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Infinite Populations, Choice and Determinacy

Abstract. This paper criticizes non-constructive uses of set theory in formal economics. The main focus is on results on preference aggregation and Arrow's theorem for infinite electorates, but the present analysis would apply as well, e.g., to analogous results in intergenerational social choice. To separate justified and unjustified uses of infinite populations in social choice, I suggest a principle which may be called the *Hildenbrand criterion* and argue that results based on unrestricted axiom of choice do not meet this criterion. The technically novel part of this paper is a proposal to use a set-theoretic principle known as the axiom of determinacy (AD), not as a replacement for Choice, but simply to eliminate applications of set theory violating the Hildenbrand criterion. A particularly appealing aspect of AD from the point of view of the research area in question is its game-theoretic character.

Keywords: Axiom of choice, Axiom of determinacy, Multiverse, intergenerational social choice, Preference aggregation, Arrow's impossibility theorem, Social welfare analysis.

1. Introduction

Formal economists and social choice researchers do consider infinitely large populations and, on occasions, even infinite voting electorates. There are several potential reasons for doing so, some of which are critically discussed in the present paper. However, such populations raise several philosophical and methodological concerns. Potential axioms, principles and methods employed to handle infinite collections, even when they appear to directly generalize those valid in the finite case, may have little computational content. What is less known in the social choice community is that axioms trivially valid in the finite may even directly contradict each other when assumed to hold for arbitrary infinite classes; in fact, we will see a prominent example of such a phenomenon in this work.

Just like in other areas of mathematics, the axiom of choice (AC) tends to be, to risk a pun, the axiom of choice for most economists. But it seems most often inherited by the sheer force of inertia, despite the fact that the nonconstructive parts of set theory are accepted by mathematicians less broadly than commonly believed; see Appendix B for a critical overview. In

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Special Issue: Logics for Social Behaviour
Edited by **Alessandra Palmigiano and Marcus Pivato**

Studia Logica (2018) 106: 969–999
https://doi.org/10.1007/s11225-017-9730-3

fact, criticism of their use has been often also stated (to a varying degree of explicitness) in the social choice literature [24,29,51] (see Sections 2 and 3). Already one of the earliest references considering an infinite population of agents, a 1964 paper by Aumann, mentions that

> the measurability assumption ... is of technical signicance only and constitutes no real economic restriction. Nonmeasurable sets are extremely pathological; it is unlikely that they would occur in the context of an economic model [1, p. 44].

Much more recently, Zame asserted that

> the full power of the Axiom of Choice is almost never used in formal economics or in classical analysis for that matter. What is used is a much weaker axiom, the Axiom of Dependent Choice [51, p. 195].

Such explicit admissions that unrestricted AC does not appear a suitable axiom for formal economics or social choice theory did not prevent controversies caused by problematic uses thereof. Zame's paper itself [51] was arguing that an intergenerational *ethical preference relation*, whose existence relative to ZFC had been previously established by Svensson [50], not only cannot be shown to exist on the basis of $\mathsf{ZF} + \mathsf{DC}$ (Zermelo-Fraenkel set theory with the Axiom of Dependent Choice, a.k.a. Principle of Dependent Choices), but is nonmeasurable and undefinable even relatively to full ZFC (more on this in Section 7). An even richer source of examples is provided by references on supposed infinitary solutions of Arrow's impossibility in voting theory. Fishburn's [10] claim of existence of such solutions (again, relative to ZFC) provoked several decades of discussions in the literature, presented in Section 3 of this paper. Kirman and Sonderman [24] used measure-theoretic arguments and stone-Čech compactifications to argue that social welfare functions for infinite electorates found by Fishburn are actually "invisible dictators". Lauwers and van Liedekerke claimed that aggregation procedures corresponding to nonprincipal ultrafilters are not appropriate in the context of social choice and "exhibit an insuperable arbitrariness in construction" [29, p. 236]. Mihara [33] added that such "non-dictatorial" social welfare functions are not computable (see, however, Footnote 15).

I want to contribute to this discussion and strengthen the case for caution with two suggestions, one methodological and one technical. On the methodological—or perhaps even philosophical—front, I isolate in Section 2 the *Hildenbrand criterion* from an early paper [19] regarding valid and fruitful uses of infinite populations in theoretical work. My claim is that results based on powerful consequences of AC do not meet this criterion.

On the technical front, I am proposing that the Axiom of Determinacy (AD, recalled in Section 4) is a good tool to check conformity with the Hildenbrand criterion. This axiom is perhaps the most spectacular example of an elegant principle with interesting set-theoretical consequences contradicting full AC, while being compatible with its more benign corollaries.

Just to avoid a major misunderstanding: building AD-based social choice theory inconsistent with AC is not what I am proposing here. Such results on infinite populations could be regarded at most as curious thought experiments.[1] The point here is that I see little reason to treat differently social choice results which require consequences of AC so powerful that they cannot be squared with alternative yet meaningful principles like Determinacy. As discussed in Sections 2–3, even the very consideration of infinite populations, especially in the context of preference aggregation, requires some serious analysis, and this is precisely where the Hildenbrand criterion is intended to be a normative guide. And assuming that these hypothetical infinite electorates obey axioms like full AC reminds one of questions medieval scholastics have often been ridiculed as being supposedly preoccupied with: "How many nonmeasurable coalitons of voters can dance on the head of a pin?"

In other words, I claim that principles of set theory safe to use in economics and social sciences, especially in areas like preference aggregation, should be consistent with *both* ZFC *and* ZF + AD, DC being a perfect example.[2] We have already seen that the authors working in the area acknowledge DC is all one needs in meaningful economical applications. It might be the case that infinite populations, or even infinite electorates are legitimate objects of study, but then one should focus on a certain reasonable "common core" without unduly preferring or discriminating against any of competing mathematical ontologies.

Perhaps the most interesting feature of AD-based (counter-)arguments is that the axiom has a clear game-theoretic flavour, which should be of natural appeal to researchers in the area. Zame [51], Kirman and Sonderman [24] or Lauwers and van Liedekerke [29] provided eloquent and accurate criticism of applicability of results based on unrestricted AC, but their use of set-theoretical independence proofs, techniques like forcing and the structure of ZF- and ZFC-definable sets may feel somewhat too arcane for most

[1] As discussed in Section B.2, the recently proposed *multiverse* approach to set-theoretic foundations [14] would cast an interesting light on such thought experiments.

[2] Note that I'm saying "consistent with" rather than "derivable from"—otherwise, as discussed in Section 5, even AC_ω would be excluded!

economists. It is just not plausible to expect that experts in, say, preference aggregation would collectively devote their time to studying non-standard models of ZF. By contrast, examining the (non-)existence of a winning strategy should feel like a rather natural thing to do. I am illustrating this point in detail in Section 6 by showing how an AD-based "strategy-stealing" argument short-circuits the claim that allowing infinite electorates somehow resolves Arrow's Impossibility Theorem. As briefly discussed in Section 7, similar arguments are available regarding, e.g., intergenerational equity.

The paper is equipped with two appendices. Appendix A recalls basic information about ZFC. Appendix B summarizes traditional and less traditional lines of attack on AC.[3]

2. The Hildenbrand Criterion

It is natural to suppose that areas of mathematics relevant to the social choice theory should be of finitary and effective character: combinatorics, algorithmics, complexity, finite model theory. First and foremost, human groups, societies and populations are finite, if arbitrarily large. Equally importantly, the fundamental research interest in this particular area seems to lie in concrete algorithms to align preferences, maximize welfare, optimize voting procedures, etc. Results relying on strong consequences of AC like BPI by their very nature cannot offer any usable algorithms.

The latter point is particularly important here. Let us begin, however, by a short discussion of the former one, i.e., finiteness of human populations. An early paper by Aumann provided the following argument for considering *uncountably* many agents in economical papers, in that particular case when studying an idealized notion of "perfect competition":

> The essential idea ... is that the economy under consideration has a "very large" number of participants, and that the influence of each individual participant is "negligible" ... [I]n economics, as in the physical sciences, the study of the ideal state has proved very fruitful, though in practice it is, at best, only approximately achieved ... We submit that the most natural model for this purpose contains a continuum

[3]These appendices were initially conceived as a part of the main paper. The criticism of nonconstructive methods in formal economics I am presenting here should be seen as a part of a larger mathematical tradition, and I am not sure whether the material contained therein is sufficiently well-known among economists. However, its sheer size would distract from the main line of argument.

> of participants, similar to the continuum of points on a line or the continuum of particles in a fluid. Very succinctly, the reason for this is that one can integrate over a continuum, and changing the integrand at a single point does not affect the value of the integral, that is, *the actions of a single individual are negligible* [1, p. 39, emphasis mine–T.L.].

Whether or not one agrees with this reasoning (for a more detailed analysis of continual models of perfect competition cf., e.g., Ostroy and Zame [40]), classical references deserve credit for such explicit and elaborate arguments for counterintuitive modelling assumptions. In a subsequent work, Hildenbrand elaborates on usefuleness of infinitely large, even uncountable sets of agents, but making a very important qualification:

> Instead of considering a sequence of economies and looking for an asymptotic identity one may reason "in the limit," i.e., one considers economic systems with more than finitely many participants and proves that the identity holds in this case. . . . But, as an economist, our interest in these ideal economies is proportional to *how much new information can be derived for large but finite economies*. In other words, *the relevance of the ideal case to the finite case has to be established* [19, p. 162, emphasis mine–T.L.].

The principle that *the relevance of the ideal case to the finite case has to be established* deserves the name of the *Hildenbrand criterion* (although perhaps the *Hildenbrand–Aumann criterion* would be more adequate). Let us develop it in more detail: even if a theoretical economist decides to consider infinite populations of agents as "limit" or "ideal" generalizations of very large finite ones, the results obtained via such an idealization process should remain *effective*. *Moving to the limit* should mean precisely this and nothing more: making the rôle of any particular individual, or perhaps even any particular generation, infinitesimally negligible. It cannot be an excuse to introduce by a sleight of hand—or, if one prefers, a magical or metaphysical trick—a pseudo-solution, which by nature cannot correspond to any meaningful algorithm or definable strategy.

The criterion seems even more central when attention is shifted from modelling economical competition to typical concerns of social choice: preference aggregation or welfare analysis. Moreover, it appears applicable regardless of whether all the agents are assumed to exist at the same time, or whether infinity comes from the assumption of an infinite time horizon. This is what happens in *intergenerational* (or *intertemporal*) social choice.

Studying, for example, infinitely repeating patterns of behaviour in games or *utility streams* (cf., e.g., [51]) seems a natural enterprise. Conceptually, the underlying assumption boils down to the statement: "assume that the game can be considered as infinite for all practical purposes", which is certainly in keeping with the Hildenbrand criterion. On the other hand, considering, e.g., *nonmeasurable* families of subsets of the *totality of all generations* violates both the letter and the spirit of this criterion. In other words, the intertemporal context once again is calling for effective methods and describable algorithms to ensure intergenerational equity. This point was made, e.g., by Zame [51] (see also Lauwers [28,28] and Section 7 below).

Still somewhat different, yet again related motivation for considering infinite collections of agents can be provided by investigations of social choice under uncertainty, with an infinite set of possible "states of the world", each with its own (conceptual) inhabitants.[4] Such considerations figure less prominently in the papers discussed here, though an explicit connection with reasoning under uncertainty has been made, e.g., by Mihara [33]. All the above arguments regarding the relationship of the infinite to the finite, effectiveness, computability and constructivity seem to apply here as well.

Altogether, we can summarize with two passages quoted already in Section 1: *Nonmeasurable sets are extremely pathological; it is unlikely that they would occur in the context of an economic model* [1] and *the full power of* AC *is almost never used in formal economics or in classical analysis for that matter. What is used is a much weaker axiom, the Axiom of Dependent Choice* [51]. It does seem there are very good reasons for this state of affairs.

3. Infinite Electorates and the Delusion of Possibility

A particular interest of this work lies with a somewhat problematic group of papers, where one can easily see why considering an infinity of agents is/was an interesting mathematical exercise, yet where any assumptions regarding laws obeyed by such infinite populations seem to require particular care. I mean the line of work dealing with results such as those discussed in Section 6: dealing with the fate of Arrow's celebrated Impossibility Theorem in the infinite [2,7,8,10,16–18,24,29,33–35].

Is there an analogy with the work of Aumann and other authors on perfect competition? In other words, does one assume an infinity of voters

[4] A more detailed analysis of possible interpretations of infinite products in the context of utility functions—intertemporal choice, choice under uncertainty, variable population social choice and combinations thereof—can be found in a recent paper by Pivato [44].

in order to make choices of each particular individual entirely negligible in the sense of, e.g., being removable via integration? The first obvious contrast is that calculus and integration play a very minor role in this line of work; in fact, the only notion of calculus that seems to enter the picture is the very notion of measurability, quite unlike the papers on perfect competition [1, 19, 40]. More importantly, already one of the first papers in the area [24] argued that even from a purely mathematical point of view, such analogies are spurious, as we will see in detail below. Does the motivation come then from intertemporal social choice? Or social choice under uncertainty? In both of these settings, it is hard to see the relevance of any type of a voting model. It does not make much sense to consider people from the whole infinite collection of different generations or different possible states of the world as a single electorate. Even using the word "dictatorship" in such a context feels somewhat preposterous. Aggregation of utility functions in such contexts appears to have precious little connection with voting.

As far as I have been able to establish, the story of infinite electorates in the context of Arrow's theorem begun with a short paper by Fishburn [10]. His four-page note claimed to aim simply at a concise proof of Arrow's result, with the opening paragraph stating that

> my intention here is to concentrate on the mathematical rather than the interpretive aspects.

This is perfectly understandable. But even from a purely mathematical point of view, the author's own description of results contains an important omission:

> The proof in this paper begins with the nondictatorship condition and shows that this and the other conditions imply that the set of voters must be infinite. A finite set of voters is essential to Arrow's theorem. The final section of the paper demonstrates that if the set of voters is allowed to be infinite then Arrow's other conditions are simultaneously compatible.

The accuracy of the last sentence depends on the assumed meaning of "compatibility". To be sure, the proof of the Possibility Theorem itself later in the paper does explicitly mention its dependence on Zorn's Lemma. More specifically, the proof relies on the fact that ZFC guarantees the existence of a finitely additive, two-valued probability measure defined on all subsets of the infinite set of voters such that every singleton is of measure zero;

that is, a free ultrafilter. But this is not reflected in Fishburn's introductory discussion of the results quoted above. Hence, the paper may be read as implying that the very move to infinite electorates, without as much as touching upon the motivation for considering such entities, makes Arrow's impossibility magically and automatically turn into a possibility.

This was bound to provoke a reaction—and it was provided two years later by Kirman and Sondermann [24], the paper usually credited as the main reference connecting ultrafilters with Arrow's result. Such a claim is not incorrect, but can be misinterpreted. As stated in the preceding paragraph, the probability measure used by Fishburn [10] is just another way of presenting a free ultrafilter.[5] The technical pillar of their paper [24] is indeed a theorem establishing 1-1 correspondence between social welfare functions satisfying Arrowian axioms and ultrafilters. Kirman and Sondermann, however, proved this result with a specific, negative purpose: to argue that the "possibility" offered by the move to infinite electorates is illusory.

Interestingly enough, their criticism was not aimed at set-theoretic principles used in the proof, at least not directly. What they focused on was to what extent—even assuming ZFC-based metatheory—Fishburn's results meets what I called the *Hildenbrand criterion* in Section 2:

> Comparing Fishburn's result with that of Arrow, it seems that the existence problem for social welfare functions looks essentially different when one essays the step from the finite to the infinite, as has been successfully done by Aumann [1] and others. Since we have limiting results, which show that Aumann's theorems for a continuum of economic agents are also approximately true for finite but large economies, we might ask the following question: Does Fishburn's result allow us to make a similar deduction for large set of individuals? The answer is no! [24, p. 268–269]

Kirman and Sondermann justify this claim with two mathematical results. The first of them [24, Theorem 5] provides a connection with the theory of atomless measure spaces, central to the work of Aumann or Hildenbrand. It is argued that an appropriate reformulation of dictatorship in such a setting would require it to be exercised not *by one individual alone, but rather*

[5] It should be noticed that even Kirman and Sondermann themselves do not claim the credit for the observation that Fishburn's *finitely additive, two-valued probability measure* is a barely disguised free ultrafilter. Their acknowledgements state this has been suggested to them in two separate and independent conversations, one of them being with Peter C. Fishburn himself.

by an "arbitrarily" small group of agents—arbitrarily small, that is, in the sense of measure theory. And, surely enough, any solution based on free (nonprincipal) ultrafilters is dictatorial in this sense. Pushing the argument still further, they observe that the topological notion of Stone-Čech compactification allows one to turn such infinitesimally small dictatorial groups arising from a free ultrafilter into an "invisible dictator" in the limit.

Kirman and Sondermann [24] started a long line of research. See Monjardet [35] for an overview of the early literature. In a more recent period, the interest was renewed by a 1995 paper by Lauwers and Van Liedekerke [18, 29]. Further authors include Chichilnisky and Heal [8], Mihara [33, 34], Brunner and Mihara [7], Herzberg and Eckert [16, 17], Bedrosian et al. [2], and this list is by no means exhaustive. Nevertheless, it can be safely said that these newer references have not undermined Kirman and Sondermann's original conclusions regarding the relationship between results based on unrestricted AC and the Hildenbrand criterion. In the words of Lauwers and Van Liedekerke:

> Our results on ultrafilters are discouraging. In the context of social choice, aggregation by means of ultraproducts is not appropriate. Aggregation procedures should filter out properties that hold for "almost all" individuals. Ultraproducts, however, also let pass properties that hold for a fraction of the whole set of individuals. Besides that, they exhibit an insuperable arbitrariness in construction. It follows that ultrafilters provide no good intuition to the notion of "almost all" [29, p. 235–236].

4. Axiom of Determinacy

It is time to present a technique which I believe is helpful in eliminating applications of set theory not conforming to the Hildenbrand criterion. As illustrated by the overview in Appendix B, most of the criticism of ZFC presented in the available literature seems to suggest *weakening* it: either keeping just a weak form of AC (if not removing it altogether) or even—in the case of intuitionistic, type-, topos- or category-theoretic approaches—attacking the propositional base and removing classical tautologies. However, it is also possible to extend ZF with meaningful principles *contradicting* AC. This is precisely the story of the Axiom of Determinacy[6] (AD) proposed by Mycielski and Steinhaus in 1962 [38] and studied ever since [20–22, 36, 37, 39].

[6] Early references used the name *Axiom of Determinateness* instead.

Consider any set X and $A \subseteq X^\omega$, i.e., any given subset of the space of infinite sequences of elements of A. Note here that if X is taken to be ω, X^ω can be identified with $\mathbb{R}$ via a standard argument. With every such X and A, we can associate naturally an infinite two-person game $G_X(A)$ with perfect information, where two players $\forall$ and $\exists$ take turns to choose elements of X. The $\exists$-player, who makes the first move, wins if the sequence created this way belongs to A; otherwise the game is won by the $\forall$-player. In other words, A is what is usually called the *payoff* for $G_X(A)$. If either of the two players has a winning strategy, $G_X(A)$ is *determined*. AD says simply that *all games of the form* $G_\omega(A)$*, i.e., whose payoffs are (identifiable with) subsets of* $\mathbb{R}$*, are determined.*

The axiom almost immediately kills off almost the whole unsavoury bestiary of pathological sets associated with unrestricted AC. Contrast the result below with those summarized in Section A:

Theorem 4.1. ([21], Proposition 27.9) *Under* $\mathsf{ZF}+\mathsf{AD}$*, every set of reals is Lebesgue measurable, has the Baire property and the perfect set property.*[7]

The above result readily implies that under ZF + AD, every ultrafilter over ω is principal, using the proof of Theorem 9.5 (i.e., non-measurability of nonprincipal ultrafilters over ω conceived as sets of reals). However, there is a more direct game-theoretic "strategy-stealing" argument presented, e.g., by Kanamori [21, Proposition 28.1]. In more detail, one could use such an ultrafilter to create the payoff of a game where players pick (disjoint) finite subsets of ω. The assumption that this game is determined would lead to a contradiction: a winning strategy for the $\forall$-player could be easily converted into a winning strategy for the $\exists$-player and vice versa. Instead of presenting such an argument here, we postpone it to Section 6, where it is going to be directly applied in the context of Arrow's impossibility result.

A striking fact about determinacy is that one can see it as an infinitary generalization of the *De Morgan law for quantifiers*, see, e.g., [23, Section 20.D]. To be more specific, consider any $A \subseteq X^\omega$. The existence of a winning strategy for the $\exists$-player can be reformulated as $\exists x_0.\forall x_1.\exists x_2.\dots(x_i)_{i\in\omega} \in A$, and the existence of a winning strategy for the $\forall$-player as $\forall x_0.\exists x_1.\forall x_2.\dots(x_i)_{i\in\omega} \notin A$. Thus, determinacy of $G_X(A)$ can be rewritten as

$$\neg\exists x_0.\forall x_1.\exists x_2.\dots A(x_0,x_1,x_2\dots) \leftrightarrow \forall x_0.\exists x_1.\forall x_2.\dots\neg A(x_0,x_1,x_2\dots).$$

[7] Kanamori's monograph [21, p. 379], which is only quoted as a convenient and comprehensive source, states that this theorem is a compilation of results by Mycielski and Świerczkowski [39], Mazur, Banach and Davis.

It is worth contrasting this with the fact that AC can also be seen as an infinitary generalization of a finitary law; namely, the law of excluded middle [9,13] (see Section B.3). It turns out that natural generalizations of finitary laws can conflict with each other in suitably large infinite domains.

Kanamori [21, Chapter 27] puts determinacy in the context of interactions between foundational studies and game theory going back to Zermelo and König. Researchers in social choice are likely to be aware of at least some of the early work in that area, in particular, the seminal work of von Neumann and Morgenstern connecting games and economic behaviour. A more direct stimulus was provided by *renewed interest in infinite games*, especially among Polish researchers, in the 1950's [21, p. 377]. But a set-theoretic motivation came from the study of regularity properties and results like Theorem 4.1.

5. Consistency Strength: AD, AC, DC and $L(\mathbb{R})$

Mycielski and Steinhaus themselves [38] were careful to state that they did not question the validity of AC in the "absolute" *universum of sets*.[8] They proposed it describes a *smaller universum* of "determined sets". Soon afterwards, Mycielski [36] made this idea more specific suggesting the rôle largely played by AD since then in most set-theoretic references: as a candidate for a proposition valid in an *inner model* containing $\mathbb{R}$. Solovay and Takeuti noted that the most natural candidate is the smallest such model, i.e., $L(\mathbb{R})$ [21, p. 378]; see Appendix A for this notation. However, this has to be seen in the light of the fact that (as discussed in Section B.2) contemporary set theory leaves little ground for believing in the existence of a single distinguished "absolute" universum of sets V and still less ground for believing that even if such a universum existed, we would be able to identify it. AD appears to be delimiting a rather attractive area of the *set-theoretic multiverse* [14]; see Section B.2 for more about this notion.

From the point of view of consistency strength, AD is a much more powerful axiom than AC. On the other hand, it is also consistent with weak versions of AC. Detailed information and more references can be found in [21, Chapter 6], but here are the basic results:

[8] Mycielski [36] called the inconsistency of AD with AC a "sad fact".

THEOREM 5.1.

- AD *implies that every countable family of non-empty subsets of* $\mathbb{R}$ *has a choice function (Mycielski* [36]*).*[9]
- *If the model of* ZF *based on* $L(\mathbb{R})$ *is assumed to satisfy* AD*, then* DC *holds therein (Kechris* [22]*).*
- *In* ZFC *enriched with axioms ensuring the existence of infinitely many Woodin cardinals with a measurable cardinal*[10] *above them,* $L(\mathbb{R})$ *is a model of* ZF + AD *(Woodin, see* [21, Theorem 32.14]*).*
- ZF + AD *is equiconsistent with* ZFC *enriched with axioms ensuring the existence of infinitely many Woodin cardinals. (Woodin, see* [21, Theorem 32.16]*).*

These results should be understood properly. For example, the first two clauses do not say that AD *implies* AC_ω, much less DC (cf. Lemma 9.1). Kechris [22] describes a construction by Woodin, which assumes that $L(\mathbb{R})$ taken as a model of ZF satisfies AD and constructs its "generic extension" where AC_ω fails. The fact that a sentence ϕ is assumed to hold in a given model of ZF does *not* mean that this model cannot "believe in the existence" of another model of ZF where $\neg\phi$ holds. This is precisely the point of the notion of "consistency strength". For the same reason, the last-but-one and the last clause of Theorem 5.1 are distinct.

At any rate, set-theorists find the idea that AD should hold in $L(\mathbb{R})$ rather natural. Yes, constructing a model of set theory where all subsets of $\mathbb{R}$ are measurable requires assuming consistency of an inaccessible cardinal [45,47], and constructing one where AD holds requires further assumptions as in the last two clauses of Theorem 5.1. One, however, would be hard-pressed to find a set-theorist today who would reject all large cardinals.

But then, even if for some reason one needs to consider infinite electorates, or indeed infinite populations of any kind, how is it possible to know whether the ambient setting is $V = L(\mathbb{R})$, presumably satisfying AD, or rather, say, Gödel's constructible universe $V = L$, where AC holds? It is safe to assume that DC is available—even under AD, models without dependent choice are rather deviant. Beyond that, however, why would such an infinite population obey choice or rather determinacy? Let us see what would happen under the

[9]Also proved independently by Świerczkowski and by Scott.

[10]Steel [49] provides perhaps the most accessible and self-contained presentation of *Woodin cardinals* and *measurable cardinals* I was able to find; for lack of space, the reader is referred there for definitions.

latter assumption. Even if one is a strong believer in choice, it may be at least an educating thought experiment.[11]

6. AD and Infinite Electorates

Our main case study concerns tailoring the game-theoretic argument for the non-existence of free ultrafilters over ω to Arrow's result as reconstructed by Kirman and Sonderman [24] (see also Fishburn [10]). Let Voters, $\mathsf{Options}$ be arbitrary sets and $\mathcal{PO}(\mathsf{Options})$ be the set of *preference orders* on $\mathsf{Options}$, i.e., those $R \subseteq \mathsf{Options} \times \mathsf{Options}$ satisfying

asymmetry $\mathsf{a}R\mathsf{b}$ implies not $\mathsf{b}R\mathsf{a}$,

negative transitivity for any $\mathsf{a}, \mathsf{b}, \mathsf{c} \in \mathsf{Options}$, $\mathsf{a}R\mathsf{b}$ implies $\mathsf{a}R\mathsf{c}$ or $\mathsf{c}R\mathsf{b}$.

As noted by, e.g., Kirman and Sonderman [24], these conditions imply transitivity. Furthermore, denote $\mathsf{Situations} := \mathsf{Voters} \to \mathcal{PO}(\mathsf{Options})$ and $\mathsf{SWFs} := \mathsf{Situations} \to \mathcal{PO}(\mathsf{Options})$, where arrows denote corresponding function spaces and "SWF" stands for "Social Welfare Function". Some additional conventions for $\mathsf{a}, \mathsf{b} \in \mathsf{Options}$, $v \in \mathsf{Voters}$, $U \subseteq \mathsf{Voters}$, $f, g \in \mathsf{Situations}$ and $\sigma \in \mathsf{SWFs}$:

$$
\begin{array}{rcl}
\mathsf{a}f[U]\mathsf{b} & \text{if} & \text{for all } v \in U,\ \mathsf{a}f(v)\mathsf{b}, \\
f(v) =_{\mathsf{a},\mathsf{b}} g(v) & \text{if} & (\mathsf{a}f(v)\mathsf{b} \text{ iff } \mathsf{a}g(v)\mathsf{b}) \text{ and } (\mathsf{b}f(v)\mathsf{a} \text{ iff } \mathsf{b}g(v)\mathsf{a}), \\
f =_{\mathsf{a},\mathsf{b}} g & \text{if} & \text{for all } v \in \mathsf{Voters},\ f(v) =_{\mathsf{a},\mathsf{b}} g(v).
\end{array}
$$

There is one rather trivial axiom being imposed on $\mathsf{Options}$:

Options (A1) $|\mathsf{Options}| \geq 3$.

Furthermore, the following axioms are imposed on SWFs for all $\mathsf{a}, \mathsf{b} \in \mathsf{Options}$, $f, g \in \mathsf{Situations}$ and $\sigma \in \mathsf{SWFs}$:

Unanimity (A3) $\mathsf{a}f[\mathsf{Voters}]\mathsf{b}$ implies $\mathsf{a}\sigma(f)\mathsf{b}$,

Independence (A4) $f =_{\mathsf{a},\mathsf{b}} g$ implies $\sigma(f) =_{\mathsf{a},\mathsf{b}} \sigma(g)$.

[11] Hamkins [14, Footnote 1] points out that already in the 1960's, Mostowski claimed that there are *several essentially different notions of set which are equally admissible as an intuitive basis for set theory* with Kalmár adding that *I guess that in the future we shall say as naturally "let us take a set theory," as we take now a group G or a field F*. More information on historical and contemporary aspects of this discussion can be found in Appendix B.

Given any $\sigma \in \mathsf{SWFs}$, define

$$\begin{aligned}
\mathcal{U}_\sigma &:= \{U \subseteq \mathsf{Voters} \mid \exists \mathsf{a},\mathsf{b} \in \mathsf{Options} \\
&\qquad \exists f \in \mathsf{Situations}.\mathsf{a}f[U]\mathsf{b} \,\&\, \mathsf{b}f[\mathsf{Voters}-U]\mathsf{a} \,\&\, \mathsf{a}\sigma(f)\mathsf{b}\},\\
\mathcal{U}'_\sigma &:= \{U \subseteq \mathsf{Voters} \mid \exists \mathsf{a},\mathsf{b} \in \mathsf{Options} \\
&\qquad \forall f \in \mathsf{Situations}.(\mathsf{a}f[U]\mathsf{b} \,\&\, \mathsf{b}f[\mathsf{Voters}-U]\mathsf{a} \Rightarrow \mathsf{a}\sigma(f)\mathsf{b})\},\\
\mathcal{U}''_\sigma &:= \{U \subseteq \mathsf{Voters} \mid \forall \mathsf{a},\mathsf{b} \in \mathsf{Options} \\
&\qquad \forall f \in \mathsf{Situations}.(\mathsf{a}f[U]\mathsf{b} \,\&\, \mathsf{b}f[\mathsf{Voters}-U]\mathsf{a} \Rightarrow \mathsf{a}\sigma(f)\mathsf{b})\}.
\end{aligned}$$

Lemma 6.1. *([24]) For any $\sigma \in \mathsf{SWFs}$ satisfying unanimity (A3) and independence (A4) above:*

- *$\mathcal{U}_\sigma = \mathcal{U}'_\sigma = \mathcal{U}''_\sigma$;*
- *whenever $\mathsf{Options}$ satisfies (A1), $\mathcal{U}_\sigma$ $(= \mathcal{U}'_\sigma = \mathcal{U}''_\sigma)$ is an ultrafilter on Voters and furthermore:*
- *$\mathcal{U}_\sigma$ is the unique ultrafilter $\mathcal{U}$ with the property*

$$\forall U \in \mathcal{U}, \mathsf{a},\mathsf{b} \in \mathsf{Options}, f \in \mathsf{Situations}.(\mathsf{a}f[U]\mathsf{b} \Rightarrow \mathsf{a}\sigma(f)).$$

- *$\mathcal{U}_\sigma$ is principal iff σ satisfies*

  Dictatorship (non-A5) *there is $v_0 \in \mathsf{Voters}$ such that for any $f \in \mathsf{Situations}$, $\mathsf{a},\mathsf{b} \in \mathsf{Options}$, $\mathsf{a}f(v_0)\mathsf{b}$ implies $\mathsf{a}\sigma(f)\mathsf{b}$.*

Proof. See Kirman and Sondermann [24], correspondingly for each item: Lemma 6.A, Lemma 6.B, Theorem 1(i), Proposition 2. ■

Theorem 6.2. *For any countable set of Voters, any $\mathsf{Options}$ satisfying (A1) and any σ satisfying (A3) and (A4), $\mathsf{ZF}+\mathsf{AD}$ implies the Dictatorship (non-A5) condition.*

We can redo more or less verbatim the perspicuous game-theoretic proof of Proposition 28.1 in Kanamori [21]:

Proof. If Voters is finite, $\mathcal{U}_\sigma$ must be a principal ultrafilter over Voters, which implies the result by Lemma 6.1. For an infinite countable set of Voters, assume that Dictatorship does not hold. Define a game where the players pick finite, mutually disjoint subsets of Voters. The $\exists$-player wins if the sum $\mathtt{E}$ of all Voters chosen by her[12] belongs to $\mathcal{U}_\sigma$, i.e., if it holds that

$$\forall \mathsf{a},\mathsf{b} \in \mathsf{Options}, f \in \mathsf{Situations}.(\mathsf{a}f[\mathtt{E}]\mathsf{b} \,\&\, \mathsf{b}f[\mathsf{Voters}-\mathtt{E}]\mathsf{a} \Rightarrow \mathsf{a}\sigma(f)\mathsf{b}).$$

[12]It is conventional to refer to the $\exists$-player as female and to the $\forall$-player as male.

A play of this game is of the form $\mathtt{e}_0, \mathtt{a}_0, \mathtt{e}_1, \mathtt{a}_1 \ldots$ where $\mathtt{e}_i$ and $\mathtt{a}_i$ are finite, mutually disjoints subsets of Voters chosen, respectively, by $\exists$- and $\forall$-players. The sums of these choices will be denoted as $\mathtt{E} = \bigcup_{i\in\omega} \mathtt{e}_i$ and $\mathtt{A} = \bigcup_{i\in\omega} \mathtt{a}_i$. A strategy for the $\exists$-player is thus a function

$$\tau_\exists : \bigcup_{n\in\omega} (\mathcal{P}_{\mathsf{fin}}(\mathsf{Voters}))^{2n} \to \mathcal{P}_{\mathsf{fin}}(\mathsf{Voters}),$$

and the $\exists$-player plays according to $\tau_\exists$ if the play of the game is of the form

$$\tau_\exists\emptyset, \mathtt{a}_0, \tau_\exists\langle\emptyset, \mathtt{a}_0\rangle, \mathtt{a}_1, \tau_\exists\langle\tau_\exists\emptyset, \mathtt{a}_0, \tau_\exists\langle\emptyset, \mathtt{a}_0\rangle, \mathtt{a}_1\rangle \ldots$$

Analogously, a strategy for the $\forall$-player is thus a function

$$\tau_\forall : \bigcup_{n\in\omega} (\mathcal{P}_{\mathsf{fin}}(\mathsf{Voters}))^{2n+1} \to \mathcal{P}_{\mathsf{fin}}(\mathsf{Voters}),$$

and the $\forall$-player plays according to $\tau_\forall$ if the play of the game is of the form

$$\mathtt{e}_0, \tau_\forall\langle\mathtt{e}_0\rangle, \mathtt{e}_1, \tau_\forall\langle\mathtt{e}_0, \tau_\forall\langle\mathtt{e}_0\rangle, \mathtt{e}_1\rangle \ldots$$

We will show that both the assumption of the existence of a winning $\tau_\exists$ and the assumption of the existence of a winning $\tau_\forall$ lead to a contradiction.

Assume first a winning $\tau_\exists$ exists. Define

$$\tau'_\forall\langle s_0, s_1 \ldots, s_{2n}\rangle := \tau_\exists\langle s_1 \ldots, s_{2n}\rangle - s_0.$$

Let now $\mathtt{e}_0, \mathtt{a}_0, \mathtt{e}_1, \mathtt{a}_1 \ldots$ be a play according to $\tau'_\forall$. By assumption on $\tau_\exists$, we have that $\mathtt{e}_0 \cup \mathtt{A} \in \mathcal{U}_\sigma$. As $\mathcal{U}_\sigma$ is nonprincipal, we have $\mathtt{A} \in \mathcal{U}_\sigma$. Thence, as $\mathtt{A} \cap \mathtt{E} = \emptyset$, it cannot be the case that $\mathtt{E} \in \mathcal{U}_\sigma$ and thus $\tau'_\forall$ is a winning strategy; a contradiction.

Assume now a winning $\tau_\forall$ exists. Note first that a slightly tweaked version of it is a winning strategy too:

$$\tau^+_\forall\langle s_0, s_1 \ldots, s_{2n}\rangle := \begin{cases} \tau_\forall\langle s_0, s_1, \ldots, s_{2n}\rangle \cup \{n\} & \text{if } n \notin s_0 \cup s_1 \ldots \cup s_{2n} \\ \tau_\forall\langle s_0, s_1, \ldots, s_{2n}\rangle & \text{otherwise.} \end{cases}$$

This new strategy satisfies furthermore the condition that in any play $\mathtt{e}_0, \mathtt{a}_0, \mathtt{e}_1, \mathtt{a}_1 \ldots$ according to it, we have that $\mathtt{A} \cup \mathtt{E} = \mathsf{Voters}$. As $\mathcal{U}_\sigma$ is an ultrafilter, $\tau_\forall$ is winning and $\mathtt{A} \cap \mathtt{E} = \emptyset$, we have that $\mathtt{A} \in \mathcal{U}_\sigma$. But now one can turn $\tau^+_\forall$ into a winning strategy for the $\exists$-player simply by augmenting the input with $\emptyset$, a contradiction. ■

As follows from the discussion in Section 5, the above result can be used even in the setting of ZFC, although this requires rather strong assumptions about infinite cardinals:

Corollary 6.3. *In* ZFC *enriched with axioms ensuring the existence of infinitely many Woodin cardinals with a measurable cardinal above them, $L(\mathbb{R})$ is a model of* ZF + DC *such that for any countable set of* Voters*, any* Options *satisfying (A1) and any σ satisfying (A3) and (A4), the Dictatorship (non-A5) condition holds.*

Proof. Follows from Theorems 5.1 and 6.2. ∎

Related results were obtained by Brunner and Mihara [7].

A careful reader may have noticed that the statements of Theorem 6.2 and Corollary 6.3 are restricted to countable sets of voters. In discussing potential insights AD may bring to social choice, one should be careful to point out that AD does not kill *all* nonprincipal ultrafilters. There are *uncountable* sets for which AD allows the existence of free ultrafilters. In this case it would seem to suggest, e.g., that by introducing suitable *uncountably* large collection of voters, we could still avoid, or at least pretend to avoid, Arrow's impossibility result. But the discussion presented in this paper hopefully shows that such claims would require a more detailed justification. AD is a good tool to verify conformity with the Hildenbrand criterion, but it is not infallible. Consistency with both ZFC and ZF + AD is a necessary, but not sufficient condition.

7. AD and Intergenerational Equity

The story of Fishburn's possibility claim for infinite electorates and its subsequent deconstruction is paralleled by that of Svensson's [50] possibility argument in intergenerational equity and its later unpicking, in particular an exhaustive examination by Zame [51]. Recall that this line of work dealt with the *space of utility streams* $X = [0, 1]^\omega$. The goal was to find a strict ordering $\succ$ on such a space which is an *ethical preference relation*, that is:

- *displays intergenerational equity*, i.e., is invariant under finite permutations of ω,
- *respects the weak Pareto ordering*, i.e., if $x_n > y_n$ for all n, then $x \succ y$,
- is *linear* or *total*, i.e., makes any two elements comparable; the authors in this line of work often choose to use the word *complete* instead.

Here is a recap of Svensson's argument provided by Zame:

define an incomplete preference relation $\succ$ on X in the following way: $y \succ x$ exactly when there is a finite permutation y^* of y (that is a reordering of finitely many of the terms of y) such that $y^* > x$ (i.e., $y^* \geq x$ and $y^* \neq x$). The relation $\succ$ is irreflexive, transitive, displays intergenerational equity and respects the Pareto ordering, but it is incomplete: some—indeed, many—pairs of utility streams are not comparable. However we can use Szpilrajn's (1930) extension lemma to find an extension $\succeq$ of $\succ$ to a complete transitive preference relation on X. This extension $\succeq$ automatically displays intergenerational equity and respects the Pareto ordering; i.e., it is an ethical preference relation [51, p. 188–189].

The problem is that Szpilrajn's Extension Lemma essentially relies on AC. Zame [51] shows that the existence of such an ethical preference relation is independent from $\mathsf{ZF} + \mathsf{DC}$ and that no such relation can be shown to be *definable* in ZFC. Analysis of Zame's argument shows that AD can also be used to dispel the illusion of intergenerational equity:

THEOREM 7.1. *The existence of an ethical preference relation on the space of utility streams $X = [0,1]^\omega$ is incompatible with* $\mathsf{ZF} + \mathsf{AD}$.

PROOF. Zame [51, Theorem 2] shows that a graph of an ethical preference relation would be a nonmeasurable subset of X and as noted in the same reference [51, p. 196, proof of Theorem 3] one can use such a subset to construct a nonmeasurable subset of $\mathbb{R}$. As stated in Theorem 4.1, the existence of nonmeasurable sets of reals is excluded by AD. ∎

An interesting exercise for the reader is to provide a more direct, game-theoretic proof in the spirit of Section 6. Let us add that issues of efficiency and constructibility seems to attract attention in contemporary research on intergenerational equity: apart from the work of Zame and papers quoted therein, see also, e.g., Lauwers [27,28]. It is worth mentioning that both Zame [51] and Lauwers [27] explicitly refer to Solovay's [47] early construction of a model of ZF where every subset of $\mathbb{R}$ is Lebesgue measurable.

8. Conclusions

As has been stated above, the goal is not to present AD as some kind of philosopher's stone or as an ultimate solution to problems of infinity in social choice. The Hildenbrand criterion is independent of the actual or potential rôle AD may play here. The existence of a powerful set-theoretic principle which—despite being inconsistent with AC—stems from a natural game-theoretic motivation, generalizes an obvious finitary law and does not seem to lead to any paradoxical constructions[13] should be simply taken as yet another piece of evidence for the Hildenbrand criterion.[14] It would be disputable from a philosophical point of view, if rather interesting mathematically, to build formal economics using consequences of $\mathsf{ZF} + \mathsf{AD}$ inconsistent with AC. But it seems even more problematic to rely on theorems of ZFC which are neither consistent with AD nor acceptable from the point of view of other lines of criticism sketched in Appendix B. Regardless of reasons for considering infinite populations of agents, a decision whether such populations are governed by choice or by determinacy seems entirely arbitrary. Such assumptions are of a metaphysical rather than scientific character. Only mild forms of AC like DC (and hence AC_ω) seem safe.

The advantage of AD from our point of view is simply that it offers economists a convenient, though not infallible tool of verifying whether a proposed result conforms to the Hildenbrand criterion. I hope that this paper illustrates that AD is easier to use for this purpose than relying on model-theoretic techniques like forcing, though resorting to the latter may still be necessary in corner cases.

BPI and unrestricted ZFC have some interesting applications in economics through nonstandard analysis (cf., e.g., [51, Footnote 1]). It would be interesting to study how these applications relate to our criterion. There is, however, very little effort towards seriously motivating or consciously defending the use of AC to guarantee possibility results on preference aggregation or intergenerational equity.[15] Quite to the contrary: we have seen above that skepticism in this respect has been, in fact, repeatedly expressed

[13] Of course, the claim of the absence of paradoxical constructions under AD cannot be accepted if one grew so accustomed to AC that the absence of certain Hamel bases, of some free ultrafilters or of linear ordering on cardinal numbers begins to appear paradoxical in its own right. But I can think at the moment of only one consequence of living in an universe where all subsets of $\mathbb{R}$ are Lebesgue measurable which may seem quite controversial to a mathematician who is not already *entirely* sold on AC. Namely, in such an universe $\mathbb{R}/\mathbb{Q}$ may have *more* elements than $\mathbb{R}$ itself; the former set cannot be even linearly ordered [46].

[14] As pointed out by one of referees, it should be stressed that AD itself is no more constructive than AC. If AD offered any way to *construct* a winning strategy or even to *decide* which player has a winning strategy, we should probably take it as an axiom of set theory no less fundamental than those of ZF, or ensure that other axioms entail it. This is not what I am suggesting here.

[15] To a certain extent, one can probably include here, e.g., Mihara [34], which argues that "invisible dictators" can be seen as computable (on measurable sets!) in a broader sense, namely that of *computability with an oracle*. Such an oracle is able to "decide" an undecidable problem. The paper does not address the Hildenbrand criterion or the objections raised in the first paragraph of Section 3 above.

in references since the 1960s till the present day. Yet somehow nonprincipal ultrafilters and nonmeasurable coalitions of agents continue to appear in the more technical literature without much justification. This is a state of affairs I find hard to understand. It does seem that a systematic discussion of foundational issues in formal economics, especially in social choice, is much overdue. My broader hope is that the present paper helps to build up the critical mass necessary to trigger such a discussion.

Acknowledgements. I would like to thank Alessandra Palmigiano and Marcus Pivato for an invitation to write this article. I would also like to acknowledge Marcus for discussions regarding early ideas for this paper, the status of infinite collections of agents and for suggesting several references on intertemporal choice. Moreover, I would like to thank Apostolos Tzimoulis for discussions regarding free ultrafilters over uncountable sets under AD (only partially reflected here) and for the reference to Hamkins' work. In addition, I thank the referees for their criticism; hopefully, the argument is clearer now. I am very grateful to Wesley H. Holliday for careful reading of and detailed comments on the near-final version. Finally, credit is due to Erwin R. Catesbeiana for his reverberating warnings about the tension between choice and determinacy.

A. ZF and AC

Zermelo-Fraenkel set theory ZF is formulated in the first-order language $\mathcal{L}_{\mathsf{ZF}}$ with one binary primitive $\in$. I will use numerous standard abbreviations like $x = \{y, z\}, \emptyset, x \cap y = \emptyset, \exists x \in y.\phi(x), \exists! x \in y.\phi(x), x \subseteq y$, etc.

The axioms are as follows:

Axiom of Extensionality $\forall xy.(\forall z.z \in x \leftrightarrow z \in y) \to x = y$.

Axiom of Regularity $\forall x.x \neq \emptyset \to \exists y \in x.x \cap y = \emptyset$.

Axiom of Union $\forall x \exists y \forall z.z \in y \leftrightarrow \exists v \in x.z \in v$.

Axiom of Powerset $\forall x \exists y \forall z.z \in y \leftrightarrow \forall v \in z.v \in x$.

Axiom of Infinity $\exists x.\emptyset \in x \land \forall y.y \in x \to y \cup \{y\} \in x$.

Axiom scheme of Restricted Comprehension (also known as *Specification* or *Separation*) For any $\mathcal{L}_{\mathsf{ZF}}$ formula $\phi(x, \overline{z})$ with no occurrences of y, the following is an axiom:

$$\forall x \exists y \forall z.z \in y \leftrightarrow (z \in x \land \phi(x, \overline{z})).$$

Axiom scheme of Replacement (also known as *Collection*) For any $\mathcal{L}_{\mathsf{ZF}}$ formula $\phi(x, y, \overline{z}, w)$ with no occurrences of v, the following is an axiom:

$$\forall w\overline{z}.(\forall x \in w\exists! y.\phi(x, y, \overline{z}, w)) \to \exists v\forall x \in w\exists y \in v.\phi(x, y, \overline{z}, w).$$

The *Zermelo–Fraenkel set theory with Choice* (ZFC) extends the above set of axioms with

Axiom of Choice (AC)

$$\forall w.(\forall x \in w.x \neq \emptyset \wedge \forall y \in w.x \neq y \to x \cap y = \emptyset) \to \\ \exists c\forall x \in w.\exists! y \in x.y \in c).$$

There are several postulates often added as axioms which are nevertheless derivable from the system above, in particular the Axiom of Pairing or the Axiom of Empty Set. A rich ontology can be developed on the basis of this seemingly spartan system—natural numbers, real numbers, functions, relations, ordinals, cardinals, cartesian products, power sets, etc.

Details can be found in numerous references, e.g., [12,20,21,25].

ZF and its extensions are grounded in the *iterative conception of set* [6,11,41]. But this conception and its underlying *stage theory* [6] do not provide equal support to every axiom.

Consider, for example, the quantified variable z in the statement of the Axiom of Powerset. What is it ranging over? That is, what subsets of a given set are supposed to exist apart from those guaranteed by the Axiom of Restricted Comprehension, i.e., determined by a formula of $\mathcal{L}_{\mathsf{ZF}}$? Cantor, the founding father of set theory, famously defined a set as "a totality of definite elements that can be combined into a whole by a *law*" (cf. [6, p. 215]). It would seem that such a law should be expressible by finitary means.

The strength and the controversial character of AC stems from the fact that it provides an abundant supply of sets not determined by any kind of formula, property or algorithm. Let us restate its contents in plain words:

> For any family w of non-empty, pairwise disjoint sets, there exists a "choice set" c such that for any set $x \in w$, $|x \cap c| = 1$.

Obviously, such a choice set can be replaced by the corresponding *choice function*, which sends each $x \in w$ to an element of $x \cap c$; we could denote this element as $c(x)$. This is in fact the form in which AC is often stated: as ensuring the existence of a choice function for an arbitrary family of non-empty sets (not necessarily pairwise disjoint). Using other axioms of ZF, such a formulation can be easily proved equivalent to the one given above.

Note that if we have defined a well-ordering $\prec$ on $\bigcup w$, there is no need to resort to AC to ensure the existence of a choice function. We could explicitly define it by taking for each x its $\prec$-smallest element as $c(x)$. This is why AC is implied by the Well-Ordering Principle. And, as discovered already by Zermelo in the earliest work using AC, the converse is true as well.

The problem is that in many cases, no such well-ordering (or such a choice function) can be constructed or defined, even in a very liberal sense of "defining" or "constructing". And accepting as an axiom the existence of such objects is an article of faith with far-reaching and not necessarily positive consequences.

Before we discuss further objections to AC in Appendix B, let us recall that not all of its applications require all of its strength. To begin with, if w is a *finite* family of non-empty (possibly infinite) sets, then the existence of a choice function is provable in plain ZF. So the weakest form of AC of any interest is

> **Axiom of Countable Choice** (AC_ω) Every countable family of non-empty sets has a choice function.

A somewhat stronger principle was proposed by Bernays in 1942 [5].

> **Axiom of Dependent Choice** (DC) For any $X \neq \emptyset$ and any $R \subseteq X \times X$ such that $\forall x \exists y. xRy$, there exists $f : \omega \to X$ s.t. $\forall n \in \omega. f(x_n) R f(x_{n+1})$.

Lemma 9.1. $\mathsf{ZF} + \mathsf{DC} \vdash \mathsf{AC}_\omega$.

Proof. Let $W := \{W_n\}_{n\in\omega}$ be a family of non-empty sets. Consider $X := \{(w_1, \ldots w_n) \mid n \in \omega, \forall i \leq n, w_i \in W_i\}$. Given any $\overline{w} = (w_1, \ldots w_m), \overline{v} = (v_1, \ldots v_n) \in X$, define $\overline{w} R \overline{v}$ if $n = m + 1$ and for any $i \leq n$, $w_i = v_i$. We obtain a choice function for W by a direct application of DC. ∎

Lemma 9.2. *The following facts are provable in* $\mathsf{ZF} + \mathsf{AC}_\omega$*:*

- *Every infinite set has a countable subset.*
- *The union of countably many countable sets is countable.*
- *Cauchy-style ($\epsilon - \delta$) and Heine-style (limits of sequences) definitions of continuity, closedness and compactness are equivalent.*
- *Every subspace of a separable metric space is separable.*
- *Lebesgue measure and meager sets have the property of countable additivity.*

Proof. See Jech [20, Chapter 2] for details and references. ∎

LEMMA 9.3. *The following facts are provable in* $\mathsf{ZF}+\mathsf{DC}$*:*

- *A linearly ordered set is well-founded iff it contains no infinite descending sequence.*
- *Urysohn's Lemma: if* X, Y *are disjoint closed sets in a* T_4*-space* S*, then there is a continuous function from* S *to* $[0,1]$ *which takes the value 1 everywhere in* X *and 0 everywhere in* Y*.*

PROOF. The first item is a direct application of DC. For the second, see Problem 2.26 in Jech's book [20]. ∎

There are many consequences of AC which, while not as strong as the axiom itself, already go far beyond what can be proved from benign principles like DC. For an important axiom of this kind, we need to recall the notion of an ultrafilter. For any boolean algebra $\mathfrak{A} := (A, \wedge, \neg, \top)$, a subset $F \subseteq A$ is a *filter* if for any $a, b \in A$, $a, b \in F$ iff $a \wedge b \in F$, and a filter F is an *ultrafilter* if moreover for any $a \in A, |F \cap \{a, \neg a\}| = 1$. In the special case when the algebra $\mathfrak{A}$ is of the form 2^S for some S, one speaks of ultrafilters *over* S. Ultrafilters over S always exist: for any $s \in S$, its *principal ultrafilter* $\{X \subseteq S \mid s \in X\}$ is an example. Ultrafilters which are *not* of this form are called *free* or *nonprincipal*. One can immediately show that nonprincipal ultrafilters cannot exist over finite sets. For an infinite S, a necessary and sufficient condition for an ultrafilter to be free is to contain the *Fréchet filter*, i.e., the collection of cofinite subsets of S. As it turns out, however, ZF—even extended with AC_ω or, still further, with DC—has nothing to say about the existence of free ultrafilters over infinite sets. We need to go closer towards full AC to prove this. Here are corresponding set-theoretic principles.[16]

Strong BPI Every proper filter in a boolean algebra can be extended to an ultrafilter.

Weak BPI Every boolean algebra contains an ultrafilter.

THEOREM 9.4.
- *Over* ZF*, the weak and the strong variant of* BPI *are equivalent.*
- $\mathsf{ZFC} \vdash \mathsf{BPI}$.
- $\mathsf{ZF}+\mathsf{DC} \nvdash \mathsf{BPI}$.
- $\mathsf{ZF}+\mathsf{BPI} \nvdash \mathsf{AC}$.

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[16]The traditionally used abbreviation BPI comes from *Boolean Prime Ideal*, as the principle was often formulated in terms of *ideals* rather than using the dual notion of *filter*.

Proof. All of these results are well-known in set theory. See, for example, Jech [20] for more information and references. ■

Finally, let us consider perhaps the most famous of controversial consequences of AC. We do not need the full strength of AC for this. In fact, free ultrafilters do the job.

Theorem 9.5. ZF + BPI *implies the existence of a non-Lebesgue measurable set.*

Proof. As noted, e.g., in Kanamori [21, p. 384], this already follows from a 1938 result by Sierpiński that a free ultrafilter over ω understood as a set of reals is not Lebesgue measurable. ■

Of course, the very first proof of ZFC implying the existence of a non-measurable set was provided by Vitali almost immediately after Zermelo's famous well-ordering paper.

AC is not derivable from the axioms of ZF (see Section B.2). But Gödel showed how to construct a model of ZFC assuming only the axioms of ZF, thus settling the issue of consistency. Given any set A, define the collection $L(A)$ of all sets *constructible from* A by transfinite recursion of the operation of forming sets *definable* in $\mathcal{L}_{\mathsf{ZF}}$, starting from the transitive closure of $\{A\}$ under $\in$, which we can denote as $cl_{\in}(A)$. If $L(A)$ contains a well-ordering of $cl_{\in}(A)$, it can be well-ordered and thus we obtain a model satisfying ZFC. In particular, Gödel's *constructible universe* is defined as $L(\emptyset)$ and denoted simply as L.

B. Traditional and Constructive Criticism of AC

Graduate-level courses and most popular expositions of set theory these days tend to present full AC as something barely questionable, to the point of being nearly self-evident. Nothing can be further from the truth. Historically, AC attracted very strong opposition from many leading mathematicians and these objections had much more substance than it is presently admitted. Furthermore, claims about the present total acceptance of AC are also overstated. In most areas close to actual foundations of mathematics—category theory, type theory, proof theory, theoretical computer science—AC certainly does not get a free pass. Let us briefly overview some of this opposition, including both well-known and less commonly mentioned arguments. They are ordered, roughly, in historical order, from the past to the present times.

B.1. Historical Opposition

It is often forgotten that the Axiom of Choice was regarded as problematic for many decades after its introduction:[17]

> at the moment (1904) when the axiom, explicitly formulated, was used by Zermelo to prove and confirm one of the earliest assertions of Cantor, viz. the well-ordering theorem, mathematical journals were *flooded with critical notes rejecting the proof* [emphasis mine–T.L.], mostly arguing that our axiom was either illegitimate or meaningless [12, p. 82].

That is, even if contemporary introductions tend to present AC as an obvious invention, the initial reaction of the mathematical community was the exact opposite.[18] Furthermore, Fraenkel et al. admit that

> [i]t may surprise scholars working in the field of abstract or applied set theory that even after more than half a century of utilizing the axiom of choice and the well-ordering theorem, a number of first-rate mathematicians (especially French) have not essentially changed their distrustful attitude; not even such as have been working most successfully in the domain of point sets and of real functions [12, p. 83].

Borel is an example of such a first-rate French mathematician distrustful towards full-blown AC (despite, let us add, being willing to admit its countable version AC_ω). Other famous mathematicians critical from the very beginning of careless usages of AC included, e.g., Baire, Lebesgue, Pasch and

[17] For a long time, these controversies were reflected even in standard monographs, including Fraenkel et al. monumental 1973 work [12] quoted in this section. For another convincing example, the reader can consult Kuratowski and Mostowski [25]. Even though it was written at a time when the community had grown accustomed to AC, the results depending on it were still clearly distinguished typographically (to be precise, marked with the $^\circ$ sign), a custom they claim to be introduced and consistently observed by Sierpiński [25, p. 54]. According to Kuratowski and Mostowski, early controversies surrounding AC have shown that there is no single "intuitive" notion of a set, a point interestingly convergent with much more recent views of Hamkins [14] discussed in Section B.2 below. See also Footnote 11.

[18] It is instructive to compare such presentations of ZFC with, say, those of relativity theory or quantum mechanics. In these cases, nobody is trying to gloss over the initial apprehension and skepticism of the community, or pretend such a reaction is unnatural, also for a contemporary student of mathematical physics. Niels Bohr famously claimed that *[a]nyone who is not shocked by quantum theory has not understood it*. Classroom introductions to foundations of mathematics and the question of constructive means of proof should aim to elicit a similar reaction.

Lusin [12, p. 216]; in hindsight, one can also add the name of Peano [12, p. 57]. Brouwer made a more radical attempt to break with set-theoretical foundations (cf., e.g., [12, Chapter IV]). While his original philosophical positions are of mostly historical importance, the *intuitionistic logic* he inspired remains relevant in the present context, as we will see in Section B.3.

B.2. Independence of AC

Regardless of the attempts at building foundations alternative to first-order set theories like those discussed in Section B.3, two fundamental revelations shook the confidence of the mathematical community in the 1960's. On the one hand, AC turned out to be *independent* from the remaining axioms of ZF: there are models where the ZF axioms hold, yet AC does not. On the other hand, even full ZFC does not settle the central question which motivated Cantor: the cardinality of $\mathbb{R}$. In other words, the Continuum Hypothesis is independent from ZFC, just like AC is independent from ZF.

Furthermore, if one is willing to replace *pure* ZF with ZFA, i.e., set theory allowing *atoms* or *urelements*—whose presence seems natural in most applications—the apparatus developed by Cohen and subsequent authors becomes redundant. The independence of AC from ZFA can be shown using much earlier *permutation models*: see, e.g., [20, Chapter 4] for an overview.[19]

If one thinks of set theory as describing some *intended model* in which to develop the entirety of mathematics, such independence results seem to pose a peculiar philosophical problem.

Gödel's platonic ideal would require investigating candidates for further axioms about the one-and-only "universe of sets", hoping that our intuitions, increasingly refined by an ever-growing body of results, will somehow eventually distinguish genuine axioms from spurious candidates. And perhaps even some researchers working presently on large cardinals and related subjects could agree with such a rough description of their long-term goals.

However, there is also growing awareness that this perspective is untenable and divorced from the actual state of affairs in set theory. It is not exactly a brand new revelation (cf. Footnotes 11 and 17), but is probably most vocally and consistently expressed by Hamkins in his recent work proposing an alternative, *multiverse* (as opposed to *universe*) point of view

[19]Such models are presently enjoying renewed attention as *nominal sets* in the context of reasoning about syntax with binding. See [42, 43] for an up-to-date overview. Permutation models and nominal sets seem to be relevant in the context of discussion of the *Pareto-compatible permutations* and *anonimity conditions* of Lauwers [28].

> which holds that there are diverse distinct concepts of set, each instantiated in a corresponding set-theoretic universe, which exhibit diverse set-theoretic truths. Each such universe exists independently in the same Platonic sense that proponents of the universe view regard their universe to exist. Many of these universes have been already named and intensely studied in set theory [14, p. 416–417].

As explicitly admitted by Hamkins, on this view in particular

> [t]here seems to be no reason to restrict inclusion only to ZFC models, as we can include models of weaker theories ZF, ZF^-, KP and so on, perhaps even down to second order number theory, as this is set-theoretic in a sense [14, p. 436].

A further question can be raised regarding the status of AC in relation to the conceptual foundation of ZF and its extensions: the *iterative conception of set* [6,11,41]. In the first reference systematically reconstructing this conception via its underlying *stage theory*, Boolos admitted that

> it seems that, unfortunately, the iterative conception is *neutral with respect to* AC [emphasis mine—T.L.] ...no additional axiom, which would decide choice, can be inferred from the rough description without the assumption of the axiom of choice itself, or some equally uncertain principle, in the inference [6, p. 230].

B.3. Topos-theoretic and Modern Type-theoretic Perspectives

Since the 1950's, the category-theoretic approach to foundations of mathematics has dramatically grown in importance. And the researchers working in this area tend to be rather skeptical and often even hostile with respect to ZFC. On the one hand, the ontology of ZFC is simply not rich enough for a smooth development of category theory. On the other hand, careless usage of choice raises opposition among category theorists, for both aesthetic and foundational reasons.[20]

[20] For a discussion of constructivist and intuitionistic aspects of category theory, the reader can consult, e.g., McLarty [32] or Bell [4]. Lambek and Scott [26] not only present the interpretation of intuitionistic logic in terms of the internal language of toposes as discussed here, but show another categorical interpretation of Heyting's formalism in terms of *cartesian closed categories*. This is of less relevance for us here, but it is worth noting that whenever one chooses to interpret logical connectives in a suitably general categorical setting, one obtains intuitionistic rather than classical logic.

In category-theoretic terms, a good candidate for a mathematical universe is provided by the notion of a *topos*. As pointed out by Hellman [15], one of the most systematic and philosophically conscious attempts to provide a topos-based alternative to set-theoretic foundations can be found in the work of Bell on *local set theories* [4]. It is worth noting here that both Bell's *many topoi* view based on local set theories and the *modal structuralism* advocated by Hellman seem to share many common features with Hamkin's *multiverse* perspective mentioned in Section B.2 above.

Any topos-based approach to foundations is by nature more general than any approach taking ZF as a starting point, not to mention ZFC. The very first difference is that the internal logic of the topos is *intuitionistic*, as opposed to the classical logic underlying ZF. And AC is, from this point of view, even more restrictive: a famous result of Diaconescu [9] shows that AC can only hold in those toposes whose internal logic is classical. Needless to say, the converse implication does not hold: there are boolean toposes where AC is not valid. From a topos-theoretic perspective one can see AC as a strong generalization of the law of excluded middle.

Furthermore, toposes themselves can be seen as a special case of (models of) higher-order intuitionistic *type theories*.[21] The type-theoretic approach is motivated not only by purely mathematical or philosophical considerations, but also by *computational* aspects, rooted in the *Curry-Howard correspondence* [31,48]. Type theories provide conceptual and technical foundations of state-of-the-art proof assistants, in particular Coq, Agda or Lean.

Topos-related type theories are distinguished by their *extensionality* and *impredicativity*. Perhaps the most famous proposal for a constructive type theory which is, in contrast, both intensional and predicative comes from Martin-Löf [31]. The same author offered a penetrating critique of AC, distinguishing its *intensional* and *extensional* version [30]. This reference clarifies the apparent contrast between the firm opposition of early constructivists like Baire, Borel, Lebesgue and Brouwer himself and the seeming acceptance of some form of AC by later authors like Bishop (not to mention a similar principle in Martin-Löf's earlier writings). The *intensional* form of AC is acceptable constructively via the Brouwer-Heyting-Kolmogorov interpretation. But this form is a rather trivial statement when interpreted

[21] "Topoi, generally, can be realized, up to a precisely specified notion of categorical equivalence, as models of higher-order type theories based on intuitionistic logic. Indeed, one specifies a class of type-theoretic languages and theories (Bell's *local set theories*) and shows that any topos has such a theory associated with it which in turns determines that topos up to categorical equivalence" [15].

in set-theoretic terms; the interesting and problematic form is the *extensional* one. Needless to say, the extensional formulation is needed for the non-constructive set-theoretic results we have been concerned with in this paper. It is also the one which can be seen as a strong form of excluded middle, even beyond the topos-theoretic context (see, e.g., [3,9,13,30] for more information).

In conclusion, it is worth recalling the point made in Section 4: just as (the disputable extensional form of) AC is a strong generalization of the law of excluded middle, AD is a strong generalization of the De Morgan law for quantifiers. Apparently, generalizations of finitary laws can conflict with each other in suitably large domains—and our intuitions and inherited mathematical prejudices are far from being reliable criteria of their validity.

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