Let $\mathfrak{R} = \langle R, \sqsubseteq,)\kern-0.3em(\rangle$ be a non-trivial G-structure. Then:*

1. *Both $)\kern-0.3em(\neq \emptyset$ and $\complement \neq R \times R$.*

2. *If $\mathfrak{R}$ satisfies* (w∃sum) *and has at least one atomless element, then $\emptyset \subsetneq)\kern-0.3em(\subsetneq \wr$ and $\bigcirc \subsetneq \complement \subsetneq R \times R$.*

Therefore for all members of the class $\mathbf{G}+$(w∃sum) (and the more so of $\mathbf{G_c}$) the counterpart of Proposition 4.2 holds only for atomic structures. Thus, in (standard) G-structures (see Theorem 6.11) separation is different from disjointness: $)\kern-0.3em(\neq \wr$. However, it is not excluded that in some G-structures we have $)\kern-0.3em(= \wr$. It will hold true, for example, for all atomic G-structures of the form $\langle R, \sqsubseteq, \wr \rangle$.

PROPOSITION 6.8. *Every atomic separation structure $\mathfrak{R} = \langle R, \sqsubseteq, \wr \rangle$ is a G-structure.*

PROOF. First, we show that $\mathfrak{R}$ satisfies $(G_{\circ})$. Let $x, y \in R$ be such that $x \circ y$. Then in R there are the product $x \sqcap y$ and $a \in \text{At}_{\mathfrak{R}}$ such that $a \sqsubseteq x \sqcap y$. But $\{a\} \in \mathbf{Q}_{\mathfrak{R}}$, by Lemma 5.5. Second, since $\mathsf{C} = \circ$, the condition '$x \mathsf{C} y \wedge x \wr y$' is false for all $x, y \in R$. Hence $(G_{\wr})$ also holds. ∎

The following general result will be used in the sequel:

THEOREM 6.9. *Let $\mathcal{T} = \langle S, \mathcal{O} \rangle$ be a T_1-space such that for any point $p \in S$ there is a base $\mathscr{B}^p$ at p satisfying* (R1). *Then* $\mathtt{sep}\mathcal{T}$ *belongs to* $\mathbf{G_c}$.

PROOF. By Proposition 5.12, for any $p \in S$, the family $r\mathscr{B}^p := \{\text{Int Cl } B \mid B \in \mathscr{B}^p\}$ is a base at p and it is a pre-point in $\mathtt{sep}\mathcal{T}$.

For $(G_{\circ})$: Suppose that for $U, V \in r\mathcal{O}^+$ we have $U \circ V$, i.e., $U \cap V \neq \emptyset$. Let $p \in U \cap V \in r\mathcal{O}$. Then for some $Z \in r\mathscr{B}^p$ we have $Z \subseteq U \cap V$, i.e., $Z \sqsubseteq U \sqcap V$.

For $(G_{\wr})$: Suppose that for $U, V \in r\mathcal{O}^+$ we have $U \mathsf{C} V$, i.e., $\text{Cl } U \cap \text{Cl } V \neq \emptyset$. Let $p \in \text{Cl } U \cap \text{Cl } V$. Then for any $A \in r\mathscr{B}^p$: $A \cap U \neq \emptyset \neq Z \cap V$; i.e., $Z \circ U$ and $Z \circ V$. ∎

By Theorems 6.9 and 5.13 we obtain:

THEOREM 6.10. *If $\mathcal{T}$ is a first-countable Tychonoff space, then* $\mathtt{sep}\mathcal{T}$ *belongs to* $\mathbf{G_c}$.

Of course, for any $n > 0$, the finitely dimensional topological Euclidean space $\mathcal{E}^n$ (see p. 33) is a Tychonoff space. Thus, by Theorem 6.10, we have:

COROLLARY 6.11. *For any $n > 0$, the structure* $\mathtt{sep}\mathcal{E}^n$ *belongs to* $\mathbf{G_c}$, *satisfies* (IA), *and has c.c.c.*

The following is another explanation of this conclusion. Thanks to Corollary 5.18, for any $p \in \mathbb{R}^n$ the family $\mathscr{B}^p$ of all open balls with center at p is a base at p, which is a pre-point in $\mathtt{sep}\mathcal{E}^n$, and thus satisfies (R1). So we can refert Theorem 6.9.

A.   Appendix: Definitions and Facts from Topology

The set of all real numbers is denoted by $\mathbb{R}$, ω (resp. ω^+) denotes the set of all (resp. all positive) natural numbers.

Let $\mathcal{T} := \langle S, \mathscr{O} \rangle$ be a topological space. Let $\mathscr{C}$ be the family of all *closed sets* of $\mathcal{T}$, and let Int and Cl be standard *interior* and *closure* operations of $\mathcal{T}$. For any $V \in \mathscr{O}$ we have $V \subseteq \operatorname{Int} \operatorname{Cl} V$ and $\operatorname{Cl} V = \operatorname{Cl} \operatorname{Int} \operatorname{Cl} V$. For a given $p \in S$ we put $\mathscr{O}^p := \{U \in \mathscr{O} \mid p \in U\}$.

In the standard way we define T_1, T_2 (of *Hausdorff*), $\mathrm{T}_{2\frac{1}{2}}$ (or *Urysohn*), T_3 (or *regular*), $\mathrm{T}_{3\frac{1}{2}}$ (or *completely regular*, or a *Tychonoff*), T_4 (or *normal*) and *perfectly normal* spaces (we include T_1 in the definitions of T_3–T_4).

We define a *base* $\mathscr{B}$ of $\mathcal{T}$ in the standard way. For any point p, a family $\mathscr{B}^p \in \mathcal{P}(\mathscr{O})$ is called a *base for $\mathcal{T}$ at p* iff $\mathscr{B}^p \subseteq \mathscr{O}^p$ and for any $U \in \mathscr{O}^p$ there exists $V \in \mathscr{B}^p$ such that $V \subseteq U$. If $\mathscr{B}$ is a base for $\mathcal{T}$ then for any $p \in S$ the family $\mathscr{B}^p := \{U \in \mathscr{F} \mid p \in U\}$ is a base at p. On the other hand, if for any $p \in S$ a base $\mathscr{B}^p$ at p is given, then the union $\bigcup_{p \in S} \mathscr{B}^p$ is a base for $\mathcal{T}$.

A subset U of S is *regular open of $\mathcal{T}$* iff $U = \operatorname{Int} \operatorname{Cl} U$. Let $\mathrm{r}\mathscr{O}$ be the family of all regular open sets of $\mathcal{T}$. For all $U, V \in \mathrm{r}\mathscr{O}$: $U \subseteq V$ iff $\operatorname{Cl} U \subseteq \operatorname{Cl} V$.

The space $\mathcal{T}$ is *semiregular* iff $\mathcal{T}$ has a base consisting of regular open sets.[22] We say that $\mathcal{T}$ is *weakly regular* iff $\mathcal{T}$ is semiregular and for any $U \in \mathscr{O}^+$ there is $V \in \mathscr{O}^+$ such that $\operatorname{Cl} V \subseteq U$ iff $\mathcal{T}$ is semiregular and for any $U \in \mathscr{O}^+$ there is $V \in \mathrm{r}\mathscr{O}^+$ such that $\operatorname{Cl} V \subseteq U$. Not all semiregular Urysohn spaces are regular, nor all connected. Semiregular Hausdorff spaces are weakly regular, but it is known that all regular spaces are weakly regular:

LEMMA A.1. *Let $\mathcal{T} = \langle S, \mathscr{O} \rangle$ be a regular space.*

1. *If $\mathscr{B}^p$ is a base at point p, then $\mathrm{r}\mathscr{B}^p := \{\operatorname{Int} \operatorname{Cl} B \mid B \in \mathscr{B}^p\}$ is a base at p, too.*

2. *The sum $\bigcup_{p \in S} \mathrm{r}\mathscr{B}^p$ is a base for $\mathcal{T}$, where for any $p \in S$ a family $\mathscr{B}^p$ is any base at p. So $\mathcal{T}$ is semiregular.*

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[22]Engelking [5] assumes that semiregular spaces are also Hausdorff spaces. We read ([5], p. 58): "Note that there exist T_1-spaces in which open domains [regular open sets] form a base but which are not T_2-spaces." Not all Hausdorff spaces are semiregular (see e.g. [5], p. 58).

3. $\mathcal{T}$ *is weakly regular.*

PROOF. *Ad* 1. If $U \in \mathscr{O}^p$ then there are $B \in \mathscr{B}^p$ and $V \in \mathscr{O}^p$ such that $B \subseteq V \subseteq \mathrm{Cl}\, V \subseteq U$. So $B \subseteq \mathrm{Cl}\, B \subseteq U$. Hence $B \subseteq \mathrm{Int}\,\mathrm{Cl}\, B \subseteq \mathrm{Cl}\, B = \mathrm{Cl}\,\mathrm{Int}\,\mathrm{Cl}\, B \subseteq U$. Thus, the family $\{\mathrm{Int}\,\mathrm{Cl}\, B \mid B \in \mathscr{B}^p\}$ is a base at p.

Ad 2. Since every point $p \in S$ has a local base (e.g., $\mathscr{O}^p$), by 1 the sum $\bigcup_{p \in S} \mathrm{r}\mathscr{B}^p$ is a base for $\mathcal{T}$, so $\mathrm{r}\mathscr{O}^+$ is a base for $\mathcal{T}$ either.

Ad 3. Obvious. ∎

LEMMA A.2. *Let $\mathcal{T}$ be a T_1-space such that for any point p there is a base $\mathscr{B}^p$ at p satisfying* (R1). *Then*:

1. $\mathcal{T}$ *is regular.*

2. *For any point p the family $\mathrm{r}\mathscr{B}^p := \{\mathrm{Int}\,\mathrm{Cl}\, B \mid B \in \mathscr{B}^p\}$ is a base at p and satisfies* (R1).

PROOF. *Ad* 1. Fix $U \in \mathscr{O}^p$. Let $B \in \mathscr{B}^p$ be such that $B \subseteq U$. If $B = \{p\}$, then $\mathrm{Cl}\, B \subseteq U$. Assume there is $q \in B\backslash\{p\}$ and let $M \in \mathscr{O}^p\backslash\mathscr{O}^q$. Then $\mathscr{O}^p \ni M \cap B \subsetneq B$ and there is $B_0 \in \mathscr{B}^p$ such that $B_0 \subseteq M \cap B$. It cannot be the case that $\mathrm{Cl}\, B \subseteq B_0$, since this would entail that $\mathrm{Cl}\, B \subsetneq B$, so $\mathrm{Cl}\, B_0 \subseteq B$ by (R1). In consequence $\mathrm{Cl}\, B_0 \subseteq U$.

Ad 2. By 1 and Lemma A.1(1), for any $p \in S$ the family $\mathrm{r}\mathscr{B}^p := \{\mathrm{Int}\,\mathrm{Cl}\, B \mid B \in \mathscr{B}^p\}$ is a base at p. Moreover, if $\mathscr{B}^p$ satisfies (R1) then $\{\mathrm{Int}\,\mathrm{Cl}\, B \mid B \in \mathscr{B}^p\}$ satisfies (R1) either. Indeed, for all $U, V \in \mathscr{O}$, if $\mathrm{Cl}\, U \subseteq V$ then $\mathrm{Cl}\,\mathrm{Int}\,\mathrm{Cl}\, U = \mathrm{Cl}\, U \subseteq V \subseteq \mathrm{Int}\,\mathrm{Cl}\, V$. ∎

The open interval in $\mathbb{R}$ with end-points a and b, where $a < b$, is denoted by (a, b), and the closed interval with end-points a and b is denoted by $[a, b]$; half-open intervals are denoted by $(a, b]$ and $[a, b)$, respectively. For the set $\mathbb{R}$ we take the *natural topology*. On any closed interval $[a, b]$ we take also the *natural topology* which is the family consisting of all sets of the form $[a, b] \cap U$, where U is a open set with respect to the natural topology on $\mathbb{R}$. Both $\mathbb{R}$ and $[a, b]$ with natural topologies constitute second-countable topological spaces (see e.g. [5], p. 16).

$\mathcal{T}$ is *first-countable* iff every one of its points has a countable base. Moreover, $\mathcal{T}$ is *second-countable* iff it has a countable base. Every second-countable regular space is perfectly normal (see e.g. [5], pp. 44 and 45]). Hence, by Lemma A.2:

LEMMA A.3. *Let $\mathcal{T}$ be a second-countable T_1-space such that for any point p there is a base $\mathscr{B}^p$ at p satisfying* (R1). *Then $\mathcal{T}$ is perfectly normal.*

LEMMA A.4 ([**19**]). *If $\mathcal{T} = \langle S, \mathcal{O} \rangle$ is a first-countable Tychonoff space then for any $p \in S$ there is a continuous mapping $f \colon S \to [0,1]$ such that $f(p) = 0$, $\{ f^{-1}[[0, \frac{1}{2^n})] \mid n \in \omega^+ \}$ is a base at p, and $\{p\} = \bigcap_{n \in \omega^+} f^{-1}[[0, \frac{1}{2^n})]$.*

PROOF. Let $p \in S$ and $\{ B_n \mid n \in \omega^+ \}$ be a countable base at $p \in S$ such that $B_n \neq S$, for all $n \in \omega^+$. For all $n \in \omega^+$ we put $C_n := S \backslash B_n$; so $\emptyset \neq C_n \in \mathscr{C}$ and $p \notin C_n$. Then for all $n \in \omega^+$, we have a continuous mapping $f_n \colon S \to [0,1]$ such that $f_n(p) = 0$ and $f_n[C_n] = \{1\}$. We define the continuous function $f \colon S \to [0,1]$ such that $f(q) := \sum_{n=1}^{\infty} \frac{f_n(q)}{2^n}$, for any $q \in S$. The collection of all sets $U_n := f^{-1}[[0, \frac{1}{2^n})]$ (with $n \in \omega^+$) is a base at p. To see this, we take $V \in \mathcal{O}^p$. Then, by assumption, for some $k \in \omega$ we have $B_k \subseteq V$. But $U_k \cap C_k = \emptyset$, because for any $q \in C_k$ we have $f(q) > \frac{1}{2^k}$, and therefore $U_k \subseteq B_k \subseteq V$.

Of course, $f(p) = 0$ and for any $q \in S$: $q \in \bigcap_{n \in \omega^+} f^{-1}[[0, \frac{1}{2^n})]$ iff $f(q) = 0$. So $p \in \bigcap_{n \in \omega^+} f^{-1}[[0, \frac{1}{2^n})]$, and p is the only point in the intersection, since $\mathcal{T}$ is T_1. ∎

Let us remind that *antichains* of open sets of a given topological space are families of pairwise disjoint open sets (algebraically speaking, these are antichains in the lattice of open sets of the space). We say that a given topological space has *the countable chain condition* (abbrv.: c.c.c.) iff every antichain of its open sets is countable.

LEMMA A.5. *Every second-countable topological space has c.c.c.*

For $\mathcal{T} = \langle S, \mathcal{O} \rangle$, let us generate the topology $\mathcal{O}_{\mathrm{sr}}$ on S by means of the base consisting of all sets from $\mathrm{r}\mathcal{O}$, i.e., $\mathcal{O}_{\mathrm{sr}} := \mathcal{O}_{\mathrm{r}\mathcal{O}}$. The topological space $\mathcal{T}_{\mathrm{sr}} = \langle S, \mathcal{O}_{\mathrm{sr}} \rangle$ is called the *semi-regularization* of $\mathcal{T}$, since:

LEMMA A.6 ([**4**,**5**]). *The space $\mathcal{T}_{\mathrm{sr}}$ is semiregular and has the same open regular sets as $\mathcal{T}$, i.e., we have $\mathrm{r}\mathcal{O} = \mathrm{r}\mathcal{O}_{\mathrm{sr}}$.*

If $\mathcal{T} = \langle S, \mathcal{O} \rangle$ is semiregular then $\mathcal{T}_{\mathrm{sr}} = \mathcal{T}$, since $\mathrm{r}\mathcal{O}$ is a base for $\mathcal{T}$.

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