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# Interpolation in 16-Valued Trilattice Logics

**Abstract.** In a recent paper we have defined an analytic tableau calculus $\mathsf{PL_{16}}$ for a functionally complete extension of Shramko and Wansing's logic based on the trilattice $SIXTEEN_3$. This calculus makes it possible to define syntactic entailment relations that capture central semantic relations of the logic—such as the relations $\models_t$, $\models_f$, and $\models_i$ that each correspond to a lattice order in $SIXTEEN_3$; and $\models$, the intersection of $\models_t$ and $\models_f$. It turns out that our method of characterising these semantic relations—as intersections of auxiliary relations that can be captured with the help of a single calculus—lends itself well to proving interpolation. All entailment relations just mentioned have the interpolation property, not only when they are defined with respect to a functionally complete language, but also in a range of cases where less expressive languages are considered. For example, we will show that $\models$, when restricted to $\mathcal{L}_{tf}$, the language originally considered by Shramko and Wansing, enjoys interpolation. This answers a question that was recently posed by M. Takano.

*Keywords*: Interpolation, 16-Valued logic, Trilattice $SIXTEEN_3$, Multiple tree calculus.

## Introduction

In Muskens and Wintein [4] we have presented an analytic tableau calculus $\mathsf{PL_{16}}$ for a functionally complete extension of the logic considered in Shramko and Wansing [8]. Both Shramko and Wansing's original logic and our extension are based on the trilattice $SIXTEEN_3$ and $\mathsf{PL_{16}}$ can capture three semantic entailment relations, $\models_t$, $\models_f$, and $\models_i$, that each correspond to one of $SIXTEEN_3$'s three lattice orderings.[1] The calculus has a relatively simple formulation—only one rule scheme is needed for each of the three negations present in the logic, while each of the three conjunctions and each of the three disjunctions comes with two rule schemes.

In this paper we build upon [4] and study interpolation in Shramko and Wansing's trilattice logics. Using what is essentially Maehara's method we will prove a variant of his lemma for $\mathsf{PL_{16}}$. Interpolation theorems for $\models_t$, $\models_f$, $\models_i$, and the intersection $\models$ of $\models_t$ and $\models_f$ readily follow if these notions

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[1] The relations $\models_t$ and $\models_f$ were already present in Shramko and Wansing [8], $\models_i$ is an obvious analogon.

are interpreted as relations between sentences of the functionally complete language $\mathcal{L}_{tfi}$. We will also consider restrictions of these relations to those sublanguages of $\mathcal{L}_{tfi}$ that have the property that if one of the conjunctions or disjunctions of the language is present then so is its dual. All these restrictions enjoy interpolation. In particular, $\models$ is shown to have the (perfect) interpolation property on Shramko and Wansing's original language $\mathcal{L}_{tf}$, which answers a question by Takano [11].

The rest of the paper will be set up as follows. We will first give concise definitions of $SIXTEEN_3$, of the functionally complete language $\mathcal{L}_{tfi}$ and its semantics, and of the tableau system $\mathsf{PL_{16}}$. Once the stage is set in this way we will state and prove our interpolation results—first for logics based on $\mathcal{L}_{tfi}$ and then for the restrictions. A short conclusion will end the paper.

## 1. The Trilattice $SIXTEEN_3$

The introduction of $SIXTEEN_3$ in Shramko and Wansing [8] was motivated by a wish to generalise the well-known four-valued Belnap–Dunn logic (Belnap [1,2], Dunn [3]). The latter is based on the values $\mathbf{T} = \{1\}$ (true and not false), $\mathbf{F} = \{0\}$ (false and not true), $\mathbf{N} = \varnothing$ (neither true nor false), and $\mathbf{B} = \{0,1\}$ (both true and false) and can be viewed as a generalisation of classical logic—a move from $\{0,1\}$ with its usual ordering to $\mathcal{P}(\{0,1\})$ with *two* lattice orders. Shramko and Wansing in fact repeat this move, going from the set of truth-values $\mathcal{P}(\{0,1\}) = \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$ to its power set $\mathcal{P}(\mathcal{P}(\{0,1\}))$, now with *three* lattices. While the four-valued logic is meant to model the reasoning of a computer that is fed potentially incomplete or conflicting information, the 16-valued logic that results models networks of such computers (for more complete information, see the papers cited above, Wansing [12], or Shramko and Wansing [9], for example).

While the logic is thus based on $\mathcal{P}(\{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\})$ and can have a direct formulation on the basis of this set of truth-values, it is in fact slightly more convenient to follow Odintsov [5], who represents subsets of $\{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$ with the help of matrices of the following form.

$$\begin{vmatrix} n & f \\ t & b \end{vmatrix}$$

Here each element of the matrix is a 0 or a 1 and signals the presence or the absence of an element of $\{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$. Rivieccio [7] linearises this notation, obtaining the more manageable $\langle b, f, t, n \rangle$. We shall follow him in this and define $\mathbf{16}$ as $\{0,1\}^4$. Any $\mathcal{A} \subseteq \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$ will be represented by

a quadruple $\langle S_{\mathbf{B}}, S_{\mathbf{F}}, S_{\mathbf{T}}, S_{\mathbf{N}} \rangle \in \mathbf{16}$ such that $S_X = 1$ iff $X \in \mathcal{A}$, for $X \in \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$. With this representation in place, the three lattice orderings of the trilattice can be defined as follows (we let $\leq_f$ be the inverse of the relation originally defined in [8], so that it becomes a *non*falsity ordering, not a falsity ordering, see also [5,6]).

DEFINITION 1. Let $\leq$ be the usual order on $\{0, 1\}$. Define the orderings $\leq_t$, $\leq_f$, and $\leq_i$ on $\mathbf{16}$ by letting, for each $S = \langle S_{\mathbf{B}}, S_{\mathbf{F}}, S_{\mathbf{T}}, S_{\mathbf{N}} \rangle$ and $S' = \langle S'_{\mathbf{B}}, S'_{\mathbf{F}}, S'_{\mathbf{T}}, S'_{\mathbf{N}} \rangle \in \mathbf{16}$:

$$S \leq_t S' \quad \text{iff} \quad S_{\mathbf{B}} \leq S'_{\mathbf{B}}, \ S'_{\mathbf{F}} \leq S_{\mathbf{F}}, \ S_{\mathbf{T}} \leq S'_{\mathbf{T}}, \ S'_{\mathbf{N}} \leq S_{\mathbf{N}}$$

$$S \leq_f S' \quad \text{iff} \quad S'_{\mathbf{B}} \leq S_{\mathbf{B}}, \ S'_{\mathbf{F}} \leq S_{\mathbf{F}}, \ S_{\mathbf{T}} \leq S'_{\mathbf{T}}, \ S_{\mathbf{N}} \leq S'_{\mathbf{N}}$$

$$S \leq_i S' \quad \text{iff} \quad S_{\mathbf{B}} \leq S'_{\mathbf{B}}, \ S_{\mathbf{F}} \leq S'_{\mathbf{F}}, \ S_{\mathbf{T}} \leq S'_{\mathbf{T}}, \ S_{\mathbf{N}} \leq S'_{\mathbf{N}}$$

Figure 1 depicts the orderings $\leq_t$ and $\leq_f$ on $\mathbf{16}$, while Figure 2 shows $\leq_i$ and the intersection $\leq_t \cap \leq_f$. The node names employed in these pictures belong to the object language defined in Table 1 below (with tb denoting $\langle 1, 0, 1, 0 \rangle$, for example).

While the definition above provides lattice orderings, the next definition gives the lattices via their meet and join operations. The official definition of $SIXTEEN_3$ is based upon these operations.

DEFINITION 2. Let $\vee$ and $\wedge$ be the usual join and meet on $\{0, 1\}$. The operations $\sqcap_t$, $\sqcup_t$, $\sqcap_f$, $\sqcup_f$, $\sqcap_i$, and $\sqcup_i$ on $\mathbf{16}$ are defined by letting, for each $S = \langle S_{\mathbf{B}}, S_{\mathbf{F}}, S_{\mathbf{T}}, S_{\mathbf{N}} \rangle$ and $S' = \langle S'_{\mathbf{B}}, S'_{\mathbf{F}}, S'_{\mathbf{T}}, S'_{\mathbf{N}} \rangle \in \mathbf{16}$:

$$S \sqcap_t S' = \langle S_{\mathbf{B}} \wedge S'_{\mathbf{B}}, S_{\mathbf{F}} \vee S'_{\mathbf{F}}, S_{\mathbf{T}} \wedge S'_{\mathbf{T}}, S_{\mathbf{N}} \vee S'_{\mathbf{N}} \rangle,$$

$$S \sqcup_t S' = \langle S_{\mathbf{B}} \vee S'_{\mathbf{B}}, S_{\mathbf{F}} \wedge S'_{\mathbf{F}}, S_{\mathbf{T}} \vee S'_{\mathbf{T}}, S_{\mathbf{N}} \wedge S'_{\mathbf{N}} \rangle,$$

$$S \sqcap_f S' = \langle S_{\mathbf{B}} \vee S'_{\mathbf{B}}, S_{\mathbf{F}} \vee S'_{\mathbf{F}}, S_{\mathbf{T}} \wedge S'_{\mathbf{T}}, S_{\mathbf{N}} \wedge S'_{\mathbf{N}} \rangle,$$

$$S \sqcup_f S' = \langle S_{\mathbf{B}} \wedge S'_{\mathbf{B}}, S_{\mathbf{F}} \wedge S'_{\mathbf{F}}, S_{\mathbf{T}} \vee S'_{\mathbf{T}}, S_{\mathbf{N}} \vee S'_{\mathbf{N}} \rangle,$$

$$S \sqcap_i S' = \langle S_{\mathbf{B}} \wedge S'_{\mathbf{B}}, S_{\mathbf{F}} \wedge S'_{\mathbf{F}}, S_{\mathbf{T}} \wedge S'_{\mathbf{T}}, S_{\mathbf{N}} \wedge S'_{\mathbf{N}} \rangle,$$

$$S \sqcup_i S' = \langle S_{\mathbf{B}} \vee S'_{\mathbf{B}}, S_{\mathbf{F}} \vee S'_{\mathbf{F}}, S_{\mathbf{T}} \vee S'_{\mathbf{T}}, S_{\mathbf{N}} \vee S'_{\mathbf{N}} \rangle.$$

The trilattice $SIXTEEN_3$ is defined to be $\langle \mathbf{16}, \sqcap_t, \sqcup_t, \sqcap_f, \sqcup_f, \sqcap_i, \sqcup_i \rangle$.

It is easily checked that, for each $x \in \{t, f, i\}$, the function $\sqcap_x$ just defined is meet in the $\leq_x$ ordering, while $\sqcup_x$ is the corresponding join.

$SIXTEEN_3$ can be further enriched with the following operations.

DEFINITION 3. For each $S = \langle S_{\mathbf{B}}, S_{\mathbf{F}}, S_{\mathbf{T}}, S_{\mathbf{N}} \rangle \in \mathbf{16}$ the operations $-_t$, $-_f$, and $-_i$, are defined as follows.

$$-_t S = \langle S_{\mathbf{F}}, S_{\mathbf{B}}, S_{\mathbf{N}}, S_{\mathbf{T}} \rangle$$
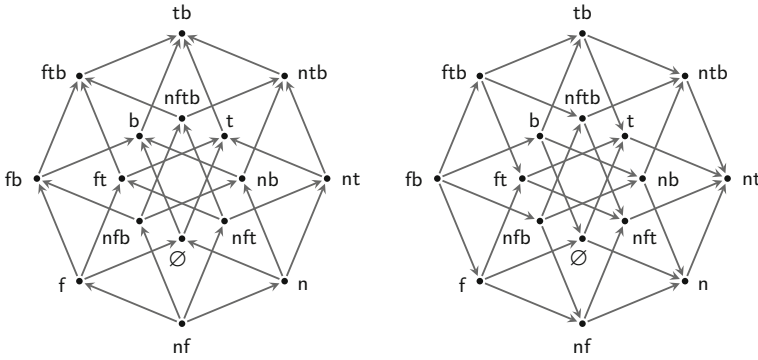
Figure 1. The trilattice $SIXTEEN_3$ with the truth order $\leq_t$ (*left*) and with the nonfalsity order $\leq_f$ (*right*). Vertices are accompanied by $\mathcal{L}_{tfi}$ formulas denoting them. The *top* and *bottom* elements of $\leq_t$ are tb and nf, while those of $\leq_f$ are nt and fb
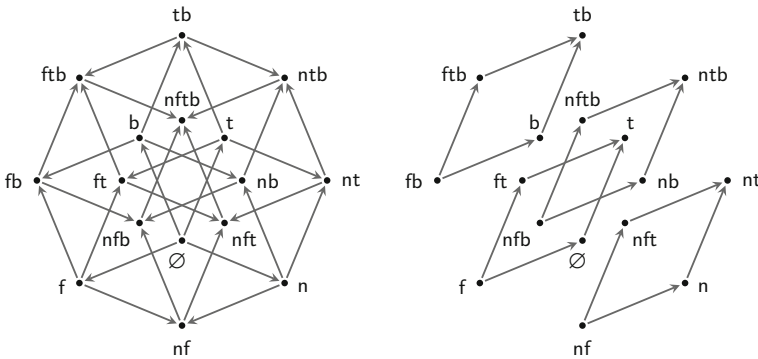


Figure 2. The trilattice $SIXTEEN_3$ with the information order $\leq_i$ (*left*; *top*: nftb, *bottom*: ∅) and with the intersection of truth and nonfalsity orders $\leq_t \cap \leq_f$ (*right*)

$$-_f S = \langle S_\mathbf{T}, S_\mathbf{N}, S_\mathbf{B}, S_\mathbf{F} \rangle$$
$$-_i S = \langle 1 - S_\mathbf{N}, 1 - S_\mathbf{T}, 1 - S_\mathbf{F}, 1 - S_\mathbf{B} \rangle$$

It is worthwhile to observe that, for each pairwise distinct $x, y \in \{t, f, i\}$, the following contraposition, monotonicity, and involution properties hold.

$$a \leq_x b \Longrightarrow -_x b \leq_x -_x a$$
$$a \leq_y b \Longrightarrow -_x a \leq_y -_x b$$
$$a = -_x -_x a$$

Table 1. Formulas denoting elements of **16**

| Form. | Definition | Value |
|---|---|---|
| tb | $\neg p_0 \vee_t p_0$ | $\langle 1,0,1,0 \rangle$ |
| nf | $\neg p_0 \wedge_t p_0$ | $\langle 0,1,0,1 \rangle$ |
| nt | $\neg p_0 \vee_f p_0$ | $\langle 0,0,1,1 \rangle$ |
| fb | $\neg p_0 \wedge_f p_0$ | $\langle 1,1,0,0 \rangle$ |
| nftb | $\neg p_0 \vee_i p_0$ | $\langle 1,1,1,1 \rangle$ |
| $\varnothing$ | $\neg p_0 \wedge_i p_0$ | $\langle 0,0,0,0 \rangle$ |
| b | tb $\wedge_f \varnothing$ | $\langle 1,0,0,0 \rangle$ |
| ntb | tb $\vee_f$ nftb | $\langle 1,0,1,1 \rangle$ |
| f | nf $\wedge_f \varnothing$ | $\langle 0,1,0,0 \rangle$ |
| nft | nf $\vee_f$ nftb | $\langle 0,1,1,1 \rangle$ |
| t | tb $\vee_f \varnothing$ | $\langle 0,0,1,0 \rangle$ |
| n | nf $\vee_f \varnothing$ | $\langle 0,0,0,1 \rangle$ |
| ftb | tb $\wedge_f$ nftb | $\langle 1,1,1,0 \rangle$ |
| nfb | nf $\wedge_f$ nftb | $\langle 1,1,0,1 \rangle$ |
| nb | b $\wedge_t$ ntb | $\langle 1,0,0,1 \rangle$ |
| ft | t $\wedge_t$ ftb | $\langle 0,1,1,0 \rangle$ |

Here $\neg$ abbreviates $\sim_t \sim_f \sim_i$.

## 2. The Language $\mathcal{L}_{tfi}$ and its Semantics

The language $\mathcal{L}_{tfi}$ is defined by the following BNF form (where $p$ comes from some countably infinite set of propositional constants).

$$\varphi ::= p \mid \sim_t \varphi \mid \sim_f \varphi \mid \sim_i \varphi \mid \varphi \wedge_t \varphi \mid \varphi \wedge_f \varphi \mid \varphi \wedge_i \varphi \mid$$
$$\varphi \vee_t \varphi \mid \varphi \vee_f \varphi \mid \varphi \vee_i \varphi$$

This language receives an interpretation as follows.

DEFINITION 4. A *valuation function* is a function $V$ from the sentences of $\mathcal{L}_{tfi}$ to **16** such that

$$V(\varphi \wedge_t \psi) = V(\varphi) \sqcap_t V(\psi); \qquad V(\sim_t \varphi) = -_t V(\varphi);$$
$$V(\varphi \wedge_f \psi) = V(\varphi) \sqcap_f V(\psi); \qquad V(\sim_f \varphi) = -_f V(\varphi);$$
$$V(\varphi \wedge_i \psi) = V(\varphi) \sqcap_i V(\psi); \qquad V(\sim_i \varphi) = -_i V(\varphi);$$
$$V(\varphi \vee_t \psi) = V(\varphi) \sqcup_t V(\psi); \qquad V(\varphi \vee_f \psi) = V(\varphi) \sqcup_f V(\psi);$$
$$V(\varphi \vee_i \psi) = V(\varphi) \sqcup_i V(\psi).$$

$\mathcal{L}_{tfi}$ sentences $\varphi$ and $\psi$ are *logically equivalent* if $V(\varphi) = V(\psi)$, for all $V$.

Muskens and Wintein [4] show that $\mathcal{L}_{tfi}$ is functionally complete. Indeed, it is possible to denote each of the elements of **16** with the help of an $\mathcal{L}_{tfi}$ sentence, as in the following definition.

DEFINITION 5. Let $p_0$ be some fixed propositional constant. The formulas in the first column of Table 1 will be defined by the corresponding entries in the second column. For any of these abbreviations $\xi$ and any $p$, we will write $\xi^p$ for the result of replacing each $p_0$ in $\xi$ by $p$.

It is not difficult to verify that, for any valuation $V$, any $\xi$ in the first column of Table 1, and any $p$, $V(\xi^p)$ equals the corresponding entry in the third column.

We now come to the definition of the semantic consequence relations. As was already announced in the introduction, the relations $\models_t$, $\models_f$, and $\models_i$ are directly based upon $\leq_t$, $\leq_f$, and $\leq_i$ respectively, while $\models$ is the intersection of $\models_t$ and $\models_f$.

DEFINITION 6. Let the relations $\models_t$, $\models_f$, $\models_i$, and $\models$ be defined as follows.

$$\varphi \models_t \psi \Longleftrightarrow V(\varphi) \leq_t V(\psi), \text{ for all valuations } V$$
$$\varphi \models_f \psi \Longleftrightarrow V(\varphi) \leq_f V(\psi), \text{ for all valuations } V$$
$$\varphi \models_i \psi \Longleftrightarrow V(\varphi) \leq_i V(\psi), \text{ for all valuations } V$$
$$\varphi \models \psi \Longleftrightarrow \varphi \models_t \psi \text{ and } \varphi \models_f \psi$$

Further decomposition of these relations is in fact possible and useful. This decomposition will be in terms of the relations $\models_{\mathbf{B}}$, $\models_{\mathbf{F}}$, $\models_{\mathbf{T}}$, and $\models_{\mathbf{N}}$, defined below. We follow the convention that, for any $V$ and $\varphi$, $V_{\mathbf{B}}(\varphi)$ refers to the first element of $V(\varphi)$, $V_{\mathbf{F}}(\varphi)$ to its second element, $V_{\mathbf{T}}(\varphi)$ to its third, and $V_{\mathbf{N}}(\varphi)$ to its fourth (so that $V(\varphi) = \langle V_{\mathbf{B}}(\varphi), V_{\mathbf{F}}(\varphi), V_{\mathbf{T}}(\varphi), V_{\mathbf{N}}(\varphi)\rangle$).

DEFINITION 7. For each $x \in \{\mathbf{T}, \mathbf{B}, \mathbf{F}, \mathbf{N}\}$, define the auxiliary entailment relation $\models_x$ by letting, for each two $\mathcal{L}_{tfi}$ sentences $\varphi$ and $\psi$, $\varphi \models_x \psi$ iff for all $V$: $V_x(\varphi) \leq V_x(\psi)$.

It is not difficult to see, on the basis of these definitions and the ones in Definition 1, that the equivalences in the following proposition hold.

PROPOSITION 1.

$$\varphi \models_t \psi \quad \Longleftrightarrow \quad \varphi \models_{\mathbf{B}} \psi,\ \psi \models_{\mathbf{F}} \varphi,\ \varphi \models_{\mathbf{T}} \psi,\ \psi \models_{\mathbf{N}} \varphi$$
$$\varphi \models_f \psi \quad \Longleftrightarrow \quad \psi \models_{\mathbf{B}} \varphi,\ \psi \models_{\mathbf{F}} \varphi,\ \varphi \models_{\mathbf{T}} \psi,\ \varphi \models_{\mathbf{N}} \psi$$
$$\varphi \models_i \psi \quad \Longleftrightarrow \quad \varphi \models_{\mathbf{B}} \psi,\ \varphi \models_{\mathbf{F}} \psi,\ \varphi \models_{\mathbf{T}} \psi,\ \varphi \models_{\mathbf{N}} \psi$$
$$\varphi \models \psi \quad \Longleftrightarrow \quad \varphi \models_{\mathbf{B}} \psi,\ \psi \models_{\mathbf{B}} \varphi,\ \psi \models_{\mathbf{F}} \varphi,\ \varphi \models_{\mathbf{T}} \psi,\ \varphi \models_{\mathbf{N}} \psi,\ \psi \models_{\mathbf{N}} \varphi$$

## 3. The Calculus $\mathsf{PL_{16}}$ and Satisfiability

In order to capture these semantic entailment relations, Muskens and Wintein [4] define the calculus $\mathsf{PL_{16}}$. Entries in this calculus are *signed formulas* $\mathsf{x} : \varphi$, where $\varphi$ is an $\mathcal{L}_{tfi}$ formula and $\mathsf{x}$ is one of the signs $\mathsf{b}, \mathsf{f}, \mathsf{t}, \mathsf{n}, \overline{\mathsf{b}}, \overline{\mathsf{f}}, \overline{\mathsf{t}}$, and $\overline{\mathsf{n}}$. While the role of these signed sentences in the calculus is a purely formal one, they also have an intuitive meaning. $\mathsf{b} : \varphi$, for example, can be read as saying that the first (i.e. **B**) component of the value of $\varphi$ is 1; that of $\overline{\mathsf{b}} : \varphi$ is that it is 0. The other signs can be interpreted similarly.

DEFINITION 8. The following are expansion rules of the calculus $\mathsf{PL_{16}}$.

$$\frac{\mathsf{x} : \varphi \wedge_t \psi}{\mathsf{x} : \varphi, \ \mathsf{x} : \psi} \ (\wedge_t^1)$$
where $\mathsf{x} \in \{\overline{\mathsf{n}}, \overline{\mathsf{f}}, \mathsf{t}, \mathsf{b}\}$

$$\frac{\mathsf{x} : \varphi \wedge_t \psi}{\mathsf{x} : \varphi \mid \mathsf{x} : \psi} \ (\wedge_t^2)$$
where $\mathsf{x} \in \{\mathsf{n}, \mathsf{f}, \overline{\mathsf{t}}, \overline{\mathsf{b}}\}$

$$\frac{\mathsf{x} : \varphi \wedge_f \psi}{\mathsf{x} : \varphi, \ \mathsf{x} : \psi} \ (\wedge_f^1)$$
where $\mathsf{x} \in \{\mathsf{n}, \overline{\mathsf{f}}, \mathsf{t}, \overline{\mathsf{b}}\}$

$$\frac{\mathsf{x} : \varphi \wedge_f \psi}{\mathsf{x} : \varphi \mid \mathsf{x} : \psi} \ (\wedge_f^2)$$
where $\mathsf{x} \in \{\overline{\mathsf{n}}, \mathsf{f}, \overline{\mathsf{t}}, \mathsf{b}\}$

$$\frac{\mathsf{x} : \varphi \wedge_i \psi}{\mathsf{x} : \varphi, \ \mathsf{x} : \psi} \ (\wedge_i^1)$$
where $\mathsf{x} \in \{\mathsf{n}, \mathsf{f}, \mathsf{t}, \mathsf{b}\}$

$$\frac{\mathsf{x} : \varphi \wedge_i \psi}{\mathsf{x} : \varphi \mid \mathsf{x} : \psi} \ (\wedge_i^2)$$
where $\mathsf{x} \in \{\overline{\mathsf{n}}, \overline{\mathsf{f}}, \overline{\mathsf{t}}, \overline{\mathsf{b}}\}$

$$\frac{\mathsf{x} : \varphi \vee_t \psi}{\mathsf{x} : \varphi, \ \mathsf{x} : \psi} \ (\vee_t^1)$$
where $\mathsf{x} \in \{\mathsf{n}, \mathsf{f}, \overline{\mathsf{t}}, \overline{\mathsf{b}}\}$

$$\frac{\mathsf{x} : \varphi \vee_t \psi}{\mathsf{x} : \varphi \mid \mathsf{x} : \psi} \ (\vee_t^2)$$
where $\mathsf{x} \in \{\overline{\mathsf{n}}, \overline{\mathsf{f}}, \mathsf{t}, \mathsf{b}\}$

$$\frac{\mathsf{x} : \varphi \vee_f \psi}{\mathsf{x} : \varphi, \ \mathsf{x} : \psi} \ (\vee_f^1)$$
where $\mathsf{x} \in \{\overline{\mathsf{n}}, \mathsf{f}, \overline{\mathsf{t}}, \mathsf{b}\}$

$$\frac{\mathsf{x} : \varphi \vee_f \psi}{\mathsf{x} : \varphi \mid \mathsf{x} : \psi} \ (\vee_f^2)$$
where $\mathsf{x} \in \{\mathsf{n}, \overline{\mathsf{f}}, \mathsf{t}, \overline{\mathsf{b}}\}$

$$\frac{\mathsf{x} : \varphi \vee_i \psi}{\mathsf{x} : \varphi, \ \mathsf{x} : \psi} \ (\vee_i^1)$$
where $\mathsf{x} \in \{\overline{\mathsf{n}}, \overline{\mathsf{f}}, \overline{\mathsf{t}}, \overline{\mathsf{b}}\}$

$$\frac{\mathsf{x} : \varphi \vee_i \psi}{\mathsf{x} : \varphi \mid \mathsf{x} : \psi} \ (\vee_i^2)$$
where $\mathsf{x} \in \{\mathsf{n}, \mathsf{f}, \mathsf{t}, \mathsf{b}\}$

$$\frac{\mathsf{x} : \sim_t \varphi}{\mathsf{y} : \varphi} \ (\sim_t) \quad \text{where } \{\mathsf{x}, \mathsf{y}\} \in \{\{\mathsf{n}, \mathsf{t}\}, \{\mathsf{f}, \mathsf{b}\}, \{\overline{\mathsf{n}}, \overline{\mathsf{t}}\}, \{\overline{\mathsf{f}}, \overline{\mathsf{b}}\}\}$$

$$\frac{\mathsf{x} : \sim_f \varphi}{\mathsf{y} : \varphi} \,(\sim_f) \quad \text{where } \{\mathsf{x},\mathsf{y}\} \in \{\{\mathsf{n},\mathsf{f}\},\{\mathsf{t},\mathsf{b}\},\{\overline{\mathsf{n}},\overline{\mathsf{f}}\},\{\overline{\mathsf{t}},\overline{\mathsf{b}}\}\}$$

$$\frac{\mathsf{x} : \sim_i \varphi}{\mathsf{y} : \varphi} \,(\sim_i) \quad \text{where } \{\mathsf{x},\mathsf{y}\} \in \{\{\mathsf{n},\overline{\mathsf{b}}\},\{\mathsf{f},\overline{\mathsf{t}}\},\{\overline{\mathsf{n}},\mathsf{b}\},\{\overline{\mathsf{f}},\mathsf{t}\}\}$$

The general form of these rules is $\vartheta/B_1,\ldots,B_n$, where $\vartheta$ is a signed sentence, called the *top formula* of the rule, and each $B_i$ is a set of signed sentences, called a *set of bottom formulas* of the rule. For example, using this general form one instantiation of the $(\wedge_i^1)$ rule can be expressed as $\mathsf{f} : \varphi \wedge_i \psi \,/\, \{\mathsf{f} : \varphi, \mathsf{f} : \psi\}$, while $\overline{\mathsf{t}} : \varphi \wedge_t \psi \,/\, \{\overline{\mathsf{t}} : \varphi\}, \{\overline{\mathsf{t}} : \psi\}$ instantiates the $(\wedge_t^2)$ rule.

On the basis of these rules *tableaux* can be obtained in the usual way (see [4] for a precise definition). A tableau branch will be *closed* if it contains signed sentences $\mathsf{x} : \varphi$ and $\overline{\mathsf{x}} : \varphi$ for $\mathsf{x} \in \{\mathsf{n},\mathsf{f},\mathsf{t},\mathsf{b}\}$, while a tableau is *closed* if all its branches are closed.

As we shall see shortly there is an intimate connection between the $\mathsf{PL_{16}}$ rules just given and the following notion of satisfiability.

DEFINITION 9. Let $\Theta$ be a set of signed $\mathcal{L}_{tfi}$ sentences and let $V$ be an $\mathcal{L}_{tfi}$ valuation. $V$ *satisfies* $\Theta$ iff the following statements hold.

$$\mathsf{t} : \varphi \in \Theta \Rightarrow V_{\mathbf{T}}(\varphi) = 1 \qquad \overline{\mathsf{t}} : \varphi \in \Theta \Rightarrow V_{\mathbf{T}}(\varphi) = 0$$
$$\mathsf{f} : \varphi \in \Theta \Rightarrow V_{\mathbf{F}}(\varphi) = 1 \qquad \overline{\mathsf{f}} : \varphi \in \Theta \Rightarrow V_{\mathbf{F}}(\varphi) = 0$$
$$\mathsf{n} : \varphi \in \Theta \Rightarrow V_{\mathbf{N}}(\varphi) = 1 \qquad \overline{\mathsf{n}} : \varphi \in \Theta \Rightarrow V_{\mathbf{N}}(\varphi) = 0$$
$$\mathsf{b} : \varphi \in \Theta \Rightarrow V_{\mathbf{B}}(\varphi) = 1 \qquad \overline{\mathsf{b}} : \varphi \in \Theta \Rightarrow V_{\mathbf{B}}(\varphi) = 0$$

A set of signed sentences will be called *satisfiable* if some $V$ satisfies it, *unsatisfiable* otherwise.

It is shown in [4] that a finite set of sentences is unsatisfiable if and only if it has a closed tableau. In this paper we will stay entirely on the semantic side of this equation, but will make use of the following relation between the $\mathsf{PL_{16}}$ rules and satisfiability. It follows from an easy inspection of the relevant definitions.

PROPOSITION 2. *Let* $\vartheta/B_1,\ldots,B_n$ *be an instantiation of a* $\mathsf{PL_{16}}$ *rule and let* $V$ *be a valuation. Then* $V$ *satisfies* $\vartheta$ *iff* $V$ *satisfies some* $B_i$ *($1 \le i \le n$). Hence if* $\Theta$ *is a set of signed* $\mathcal{L}_{tfi}$ *sentences and* $\vartheta/B_1,\ldots,B_n$ *is a* $\mathsf{PL_{16}}$ *rule, then* $\Theta \cup \{\vartheta\}$ *is unsatisfiable iff* $\Theta \cup B_i$ *is unsatisfiable for all* $i$.

We also note that the following connection between unsatisfiability and our auxiliary entailment relations obtains.

PROPOSITION 3. *Let $\varphi$ and $\psi$ be $\mathcal{L}_{tfi}$ sentences. Then*

$$\varphi \models_{\mathbf{T}} \psi \Longleftrightarrow \{\mathsf{t} : \varphi, \bar{\mathsf{t}} : \psi\} \text{ is unsatisfiable;}$$

$$\varphi \models_{\mathbf{F}} \psi \Longleftrightarrow \{\mathsf{f} : \varphi, \bar{\mathsf{f}} : \psi\} \text{ is unsatisfiable;}$$

$$\varphi \models_{\mathbf{N}} \psi \Longleftrightarrow \{\mathsf{n} : \varphi, \bar{\mathsf{n}} : \psi\} \text{ is unsatisfiable;}$$

$$\varphi \models_{\mathbf{B}} \psi \Longleftrightarrow \{\mathsf{b} : \varphi, \bar{\mathsf{b}} : \psi\} \text{ is unsatisfiable.}$$

## 4.  A Maehara Style Theorem and Interpolation in $\mathcal{L}_{tfi}$

Interpolation theorems usually come in two flavours, depending on whether the logical language that was defined is capable of naming truth-values with the help of zero-place connectives or not. Classical propositional logic, for example, has the property that whenever $\varphi \models^2 \psi$ (with $\models^2$ the classical entailment relation), there is an interpolant $\chi$ such that $\varphi \models^2 \chi$, $\chi \models^2 \psi$, and all propositional letters occurring in $\chi$ also occur in both $\varphi$ and $\psi$. If the language that was defined contains $\bot$ or $\top$ as zero-place connectives, that is, otherwise a condition is needed that excludes cases where $\varphi$ and $\psi$ have no propositional letters in common. The usual condition is that $\varphi$ is not a contradiction and that $\psi$ is not a tautology.

A similar condition will not always work here. Consider the relation $\models_t$ and let $p$ and $q$ be two (distinct) propositional letters. Then $\mathsf{f}^p \models_t \mathsf{ftb}^q$ clearly holds, $\mathsf{f}^p$ is not a contradiction in any sense ($\mathsf{f}^p \not\models_t \mathsf{nf}$ for example), $\mathsf{ftb}^q$ is not a tautology ($\mathsf{tb} \not\models_t \mathsf{ftb}^q$), but since there are no formulas that do not contain any propositional letters there cannot be an interpolant. One obvious way to get rid of this somewhat artificial conundrum would be to reintroduce, say, $\mathsf{tb}$ as a zero-place connective, but here we will stick to our earlier set-up of the language in [4] and will state conditions on interpolation where necessary. These conditions will be stated in terms of the existence of shared vocabulary.

We will prove a general Maehara-style theorem in this section, but will first prepare the ground and start with laying down conventions with respect to signs.

DEFINITION 10. If $\mathsf{x} \in \{\mathsf{n}, \mathsf{f}, \mathsf{t}, \mathsf{b}\}$ then $\bar{\mathsf{x}}$ is the *opposite* of $\mathsf{x}$, and $\mathsf{x}$ is the *opposite* of $\bar{\mathsf{x}}$. The opposite of any sign $\mathsf{x} \in \{\mathsf{n}, \mathsf{f}, \mathsf{t}, \mathsf{b}, \bar{\mathsf{n}}, \bar{\mathsf{f}}, \bar{\mathsf{t}}, \bar{\mathsf{b}}\}$ will be denoted by $\mathsf{x}'$. If $\mathcal{S}$ is any set of signs, then $\{\mathsf{x}' \mid \mathsf{x} \in \mathcal{S}\}$ will be denoted as $\mathcal{S}'$ and will also be called the *opposite* of $\mathcal{S}$. A signed sentence $\mathsf{x} : \varphi$ will be called $\mathcal{S}$-*signed* or *signed in $\mathcal{S}$* if $\mathsf{x} \in \mathcal{S}$ and a set of signed sentences $\Theta$ will be said to be $\mathcal{S}$-*signed* or *signed in $\mathcal{S}$* if each of its elements is signed in $\mathcal{S}$.
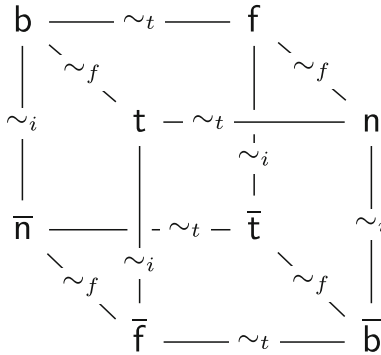
Figure 3. A cube summarizing the negation rules of $\mathsf{PL_{16}}$. If $\mathsf{x}$ and $\mathsf{y}$ are vertices connected with an edge labeled $\sim_k$ then $\mathsf{y} : \varphi$ can be obtained from $\mathsf{x} : \sim_k \varphi$ with the help of $(\sim_k)$

We will formulate our theorem not just for the functionally complete language, but also for (virtually) all sublanguages of $\mathcal{L}_{tfi}$. Languages will be identified with their basic set of connectives, as usual.

Note that the only rules in $\mathsf{PL_{16}}$ that change the signs of signed formulas are the negation rules $(\sim_t)$, $(\sim_f)$, and $(\sim_i)$. In Figure 3 we have summarised them. The eight signs of the calculus form the nodes of a labelled graph that is arranged in such a way that whenever $\mathsf{x}$ and $\mathsf{y}$ are vertices connected with an edge labeled $\sim_k$, any signed sentence $\mathsf{x} : \varphi$ can be obtained from $\mathsf{y} : \sim_k \varphi$ with the help of rule $(\sim_k)$—and vice versa, the graph is undirected. We see at a glance, for example, that $\mathsf{b} : \varphi$ can be obtained from $\bar{\mathsf{t}} : \sim_t \sim_i \varphi$, since there is a path from $\mathsf{b}$ to $\bar{\mathsf{t}}$ labelled $\sim_t \sim_i$.

There clearly are an infinite number of paths between any two nodes $\mathsf{x}$ and $\mathsf{y}$, but we find it expedient to define canonical short paths between them and canonical strings of negations labelling these paths.

DEFINITION 11. We denote the empty string with $\epsilon$. Define $\mathcal{C}$ to be the following set of strings of negations.

$$\{\epsilon, \sim_t, \sim_f, \sim_i, \sim_t \sim_f, \sim_t \sim_i, \sim_f \sim_i, \sim_t \sim_f \sim_i\}$$

If $\tau \in \mathcal{C}$ then $\tau$ is called a *canonical string of negations*. Consider Figure 3 and let $\mathsf{x}$ and $\mathsf{y}$ be signs. There is a unique $\sigma \in \mathcal{C}$ labelling a path in Figure 3 from $\mathsf{x}$ to $\mathsf{y}$. $\sigma$ is called the *(canonical)* $\mathsf{x}, \mathsf{y}$-*string*.

If in Figure 3 there is a path labelled $\sim_k \sim_\ell$ from $\mathsf{x}$ to $\mathsf{y}$ ($k, \ell \in \{t, f, i\}$), there is also a path labelled $\sim_\ell \sim_k$ from $\mathsf{x}$ to $\mathsf{y}$. Also, if there is a path labelled $\sim_k \sim_k$ from $\mathsf{x}$ to $\mathsf{y}$, then $\mathsf{x} = \mathsf{y}$. It follows that if there is any string of negations from a language $\mathcal{L} \subseteq \mathcal{L}_{tfi}$, labelling a path from $\mathsf{x}$ to $\mathsf{y}$, there

Table 2. The partition $\{[\mathsf{x}]_{\mathcal{L}} \mid \mathsf{x} \text{ is a sign}\}$ for the eight possible values of $\mathcal{L} \cap \{\sim_t, \sim_f, \sim_i\}$

| $\mathcal{L} \cap \{\sim_t, \sim_f, \sim_i\}$ | $\{[\mathsf{x}]_{\mathcal{L}} \mid \mathsf{x} \text{ is a sign}\}$ |
|---|---|
| $\{\sim_t, \sim_f, \sim_i\}$ | $\{\{\mathsf{n}, \mathsf{f}, \mathsf{t}, \mathsf{b}, \overline{\mathsf{n}}, \overline{\mathsf{f}}, \overline{\mathsf{t}}, \overline{\mathsf{b}}\}\}$ |
| $\{\sim_t, \sim_f\}$ | $\{\{\mathsf{n}, \mathsf{f}, \mathsf{t}, \mathsf{b}\}, \{\overline{\mathsf{n}}, \overline{\mathsf{f}}, \overline{\mathsf{t}}, \overline{\mathsf{b}}\}\}$ |
| $\{\sim_t, \sim_i\}$ | $\{\{\mathsf{n}, \overline{\mathsf{f}}, \mathsf{t}, \overline{\mathsf{b}}\}, \{\overline{\mathsf{n}}, \mathsf{f}, \overline{\mathsf{t}}, \mathsf{b}\}\}$ |
| $\{\sim_f, \sim_i\}$ | $\{\{\mathsf{n}, \mathsf{f}, \overline{\mathsf{t}}, \overline{\mathsf{b}}\}, \{\overline{\mathsf{n}}, \overline{\mathsf{f}}, \mathsf{t}, \mathsf{b}\}\}$ |
| $\{\sim_t\}$ | $\{\{\mathsf{n}, \mathsf{t}\}, \{\mathsf{f}, \mathsf{b}\}, \{\overline{\mathsf{n}}, \overline{\mathsf{t}}\}, \{\overline{\mathsf{f}}, \overline{\mathsf{b}}\}\}$ |
| $\{\sim_f\}$ | $\{\{\mathsf{n}, \mathsf{f}\}, \{\mathsf{t}, \mathsf{b}\}, \{\overline{\mathsf{n}}, \overline{\mathsf{f}}\}, \{\overline{\mathsf{t}}, \overline{\mathsf{b}}\}\}$ |
| $\{\sim_i\}$ | $\{\{\mathsf{n}, \overline{\mathsf{b}}\}, \{\mathsf{f}, \overline{\mathsf{t}}\}, \{\overline{\mathsf{n}}, \mathsf{b}\}, \{\overline{\mathsf{f}}, \mathsf{t}\}\}$ |
| $\varnothing$ | $\{\{\mathsf{n}\}, \{\mathsf{f}\}, \{\mathsf{t}\}, \{\mathsf{b}\}, \{\overline{\mathsf{n}}\}, \{\overline{\mathsf{f}}\}, \{\overline{\mathsf{t}}\}, \{\overline{\mathsf{b}}\}\}$ |

Note that the first partition corresponds to the cube in Figure 3 as a whole, the next three partitions each correspond to opposing faces of that cube, the following three to sets of edges, and the last to its set of vertices

is also a canonical $\mathsf{x}, \mathsf{y}$-string of $\mathcal{L}$ negations. Another observation is that, for any $\mathsf{x}$ and $\mathsf{y}$, the $\mathsf{x}, \mathsf{y}$-string is identical to the $\mathsf{x}', \mathsf{y}'$-string. The following proposition is easily seen to be true.

PROPOSITION 4. *Let* $\mathsf{x}$ *and* $\mathsf{y}$ *be signs, let* $p$ *be a propositional letter, and let* $\sigma$ *be the* $\mathsf{x}, \mathsf{y}$-*string. Then, for all* $V$, $V$ *satisfies* $\mathsf{x} : p$ *iff* $V$ *satisfies* $\mathsf{y} : \sigma p$, *while* $V$ *satisfies* $\mathsf{x}' : p$ *iff* $V$ *satisfies* $\mathsf{y}' : \sigma p$.

Of course, if one or more negations are not present in $\mathcal{L} \subseteq \mathcal{L}_{tfi}$, there may be no $\mathsf{x}, \mathsf{y}$-string of $\mathcal{L}$ negations (and hence no path labelled with negations from $\mathcal{L}$ at all) between two given nodes. We introduce the notion of $\mathcal{L}$-reachability.

DEFINITION 12. Let $\mathcal{L} \subseteq \mathcal{L}_{tfi}$, and let $\mathsf{x}$ and $\mathsf{y}$ be signs. $\mathsf{x}$ and $\mathsf{y}$ are in the $\mathcal{L}$-*reachability* relation if the $\mathsf{x}, \mathsf{y}$-string contains only negations from $\mathcal{L}$.

$\mathcal{L}$-reachability clearly is an equivalence relation. For each $\mathcal{L}$ and each sign $\mathsf{x}$, let $[\mathsf{x}]_{\mathcal{L}}$ be the set $\{\mathsf{y} \mid \mathsf{y} \text{ is } \mathcal{L}\text{-reachable from } \mathsf{x}\}$. For ease of reference, Table 2 gives an overview of the various partitions $\{[\mathsf{x}]_{\mathcal{L}} \mid \mathsf{x} \text{ is a sign}\}$. Note that $\mathcal{S} \in \{[\mathsf{x}]_{\mathcal{L}} \mid \mathsf{x} \text{ is a sign}\}$ if and only if $\mathcal{S}' \in \{[\mathsf{x}]_{\mathcal{L}} \mid \mathsf{x} \text{ is a sign}\}$, for all $\mathcal{L}$.

We define a general notion of interpolant. In the following, as in the rest of the paper, $\mathsf{Voc}(\varphi)$ will be used for the set of propositional letters occurring in $\varphi$ and $\mathsf{Voc}(\Theta)$ will be the set of propositional letters occurring in signed sentences in $\Theta$.

DEFINITION 13. Let $\mathcal{L} \subseteq \mathcal{L}_{tfi}$, let $\Theta_1$ and $\Theta_2$ be sets of signed $\mathcal{L}$-sentences, let $\mathsf{z}$ be any sign, and let $p$ be a proposition letter. An $\mathcal{L}$-sentence $\chi$ is called

a $\mathsf{z}, p$-*interpolant of* $\Theta_1$ *and* $\Theta_2$ *in* $\mathcal{L}$ if $\Theta_1 \cup \{\mathsf{z}' : \chi\}$ and $\Theta_2 \cup \{\mathsf{z} : \chi\}$ are unsatisfiable while $\mathsf{Voc}(\chi) \subseteq (\mathsf{Voc}(\Theta_1) \cap \mathsf{Voc}(\Theta_2)) \cup \{p\}$. If, moreover, $\mathsf{Voc}(\chi) \subseteq \mathsf{Voc}(\Theta_1) \cap \mathsf{Voc}(\Theta_2)$ then $\chi$ is called a $\mathsf{z}$-*interpolant of* $\Theta_1$ *and* $\Theta_2$ *in* $\mathcal{L}$.

We now state and prove a general theorem for the calculus. The proof is in fact an adaptation of Maehara's method—most often used in the context of Gentzen sequent calculi—to the present setting.

THEOREM 1. *(Maehara Theorem) Let* $\mathcal{L} \subseteq \mathcal{L}_{tfi}$ *and* $\mathcal{L} \neq \{\sim_t, \sim_f, \sim_i\}$. *Let* $\mathcal{S} \in \{[\mathsf{x}]_{\mathcal{L}} \mid \mathsf{x} \text{ is a sign}\}$ *and let* $\Theta_1$ *be a set of* $\mathcal{S}$-*signed sentences, while* $\Theta_2$ *is a set of* $\mathcal{S}'$-*signed sentences, and* $\Theta_1 \cup \Theta_2$ *is unsatisfiable. Let* $\mathsf{z} \in \mathcal{S}$ *and let* $p$ *be a proposition letter. Then there is a* $\mathsf{z}, p$-*interpolant of* $\Theta_1$ *and* $\Theta_2$ *in* $\mathcal{L}$. *Hence if* $\mathsf{Voc}(\Theta_1) \cap \mathsf{Voc}(\Theta_2) \neq \varnothing$ *there is a* $\mathsf{z}$-*interpolant of* $\Theta_1$ *and* $\Theta_2$ *in* $\mathcal{L}$. *For languages* $\mathcal{L}$ *such that* $\{\sim_t, \sim_f, \sim_i\} \not\subseteq \mathcal{L}$ *the condition* $\mathsf{Voc}(\Theta_1) \cap \mathsf{Voc}(\Theta_2) \neq \varnothing$ *is satisfied and there is a* $\mathsf{z}$-*interpolant of* $\Theta_1$ *and* $\Theta_2$ *in* $\mathcal{L}$.

PROOF. We will proceed by induction on the number of connectives occurring in signed sentences in $\Theta_1 \cup \Theta_2$. For the base step, assume that $\Theta_1 \cup \Theta_2$ only contains signed propositional letters.

In general, if a set of signed sentences $\Xi$ has only elements of the form $\mathsf{y} : q$, with $q$ a propositional variable, and, for no $q$ and $\mathsf{y}$, $\{\mathsf{y} : q, \mathsf{y}' : q\} \subseteq \Xi$, then $\Xi$ is easily shown to be satisfiable. By contraposition we find that $\{\mathsf{x} : r, \mathsf{x}' : r\} \subseteq \Theta_1 \cup \Theta_2$, for some $\mathsf{x}$ and $r$.

We consider two main subcases and in each define a $\mathsf{z}, p$-interpolant $\chi$.

I. $\mathsf{x} : r \in \Theta_1$ and $\mathsf{x}' : r \in \Theta_2$, for some $\mathsf{x}$ and $r$. In this case we can let $\chi = \sigma r$, where $\sigma$ is the $\mathsf{x}, \mathsf{z}$-string. Since $\mathsf{x}$ and $\mathsf{z}$ are both elements of $\mathcal{S}$, $\sigma$ only contains negation symbols from $\mathcal{L}$. Note that in this case $\chi$ is in fact a $\mathsf{z}$-interpolant of $\Theta_1$ and $\Theta_2$ in $\mathcal{L}$.

II. $\{\mathsf{x} : r, \mathsf{x}' : r\} \subseteq \Theta_1$ or $\{\mathsf{x} : r, \mathsf{x}' : r\} \subseteq \Theta_2$, for some $\mathsf{x}$ and $r$. Then $\mathcal{S} \cap \mathcal{S}' \neq \varnothing$, from which we can conclude that $\{\sim_t, \sim_f, \sim_i\} \subseteq \mathcal{L}$. Since $\mathcal{L} \neq \{\sim_t, \sim_f, \sim_i\}$, $\mathcal{L}$ must contain at least one conjunction or disjunction and so either tb, or nf, or nt, or fb, or nftb, or $\varnothing$ is definable in $\mathcal{L}$ (compare Table 1). In the first case (in which $\vee_t \in \mathcal{L}$) we can consider the following further subcases.

   (a) If $\{\mathsf{x} : r, \mathsf{x}' : r\} \subseteq \Theta_1$ and $\mathsf{z} \in \{\mathsf{t}, \mathsf{b}, \bar{\mathsf{n}}, \bar{\mathsf{f}}\}$, let $\chi = \sim_t \mathsf{tb}^p$;
   (b) If $\{\mathsf{x} : r, \mathsf{x}' : r\} \subseteq \Theta_1$ and $\mathsf{z} \in \{\bar{\mathsf{t}}, \bar{\mathsf{b}}, \mathsf{n}, \mathsf{f}\}$, let $\chi = \mathsf{tb}^p$;
   (c) If $\{\mathsf{x} : r, \mathsf{x}' : r\} \subseteq \Theta_2$ and $\mathsf{z} \in \{\mathsf{t}, \mathsf{b}, \bar{\mathsf{n}}, \bar{\mathsf{f}}\}$, let $\chi = \mathsf{tb}^p$;
   (d) If $\{\mathsf{x} : r, \mathsf{x}' : r\} \subseteq \Theta_2$ and $\mathsf{z} \in \{\bar{\mathsf{t}}, \bar{\mathsf{b}}, \mathsf{n}, \mathsf{f}\}$, let $\chi = \sim_t \mathsf{tb}^p$.

In each of these subcases $\mathsf{Voc}(\chi) \subseteq (\mathsf{Voc}(\Theta_1) \cap \mathsf{Voc}(\Theta_2)) \cup \{p\}$, while $\Theta_1 \cup \{\mathsf{z}' : \chi\}$ and $\Theta_2 \cup \{\mathsf{z} : \chi\}$ are unsatisfiable. The cases where conjunctions or disjunctions other than $\vee_t$ are present in $\mathcal{L}$ are entirely similar and left to the reader.

For the induction step, assume that $\Theta_1$ and $\Theta_2$ satisfy the constraints mentioned in the theorem, while the unsatisfiable $\Theta_1 \cup \Theta_2$ contains $n + 1$ connectives and the theorem holds for all $\Theta_1'$ and $\Theta_2'$ such that $\Theta_1' \cup \Theta_2'$ contains at most $n$ connectives. Let $\vartheta \in \Theta_1 \cup \Theta_2$ be a signed sentence containing at least one connective. There is a unique tableau rule $\rho$ such that $\vartheta$ is an instantiation of its top formula. We prove the induction step by cases, taking into account 1) which rule $\rho$ matches $\vartheta$ and 2) whether $\vartheta \in \Theta_1$ or $\vartheta \in \Theta_2$. This gives 30 cases, but they cluster in two similarity groups. Note that all rules have the property that if their top formula is an $\mathcal{L}$ sentence signed in $\mathcal{S}$, their bottom formulas will also be signed in $\mathcal{S}$.

If $\rho = (\wedge_t^1)$ and $\vartheta \in \Theta_1$, then $\vartheta$ has the form $\mathsf{x} : \varphi \wedge_t \psi$. Since $\Theta_1 \cup \Theta_2$ is unsatisfiable, $(\Theta_1 \backslash \{\vartheta\}) \cup \{\mathsf{x} : \varphi, \ \mathsf{x} : \psi\} \cup \Theta_2$ is also unsatisfiable by Proposition 2. Since the latter contains $n$ connectives, induction provides a $\mathsf{z}, p$-interpolant $\chi$ of $(\Theta_1 \backslash \{\vartheta\}) \cup \{\mathsf{x} : \varphi, \ \mathsf{x} : \psi\}$ and $\Theta_2$ in $\mathcal{L}$. Hence the sets $(\Theta_1 \backslash \{\vartheta\}) \cup \{\mathsf{x} : \varphi, \ \mathsf{x} : \psi, \ \mathsf{z}' : \chi\}$ and $\Theta_2 \cup \{\mathsf{z} : \chi\}$ are unsatisfiable. But then $\Theta_1 \cup \{\mathsf{z}' : \chi\}$ is unsatisfiable by Proposition 2. We conclude that $\chi$ is also a $\mathsf{z}, p$-interpolant for $\Theta_1$ and $\Theta_2$ in $\mathcal{L}$. The case that $\vartheta \in \Theta_2$ is entirely similar. In case $\rho$ is any of the rules $(\sim_t)$, $(\sim_f)$, $(\sim_i)$, $(\wedge_f^1)$, $(\wedge_i^1)$, $(\vee_t^1)$, $(\vee_f^1)$, or $(\vee_i^1)$, a $\mathsf{z}, p$-interpolant in $\mathcal{L}$ is obtained in a similar way.

If $\rho = (\wedge_t^2)$ and $\vartheta \in \Theta_1$, then $\vartheta$ again has the form $\mathsf{x} : \varphi \wedge_t \psi$. This time the unsatisfiability of $\Theta_1 \cup \Theta_2$ implies that $(\Theta_1 \backslash \{\vartheta\}) \cup \{\mathsf{x} : \varphi\} \cup \Theta_2$ and $(\Theta_1 \backslash \{\vartheta\}) \cup \{\mathsf{x} : \psi\} \cup \Theta_2$ are unsatisfiable. The induction hypothesis gives $\chi_1$ and $\chi_2$ so that the following are unsatisfiable.

a. $(\Theta_1 \backslash \{\vartheta\}) \cup \{\mathsf{x} : \varphi, \ \mathsf{z}' : \chi_1\}$

b. $\Theta_2 \cup \{\mathsf{z} : \chi_1\}$

c. $(\Theta_1 \backslash \{\vartheta\}) \cup \{\mathsf{x} : \psi, \ \mathsf{z}' : \chi_2\}$

d. $\Theta_2 \cup \{\mathsf{z} : \chi_2\}$

Since a. and c. are unsatisfiable, e. and f. below are too, and from this we deduce that g. is unsatisfiable.

e. $(\Theta_1 \backslash \{\vartheta\}) \cup \{\mathsf{x} : \varphi, \ \mathsf{z}' : \chi_1, \ \mathsf{z}' : \chi_2\}$

f. $(\Theta_1 \backslash \{\vartheta\}) \cup \{\mathsf{x} : \psi, \ \mathsf{z}' : \chi_1, \ \mathsf{z}' : \chi_2\}$

g. $\Theta_1 \cup \{\mathsf{z}' : \chi_1, \ \mathsf{z}' : \chi_2\}$

There are now two possibilities. The first is that $z$ is one of the signs mentioned in the side condition of $(\wedge_t^2)$, i.e. $z \in \{n, f, \bar{t}, \bar{b}\}$. Then $z' \in \{\bar{n}, \bar{f}, t, b\}$, i.e. $z$ is one of the signs mentioned in the side condition of $(\wedge_t^1)$. Using $(\wedge_t^1)$ we see that h. is unsatisfiable since g. is and using $(\wedge_t^2)$ it follows that i. is unsatisfiable because b. and d. are. We conclude that $\chi_1 \wedge_t \chi_2$ is a $z, p$-interpolant of $\Theta_1$ and $\Theta_2$ in this case.

  h. $\Theta_1 \cup \{z' : \chi_1 \wedge_t \chi_2\}$

  i. $\Theta_2 \cup \{z : \chi_1 \wedge_t \chi_2\}$

If, on the other hand, $z \in \{\bar{n}, \bar{f}, t, b\}$, we reason as follows. Since $x \in \{n, f, \bar{t}, \bar{b}\}$, while $x \in \mathcal{S}$ and $z \in \mathcal{S}$, it must be the case that $\sim_t \in \mathcal{L}$. This means that $\sim_t(\sim_t \chi_1 \wedge_t \sim_t \chi_2)$, a sentence equivalent to $\chi_1 \vee_t \chi_2$ (note that we have not assumed that $\vee_t \in \mathcal{L}$), is an $\mathcal{L}$ sentence. Using $(\vee_t^1)$ we conclude from g. that $\Theta_1 \cup \{z' : \chi_1 \vee_t \chi_2\}$ is unsatisfiable, while from b. and d. it follows with the help of $(\vee_t^2)$ that $\Theta_2 \cup \{z : \chi_1 \vee_t \chi_2\}$ is. Therefore the sets j. and k. are unsatisfiable and hence $\sim_t(\sim_t \chi_1 \wedge_t \sim_t \chi_2)$ is the $z, p$-interpolant that was sought after.

  j. $\Theta_1 \cup \{z' : \sim_t(\sim_t \chi_1 \wedge_t \sim_t \chi_2)\}$

  k. $\Theta_2 \cup \{z : \sim_t(\sim_t \chi_1 \wedge_t \sim_t \chi_2)\}$

We conclude that either $\sim_t(\sim_t \chi_1 \wedge_t \sim_t \chi_2)$ or $\chi_1 \wedge_t \chi_2$ is a $z, p$-interpolant of $\Theta_1$ and $\Theta_2$ in $\mathcal{L}$.

    The case in which $\rho = (\wedge_t^2)$ and $\vartheta \in \Theta_2$ leads to very similar reasoning and in case $\rho$ is $(\wedge_f^2)$, $(\wedge_i^2)$, $(\vee_t^2)$, $(\vee_f^2)$, or $(\vee_i^2)$, $z, p$-interpolants can be found in ways analogous to that in the $(\wedge_t^2)$ case.

    Note that if $\{\sim_t, \sim_f, \sim_i\} \not\subseteq \mathcal{L}$ the $z, p$-interpolant that is constructed is a $z$-interpolant, so that $\Theta_1$ and $\Theta_2$ must have common vocabulary. ∎

    Let us turn to the entailment relations we are interested in and to the auxiliary relations in terms of which they are characterised. We first define what it means for these relations to have the interpolation property on a sublanguage of $\mathcal{L}_{tfi}$.

DEFINITION 14. Let $\mathcal{R} \in \{\models_{\mathbf{T}}, \models_{\mathbf{F}}, \models_{\mathbf{N}}, \models_{\mathbf{B}}, \models_t, \models_f, \models_i, \models\}$ and let $\mathcal{L}$ be a sublanguage of $\mathcal{L}_{tfi}$. We say that $\mathcal{R}$ *has the interpolation property on* $\mathcal{L}$ if, for any $\varphi, \psi \in \mathcal{L}$ such that $\varphi \mathcal{R} \psi$ and $\mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi) \neq \varnothing$, there is a $\chi \in \mathcal{L}$ with $\mathsf{Voc}(\chi) \subseteq \mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi)$ such that $\varphi \mathcal{R} \chi$ and $\chi \mathcal{R} \psi$.

    $\mathcal{R}$ is said to *have the perfect interpolation property on* $\mathcal{L}$ if the condition that $\mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi) \neq \varnothing$ can be dropped, i.e. if, for any $\varphi, \psi \in \mathcal{L}$ such that

$\varphi \mathcal{R} \psi$, there is a $\chi \in \mathcal{L}$ with $\mathsf{Voc}(\chi) \subseteq \mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi)$ such that $\varphi \mathcal{R} \chi$ and $\chi \mathcal{R} \psi$.

The auxiliary relations $\models_{\mathbf{T}}$, $\models_{\mathbf{F}}$, $\models_{\mathbf{N}}$, and $\models_{\mathbf{B}}$ indeed have the interpolation property on all sublanguages $\mathcal{L}$ of $\mathcal{L}_{tfi}$ (note that for the functionally complete language itself this also follows from Takano [10]). If at least one of the negations is missing from $\mathcal{L}$, they have the perfect interpolation property.

LEMMA 1. *Let $\mathcal{L} \subseteq \mathcal{L}_{tfi}$ and let $x \in \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$. Then $\models_x$ has the interpolation property on $\mathcal{L}$. If $\{\sim_t, \sim_f, \sim_i\} \not\subseteq \mathcal{L}$, $\models_x$ has the perfect interpolation property on $\mathcal{L}$.*

PROOF. Let $\varphi$ and $\psi$ be $\mathcal{L}$-sentences such that $\mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi) \neq \varnothing$ and $\varphi \models_{\mathbf{T}} \psi$. Then $\{\mathsf{t} : \varphi, \bar{\mathsf{t}} : \psi\}$ is unsatisfiable. If $\mathcal{L} = \{\sim_t, \sim_f, \sim_i\}$, then $\varphi$ and $\psi$ must have their only proposition letter in common and $\varphi$ is an interpolant. Otherwise, Theorem 1 provides a $\chi$ in $\mathcal{L}$ with $\mathsf{Voc}(\chi) \subseteq \mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi)$ such that $\{\mathsf{t} : \varphi, \bar{\mathsf{t}} : \chi\}$ and $\{\mathsf{t} : \chi, \bar{\mathsf{t}} : \psi\}$ are unsatisfiable, whence $\varphi \models_{\mathbf{T}} \chi$ and $\chi \models_{\mathbf{T}} \psi$. The other three cases are entirely similar. If $\{\sim_t, \sim_f, \sim_i\} \not\subseteq \mathcal{L}$, Theorem 1 allows dropping the assumption that $\mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi) \neq \varnothing$. ∎

Can this result be extended to the entailment relations $\models_t$, $\models_f$, $\models_i$, and $\models$ that we are after? The answer is that in many cases we can find interpolants for these entailment relations that are certain truth-functional combinations of interpolants for the auxiliary relations in terms of which they can be analysed. Before we show the general procedure, let us first make a few simple observations. The first has to do with perfect interpolation.

LEMMA 2. *If $\mathcal{R} \in \{\models_t, \models_f, \models_i, \models\}$ has the interpolation property on $\mathcal{L} \subseteq \mathcal{L}_{tfi}$ and $\{\sim_t, \sim_f, \sim_i\} \not\subseteq \mathcal{L}$ then $\mathcal{R}$ has the perfect interpolation property on $\mathcal{L}$.*

PROOF. Let $\mathcal{R}$ be as described. Suppose $\varphi$ and $\psi$ are $\mathcal{L}$ sentences such that $\varphi \mathcal{R} \psi$. Then $\varphi \models_{\mathbf{T}} \psi$ and Lemma 1 gives an interpolant $\chi$ such that $\mathsf{Voc}(\chi) \subseteq \mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi)$. Since no sentence can have an empty vocabulary, it follows that $\mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi) \neq \varnothing$. So, since $\mathcal{R}$ has the interpolation property on $\mathcal{L}$, it has the perfect interpolation property on $\mathcal{L}$. ∎

The second observation concerns the relation $\models$.

PROPOSITION 5. *If $\varphi \models \psi$ and either $\varphi \models_t \chi \models_t \psi$ or $\varphi \models_f \chi \models_f \psi$ then $\varphi \models \chi \models \psi$.*

PROOF. The proof makes repeated use of Proposition 1. Note that, in general, one can conclude $\varphi \models \psi$ from the conjunction of $\varphi \models_t \psi$, $\psi \models_{\mathbf{B}} \varphi$,

and $\varphi \models_\mathbf{N} \psi$. Suppose $\varphi \models \psi$ and $\varphi \models_t \chi \models_t \psi$. From $\varphi \models_t \chi \models_t \psi$ it follows that $\varphi \models_\mathbf{B} \chi \models_\mathbf{B} \psi$ and from $\varphi \models \psi$ it follows that $\psi \models_\mathbf{B} \varphi$. Hence $\psi \models_\mathbf{B} \chi \models_\mathbf{B} \varphi$. In a similar way $\varphi \models_\mathbf{N} \chi \models_\mathbf{N} \psi$ is shown, so that $\varphi \models \chi \models \psi$ can be concluded.

The case in which $\varphi \models \psi$ and $\varphi \models_f \chi \models_f \psi$ is entirely similar. ■

From this proposition the following useful lemma follows directly.

LEMMA 3. *If $\models_t$ or $\models_f$ has the (perfect) interpolation property on a language $\mathcal{L}$, then $\models$ likewise has the (perfect) interpolation property on $\mathcal{L}$.*

The following theorem gives interpolation for the language $\mathcal{L}_{tfi}$. Its proof shows how interpolants for the auxiliary entailment relations can be 'glued together' in order to obtain interpolants for the relations $\models_t$, $\models_f$, and $\models_i$.

PROPOSITION 6. *The entailment relations $\models_t$, $\models_f$, $\models_i$, and $\models$ each have the interpolation property on $\mathcal{L}_{tfi}$.*

PROOF. Suppose $\varphi \models_t \psi$, while $\mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi) \neq \varnothing$. Then $\varphi \models_\mathbf{T} \psi$, $\varphi \models_\mathbf{B} \psi$, $\psi \models_\mathbf{F} \varphi$, and $\psi \models_\mathbf{N} \varphi$. Lemma 1 shows that there are interpolants $\chi_1$, $\chi_2$, $\chi_3$, and $\chi_4$ such that $\varphi \models_\mathbf{T} \chi_1 \models_\mathbf{T} \psi$, $\varphi \models_\mathbf{B} \chi_2 \models_\mathbf{B} \psi$, $\psi \models_\mathbf{F} \chi_3 \models_\mathbf{F} \varphi$, and $\psi \models_\mathbf{N} \chi_4 \models_\mathbf{N} \varphi$. Let $p \in \mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi)$ and let $\chi$ be the sentence

$$(\chi_1 \wedge_i \mathsf{t}^p) \vee_i (\chi_2 \wedge_i \mathsf{b}^p) \vee_i (\chi_3 \wedge_i \mathsf{f}^p) \vee_i (\chi_4 \wedge_i \mathsf{n}^p) . \tag{1}$$

Note that $V(\chi) = \langle V_\mathbf{B}(\chi_2), V_\mathbf{F}(\chi_3), V_\mathbf{T}(\chi_1), V_\mathbf{N}(\chi_4) \rangle$ for any $V$. From this $\varphi \models_\mathbf{T} \chi \models_\mathbf{T} \psi$, $\varphi \models_\mathbf{B} \chi \models_\mathbf{B} \psi$, $\psi \models_\mathbf{F} \chi \models_\mathbf{F} \varphi$, and $\psi \models_\mathbf{N} \chi \models_\mathbf{N} \varphi$ follow. Hence $\varphi \models_t \chi \models_t \psi$ and $\chi$ is an interpolant for $\varphi \models_t \psi$. The proofs for $\models_f$, and $\models_i$ follow very similar lines, both using the formula schema in (1), with the $\chi_j$ possibly instantiated differently. That the statement holds for $\models$ follows from Lemma 3 above. ■

## 5.   Interpolation Results for Sublanguages of $\mathcal{L}_{tfi}$

The language $\mathcal{L}_{tfi}$ is functionally complete and hence maximally expressive given the underlying semantics. This makes it relatively easy to construct interpolants. Do less expressive languages still have the interpolation property? The question is not without interest, as it concerns languages such as $\mathcal{L}_{tf} := \{\wedge_t, \wedge_f, \vee_t, \vee_f, \sim_t, \sim_f\}$, defined in Shramko and Wansing [8], and $\mathcal{L}_{tf}^{\sim_i} := \{\wedge_t, \wedge_f, \vee_t, \vee_f, \sim_t, \sim_f, \sim_i\}$, which in [4] we have shown to be expressively equivalent to the languages $\mathcal{L}_{tf}^{\to_t}$ and $\mathcal{L}_{tf}^{\to_f}$ considered in Odintsov [5].

We will give affirmative answers for these and a range of other languages here, but will restrict attention to those sublanguages of the functionally complete one that are closed under duals in the following sense.

DEFINITION 15. Let $\mathcal{L} \subseteq \mathcal{L}_{tfi}$. $\mathcal{L}$ is *closed under duals* if $\wedge_k \in \mathcal{L} \Longleftrightarrow \vee_k \in \mathcal{L}$, for $k \in \{t, f, i\}$.

So, in all languages under consideration conjunctions and disjunctions come in pairs. Let us first discuss languages that do not contain all of these pairs. For these certain dualities arise. First a definition.

DEFINITION 16. For each sign $\mathsf{x}$ and each $k \in \{t, f, i\}$, $\mathsf{x}^{*_k}$ will denote the unique sign such that

$$\{\mathsf{x}, \mathsf{x}^{*_i}\} \in \{\{\mathsf{n}, \overline{\mathsf{b}}\}, \{\mathsf{f}, \overline{\mathsf{t}}\}, \{\overline{\mathsf{n}}, \mathsf{b}\}, \{\overline{\mathsf{f}}, \mathsf{t}\}\},$$

$$\{\mathsf{x}, \mathsf{x}^{*_t}\} \in \{\{\mathsf{n}, \mathsf{t}\}, \{\mathsf{f}, \mathsf{b}\}, \{\overline{\mathsf{n}}, \overline{\mathsf{t}}\}, \{\overline{\mathsf{f}}, \overline{\mathsf{b}}\}\},$$

$$\{\mathsf{x}, \mathsf{x}^{*_f}\} \in \{\{\mathsf{n}, \mathsf{f}\}, \{\mathsf{t}, \mathsf{b}\}, \{\overline{\mathsf{n}}, \overline{\mathsf{f}}\}, \{\overline{\mathsf{t}}, \overline{\mathsf{b}}\}\}.$$

The reader may want to compare this definition with the side conditions of the $(\sim_k)$ tableau expansion rules. On languages that do not have all conjunction/disjunction pairs some entailment relations are coextensive.

PROPOSITION 7. *Let* $\mathcal{L} \subseteq \mathcal{L}_{tfi}$ *be a language such that, for some* $k \in \{t, f, i\}$, $\mathcal{L} \cap \{\wedge_k, \vee_k\} = \varnothing$. *For any set* $\Theta$ *of signed* $\mathcal{L}$-*sentences,* $\Theta$ *is unsatisfiable iff* $\Theta^{*_k} = \{\mathsf{x}^{*_k} : \varphi \mid \mathsf{x} : \varphi \in \Theta\}$ *is unsatisfiable. Hence, if* $\varphi$ *and* $\psi$ *are* $\mathcal{L}$-*sentences, we have*

$$\begin{aligned} &\textit{if } k = i: &&\varphi \models_{\mathbf{N}} \psi \Longleftrightarrow \psi \models_{\mathbf{B}} \varphi \textit{ and } \varphi \models_{\mathbf{F}} \psi \Longleftrightarrow \psi \models_{\mathbf{T}} \varphi; \\ &\textit{if } k = t: &&\varphi \models_{\mathbf{N}} \psi \Longleftrightarrow \varphi \models_{\mathbf{T}} \psi \textit{ and } \varphi \models_{\mathbf{F}} \psi \Longleftrightarrow \varphi \models_{\mathbf{B}} \psi; \\ &\textit{if } k = f: &&\varphi \models_{\mathbf{N}} \psi \Longleftrightarrow \varphi \models_{\mathbf{F}} \psi \textit{ and } \varphi \models_{\mathbf{T}} \psi \Longleftrightarrow \varphi \models_{\mathbf{B}} \psi. \end{aligned}$$

PROOF. For each valuation $V$ and $k \in \{t, f, i\}$, let $V^{-k}$ be the valuation such that $V^{-k}(p) = -_k V(p)$, for all propositional variables $p$. A straightforward induction gives that $V^{-k}(\varphi) = -_k V(\varphi)$, for all $\varphi$ not containing $\wedge_k$ or $\vee_k$, so that $V$ satisfies $\mathsf{x} : \varphi$ iff $V^{-k}$ satisfies $\mathsf{x}^{*_k} : \varphi$, for such $\varphi$. ∎

An immediate consequence of this duality (and Proposition 1) is that certain entailment relations collapse to equivalence and as a consequence have the interpolation property.

PROPOSITION 8. *Let* $\mathcal{L} \subseteq \mathcal{L}_{tfi}$ *be a language such that* $\mathcal{L} \cap \{\wedge_k, \vee_k\} = \varnothing$ ($k \in \{t, f, i\}$). *Then* $\varphi \models_k \psi$ *implies* $V(\varphi) = V(\psi)$, *for all valuations* $V$ *and* $\mathcal{L}$ *sentences* $\varphi$ *and* $\psi$. *It follows that* $\models_k$ *enjoys interpolation on* $\mathcal{L}$.

PROOF. Let $\mathcal{L}$, $\varphi$, $\psi$, and $k$ be as described. That $\varphi \models_k \psi$ implies $V(\varphi) = V(\psi)$, for all $V$, follows from Proposition 7. Suppose $\varphi \models_k \psi$ and hence $V(\varphi) = V(\psi)$, for all $V$. Suppose that $\mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi) \neq \varnothing$. Let $p \in \mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi)$ and let $\varphi'$ be the result of replacing each $q \notin \mathsf{Voc}(\psi)$ in $\varphi$ by $p$. For any valuation $V$, let $V'$ be the valuation such that $V'(r) = V(r)$ if $r \in \mathsf{Voc}(\psi)$ and $V'(r) = V(p)$ otherwise. Then, for any $V$,

$$V(\varphi') = V'(\varphi) = V'(\psi) = V(\psi) = V(\varphi) \ .$$

It follows that $\varphi \models_k \varphi' \models_k \psi$ and that $\mathsf{Voc}(\varphi') \subseteq \mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi)$, so that $\varphi'$ is the required interpolant. ∎

From this the following proposition about the limiting case of languages only containing negations follows immediately.

PROPOSITION 9. *If $\mathcal{L} \cap \{\wedge_i, \vee_i, \wedge_t, \vee_t, \wedge_f, \vee_f\} = \varnothing$ then $\models_t$, $\models_f$, and $\models_i$ enjoy interpolation on $\mathcal{L}$.*

Another consequence of Propositions 1 and 7 is that in the absence of $\wedge_k$ and $\vee_k$ ($k \in \{t, f, i\}$) the characterisations of entailment relations $\models_\ell$, where $\ell \neq k$, can be simplified.

PROPOSITION 10. *Let $\mathcal{L} \cap \{\wedge_k, \vee_k\} = \varnothing$, as before, and let $\varphi$ and $\psi$ be $\mathcal{L}$ sentences. Then the following equivalences hold.*

$$\text{If } k = i\colon \begin{cases} \varphi \models_t \psi & \Longleftrightarrow & \varphi \models_{\mathbf{B}} \psi \text{ and } \varphi \models_{\mathbf{T}} \psi \\ \varphi \models_f \psi & \Longleftrightarrow & \psi \models_{\mathbf{B}} \varphi \text{ and } \varphi \models_{\mathbf{T}} \psi \end{cases}$$

$$\text{If } k = t\colon \begin{cases} \varphi \models_i \psi & \Longleftrightarrow & \varphi \models_{\mathbf{B}} \psi \text{ and } \varphi \models_{\mathbf{T}} \psi \\ \varphi \models_f \psi & \Longleftrightarrow & \psi \models_{\mathbf{B}} \varphi \text{ and } \varphi \models_{\mathbf{T}} \psi \end{cases}$$

$$\text{If } k = f\colon \begin{cases} \varphi \models_i \psi & \Longleftrightarrow & \varphi \models_{\mathbf{F}} \psi \text{ and } \varphi \models_{\mathbf{T}} \psi \\ \varphi \models_t \psi & \Longleftrightarrow & \psi \models_{\mathbf{F}} \varphi \text{ and } \varphi \models_{\mathbf{T}} \psi \end{cases}$$

Moreover, if *two* conjunction/disjunction pairs are missing, the only remaining entailment relation that does not collapse to equivalence will in fact be coextensive with $\models_{\mathbf{T}}$, as the following proposition shows.

PROPOSITION 11. *Let $\mathcal{L} \cap \{\wedge_i, \vee_i, \wedge_t, \vee_t, \wedge_f, \vee_f\} = \{\wedge_k, \vee_k\}$, for $k \in \{t, f, i\}$. Then $\models_t$, $\models_f$, and $\models_i$ enjoy interpolation on $\mathcal{L}$.*

PROOF. Let $\varphi$ and $\psi$ be $\mathcal{L}$ sentences. Again use Propositions 1 and 7 in order to show that $\varphi \models_k \psi \Longleftrightarrow \varphi \models_{\mathbf{T}} \psi$. That $\models_k$ has the interpolation property follows from Lemma 1. That $\models_\ell$ enjoys interpolation, for $\ell \in \{t, f, i\}$ and $k \neq \ell$, follows from Proposition 8. ∎

Propositions 9 and 11 imply that $\models_t$, $\models_f$, and $\models_i$ enjoy interpolation on all relevant $\mathcal{L}$ that have at most one conjunction/disjunction pair. So, from this point on we can focus on languages closed under duality that contain at least two conjunction/disjunction pairs.

But what if negations are missing? We have already seen that interpolation results for languages lacking one or more negations can immediately be strengthened to results about *perfect* interpolation, but now must take into account that it is no longer a given that formulas constantly denoting elements of **16** are definable. Suppose, for example, that $\mathcal{L}$ is a language not containing $\sim_i$ and $\varphi$ is an $\mathcal{L}$-sentence. Then a straightforward induction on sentence complexity gives that if $V(p) = \langle 0, 0, 0, 0 \rangle$ for every $p \in \mathsf{Voc}(\varphi)$, we also have that $V(\varphi) = \langle 0, 0, 0, 0 \rangle$. Similarly, $V(\varphi) = \langle 1, 1, 1, 1 \rangle$, if $V(p) = \langle 1, 1, 1, 1 \rangle$ for every $p \in \mathsf{Voc}(\varphi)$. It follows that no $\mathcal{L}$-formula can have a constant denotation. Since formulas with constant denotation were used to 'glue' interpolants together in Proposition 6, we need to adapt the method.

In languages that contain only a single negation we see a property similar to the one just described. Consider, for example, a language $\mathcal{L}$ that only contains the $\sim_i$ negation and let $\varphi$ be any sentence of $\mathcal{L}$. Then we see that, if $V_{\mathbf{B}}(p) = 0$ and $V_{\mathbf{N}}(p) = 1$ for every $p$ occurring in $\varphi$, we also have $V_{\mathbf{B}}(\varphi) = 0$ and $V_{\mathbf{N}}(\varphi) = 1$.

Let us analyse the situation a bit further. Here are some useful definitions.

DEFINITION 17. A *form* is a partial function $F : \{\mathbf{B}, \mathbf{F}, \mathbf{T}, \mathbf{N}\} \rightharpoonup \{0, 1\}$ with a non-empty domain. If $V$ is a valuation and $\varphi$ is a formula then $V$ is called an *F-valuation* on $\varphi$ if, for all $x \in \mathrm{dom}(F)$, $V_x(\varphi) = F(x)$. If $P$ is a set of propositional letters then $V$ is an *F-valuation* on $P$ if $V$ is an $F$-valuation on all $p \in P$. A form $F$ is *fixed* for a formula $\varphi$ if $V$ is an $F$-valuation on $\varphi$ whenever $V$ is an $F$-valuation on $\mathsf{Voc}(\varphi)$, for all $V$. $F$ is *fixed* for a language $\mathcal{L}$ if $F$ is fixed for all $\mathcal{L}$-sentences.

Table 3 gives, for each $\mathcal{L} \subseteq \mathcal{L}_{tfi}$, a collection of forms fixed for $\mathcal{L}$, depending on the value of $\mathcal{L} \cap \{\sim_t, \sim_f, \sim_i\}$. For example, $\{\langle \mathbf{B}, 0 \rangle, \langle \mathbf{N}, 1 \rangle\}$ is a form fixed for languages containing only the $\sim_i$ negation, while for languages that contain only $\sim_t$ and $\sim_f$ $\{\langle \mathbf{B}, 0 \rangle, \langle \mathbf{F}, 0 \rangle, \langle \mathbf{T}, 0 \rangle, \langle \mathbf{N}, 0 \rangle\}$ is fixed. This corresponds to two of the situations just described. The proof of the following proposition is a straightforward induction on the complexity of $\mathcal{L}$ formulas in each case.

PROPOSITION 12. *Let $\mathcal{L} \subseteq \mathcal{L}_{tfi}$. If $\mathcal{L} \cap \{\sim_t, \sim_f, \sim_i\}$ is as in the left column of Table 3, then the corresponding forms on the right are fixed for $\mathcal{L}$.*

Table 3. Languages $\mathcal{L} \subseteq \mathcal{L}_{tfi}$ and forms fixed for $\mathcal{L}$, depending on $\mathcal{L} \cap \{\sim_t, \sim_f, \sim_i\}$

| $\mathcal{L} \cap \{\sim_t, \sim_f, \sim_i\}$ | Forms fixed for $\mathcal{L}$ |
|---|---|
| $\{\sim_t, \sim_f, \sim_i\}$ | — |
| $\{\sim_t, \sim_f\}$ | $\{\langle \mathbf{B}, 0\rangle, \langle \mathbf{F}, 0\rangle, \langle \mathbf{T}, 0\rangle, \langle \mathbf{N}, 0\rangle\}$, $\{\langle \mathbf{B}, 1\rangle, \langle \mathbf{F}, 1\rangle, \langle \mathbf{T}, 1\rangle, \langle \mathbf{N}, 1\rangle\}$ |
| $\{\sim_t, \sim_i\}$ | $\{\langle \mathbf{B}, 1\rangle, \langle \mathbf{F}, 1\rangle, \langle \mathbf{T}, 0\rangle, \langle \mathbf{N}, 0\rangle\}$, $\{\langle \mathbf{B}, 0\rangle, \langle \mathbf{F}, 0\rangle, \langle \mathbf{T}, 1\rangle, \langle \mathbf{N}, 1\rangle\}$ |
| $\{\sim_f, \sim_i\}$ | $\{\langle \mathbf{B}, 0\rangle, \langle \mathbf{F}, 1\rangle, \langle \mathbf{T}, 0\rangle, \langle \mathbf{N}, 1\rangle\}$, $\{\langle \mathbf{B}, 1\rangle, \langle \mathbf{F}, 0\rangle, \langle \mathbf{T}, 1\rangle, \langle \mathbf{N}, 0\rangle\}$ |
| $\{\sim_t\}$ | $\{\langle \mathbf{B}, 0\rangle, \langle \mathbf{F}, 0\rangle\}, \{\langle \mathbf{B}, 1\rangle, \langle \mathbf{F}, 1\rangle\}$, $\{\langle \mathbf{T}, 0\rangle, \langle \mathbf{N}, 0\rangle\}, \{\langle \mathbf{T}, 1\rangle, \langle \mathbf{N}, 1\rangle\}$ |
| $\{\sim_f\}$ | $\{\langle \mathbf{B}, 0\rangle, \langle \mathbf{T}, 0\rangle\}, \{\langle \mathbf{B}, 1\rangle, \langle \mathbf{T}, 1\rangle\}$, $\{\langle \mathbf{F}, 0\rangle, \langle \mathbf{N}, 0\rangle\}, \{\langle \mathbf{F}, 1\rangle, \langle \mathbf{N}, 1\rangle\}$ |
| $\{\sim_i\}$ | $\{\langle \mathbf{B}, 0\rangle, \langle \mathbf{N}, 1\rangle\}, \{\langle \mathbf{B}, 1\rangle, \langle \mathbf{N}, 0\rangle\}$, $\{\langle \mathbf{F}, 0\rangle, \langle \mathbf{T}, 1\rangle\}, \{\langle \mathbf{F}, 1\rangle, \langle \mathbf{T}, 0\rangle\}$ |
| $\varnothing$ | $\{\langle \mathbf{B}, 1\rangle\}, \{\langle \mathbf{B}, 0\rangle\}, \{\langle \mathbf{F}, 1\rangle\}, \{\langle \mathbf{F}, 0\rangle\}$, $\{\langle \mathbf{T}, 1\rangle\}, \{\langle \mathbf{T}, 0\rangle\}, \{\langle \mathbf{N}, 1\rangle\}, \{\langle \mathbf{N}, 0\rangle\}$ |

We will use certain conjunctions and disjunctions of *literals* for 'glueing' interpolants together. Here is a definition.

DEFINITION 18. A *literal* over the propositional letter $p$ is any formula $\sigma p$, where $\sigma$ is a (possibly empty) string of negations. A literal $\sigma p$ is in *canonical form* if $\sigma \in \mathcal{C}$, where $\mathcal{C}$ is as in Definition 11. Let $\mathcal{L} \subseteq \mathcal{L}_{tfi}$. A literal over $p$ in canonical form that is also an $\mathcal{L}$-formula is called a *canonical $\mathcal{L}$-literal* over $p$. If $P$ is a set of propositional letters, we let

$$\text{Lit}_{\mathcal{L}}(P) := \{\varphi \mid \varphi \text{ is a canonical } \mathcal{L}\text{-literal over some } p \in P\} .$$

The following proposition makes a connection between values that are not fixed by some form and literals witnessing that fact.

PROPOSITION 13. *Let $\mathcal{L} \subseteq \mathcal{L}_{tfi}$ while $p$ is a propositional letter and $x \in \{\mathbf{B}, \mathbf{F}, \mathbf{T}, \mathbf{N}\}$. For each valuation $V$, one of the two following statements holds.*

(a) *For some $F$ that is fixed for $\mathcal{L}$, $V$ is an $F$-valuation on $p$ and $x \in dom(F)$.*

(b) *There is a canonical $\mathcal{L}$-literal $\lambda$ over $p$ such that $V_x(p) \neq V_x(\lambda)$.*

PROOF. Note that (a) holds in case $\mathcal{L} \cap \{\sim_t, \sim_f, \sim_i\} = \varnothing$ and that (b) holds if $\{\sim_t, \sim_f, \sim_i\} \in \mathcal{L}$ (since $V_x(p) \neq V_x(\sim_t \sim_f \sim_i p)$, for all $x \in \{\mathbf{B}, \mathbf{F}, \mathbf{T}, \mathbf{N}\}$).

In all other cases, we suppose that (a) does not hold, pick the unique form $F$ that is fixed for $\mathcal{L}$ such that $\langle x, V_x(p) \rangle \in F$, conclude that $V$ is not an $F$-valuation on $p$, and construe the desired literal that witnesses (b). We give two examples.

- Consider the case that $\mathcal{L} \cap \{\sim_t, \sim_f, \sim_i\} = \{\sim_i\}$, $x = \mathbf{B}$, and $V_{\mathbf{B}}(p) = 0$. Since $V$ is not a $\{\langle \mathbf{B}, 0 \rangle, \langle \mathbf{N}, 1 \rangle\}$-valuation on $p$ it must be the case that $V_{\mathbf{N}}(p) = 0$. We conclude that $V_{\mathbf{B}}(\sim_i p) = 1$.

- Now let $\mathcal{L} \cap \{\sim_t, \sim_f, \sim_i\} = \{\sim_t, \sim_i\}$, while $x = \mathbf{B}$, and $V_{\mathbf{B}}(p) = 0$. Since $V(p) \neq \langle 0, 0, 1, 1 \rangle$ it must be the case that either $V_{\mathbf{F}}(p) = 1$, or $V_{\mathbf{T}}(p) = 0$, or $V_{\mathbf{N}}(p) = 0$. In the first case $V_{\mathbf{B}}(\sim_t p) = 1$; in the second $V_{\mathbf{B}}(\sim_t \sim_i p) = 1$; and in the third $V_{\mathbf{B}}(\sim_i p) = 1$.

Other cases are left to the reader, but are each very similar to one of these two. ∎

While we will not use the fact, it is worthwhile to note that whenever $\mathcal{L} \cap \{\sim_t, \sim_f, \sim_i\}$ is as in the left column of Table 3 and some form $F$ is fixed for $\mathcal{L}$, $F$ is the union of corresponding forms on the right. This can be proved in a way akin to the proof of the preceding proposition. Here is a sketch. If $\{\sim_t, \sim_f, \sim_i\} \subseteq \mathcal{L}$ then no $F$ is fixed for $\mathcal{L}$ (for the reason we have just seen) and the statement is trivially true. Suppose that $\{\sim_t, \sim_f, \sim_i\} \not\subseteq \mathcal{L}$ and $F$ is not a union of forms in the entry for $\mathcal{L}$ on the right of Table 3. Then there is a $\langle x, y \rangle \in F$, such that the unique form $F'$ on the right with $\langle x, y \rangle \in F'$ is not a subset of $F$. This means that there is a $\langle x', y' \rangle \in F'$ such that $\langle x', y' \rangle \notin F$. In each concrete case it is now easy to find an $F$-valuation $V$ on some $p$ and a canonical $\mathcal{L}$ literal $\lambda$ over $p$ such that $V_x(\lambda) \neq y$, which shows that $F$ is not fixed for $\mathcal{L}$. Details are left to the reader. It is now easy to see that the forms fixed for a given $\mathcal{L}$ are exactly those unions of forms mentioned in the entry for $\mathcal{L}$ in Table 3 that are functions.

Proposition 13 can be used to show that, while it is impossible to define the top and bottom elements of the three lattices if not all negations are present, we can have approximations.

PROPOSITION 14. *Let $\mathcal{L} \subseteq \mathcal{L}_{tfi}$ be a language such that $\{\wedge_k, \vee_k\} \subseteq \mathcal{L}$, for some $k \in \{t, f, i\}$. Let $\top^k = \langle \top_{\mathbf{B}}^k, \top_{\mathbf{F}}^k, \top_{\mathbf{T}}^k, \top_{\mathbf{N}}^k \rangle$ be the top of the $k$ lattice and let $\bot^k = \langle \bot_{\mathbf{B}}^k, \bot_{\mathbf{F}}^k, \bot_{\mathbf{T}}^k, \bot_{\mathbf{N}}^k \rangle$ be its bottom. For each nonempty but finite set $P$ of propositional letters, there are $\mathcal{L}$-formulas $\tau_P^k$ and $\beta_P^k$, containing only propositional letters from $P$, such that, for each $x \in \{\mathbf{B}, \mathbf{F}, \mathbf{T}, \mathbf{N}\}$ and each valuation $V$, one of the following two statements holds.*

(a) There is an $F$ that is fixed for $\mathcal{L}$, $x \in dom(F)$ and $V$ is an $F$-valuation
    on $P$. [In this case $V_x(\tau_P^k) = V_x(\beta_P^k) = F(x)$.]

(b) $V_x(\tau_P^k) = \top_x^k$ and $V_x(\beta_P^k) = \bot_x^k$.

PROOF. Define $\tau_P^k$ as $\bigvee_k \text{Lit}_{\mathcal{L}}(P)$ and $\beta_P^k$ as $\bigwedge_k \text{Lit}_{\mathcal{L}}(P)$. Let $V$ be a valuation, let $x \in \{\mathbf{B}, \mathbf{F}, \mathbf{T}, \mathbf{N}\}$, and suppose that (a) does not hold, so that $V$ is not an $F$-valuation on $P$ for any $F$ fixed for $\mathcal{L}$ with $x \in dom(F)$. By Proposition 13 there are a $p \in P$ and a canonical $\mathcal{L}$-literal $\lambda$ over $p$ such that $V_x(p) \neq V_x(\lambda)$. Inspection of Definition 2 reveals that (b) holds. ∎

Let us stress that in the (b) case of the preceding proof it is not necessarily the case that $V(\tau_P^k) = \top^k$ or $V(\beta_P^k) = \bot^k$. Counterexamples are easily arrived at. The 'pointwise' formulation is really essential here, as it is in the applications of the proposition below.

So we have formulas that approximate the constantly denoting formulas that we want, modulo certain exceptions. Will the exceptions spoil our game? They will not and the following proposition gives the essential reason.

PROPOSITION 15. *Let $\mathcal{L} \subseteq \mathcal{L}_{tfi}$ and let $\varphi$, $\psi$, and $\chi$ be $\mathcal{L}$ formulas such that $\varphi \models_x \psi$ for some $x \in \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$, while $\mathsf{Voc}(\chi) \subseteq \mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi)$. Let $F$ be fixed for $\mathcal{L}$ and let $V$ be an $F$-valuation on $\mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi)$. Then, if $x \in dom(F)$, $V_x(\varphi) \leq V_x(\chi) \leq V_x(\psi)$.*

PROOF. We show $V_x(\varphi) \leq V_x(\chi)$. That $V_x(\chi) \leq V_x(\psi)$ is shown similarly. If $F(x) = 1$ then $V_x(\chi) = 1$ and we are done. Assume that $F(x) = 0$. Define the valuation $V'$ by letting, for each $y \in \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$ and each $p$, $V'_y(p) = F(y)$ if $p \in \mathsf{Voc}(\psi)$ and $y \in dom(F)$, while $V'_y(p) = V_y(p)$ otherwise. Then $V'$ is an $F$-valuation on $\mathsf{Voc}(\psi)$ and $V'_x(\psi) = 0$. Since $\varphi \models_x \psi$, it follows that $V'_x(\varphi) = 0$. But $V$ and $V'$ agree on $\mathsf{Voc}(\varphi)$, so $V_x(\varphi) = 0$ and the statement holds. ∎

We now have enough material to prove the remaining interpolation statements. Let us first consider the case that all conjunctions and disjunctions are present. We then get a generalisation of Proposition 6 whose proof is close to the latter's, but with the twist that it uses the considerations above in order to get the necessary 'glue'.

PROPOSITION 16. *Let $\{\wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i\} \subseteq \mathcal{L} \subseteq \mathcal{L}_{tfi}$. Then the entailment relations $\models_t$, $\models_f$, and $\models_i$ each have the interpolation property on $\mathcal{L}$.*

PROOF. Assume that $\varphi \models_t \psi$ and that $\mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi) \neq \varnothing$. Then $\varphi \models_{\mathbf{T}} \psi$, $\varphi \models_{\mathbf{B}} \psi$, $\psi \models_{\mathbf{F}} \varphi$, and $\psi \models_{\mathbf{N}} \varphi$. Lemma 1 gives us $\mathcal{L}$ interpolants $\chi_1$, $\chi_2$, $\chi_3$, and $\chi_4$ such that $\varphi \models_{\mathbf{T}} \chi_1 \models_{\mathbf{T}} \psi$, $\varphi \models_{\mathbf{B}} \chi_2 \models_{\mathbf{B}} \psi$, $\psi \models_{\mathbf{F}} \chi_3 \models_{\mathbf{F}} \varphi$, and $\psi \models_{\mathbf{N}} \chi_4 \models_{\mathbf{N}} \varphi$.

Let $P$ be short for $\mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi)$ and let $\tau_P^t$, $\tau_P^f$, $\beta_P^t$, and $\beta_P^f$ be as in Proposition 14. Let $\mathsf{b}^{\approx} := \tau_P^t \wedge_i \beta_P^f$, $\mathsf{f}^{\approx} := \beta_P^t \wedge_i \beta_P^f$, $\mathsf{t}^{\approx} := \tau_P^t \wedge_i \tau_P^f$, $\mathsf{n}^{\approx} := \beta_P^t \wedge_i \tau_P^f$, and let $\chi$ be the following sentence.

$$(\chi_1 \wedge_i \mathsf{t}^{\approx}) \vee_i (\chi_2 \wedge_i \mathsf{b}^{\approx}) \vee_i (\chi_3 \wedge_i \mathsf{f}^{\approx}) \vee_i (\chi_4 \wedge_i \mathsf{n}^{\approx})$$

With the help of Proposition 14 is easily seen that, for all valuations $V$, and all $x \in \{\mathbf{B}, \mathbf{F}, \mathbf{T}, \mathbf{N}\}$, at least one of the two following statements is true.

(a) $V$ is an $F$-valuation on $P$, for some $F$ fixed for $\mathcal{L}$ with $x \in \mathrm{dom}(F)$.

(b) $V_x(\mathsf{b}^{\approx}) = V_x(\mathsf{b})$, $V_x(\mathsf{f}^{\approx}) = V_x(\mathsf{f})$, $V_x(\mathsf{t}^{\approx}) = V_x(\mathsf{t})$, and $V_x(\mathsf{n}^{\approx}) = V_x(\mathsf{n})$.

In the (a) case $\varphi \models_x \psi$ implies $V_x(\varphi) \leq V_x(\chi) \leq V_x(\psi)$ and $\psi \models_x \varphi$ implies $V_x(\psi) \leq V_x(\chi) \leq V_x(\varphi)$ by Proposition 15. In particular, we have $V_x(\varphi) \leq V_x(\chi) \leq V_x(\psi)$, if $x = \mathbf{B}$ or $x = \mathbf{T}$, and $V_x(\psi) \leq V_x(\chi) \leq V_x(\varphi)$, if $x = \mathbf{F}$ or $x = \mathbf{N}$.

In the (b) case, note that, in view of Definition 2, only the $V_x$ values of $\mathsf{t}^{\approx}$, $\mathsf{b}^{\approx}$, $\mathsf{f}^{\approx}$, and $\mathsf{n}^{\approx}$ are relevant for the value of $V_x(\chi)$, so that we have the following.

$$V_x(\chi) = V_x((\chi_1 \wedge_i \mathsf{t}) \vee_i (\chi_2 \wedge_i \mathsf{b}) \vee_i (\chi_3 \wedge_i \mathsf{f}) \vee_i (\chi_4 \wedge_i \mathsf{n}))$$

This means that $V_x(\chi) = V_x(\chi_2)$ if $x = \mathbf{B}$, $V_x(\chi) = V_x(\chi_3)$ if $x = \mathbf{F}$, $V_x(\chi) = V_x(\chi_1)$ if $x = \mathbf{T}$, and $V_x(\chi) = V_x(\chi_4)$ if $x = \mathbf{N}$.

It can be concluded that, for all $V$, $V_x(\varphi) \leq V_x(\chi) \leq V_x(\psi)$, if $x = \mathbf{B}$ or $x = \mathbf{T}$, and $V_x(\psi) \leq V_x(\chi) \leq V_x(\varphi)$, if $x = \mathbf{F}$ or $x = \mathbf{N}$. So $\varphi \models_{\mathbf{T}} \chi \models_{\mathbf{T}} \psi$, $\varphi \models_{\mathbf{B}} \chi \models_{\mathbf{B}} \psi$, $\psi \models_{\mathbf{F}} \chi \models_{\mathbf{F}} \varphi$, and $\psi \models_{\mathbf{N}} \chi \models_{\mathbf{N}} \varphi$, i.e. $\chi$ is an interpolant for $\varphi \models_t \psi$.

It follows that $\models_t$ enjoys interpolation on $\mathcal{L}$. That $\models_f$ and $\models_i$ also do follows by almost identical argumentation. ∎

The remaining case is the one in which exactly one conjunction and its dual are absent from the language. Its proof makes essential use of Proposition 10. Otherwise it is very much like the previous proof.

PROPOSITION 17. *Let $\mathcal{L} \subseteq \mathcal{L}_{tfi}$ be a language closed under duals such that $\{\wedge_t, \vee_t, \wedge_f, \vee_f, \wedge_i, \vee_i\} \not\subseteq \mathcal{L}$, but $\{\wedge_k, \vee_k, \wedge_\ell, \vee_\ell\} \subseteq \mathcal{L}$, for some $k, \ell \in \{t, f, i\}$ and $k \neq \ell$. The entailment relations $\models_t$, $\models_f$, and $\models_i$ each have the interpolation property on $\mathcal{L}$.*

PROOF. Let $\mathcal{L}$ be as described. Consider the case that $\mathcal{L} \cap \{\wedge_i, \vee_i\} = \varnothing$, so that $\{\wedge_t, \vee_t, \wedge_f, \vee_f\} \subseteq \mathcal{L}$.

Let $\varphi$ and $\psi$ be $\mathcal{L}$ sentences and suppose $\varphi \models_t \psi$, while $\mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi) \neq \varnothing$. Then $\varphi \models_{\mathbf{T}} \psi$ and $\varphi \models_{\mathbf{B}} \psi$ hold. Lemma 1 gives $\mathcal{L}$ interpolants $\chi_1$ and

$\chi_2$ such that $\varphi \models_{\mathbf{T}} \chi_1 \models_{\mathbf{T}} \psi$ and $\varphi \models_{\mathbf{B}} \chi_2 \models_{\mathbf{B}} \psi$. Let $P$ be short for $\mathsf{Voc}(\varphi) \cap \mathsf{Voc}(\psi)$, let $\tau_P^t$, and $\beta_P^t$ be as in Proposition 14, and let $\chi$ be

$$(\chi_1 \wedge_f \tau_P^t) \vee_f (\chi_2 \wedge_f \beta_P^t) \ .$$

We show that $\chi$ is the required interpolant. Let $x \in \{\mathbf{B}, \mathbf{T}\}$ and let $V$ be a valuation. If $V$ is an $F$-valuation on $P$, for some $F$ fixed for $\mathcal{L}$ such that $x \in \mathrm{dom}(F)$, we can conclude that $\varphi \models_x \psi$ implies $V_x(\varphi) \leq V_x(\chi) \leq V_x(\psi)$ and $\psi \models_x \varphi$ implies $V_x(\psi) \leq V_x(\chi) \leq V_x(\varphi)$, as before. In particular, we have $V_x(\varphi) \leq V_x(\chi) \leq V_x(\psi)$ if $x = \mathbf{B}$ and $V_x(\varphi) \leq V_x(\chi) \leq V_x(\psi)$ if $x = \mathbf{T}$ in this case. Otherwise, we can conclude that $V_x(\chi) = V_x(\chi_2)$ if $x = \mathbf{B}$ and $V_x(\chi) = V_x(\chi_1)$ if $x = \mathbf{T}$, so that again $V_x(\varphi) \leq V_x(\chi) \leq V_x(\psi)$ if $x = \mathbf{B}$ and $V_x(\varphi) \leq V_x(\chi) \leq V_x(\psi)$ if $x = \mathbf{T}$.

It follows that $\varphi \models_{\mathbf{T}} \chi \models_{\mathbf{T}} \psi$, and $\varphi \models_{\mathbf{B}} \chi \models_{\mathbf{B}} \psi$. By Proposition 10, $\varphi \models_t \chi \models_t \psi$ and $\chi$ is an interpolant for $\varphi \models_t \psi$.

The proof for $\models_f$ is virtually identical, and also gives an interpolant of the form $(\chi_1 \wedge_f \tau_P^t) \vee_f (\chi_2 \wedge_f \beta_P^t)$. That $\models_i$ enjoys interpolation on $\mathcal{L}$ follows from Proposition 8.

This concludes the case that $\mathcal{L} \cap \{\wedge_i, \vee_i\} = \varnothing$. The two remaining cases are very similar. In case $\mathcal{L} \cap \{\wedge_f, \vee_f\} = \varnothing$ one arrives at interpolants of the form

$$(\chi_1 \wedge_t \tau_P^i) \vee_t (\chi_2 \wedge_t \beta_P^i) \ ,$$

while the case that $\mathcal{L} \cap \{\wedge_t, \vee_t\} = \varnothing$ leads to interpolants of the form

$$(\chi_1 \wedge_i \tau_P^f) \vee_i (\chi_2 \wedge_i \beta_P^f) \ .$$

In all cases $\chi_1$ and $\chi_2$ are interpolants for appropriate auxiliary entailment relations. Details are left to the reader.                                                    ∎

We sum up our results in the following theorem, which is just a combination of Propositions 9, 11, 16, 17, and Lemmas 2 and 3.

THEOREM 2. *Let $\mathcal{L} \subseteq \mathcal{L}_{tfi}$ be a language closed under duals. The entailment relations $\models_t$, $\models_f$, $\models_i$, and $\models$ each have the interpolation property on $\mathcal{L}$. In case $\{\sim_t, \sim_f, \sim_i\} \not\subseteq \mathcal{L}$, the relations $\models_t$, $\models_f$, $\models_i$, and $\models$ each have the perfect interpolation property on $\mathcal{L}$.*

The theorem affirmatively answers the question that was asked in Takano [11]—does $\models$ enjoy perfect interpolation on $\mathcal{L}_{tf}$? Concrete interpolants are easily extracted from our proofs. In particular, if $\varphi$ and $\psi$ are $\mathcal{L}_{tf}$ sentences such that $\varphi \models \psi$, we can conclude that also $\varphi \models_t \psi$. From the proof of

Proposition 17 it follows that $\varphi \models_t \chi \models_t \psi$, where $\chi$ is

$$(\chi_1 \wedge_f \tau_P^t) \vee_f (\chi_2 \wedge_f \beta_P^t) \ .$$

Here $\chi_1$ and $\chi_2$ are perfect interpolants for $\varphi \models_\mathbf{T} \psi$ and $\varphi \models_\mathbf{B} \psi$ respectively and can be extracted from the proof of Theorem 1. $\tau_P^t$ is the $\vee_t$ disjunction of all canonical $\mathcal{L}_{tf}$ literals over the (nonempty) shared vocabulary $P$ of $\varphi$ and $\psi$, while $\beta_P^t$ is a similar $\wedge_t$ conjunction. From Lemma 3 it follows that in fact $\varphi \models \chi \models \psi$, so that we have extracted the interpolant that was sought after.

## 6.    Conclusion

The analytic tableau calculus $\mathsf{PL_{16}}$ provides several propositional logics based on the trilattice $SIXTEEN_3$ with a syntactic characterisation. Entailment relations of interest are typically characterisable as intersections of certain auxiliary entailment relations and/or their converses and verifying or disproving an entailment may require the development of several tableaux.

   In this paper we have shown that several entailment relations of obvious interest enjoy interpolation. Our methods have been constructive—in concrete cases interpolants can be found by first finding interpolants for some of the relevant auxiliary entailment relations and by then glueing these together in certain ways. The method works for a language that can express all truth functions over $\mathsf{PL_{16}}$, but also for all sublanguages closed under duals. This includes the language originally considered by Shramko and Wansing [8].

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