Rohan French

A Simple Sequent Calculus for Angell's Logic of Analytic Containment

Abstract. We give a simple sequent calculus presentation of R.B. Angell's logic of analytic containment, recently championed by Kit Fine as a plausible logic of partial content.

Keywords: Sequent calculus, Analytic containment, First-degree entailment, Relevant logic.

1. Introduction

Angell [2–4] introduced a logic whose conditional was intended to represent a kind of analytic containment of the consequent in the antecedant. Working in the general tradition due to Parry [24,25] one of the more distinctive features of the logic Angell proposed was in its invalidating the principle of disjunction introduction, sometimes also referred to as 'Addition':

$$A \to (A \vee B).$$

As Angell notes, it is not at all clear that the meaning of $A \vee B$ is contained in the meaning of A. For example, if containment of meaning requires containment of 'subject matters' (in something like the sense of Yablo [28]) then if the subject matter of B is not part of the subject matter of A then we will end up with $A \vee B$ being about something other than what A is about, leading to a failure of containment of meaning. One of the distinctive and interesting features of Angell's proposed logic is that it allows us capture the idea that A is a part of the content of $A \wedge B$, while $A \vee B$ is not part of the content of A. This is something which we cannot model by understanding

[1] Other notable projects falling under the general umbrella of Parry-style treatments of 'implication as meaning containment' are the paraconsistent implication of Deutsch [11] and the treatment of implication in the story semantics of Daniels [10]. The logic of meaning containment in Brady [6] has Addition as an axiom, and thus seems to fall afoul of the argument below that this is not valid on the understanding of entailment as meaning containment.

Presented by **Heinrich Wansing**; *Received* August 17, 2016

Studia Logica (2017) 105: 971–994
DOI: 10.1007/s11225-017-9719-y

partial content in terms of classical logical consequence, this following from a more general limitative result concerning partial content in Fine [17, p. 413].

Where Angell's logic differs from those in the general tradition of Parry's Analytic Implication concerns how the two logics treat statements which contain 'implicit tautologies' (on which see Angell [4, p. 122]). In particular, Parry's logic validates the equivalence

$$((A \wedge \neg A) \wedge (B \vee \neg B)) \leftrightarrow ((A \vee \neg A) \wedge (B \wedge \neg B))$$

If we are reading '$\leftrightarrow$' as analytic equivalence, then, loosely speaking, this tells us that A being inconsistent and B having a truth value is the same as B being inconsistent and A having a truth value. One can quite plausibly argue that these two statements are not analytically equivalent, but at the very least for many applications of analytic containment what one is usually after is a more discerning notion than one with consequences like this.

Appeals to analytic containment are quite natural in giving various philosophical analyses, and so it should be unsurprising that Angell's logic has seen a number of interesting applications. On the more epistemic side of things it has been used in Belnap [5] to repair a fault in a particular account of hypothetical reasoning due to Nicholas Rescher; in Ferguson [14] to model catastrophic faults in Belnapian artificial reasoners—cases where a Belnapian computer experiences a fault in retrieving the truth-value of a formula, where truth- and falsity-values are stored at separate 'addresses'; and in Ferguson [13] in connection to work on epistemic interpretations of certain bilattices, where Angell's logic turns out to be the logic of 'being at least believed'.

Perhaps one of the more prominent recent applications of Angell's logic has been to questions about the metaphysics of content and the logical structure of metaphysical dependence. For example, Fine [19] has proposed that Angell's conditional is the correct formalisation of the notion of partial content, and is a near relative of the logic of exact truthmaker containment. Similarly, in Correia [8] it is proposed that the biconditional formulation of Angell's logic (Angell's own preferred axiomatisation, as we will see in the next section) characterises 'factual equivalence' on a worldly conception of facts.[1]

[1] Correia [9] moves away from this claim, arguing that in fact a proper subsystem of Angell's logic, which involves dropping the distribution principle $A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$ from Fine's $AC_{\leftrightarrow}$, should be understood as the correct logic of factual equivalence. Correia's main motivation here is that, as is shown in Lemma 4 of Fine [19], against the

It is clear, then, that understanding Angell's logic is important to various projects for understanding various metaphysical and semantic projects concerning truthmakers and partial content, in addition to understanding its potential applications to hypothetical reasoning, and the modeling of certain kinds of artificial agents. To this end it would be helpful if we had a detailed proof-theoretic analysis of this logic and its relatives. The present paper is intended to provide the beginnings of such an analysis. In Sect. 2 I give a brief survey of existing axiomatic proof systems for Angell's logic, emphasising the similarity from a proof-theoretic perspective between Angell's logic and Anderson and Belnap's logic of tautological entailment, as a prelude to in Sect. 3, presenting a simple sequent calculus for Angell's logic. We close in Sect. 4 by showing that this system is equivalent to an axiomatic proof system for Angell's logic due to Kit Fine.

2. Axiomatic Systems for Angell's Logic

Throughout we will be concerned only with propositional languages.[2] A *truth-functional formula* is a formula built up from a denumerable supply $p_1, p_2, \ldots$ of propositional variables using the connectives $\wedge$, $\vee$, $\neg$. An *equivalential formula* is a formula of the form $A \leftrightarrow B$ where A and B are truth-functional formulas, and finally a *containment formula* is one of the form $A \rightarrow B$ where A and B are truth-functional formulas.

In giving a proof system for analytic containment we are immediately faced with a choice of whether to axiomatise the set of all valid equivalential formulas (and thus take $\leftrightarrow$ as our sole non-truth-functional primitive), or the set of all valid containment formulas (taking $\rightarrow$ as our sole non-truth-functional primitive). Kit Fine, and Angell himself have favoured axiomatising the class of equivalential formulas, while (like Correia) we will be concerned with dealing with containment formulas. In our particular case this is because we are ultimately interested in (in Sect. 3) giving a sequent calculus where a sequent $A_1, \ldots, A_n \succ B_1, \ldots, B_n$ is to be interpreted as the containment formula $(A_1 \wedge \ldots \wedge A_n) \rightarrow (B_1 \vee \ldots \vee B_n)$. Thankfully, as Angell himself points out, we do not need to choose between these two different

background of the rest of Angell's logic, the theoremhood of this principle is, in a sense, equivalent to the admissibility of the rule which takes us from $A \leftrightarrow B$ to $\neg A \leftrightarrow \neg B$.

[2]Throughout we will use uppercase Roman letters as schematic letters for arbitrary propositional formulas, and uppercase Greek letters for multisets of propositional formulas. We will consider sequents to be pairs of multisets of formulas, and will use '$\succ$' as our sequent separator, writing the pair $\langle \Gamma, \Delta \rangle$ as $\Gamma \succ \Delta$.

choices of primitives, as we can define $A \to B$ in terms of the equivalence $A \leftrightarrow (A \wedge B)$. This gives us a way to move from talk of containment formulas to talk of equivalential formulas. We can move in the opposite direction by noting that $A \leftrightarrow B$ is equivalent to $A \to B$ and $B \to A$. It would be nice to have a more systematic overview of the relationship between the class of containment and equivalential formulas, but we will not detain ourselves with taking up such an investigation here. Instead, let us begin our brief look at the extant proof systems for Angell's logic by looking at the very succinct axiomatisation of the class of all valid equivalential formulas given in Angell [4, p. 124], which we will call $\mathbf{AC}_1$.

Angell's Axiom system $\mathbf{AC}_1$.

Axioms

AC1. $A \leftrightarrow \neg\neg A$

AC2. $A \leftrightarrow (A \wedge A)$

AC3. $(A \wedge B) \leftrightarrow (B \wedge A)$

AC4. $A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$

AC5. $A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$

Rules

$$\frac{A \leftrightarrow B \quad C(A)}{C(B)}\ R1.$$

where $C(B)$ is a formula just like $C(A)$ but with some number of instances of A in $C(A)$ replaced by B.

This axiomatisation is very economical, but as a direct result rather cumbersome to work with. Given that it is also concerned with equivalential formulas, we will investigate it no further, preferring instead to follow on to look at the axiomatisation, due to Correia [7, p. 89], of the containment formulas of Angell's logic which we will call $\mathbf{AC}_2$.

Correia's Axiom system $\mathbf{AC}_2$.

Axioms

AC1a $A \to \neg\neg A$

AC1b $\neg\neg A \to A$

AC2 $A \to A \wedge A$

AC3 $A \wedge B \to A$

AC4 $A \vee B \to B \vee A$

AC5a $A \vee (B \vee C) \to (A \vee B) \vee C$

AC5b $(A \vee B) \vee C \to A \vee (B \vee C)$

AC6a $A \vee (B \wedge C) \to (A \vee B) \wedge (A \vee C)$

AC6b $(A \vee B) \wedge (A \vee C) \to A \vee (B \wedge C)$

Rules

$$\frac{A \to B \quad B \to A}{\neg A \to \neg B}\ AC7 \qquad \frac{A \to B}{A \vee C \to B \vee C}\ AC8 \qquad \frac{A \to B \quad B \to C}{A \to C}\ AC9$$

This axiomatisation is much more revealing, although it is still cumbersome in many ways. What is particularly revealing about this axiomatisation is that we can begin to see the strong affinity between Angell's logic and Anderson and Belnap's logic E_{fde} (henceforth, simply FDE) of tautological entailments. In particular this axiomatisation is very similar to that proposed in Anderson and Belnap [1, p. 158] for FDE, essentially resulting from this system by removing the disjunction axioms (i.e. the principle of Addition) with 'minimal mutilation', adding in by hand the distribution principal AC6 (which is provable in the presentation of FDE given by Anderson and Belnap). This axiomatisation has the disadvantage of making use of an awkward contraposition rule AC7, which hides the De Morgan behaviour of Angell's logic.[3]

To this end we will ultimately work with the axiom system for containment formulas described in Fine [19, pp. 201–202]. This system builds off Fine's own axiomatic system, given in terms of equivalence formulas, by first noting that $A \leftrightarrow B$ ought to entail $A \to B$ and $B \to A$, and then requiring that we can define $A \to B$ as $A \leftrightarrow (A \wedge B)$, which means that we need to also add the axiom $(A \wedge B) \to A$ (as $(A \wedge B) \leftrightarrow (A \wedge B) \wedge A$ can easily be seen to

[3]One might wonder whether we can dispense with the $A \to B$ premise of AC7, giving us the standard contraposition rule. This rule does not preserve validity in Angell's logic, though. In particular it will take us from the valid $(p \wedge q) \to p$ to the invalid $\neg p \to \neg(p \wedge q)$, the latter (after some De Morgan manoeuvres) being an instance of the invalid principle of Addition. Making use of notation to be introduced in the next section, the effect of the other premise is to force $t(A) = t(B)$.

be a theorem of $\mathbf{AC}_1$). Following Fine we will call this system $\mathbf{AC}_{\rightarrow}$. Fine's description of this system includes some redundancies, such as the fact that we only need to posit $A \rightarrow A \wedge A$ as an axiom and not its converse, as this follows from $A \wedge B \rightarrow A$ — the distinctive new axiom of $\mathbf{AC}_{\rightarrow}$ as compared to Fine's own $AC_{\leftrightarrow}$. We give the full system below, following the policy of using Fine's labels for the corresponding equivalential principles, splitting X into Xa and Xb.

Fine's Axiom system $\mathbf{AC}_{\rightarrow}$.

Axioms

E1a. $A \rightarrow \neg\neg A$
E1b. $\neg\neg A \rightarrow A$
E2. $A \rightarrow A \wedge A$
E3. $A \wedge B \rightarrow B \wedge A$
E4a. $A \wedge (B \wedge C) \rightarrow (A \wedge B) \wedge C$
E4b. $(A \wedge B) \wedge C \rightarrow A \wedge (B \wedge C)$
E5a. $A \rightarrow A \vee A$
E5b. $A \vee A \rightarrow A$
E6. $A \vee B \rightarrow B \vee A$
E7a. $A \vee (B \vee C) \rightarrow (A \vee B) \vee C$
E7b. $(A \vee B) \vee C \rightarrow A \vee (B \vee C)$
E8a. $\neg(A \wedge B) \rightarrow \neg A \vee \neg B$
E8b. $\neg A \vee \neg B \rightarrow \neg(A \wedge B)$
E9a. $\neg(A \vee B) \rightarrow \neg A \wedge \neg B$
E9b. $\neg A \wedge \neg B \rightarrow \neg(A \vee B)$
E10a. $A \wedge (B \vee C) \rightarrow (A \wedge B) \vee (A \wedge C)$
E10b. $(A \wedge B) \vee (A \wedge C) \rightarrow A \wedge (B \vee C)$
E11a. $A \vee (B \wedge C) \rightarrow (A \vee B) \wedge (A \vee C)$
E11b. $(A \vee B) \wedge (A \vee C) \rightarrow A \vee (B \wedge C)$
E12. $A \wedge B \rightarrow B$

Rules

$$\frac{A \rightarrow B \quad B \rightarrow C}{A \rightarrow C}\ (E13) \qquad \frac{A \rightarrow B}{A \wedge C \rightarrow B \wedge C}\ (E14) \qquad \frac{A \rightarrow B}{A \vee C \rightarrow B \vee C}\ (E15)$$

Again the similarity to axiom systems for FDE should be apparent, the main difference here being that we have traded in the contraposition rule for the full collection of De Morgan principles. With an axiomatic presentation of Angell's logic in hand let us turn now to our sequent calculus.

3. A Simple Sequent Calculus for Angell's Logic

Fine [19] is predominately concerned with providing a number of different semantic treatments of Angell's logic, with the particular aim of highlighting how the logic naturally arises out of considerations of his truthmaker semantics.[4] Of particular interest to us here is the characterisation of $\mathbf{AC}_{\rightarrow}$ which Fine gives in terms of a pair of four-valued logics. One of these is the familiar semantic consequence relation for FDE, the other a slightly more exotic four-valued logic reminiscient of that determined by the matrices given in Fig. 8 of Humberstone [21, p. 640].[5] Thinking of the logic semantically in terms of this pair of matrices is rather awkward, and provides little insight into the nature of Angell's logic. A more informative way of proceeding is via the notion of what Fine calls the *valence* of a formula.

Definition 3.1. Let us define *valence*, both positive and negative, of an occurence of a sentence letter p in a formula A recursively as follows.

- p occurs positively in p
- If p occurs positively (negatively) in A, then it occurs negatively (positively) in $\neg A$
- If p occurs positively (negatively) in A or B, then it occurs positively (negatively) in $A \wedge B$ and $A \vee B$

Let us say that a sequent $\Gamma \succ \Delta$ is *valence preserving* if every sentence letter which occurs positively (respectively negatively) in some formula in Γ occurs positively (resp. negatively) in some formula in Δ; and that it is *valence anti-preserving* if every sentence letter which occurs positively (resp. negatively) in some formula in Δ occurs positively (resp. negatively) in some formula in Γ. The logic of 'partial-truth preservation' mentioned above can then be shown to be characterised by the set of all sequents which are valence preserving.

As a useful shorthand in what follows we will make use of the following function $t(\cdot)$.

$$t(A) = \{p | p \text{ occurs under the scope of an even number of negations in } A\} \cup \{\neg p | p \text{ occurs under the scope of an odd number of negations in } A\}$$

[4] Fine's primary published papers on truthmaker semantics are Fine [18,19] both of which, along with the early inspiration for the approach van Fraassen [20], should be consulted for more information on that particular, very fertile, approach to semantic content.

[5] The four-valued matrices in Fig. 8 of Humberstone [21, p. 640] constitute a matrix semantics for the notion of 'partial truth' which Fine discusses just after Theorem 27 (the matrices there being the 'truth tables along these lines' mentioned by Fine), with the proviso that we replace the table for $\wedge$ with a duplicate of that given for $\vee$. We will then have that the consequence relation semantically determined by the four-valued matrices given there is such that whenever $A \vdash C$ we have that A preserves the valence of C. For present purposes it is much simpler to work directly with the notion of preservation of valence, though, as we will see below.

Intuively we can think of $t(A)$ as being the 'subject matter' of A (or perhaps more mnemonically the 'truthmaker content of A'). We can then say that valence is anti-preserved from A to B (meaning by this that the sequent $A \succ B$ is valence anti-preserving) if $t(B) \subseteq t(A)$, and thus think of valence anti-preservation as requiring that the subject matter of B is contained in the subject matter of A. Now we have enough tools in hand in order to give a particular semantic characterisation of the first-degree fragment of Angell's logic AC, due independently to Kit Fine and Thomas Ferguson, which we will be concerned with below.

PROPOSITION 3.2. (Fine [19, p. 223], Ferguson [14, p. 1630]) *$A \rightarrow C$ is a theorem of $\mathbf{AC}_{\rightarrow}$ iff $A \succ C$ is FDE-valid and valence anti-preserving. Equivalently, given the above, iff $A \vdash_{FDE} C$ and $t(C) \subseteq t(A)$.*

The present characterisation of $\mathbf{AC}_{\rightarrow}$ gives us the materials we will use to construct our sequent calculus.

3.1. The Sequent Calculus $\mathcal{G}_{AC}$

The following sequent calculus builds off the structurally absorbed sequent calculus for FDE given in Pynko [26, pp. 446–447]. The main ingredient in our sequent calculus, which we will call $\mathcal{G}_{AC}$ is a restriction on what counts as an initial sequent (which is, in a sense, equivalent to a restriction on the rule of weakening). From here on let us write $t(\Gamma)$ for $\bigcup_{A \in \Gamma} t(A)$. The sequent calculus $\mathcal{G}_{AC}$ for Angell's logic has the following initial sequents and rules:

Initial Sequents

$$[Initial+] \quad \Gamma, p \succ p, \Delta \text{ where } t(\Delta, p) \subseteq t(\Gamma, p).$$
$$[Initial-] \quad \Gamma, \neg p \succ \neg p, \Delta \text{ where } t(\Delta, \neg p) \subseteq t(\Gamma, \neg p).$$

Double Negation

$$\frac{\Gamma, A \succ \Delta}{\Gamma, \neg\neg A \succ \Delta}\,[DNL] \qquad \frac{\Gamma \succ A, \Delta}{\Gamma \succ \neg\neg A, \Delta}\,[DNR]$$

Propositional Rules

$$\frac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta}\,[\wedge L] \qquad \frac{\Gamma \succ A, \Delta \quad \Gamma \succ B, \Delta}{\Gamma \succ A \wedge B, \Delta}\,[\wedge R]$$
$$\frac{\Gamma, A \succ \Delta \quad \Gamma, B \succ \Delta}{\Gamma, A \vee B \succ \Delta}\,[\vee L] \qquad \frac{\Gamma \succ A, B, \Delta}{\Gamma \succ A \vee B, \Delta}\,[\vee R]$$

'De Morgan' Rules

$$\frac{\Gamma, \neg A \succ \Delta \quad \Gamma, \neg B \succ \Delta}{\Gamma, \neg(A \wedge B) \succ \Delta}\ [\neg\wedge L] \qquad \frac{\Gamma \succ \neg A, \neg B, \Delta}{\Gamma \succ \neg(A \wedge B), \Delta}\ [\neg\wedge R]$$

$$\frac{\Gamma, \neg A, \neg B \succ \Delta}{\Gamma, \neg(A \vee B) \succ \Delta}\ [\neg\vee L] \qquad \frac{\Gamma \succ \neg A, \Delta \quad \Gamma \succ \neg B, \Delta}{\Gamma \succ \neg(A \vee B), \Delta}\ [\neg\vee R]$$

Structural Rules

$$\frac{\Gamma, A, A \succ \Delta}{\Gamma, A \succ \Delta}\ [WL] \qquad \frac{\Gamma \succ \Delta, A, A}{\Gamma \succ \Delta, A}\ [WR]$$

Note that if we were to let our initial sequents instead be of the form $\Gamma, A \succ A, \Delta$ we would end up with essentially Pynko's sequent calculus $\mathcal{G}_{\mathbf{B}}$. Throughout we will occasionally refer ambiguiously to the initial sequents $[Initial+]$ and $[Initial-]$ of $\mathcal{G}_{AC}$ as simply $[Initial]$. The reason we refer to the rules $[\neg\wedge]$ and $[\neg\vee]$ as 'De Morgan rules' is that it is the presence of these rules which allow us to derive the usual De Morgan laws in the absence of either the standard 'flip-flop' rules for negation, or a global rule of contraposition (taking us from $\Gamma \succ \Delta$ to $\neg\Delta \succ \neg\Gamma$, where $\neg\Gamma = \{\neg A | A \in \Gamma\}$ as normal). This has the dual advantage of being 'local', allowing us to just have standard left- and right-insertion rules for the various connectives and negated connectives, as well as aligning more closely with our chosen axiom system $\mathbf{AC}_{\rightarrow}$.

The following proposition, in conjunction with Proposition 3.2, demonstrates that the present system captures all the validities of Angell's logic.

Proposition 3.3. *The rules of $\mathcal{G}_{AC}$ all preserve the property of anti-preserving valence. That is to say, if all of the premise sequents $\Gamma_i \succ \Delta_i$ have the property that $t(\Delta_i) \subseteq t(\Gamma_i)$ then the conclusion of that rule also has that property.*

Proof. We proceed by cases, supposing that the premise sequents of a rule satisfy the condition, in order to show that the conclusion sequent of that rule also satisfies the condition.

- $[\neg\neg]$ Given that $t(\neg A) = \{\neg p | p \in t(A)\}$, it is clear that $t(\neg\neg A) = t(A)$, and so it is clear that this rule preserves the property of anti-preserving valence.
- $[\wedge]$ and $[\vee]$ For the case of the one premise rules it is easy to see that, as $t(A \wedge B) = t(A \vee B) = t(A) \cup t(B)$ that if the premise sequent anti-preserves valence then so does the conclusion sequent. For the cases of the two premise rules, $[\wedge R]$ to pick the more awkward case, that if

$t(A, \Delta) \subseteq t(\Gamma)$ and $t(B, \Delta) \subseteq t(\Gamma)$ then $t(A \wedge B, \Delta) = t(A, \Delta) \cup t(B, \Delta)$ is a subset of $t(\Gamma)$ as desired.

- $[\neg\wedge]$ and $[\neg\vee]$ Similar to the previous case.
- $[W]$ The case for the rules $[WL]$ and $[WR]$ is trivial, as $t(A, \Gamma) = t(A, A, \Gamma)$, and likewise $t(A, \Delta) = t(A, A, \Delta)$. ■

COROLLARY 3.4. *If a sequent $\Gamma \succ \Delta$ is provable in $\mathcal{G}_{AC}$ then the sequent anti-preserves valence.*

PROOF. By induction on the length of derivations, noting for the basis case that all of our Initial sequents have the property, and using Proposition 3.3 in the inductive step. ■

Note that by the above corollary and the admissibility of Cut in $\mathcal{G}_{AC}$ (proved in the appendix) that it follows that $[Cut]$ also preserves the property of anti-preserving valence.

From Corollary 3.4 it is quite clear that if $A \rightarrow B$ is valid in $\mathbf{AC}_{\rightarrow}$ then the sequent $A \succ B$ is valid in $\mathcal{G}_{AC}$. To prove the converse of this implication we will show that whenever a sequent $A \succ B$ is provable in $\mathcal{G}_{AC}$ then $A \rightarrow B$ is valid in $\mathbf{AC}_{\rightarrow}$. Prior to doing this, though, it will be helpful to prove some proof-theoretic lemmata.

3.2. Some Proof-Theoretical Lemmata

Throughout we will be lax concerning association and ordering of conjuncts and disjuncts, occasionally flagging where such operations need to be attended to as needed.

THEOREM 3.5. *The rule of cut*

$$\frac{\Gamma \succ \Delta, A \quad A, \Gamma' \succ \Delta'}{\Gamma, \Gamma' \succ \Delta, \Delta'}\ [Cut]$$

is admissible in $\mathcal{G}_{AC}$.

The proof of this theorem can be found in the appendix of the present paper.

THEOREM 3.6. *The following rules are admissible in $\mathcal{G}_{AC}$*

$$\frac{\Gamma \succ \Delta}{\Gamma \succ \Delta, A}\ [KL]^{\dagger} \qquad \frac{\Gamma \succ \Delta}{\Gamma, A \succ \Delta}\ [KR]$$

($\dagger$) with the proviso that $t(A) \subseteq t(\Gamma)$.

PROOF. Suppose that we have a derivation $\mathfrak{D}$ of height n of $\Gamma \succ \Delta$. Adding an A to the left hand side of all the sequents in $\mathfrak{D}$ will give us a derivation of $\Gamma, A \succ \Delta$ of the same length. Similarly, if we have $t(A) \subseteq t(\Gamma)$ then we can add Γ to the LHS and A to the RHS of all the sequents in $\mathfrak{D}$, the side condition guaranteeing that we still have valid instances of $[Initial]$. This will result in a derivation $\mathfrak{D}^*$ whose endsequent is $\Gamma, \Gamma \succ \Delta, A$. We can then apply $|\Gamma|$-many applications of contraction in order to conclude $\Gamma \succ \Delta, A$, as desired.[6] ■

Given the admissibility of this rule, it will be helpful to use this rule to derive the following result.[7]

DEFINITION 3.7. Given a formula A, let the *De Morgan Complexity* $dmc(A)$ be defined as follows.

- $dmc(p) = 0$
- $dmc(\neg p) = 0$
- $dmc(A \wedge B) = dmc(\neg(A \wedge B)) = 1 + dmc(A) + dmc(B)$
- $dmc(A \vee B) = dmc(\neg(A \vee B)) = 1 + dmc(A) + dmc(B)$
- $dmc(\neg\neg A) = 1 + dmc(A)$

[6]This means, unfortunately, that we do not have height-preserving admissibility, but we do have a precise upper bound on the height of the new proof–namely the height of the initial proof + $|\Gamma|$.

[7]The admissibility of $[KL]$ and $[KR]$ also allows us to show that we could equally well have used context independent rather than context sharing operational rules. For example anything we can prove using $[\neg \wedge L]$ we can also prove using:

$$\frac{\Gamma, \neg A \succ \Delta \quad \Gamma', \neg B \succ \Delta'}{\Gamma, \Gamma', \neg(A \wedge B) \succ \Delta, \Delta'} \;[\neg\wedge L^*]$$

by just contracting the duplicated occurences of Γ and Δ. Similarly, we can derive $[\neg \wedge L^*]$ in $\mathcal{G}_{AC}$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \neg A \succ \Delta}{\Gamma, \Gamma', \neg(A \wedge B), \neg A \succ \Delta}\;[KL]}{\Gamma, \Gamma', \neg(A \wedge B), \neg A \succ \Delta, \Delta'}\;[KR] \quad \dfrac{\dfrac{\Gamma', \neg B \succ \Delta'}{\Gamma, \Gamma', \neg(A \wedge B), \neg B \succ \Delta'}\;[KL]}{\Gamma, \Gamma', \neg(A \wedge B), \neg B \succ \Delta, \Delta'}\;[KR]}{\Gamma, \Gamma', \neg(A \wedge B), \neg(A \wedge B) \succ \Delta, \Delta'}\;[\neg\wedge L]}{\Gamma, \Gamma', \neg(A \wedge B) \succ \Delta, \Delta'}\;[WL]$$

Note that, in the case of multi-premise rules which introduce connectives on the left we have to weaken in the formula which we are about to introduce in order to ensure that we can apply $[KR]$.

In showing that our sequent calculus $\mathcal{G}_{AC}$ and Fine's system $\mathbf{AC}_{\rightarrow}$ are equivalent it will be helpful to make use of a slight variant of the system $\mathcal{G}_{AC}$ which restricts the available initial sequents even further.

Definition 3.8. Let the system $\mathcal{G}^{-}_{AC}$ result from replacing $[Initial+]$ and $[Initial-]$ in $\mathcal{G}_{AC}$ with $[LiteralInitial]$.

$$[LiteralInitial] \qquad q^*_1, \ldots, q^*_m, p^*_1, \ldots, p^*_n \succ p^*_1, \ldots, p^*_n$$

where each p^*_i, q^*_i are literals.

Lemma 3.9. *Any sequent that is provable in $\mathcal{G}_{AC}$ is provable in the system $\mathcal{G}^{-}_{AC}$.*

Proof. Suppose that we have an instance of $[Initial]$

$$\Gamma, p \succ p, \Delta$$

where $t(\Delta) \subseteq t(\Gamma, p)$. What we will first show is that we can replace Δ with a set of literals. To do this we proceed by induction on the sum of the De Morgan complexities of formulas in Δ. If this is zero, then every formula in Δ is a literal, as desired. Now suppose that the result holds for a summed De Morgan complexity of $\leq n$, and suppose that our instance of $[Initial]$ is of one of the following forms (all other cases being similar to these)

$(i)\ \Gamma, p \succ p, \Delta, A \vee B$; $\qquad (ii)\ \Gamma, p \succ p, \Delta, A \wedge B$; $\qquad (iii)\ \Gamma, p \succ p, \Delta, \neg\neg A$

where in all cases the RHS of the sequent is of De Morgan complexity $n+1$. In each case we can derive each of these instances of $[Initial]$ from ones whose RHS of of lower De Morgan complexity, as follows:

$$(i)\ \frac{\Gamma, p \succ p, \Delta, A, B}{\Gamma, p \succ p, \Delta, A \vee B} \qquad (ii)\ \frac{\Gamma, p \succ p, \Delta, A \quad \Gamma, p \succ p, \Delta, B}{\Gamma, p \succ p, \Delta, A \wedge B} \qquad (iii)\ \frac{\Gamma, p \succ p, \Delta, A}{\Gamma, p \succ p, \Delta, \neg\neg A}$$

Given that the upper sequents all have RHSs of at least one lower De Morgan complexity, by the induction hypothesis it follows that they can be derived from instances of initial where Δ contains only literals.

From this it follows that we can replace all instances of $[Initial]$ with instances of the following sequent schemata.

$$\Gamma, p \succ p, p^*_1, \ldots, p^*_n.$$

Now all that needs to be shown is that we can replace Γ with a set of literals. Again we proceed by induction on the sum of the De Morgan complexities of formulas in Γ. If this is zero, then every formula in Γ is a literal, as desired. Now suppose that the result holds for a summed De Morgan complexity of

$\leq n$, and suppose that our instance of $[Initial]$ is of one of the following forms (all other cases being similar to these)

$(i')\ \Gamma, A \wedge B, p \succ p, \Delta$; $\quad (ii')\ \Gamma, A \vee B, p \succ p, \Delta$; $\quad (iii')\ \Gamma, \neg\neg A, p \succ p, \Delta$

Each of these cases can be derived from sequents of lower De Morgan complexity as follows.

$$(i')\ \frac{\Gamma, A, B \succ p_1^*, \ldots, p_n^*}{\Gamma, A \wedge B \succ p_1^*, \ldots, p_n^*}\ [\wedge L]$$

$$(ii')\ \dfrac{\dfrac{\dfrac{\Gamma, A \succ q_1^*, \ldots, q_m^*}{\Gamma, A \vee B, A \succ q_1^*, \ldots, q_m^*}\ [KL]}{\Gamma, A \vee B, A \succ p_1^*, \ldots, p_n^*}\ [KR] \quad \dfrac{\dfrac{\Gamma, B \succ r_1^*, \ldots, r_k^*}{\Gamma, A \vee B, B \succ r_1^*, \ldots, r_k^*}\ [KL]}{\Gamma, A \vee B, B \succ p_1^*, \ldots, p_n^*}\ [KR]}{\dfrac{\Gamma, A \vee B, A \vee B \succ p_1^*, \ldots, p_n^*}{\Gamma, A \vee B \succ p_1^*, \ldots, p_n^*}\ [WL]}\ [\vee L]$$

where $\{q_1^*, \ldots, q_m^*\} = t(A, \Gamma) \cap \{p_1^*, \ldots, p_n^*\}$ and $\{r_1^*, \ldots, r_k^*\} = t(B, \Gamma) \cap \{p_1^*, \ldots, p_n^*\}$

$$(iii')\ \frac{\Gamma, A, p \succ p, \Delta}{\Gamma, \neg\neg A, p \succ p, \Delta}\ [DNL]$$

■

4. Adequacy

In order to show that $\mathcal{G}_{AC}$ is equivalent to $\mathbf{AC}_{\rightarrow}$ it will be helpful to appeal to the following derived rules of $\mathbf{AC}_{\rightarrow}$.

LEMMA 4.1. *The following rules are derivable in* $\mathbf{AC}_{\rightarrow}$*:*

$$\frac{A \rightarrow B \quad C \rightarrow D}{(A \wedge C) \rightarrow (B \wedge D)}\ R_1 \qquad \frac{A \rightarrow B}{A \wedge C \rightarrow B \vee C}\ R_2 \qquad \frac{A \rightarrow B \quad C \rightarrow D}{(A \vee C) \rightarrow (B \vee D)}\ R_3$$

PROOF. We begin by deriving R_1

$$\dfrac{\dfrac{A \rightarrow B}{(A \wedge C) \rightarrow (B \wedge C)}\ (E14) \quad \dfrac{\dfrac{\dfrac{}{(B \wedge C) \rightarrow (C \wedge B)}\ (E3) \quad \dfrac{C \rightarrow D}{(C \wedge B) \rightarrow (D \wedge B)}\ (E14)}{(B \wedge C) \rightarrow (D \wedge B)}\ (E13) \quad \dfrac{}{(D \wedge B) \rightarrow (B \wedge D)}\ (E3)}{(B \wedge C) \rightarrow (B \wedge D)}\ (E13)}{(A \wedge C) \rightarrow (B \wedge D)}\ (E13)$$

We next derive R_3 so that we can use it in deriving R_2.

$$\dfrac{\dfrac{A \rightarrow B}{(A \vee C) \rightarrow (B \vee C)}\ (E15) \quad \dfrac{\dfrac{\dfrac{}{(B \vee C) \rightarrow (C \vee B)}\ (E3) \quad \dfrac{C \rightarrow D}{(C \vee B) \rightarrow (D \vee B)}\ (E15)}{(B \vee C) \rightarrow (D \vee B)}\ (E13) \quad \dfrac{}{(D \vee B) \rightarrow (B \vee D)}\ (E3)}{(B \vee C) \rightarrow (B \vee D)}\ (E13)}{(A \vee C) \rightarrow (B \vee D)}\ (E13)$$

Finally we can make use of R_3 in order to derive R_2

$$\dfrac{\dfrac{}{(A \wedge C) \vee (A \wedge C) \to A \wedge C}\,E5a \qquad \dfrac{\dfrac{\dfrac{A \to B}{A \wedge C \to B \wedge C}\,(E14) \qquad \dfrac{}{B \wedge C \to B}\,(E12)}{A \wedge C \to B}\,(E13) \qquad \dfrac{}{A \wedge C \to C}\,(E12)}{(A \wedge C) \vee (A \wedge C) \to B \vee C}\,R_3}{A \wedge C \to B \vee C}\,(E13)$$

■

As the reader will note the proofs that witness the derivability of these rules in $\mathbf{AC}_{\to}$ are made rather tiresome by the fact that the rules $(E14)$ and $(E15)$ only insert their new conjunct/disjunct on the right, while we will have occasion below to want to insert the additional conjunct/disjunct on the left. To make things easier on the reader we will henceforth make use of the following two derived rules of $\mathbf{AC}_{\to}$.

$$\dfrac{A \to B}{(C \wedge A) \to (C \wedge B)}\,(E14)^l \qquad \dfrac{A \to B}{(C \vee A) \to (C \vee B)}\,(E15)^l$$

Instances of these rules are, of course, derivable by applying $(E15)$, and then applying $(E13)$ to the result along with an appropriate instance of $(E3)$.

What we will now show is that if a sequent $\Gamma \succ \Delta$ is provable in $\mathcal{G}^-_{AC}$, then $\bigwedge \Gamma \to \bigvee \Delta$ is provable in $\mathbf{AC}_{\to}$, and so given that $\Gamma \succ \Delta$ is provable in $\mathcal{G}^-_{AC}$ iff $\Gamma \succ \Delta$ is provable in $\mathcal{G}_{AC}$, it will follow that provability in $\mathcal{G}_{AC}$ and provability in $\mathbf{AC}_{\to}$ coincide.

Theorem 4.2. *Suppose that $\Gamma \succ \Delta$ is provable in $\mathcal{G}^-_{AC}$. Then the formula $\bigwedge \Gamma \to \bigvee \Delta$ is provable in $\mathbf{AC}_{\to}$.*

Proof. By induction on the length of derivations in $\mathcal{G}^-_{AC}$.
Basis: suppose that $\Gamma \succ \Delta$ is an instance of $[Literal - Initial]$, that is to say we have:

$$q^*_1, \ldots, q^*_m, p^*_1, \ldots, p^*_n \succ p^*_1, \ldots, p^*_n,$$

In order to prove this we first begin with $p^*_n \to p^*_n$, which can easily be proved in $\mathbf{AC}_{\to}$. We then repeatedly apply R_2, reassociating as required, starting with this until we have

$$(p^*_1 \wedge \ldots \wedge p^*_n) \to (p^*_1 \vee \ldots \vee p^*_n)$$

We can now appeal to $(E12)$ and $(E13)$ as follows:

$$\dfrac{\dfrac{}{(q^*_1 \wedge \ldots \wedge q^*_m) \wedge (p^*_1 \wedge \ldots \wedge p^*_n) \to p^*_1 \wedge \ldots \wedge p^*_n}\,(E12) \qquad (p^*_1 \wedge \ldots \wedge p^*_n) \to (p^*_1 \vee \ldots \vee p^*_n)}{q^*_1 \wedge \ldots \wedge q^*_m \wedge p^*_1 \wedge \ldots \wedge p^*_n \to p^*_1 \vee \ldots \vee p^*_n}\,(E13)$$

Inductive Hypothesis: for all proofs of sequents $\Gamma' \succ \Delta'$ of length $\leq n$ we have that $\bigwedge \Gamma' \to \bigvee \Delta'$ is provable in $\mathbf{AC}_{\to}$.

Inductive Step: we give a representative sample of cases. Throughout we will write γ for $\bigwedge \Gamma$ and δ for $\bigvee \Delta$ in the interests of brevity and clarity.

$[DNL]$ Suppose that we have a derivation of length $\leq n$ of the sequent $\Gamma, A \succ \Delta$, and wish to apply the rule $[DNL]$ to derive the sequent $\Gamma, \neg\neg A \succ \Delta$. By the inductive hypothesis it follows that $\gamma \wedge A \to \delta$ is provable in $\mathbf{AC}_{\to}$. We derive $\gamma \wedge \neg\neg A \to \delta$ from this as follows

$$\dfrac{\dfrac{\dfrac{}{\neg\neg A \to A}\,(E1b)}{\gamma \wedge \neg\neg A \to \gamma \wedge A}\,(E14)^l \qquad \gamma \wedge A \to \delta}{\gamma \wedge \neg\neg A \to \delta}\,(E13)$$

$[\wedge L]$ Suppose that we have a derivation of length $\leq n$ of the sequent $\Gamma, A, B \succ \Delta$ and wish to apply the rule $[\wedge L]$ to derive the sequent $\Gamma, A \wedge B \succ \Delta$. By the inductive hypothesis it follows that $\gamma \wedge A \wedge B \to \delta$ is provable in $\mathbf{AC}_{\to}$, and so the desired sequent follows by appropriate re-association (i.e. applications of $(E4a)$ and $(E4b)$).

$[\vee L]$ Suppose that we have a derivation of length $\leq n$ of the sequent $\Gamma, A \succ \Delta$ and one of length $\leq n$ of the sequent $\Gamma, B \succ \Delta$ and wish to apply the rule $[\vee L]$ to derive the sequent $\Gamma, A \vee B \succ \Delta$. By the induction hypothesis it follows that we have proofs in $\mathbf{AC}_{\to}$ of $\gamma \wedge A \to \delta$ and $\gamma \wedge B \to \delta$.

$$\dfrac{\dfrac{\dfrac{}{\gamma \wedge (A \vee B) \to (\gamma \wedge A) \vee (\gamma \wedge B)}\,(E10a) \qquad \dfrac{\gamma \wedge A \to \delta \quad \gamma \wedge B \to \delta}{(\gamma \wedge A) \vee (\gamma \wedge B) \to \delta \vee \delta}\,R_3}{\gamma \wedge (A \vee B) \to \delta \vee \delta}\,(E13) \qquad \dfrac{}{\delta \vee \delta \to \delta}\,(E5b)}{\gamma \wedge (A \vee B) \to \delta}\,(E13)$$

$[\neg \wedge L]$ Suppose that we have a derivation of length $\leq n$ of the sequent $\Gamma, \neg A \succ \Delta$ and one of length $\leq n$ of the sequent $\Gamma, \neg B \succ \Delta$ and wish to apply the rule $[\neg \wedge L]$ to derive the sequent $\Gamma, \neg(A \wedge B) \succ \Delta$. By the induction hypothesis it follows that we have proofs in $\mathbf{AC}_{\to}$ of $\gamma \wedge \neg A \to \delta$ and $\gamma \wedge \neg B \to \delta$.

$$\dfrac{\dfrac{\dfrac{}{\neg(A \wedge B) \to (\neg A \vee \neg B)}\,(E8a)}{\gamma \wedge \neg(A \wedge B) \to \gamma \wedge (\neg A \vee \neg B)}\,(E14)^l \qquad \dfrac{\dfrac{\dfrac{}{\gamma \wedge (\neg A \vee \neg B) \to (\gamma \wedge \neg A) \vee (\gamma \wedge \neg B)}\,(E10a) \qquad \dfrac{\gamma \wedge \neg A \to \delta \quad \gamma \wedge \neg B \to \delta}{(\gamma \wedge \neg A) \vee (\gamma \wedge \neg B) \to \delta \vee \delta}\,R_3}{\gamma \wedge (\neg A \vee \neg B) \to \delta \vee \delta}\,(E13) \qquad \delta \vee \delta \to \delta}{\gamma \wedge (\neg A \vee \neg B) \to \delta}\,(E13)}{\gamma \wedge \neg(A \wedge B) \to \delta}\,(E13)$$

$[\wedge R]$ Suppose that we have a derivation of length $\leq n$ of the sequent $\Gamma \succ \Delta, A$ and one of length $\leq n$ of the sequent $\Gamma \succ \Delta, B$ and wish to apply the rule $[\wedge R]$ to derive the sequent $\Gamma \succ \Delta, A \wedge B$. By the induction hypothesis it follows that we have proofs in $\mathbf{AC}_{\rightarrow}$ of $\gamma \rightarrow \delta \vee A$ and $\gamma \rightarrow \delta \vee B$.

$$\dfrac{\dfrac{\dfrac{}{\gamma \wedge \gamma \rightarrow \gamma}\,(E12) \quad \dfrac{\gamma \rightarrow \delta \vee A \quad \gamma \rightarrow \delta \vee B}{\gamma \wedge \gamma \rightarrow (\delta \vee A) \wedge (\delta \vee B)}\,R1}{\gamma \rightarrow (\delta \vee A) \wedge (\delta \vee B)}\,(R13) \quad \dfrac{}{(\delta \vee A) \wedge (\delta \vee B) \rightarrow \delta \vee (A \wedge B)}\,(E11b)}{\gamma \rightarrow \delta \vee (A \wedge B)}\,(R13)$$

[$\neg \vee L$] Suppose that we have a derivation of length $\leq n$ of the sequent $\Gamma, \neg A, \neg B \succ \Delta$ and wish to apply the rule [$\neg \vee L$] to derive the sequent $\Gamma, \neg(A \vee B) \succ \Delta$. By the induction hypothesis we have that the formula $\gamma \wedge \neg A \wedge \neg B \rightarrow \delta$ is derivable in $\mathbf{AC}_{\rightarrow}$. By re-association, then, we can derive the sequent $\gamma \wedge (\neg A \wedge \neg B) \rightarrow \delta$, and the using $(E13)$ and $(E14)$ applied to an instance of $\neg(A \vee B) \rightarrow (\neg A \wedge \neg B)$ in a similar way to that used above, derive $\gamma \wedge \neg(A \vee B) \rightarrow \delta$ as desired.

[WL] Suppose that we have a derivation of length $\leq n$ of the sequent $\Gamma, A, A \succ \Delta$ and we wish to apply [WL] to derive the sequent $\Gamma, A \succ \Delta$. By the induction hypothesis we have that the formula $\gamma \wedge A \wedge A \rightarrow \delta$ is derivable in $\mathbf{AC}_{\rightarrow}$. We can derive our conclusion as follows.

$$\dfrac{\dfrac{\dfrac{}{A \rightarrow A \wedge A}\,(E2)}{\gamma \wedge A \rightarrow \gamma \wedge A \wedge A}\,(E14)^{l} \qquad \gamma \wedge A \wedge A \rightarrow \delta}{\gamma \wedge A \rightarrow \delta}\,(E13)$$

[WR] Suppose that we have a derivation of length $\leq n$ of the sequent $\Gamma \succ \Delta, A, A$ and we wish to apply [WR] to derive the sequent $\Gamma \succ \Delta, A$. By the induction hypothesis we have that the formula $\gamma \rightarrow \delta \vee A \vee A$ is derivable in $\mathbf{AC}_{\rightarrow}$. We can derive our conclusion as follows.

$$\dfrac{\gamma \rightarrow \delta \vee A \vee A \quad \dfrac{\dfrac{}{A \vee A \rightarrow A}\,(E5b)}{\delta \vee A \vee A \rightarrow \delta \vee A}\,(E15)^{l}}{\gamma \rightarrow \delta \vee A}\,(E13)$$

■

5. Conclusion

Here we have provided a simple cut-free sequent calculus for Angell's logic of analytic containment, making explicit the logic's strong resemblance to FDE and the importance of the valence anti-preservation condition. One opportunity which the present approach to thinking about Angell's logic opens up is to use the technique we have used here—restricting initial sequents and showing that rules anti-preserve valence in the appropriate way—in order to give an account of partial content for more expressive languages. For example, how should we think about partial content for first-order languages? An obvious tactic for pursuing this is to decide what $t(\forall xFx)$ is to be, and then use this information, along with which quantificational inferences are valid in first-order FDE to determine the logic of partial content for first-order languages. One plausible treatment of $t(\forall xA(x))$ is to have it consist of t applied to every instance of $A(x)$. This proposal, an analogue of the second proposal in Fine [15, p. 178], suggests the following, infinitary, rules for the quantificational extension of $\mathcal{G}_{AC}$:

$$\frac{\Gamma \succ \Delta, A(a_1), \ldots, A(a_n), \ldots}{\Gamma \succ \Delta, \forall xA(x)} \qquad \frac{A(t), \Gamma \succ \Delta}{\forall xA(x), \Gamma \succ \Delta}$$

The main intuition driving these rules (which is on the surface of a very similar nature to those motivating the grounding conditions for the quantifiers given in Fine [16]), is that any instance $A(t)$ ought to be part of the content of $\forall xA(x)$, while $\forall xA(x)$ ought only be part of the content of the collection of all possible instances of $A(x)$ (perhaps following discussions in Fine [16, p. 61], along with a 'totality' clause). One potential problem with this account is that, if knowing a statement requires knowing its content, then this appears to suggest that knowing a universally quantified claim requires possession of names for all the objects int he domain of the quantifier. We will leave further investigation of what the best treatment of partial content for quantificational logic is for another occasion, though.

Acknowledgements. I'd like to thank the audience at the 2016 Australasian Association for Logic conference in Melbourne, as well as Lloyd Humberstone, Johannes Korbmacher, Dave Ripley, and Shawn Standefer for helpful discussions of this material. A special thanks is due to an anonymous referee for this journal who provided a number of extremely helpful comments.

Appendix: Cut Elimination

The purpose of this appendix is to show that the rule of cut

$$\frac{\Gamma \succ \Delta, A \quad A, \Gamma' \succ \Delta'}{\Gamma, \Gamma' \succ \Delta, \Delta'} \ [Cut]$$

is admissible in $\mathcal{G}_{AC}$. Despite the way in which we have presented our sequent calculus, we cannot show that Contraction is height-preserving admissible in $\mathcal{G}_{AC}$, and as a result we cannot use a cut-elimination proof in the style of Dragalin [12] or Negri and von Plato [22]. Moreover we do not have an unrestricted rule of weakening, and so cannot use a standard mix-elimination proof. Instead, we will use a variant of the mix rule, called ‘intelligent-mix’ in Paoli [23, p. 97], and show how to eliminate all instances of this rule. *Intelligent Mix*, or $[IntMix]$, is the rule

$$\frac{\Gamma \succ \Delta, A^n \quad A^m, \Gamma' \succ \Delta'}{\Gamma, \Gamma' \succ \Delta, \Delta'}\,{\scriptstyle [IntMix],A}$$

where $n, m \geq 1$ and other occurences of A, the *intmix formula*, may appear in Δ and Γ'.[8] In particular what we will show is that if a sequent is provable in $\mathcal{G}_{AC}+[IntMix]$ (henceforth $\mathcal{G}_{AC}^{im}$) then it is provable in $\mathcal{G}_{AC}$. This has as a consequence that Cut is eliminable in $\mathcal{G}_{AC} + [Cut]$, as $[IntMix]$ and $[Cut]$ are interderivable rules in our system, $[IntMix]$ being derivable from $[Cut]$ as shown below, and $[Cut]$ simply being the special case of $[IntMix]$ where $n = m = 1$.

$$\frac{\dfrac{\Gamma \succ \Delta, A^n}{\Gamma \succ \Delta, A}\,{\scriptstyle [WR]^n} \quad \dfrac{A^m, \Gamma' \succ \Delta'}{A, \Gamma' \succ \Delta'}\,{\scriptstyle [WL]^m}}{\Gamma, \Gamma' \succ \Delta, \Delta'}\,{\scriptstyle [Cut]}$$

Here we will largely follow the proof alluded to in Paoli [23, pp. 97–98], with the obvious modifications made for the present system. Before getting to the central proof, though, we will first need a few preliminary definitions (all of which are analogues of those give in Paoli [23, pp. 88–90]).

Definition 5.1. A derivation $\mathfrak{D}$ in $\mathcal{G}_{AC}^{im}$ is called an *intmix proof* iff it contains a single application of $[IntMix]$, whose conclusion is the endsequent of the derivation, and is called a *intmix free proof* if it contains no application of $[IntMix]$.

Proposition 5.2. (Circumscription of Cut Elimination) *If in $\mathcal{G}_{AC}^{im}$ any intmix proof can be transformed into an intmix free proof of the same sequent, then any arbitrary proof can be transformed into an intmix free proof.*

Proof. A minor modification of the proof of Proposition 3.2 from [23, p. 89]. ■

Definition 5.3. Let $\mathfrak{D}$ be a intmix proof whose final inference is

$$\frac{\Gamma \succ \Delta, A^n \quad A^m, \Gamma' \succ \Delta'}{\Gamma, \Gamma'^{-} \succ \Delta^{-}, \Delta'}\,{\scriptstyle [IntMix],A}$$

The *rank of a sequent S in $\mathfrak{D}$*, $r_{\mathfrak{D}}(S)$ is given as follows.

- If S belongs to a subproof of $\mathfrak{D}'$ or $\mathfrak{D}$ which has $\Gamma \succ \Delta, A^n$ as its endsequent, then $r_{\mathfrak{D}}(S)$ is one less the maximal length of of an upward path of sequents

[8] In Troelstra and Schwichtenberg [27, p. 82] this rule is called *Multicut*, and it appears to be suggested that this is Gentzen’s rule of Mix. Gentzen’s rule, though, does not allow for other occurrences of the removed formula to be present in the conclusion sequent, removing all occurences of the principal formula, rather than merely some of them.

$S_1, \ldots, S_n$ s.t. $S_n = S$ and each S_i contains at least one copy of A in the succedent.

- Symmetrically for the case in which S belongs to a subproof with $A^m, \Gamma' \succ \Delta'$ as its endsequent, but with succedent replaced by antecedent.
- $r_{\mathfrak{D}}(\Gamma, \Gamma'^- \succ \Delta^-, \Delta') = r_{\mathfrak{D}}(\Gamma \succ \Delta, A^n) + r_{\mathfrak{D}}(A^m, \Gamma' \succ \Delta')$

Definition 5.4. Given an intmix proof $\mathfrak{D}$ with intmix formula A whose endsequent is $\Gamma \succ \Delta$ the *index* of $\mathfrak{D}$ is an ordered pair $i(\mathfrak{D}) = \langle dmc(A), r_{\mathfrak{D}}(\Gamma \succ \Delta)\rangle$

Theorem 5.5. *Any intmix proof in* $\mathcal{G}^{im}_{AC}$ *can be transformed into an intmix free proof of the same sequent.*

Proof. Let $\mathfrak{D}$ be an intmix proof whose final inference is:

$$\frac{\Gamma \succ \Delta, A^n \quad A^m, \Gamma' \succ \Delta'}{\Gamma, \Gamma' \succ \Delta, \Delta'}\ {\scriptstyle [IntMix], A}$$

we proceed by induction on $i(\mathfrak{D})$, the index of $\mathfrak{D}$.

1. $[i(\mathfrak{D}) = \langle 0, 0\rangle]$ As $dmc(A) = 0$ it follows that A is a literal, and thus as $r(\mathfrak{D}) = 0$ it follows that $\Gamma \succ \Delta, A^n$ is an instance of an $[Initial]$. Then there are two subcases.
   (a) The intmix formula, A is in Γ. In this case we can derive $\Gamma, \Gamma' \succ \Delta, \Delta'$ from the right premise $A, \Gamma' \succ \Delta'$ by making use of the admissibility of $[KL]$ and $[KR]$, first repeatedly applying $[KL]$ to obtain $\Gamma, \Gamma' \succ \Delta'$ and then weakening in Δ using $[KR]$, noting that the side-condition is satisfied as $\Gamma \succ \Delta, A$ is by hypothesis derivable, and thus that $t(\Delta) \subseteq t(\Gamma)$ and hence $t(\Delta) \subseteq t(\Gamma, \Gamma')$.
   (b) Γ and Δ share a common literal. In this case we have that $\Gamma, \Gamma' \succ \Delta, \Delta'$ is an initial sequent, the valence anti-preservation requirement being taken care of by the fact that $\Gamma \succ \Delta, A$ is derivable and thus (i) $t(A) \subseteq t(\Gamma)$ and $t(\Delta') \subseteq t(\Gamma') \cup t(A)$ and so $t(\Delta') \subseteq t(\Gamma, \Gamma')$ as required.
2. $[i(\mathfrak{D}) = \langle 0, k\rangle,\ 1 \leq k]$ If $r(\mathfrak{D}) = k > 0$ then either $r_{\mathfrak{D}}(\Gamma \succ \Delta, p^{*n}) > 0$ or $r_{\mathfrak{D}}(p^{*m}, \Gamma' \succ \Delta')$, with our intmix formula p^* a literal. There are two subcases.
   (a) $[r_{\mathfrak{D}}(\Gamma \succ \Delta, p^{*n}) > 0]$ In this case we know that $\Gamma \succ \Delta, A^n$ is the conclusion of an inference where p^* is either a principal formula, or a side-formula. In the case where it is a side formula we 'push upward' occurences of $[IntMix]$, reducing the rank of the intmx in order to appeal to the induction hypothesis. We give some representative cases:
      i. Derived using $[\wedge L]$, giving us:

$$\frac{\dfrac{\Gamma, B, C \succ \Delta, p^{*n}}{\Gamma, B \wedge C \succ \Delta, p^{*n}}\ {\scriptstyle [\wedge L]} \qquad p^{*m}, \Gamma' \succ \Delta'}{\Gamma, \Gamma', B \wedge C \succ \Delta, \Delta'}\ {\scriptstyle [IntMix], p^*}$$

which is transformed to

$$\dfrac{\dfrac{\Gamma, B, C \succ \Delta, p^{*n} \quad p^{*m}, \Gamma' \succ \Delta'}{\Gamma, \Gamma', B, C \succ \Delta, \Delta'}\ [IntMix], p^*}{\Gamma, \Gamma', B \wedge C \succ \Delta, \Delta'}\ [\wedge L]$$

ii, Derived using $[\neg \vee R]$, giving us:

$$\dfrac{\dfrac{\Gamma \succ \Delta, \neg B, p^{*n} \quad \Gamma \succ \Delta, \neg C, p^{*n}}{\Gamma \succ \Delta, \neg(B \vee C), p^{*n}}\ [\neg\vee R] \qquad p^{*m}, \Gamma' \succ \Delta'}{\Gamma, \Gamma' \succ \Delta, \Delta', \neg(B \vee C)}\ [IntMix], p^*$$

which is transformed into the following derivation

$$\dfrac{\dfrac{\Gamma \succ \Delta, \neg B, p^{*n} \quad p^{*m}, \Gamma' \succ \Delta'}{\Gamma, \Gamma' \succ \Delta, \Delta', \neg B}\ [IntMix], p^* \qquad \dfrac{\Gamma \succ \Delta, \neg C, p^{*n} \quad p^{*m}, \Gamma' \succ \Delta'}{\Gamma, \Gamma' \succ \Delta, \Delta', \neg C}\ [IntMix], p^*}{\Gamma, \Gamma' \succ \Delta, \Delta', \neg(B \vee C)}\ [\neg\vee R]$$

iii. Derived using $[\neg\neg L]$, giving us:

$$\dfrac{\dfrac{\Gamma, B \succ \Delta, p^{*n}}{\Gamma, \neg\neg B \succ \Delta, p^{*n}}\ [\neg\neg L] \qquad p^{*m}, \Gamma' \succ \Delta'}{\Gamma, \Gamma', \neg\neg B \succ \Delta, \Delta'}\ [IntMix], p^*$$

which is transformed to

$$\dfrac{\dfrac{\Gamma, B \succ \Delta, p^{*n} \quad p^{*m}, \Gamma' \succ \Delta'}{\Gamma, \Gamma', B \succ \Delta, \Delta'}\ [IntMix], p^*}{\Gamma, \Gamma', \neg\neg B \succ \Delta, \Delta'}\ [\neg\neg L]$$

If p^* is principal then it must have resulted from an application of $[WR]$ yielding:

$$\dfrac{\dfrac{\Gamma \succ \Delta, p^{*n}, p^*}{\Gamma \succ \Delta, p^{*n}}\ [WR] \qquad p^{*m}, \Gamma' \succ \Delta'}{\Gamma, \Gamma' \succ \Delta, \Delta'}\ [IntMix], p^*$$

which is transformed into

$$\dfrac{\Gamma \succ \Delta, p^{*n+1} \quad p^{*m}, \Gamma' \succ \Delta'}{\Gamma, \Gamma' \succ \Delta, \Delta'}\ [IntMix], p^*$$

reducing the rank of the intmix by one.

(b) $[r_{\mathfrak{D}}(p^{*m}, \Gamma' \succ \Delta') > 0]$ Dealt with in a symmetric manner to those above.

3. $[i(\mathfrak{D}) = \langle k, 0\rangle,\ 1 \leq k]$ Since $r(\mathfrak{D}) = 0$ we have two possible cases: either one of the premises is an instance of $[Initial]$, or the intmix formula A is principal in the inferences whose conclusions are the premises of the IntMix in $\mathfrak{D}$, neither of which result from contraction. Let us first quickly deal with the first case. If both sequents are instances of $[Initial]$ we proceed as in step 1. Otherwise, one of the sequents is the result of an inference, and the intmix formula is principal

in that inference, meaning we have a case with one of the following forms:

$$\dfrac{\begin{matrix}\mathfrak{D}_l\\ \vdots\\ \Gamma \succ \Delta, A\end{matrix} \quad A, \Gamma', p^* \succ p^*, \Delta'}{\Gamma, \Gamma', p^* \succ p^*, \Delta, \Delta'}\,[IntMix], A \qquad \dfrac{\Gamma', p^* \succ p^*, \Delta', A \quad \begin{matrix}\mathfrak{D}_r\\ \vdots\\ A, \Gamma \succ \Delta\end{matrix}}{\Gamma, \Gamma', p^* \succ p^*, \Delta, \Delta'}\,[IntMix], A$$

In both cases it is easy to verify that the conclusion sequents of both Intmix proofs are instances of $[Initial]$,[9] allowing us to eliminate the Intmix.

Suppose now, then, that the intmix formula A is principal in the inferences whose conclusions are the premises of the IntMix in $\mathfrak{D}$, neither of which result from contraction. In this case we provide reductions for a representative sample of cases in which we reduce the *dmc* of the intmix formula.

(a) The intmix-formula is of the form $\neg\neg A$.

$$\dfrac{\dfrac{\Gamma' \succ A, \Delta'}{\Gamma' \succ \neg\neg A, \Delta'}\,[DNR] \quad \dfrac{\Gamma, A \succ \Delta}{\Gamma, \neg\neg A \succ \Delta}\,[DNL]}{\Gamma, \Gamma' \succ \Delta, \Delta'}\,[IntMix], \neg\neg A$$

which is transformed into the following derivation

$$\dfrac{\Gamma' \succ A, \Delta' \quad \Gamma, A \succ \Delta}{\Gamma, \Gamma' \succ \Delta, \Delta'}\,[IntMix], A$$

(b) The intmix-formula is of the form $A \wedge B$, giving us the following derivation

$$\dfrac{\dfrac{\Gamma, A, B \succ \Delta}{\Gamma, A \wedge B \succ \Delta}\,[\wedge L] \quad \dfrac{\Gamma' \succ \Delta', A \quad \Gamma' \succ \Delta', B}{\Gamma' \succ \Delta', A \wedge B}\,[\wedge R]}{\Gamma, \Gamma' \succ \Delta, \Delta'}\,[IntMix], A \wedge B$$

which is transformed into

$$\dfrac{\dfrac{\Gamma' \succ \Delta', B \quad \dfrac{\Gamma' \succ \Delta', A \quad \Gamma, A, B \succ \Delta}{\Gamma, \Gamma', B \succ \Delta, \Delta'}\,[IntMix], A}{\Gamma', \Gamma', \Gamma \succ \Delta, \Delta', \Delta'}\,[IntMix], B}{\Gamma', \Gamma \succ \Delta, \Delta'}\,[W]^*$$

(c) The intmix-formula is of the form $A \vee B$, giving us the following derivation

$$\dfrac{\dfrac{\Gamma, A \succ \Delta \quad \Gamma, B \succ \Delta}{\Gamma, A \vee B \succ \Delta}\,[\vee L] \quad \dfrac{\Gamma' \succ \Delta', A, B}{\Gamma' \succ \Delta', A \vee B}\,[\vee R]}{\Gamma, \Gamma' \succ \Delta, \Delta'}\,[IntMix] A \vee B$$

[9]For example, in the case of the Intmix with $\mathfrak{D}_r$ note that $t(\Delta) \subseteq t(\Gamma, \Gamma', p^*)$, as given the leftmost premise we have that $t(A) \subseteq t(\Gamma', p^*)$ and so $t(\Gamma, A) \subseteq t(\Gamma, \Gamma', p^*)$ as by the provability of the left-premise it follows that $t(\Delta) \subseteq t(\Gamma, A)$ we can conclude $t(\Delta) \subseteq t(\Gamma, \Gamma', p^*)$ as desired.

which is transformed into

$$\dfrac{\dfrac{\Gamma' \succ \Delta', A, B \quad \Gamma, A \succ \Delta}{\Gamma, \Gamma' \succ \Delta, \Delta', B}{\scriptstyle [IntMix],A} \quad \Gamma, B \succ \Delta}{\dfrac{\Gamma, \Gamma, \Gamma' \succ \Delta, \Delta, \Delta'}{\Gamma, \Gamma' \succ \Delta, \Delta'}{\scriptstyle [W]^*}}{\scriptstyle [IntMix],B}$$

Just as in the previous case, the intmix formulas are both of lower degree.

4. $[i(\mathfrak{D}) = \langle k, j\rangle,\ 1 \leq j, k]$ We distinguish again the $r_{\mathfrak{D}}(\Gamma \succ \Delta, A^n) > 0$ and $r_{\mathfrak{D}}(A^m, \Gamma' \succ \Delta') > 0$ cases.
   (a) $[r_{\mathfrak{D}}(\Gamma \succ \Delta, A^n) > 0]$ If the intmix formula is a side formula we push intmixes upwards again. These cases follow much like those given above. Let us suppose, then, that the intmix formula is principal. We treat some representative cases.
      i. The intmix-formula is of the form $\neg\neg A$.

$$\dfrac{\dfrac{\Gamma \succ A, \Delta}{\Gamma \succ \neg\neg A, \Delta}{\scriptstyle [DNR]} \quad \Gamma', \neg\neg A \succ \Delta'}{\Gamma, \Gamma'^{-} \succ \Delta^{-}, \Delta'}{\scriptstyle [IntMix],\neg\neg A}$$

where as $r_{\mathfrak{D}}(\Gamma \succ \Delta, \neg\neg A) > 0$ we know that $\neg\neg A \in \Delta$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \succ A, \Delta \quad \Gamma', \neg\neg A \succ \Delta'}{\Gamma, \Gamma'^{-} \succ \Delta^{-}, \Delta', A}{\scriptstyle [IntMix],\neg\neg A}(1)}{\Gamma, \Gamma'^{-} \succ \Delta^{-}, \Delta', \neg\neg A}{\scriptstyle [DNR]} \quad \Gamma', \neg\neg A \succ \Delta'}{\Gamma, \Gamma'^{-}, \Gamma'^{-} \succ \Delta^{-}, \Delta', \Delta'}{\scriptstyle [IntMix],\neg\neg A}(2)}{\Gamma, \Gamma'^{-} \succ \Delta^{-}, \Delta'}{\scriptstyle [K]}$$

As $r_{\mathfrak{D}}((\Gamma \succ \Delta, A) < r_{\mathfrak{D}}(\Gamma \succ \Delta, \neg\neg A)$, the mix (1) is over lower rank than that in our original proof. Similarly, as the rank of the left premise in mix (2) is 0, we that mix is also of lower rank.

      ii. The intmix-formula is of the form $A \wedge B$.

$$\dfrac{\dfrac{\Gamma \succ \Delta, A \quad \Gamma \succ \Delta, B}{\Gamma \succ \Delta, A \wedge B} \quad A \wedge B, \Gamma' \succ \Delta'}{\Gamma, \Gamma'^{-} \succ \Delta^{-}, \Delta'}{\scriptstyle [IntMix],A\wedge B}$$

again as $r_{\mathfrak{D}}(\Gamma \succ \Delta, A \wedge B) > 0$ we know that $A \wedge B \in \Delta$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \succ \Delta, A \quad A \wedge B, \Gamma' \succ \Delta'}{\Gamma, \Gamma'^{-} \succ \Delta^{-}, \Delta', A}{\scriptstyle [IntMix],A\wedge B} \quad \dfrac{\Gamma \succ \Delta, B \quad A \wedge B, \Gamma' \succ \Delta'}{\Gamma, \Gamma'^{-} \succ \Delta^{-}, \Delta', B}{\scriptstyle [IntMix],A\wedge B}}{\Gamma, \Gamma'^{-} \succ \Delta^{-}, \Delta', A \wedge B} \quad A \wedge B, \Gamma' \succ \Delta'}{\Gamma, \Gamma'^{-}, \Gamma'^{-} \succ \Delta^{-}, \Delta', \Delta'}{\scriptstyle [IntMix],A\wedge B}}{\Gamma, \Gamma'^{-} \succ \Delta^{-}, \Delta'}{\scriptstyle [K]}$$

The upper two intmixes are of lower rank, as their left premise is now of lower rank. The final intmix is of lower rank as its left premise now has rank 0. In both cases, then we can apply the induction hypothesis as desired.

(b) $[r_{\mathfrak{D}}(A^m, \Gamma' \succ \Delta') > 0]$ is dealt with in a symmetric manner. ■

Corollary 5.6. *Cut is admissible in* $\mathcal{G}_{AC}$.

References

[1] Anderson, A. R., and N. D. Belnap, *Entailment: The Logic of Relevance and Neccessity, Vol. I.* Princeton University Press, 1975.

[2] Angell, R., Three Systems of First Degree Entailment, *Journal of Symbolic Logic* 47: 147, 1977.

[3] Angell, R., Analytic truth-tables, *Journal of Symbolic Logic* 46(3): 677, 1981.

[4] Angell, R., Deducibility, Entailment and Analytic Containment. In J. Norman and R. Sylvan (Eds.), *Directions in Relevant Logic*, Chapter 8, Dordrecht: Kluwer, 1989, pp. 119–144.

[5] Belnap, N. D., Rescher's Hypothetical Reasoning: An Amendment. In E. Sosa (Ed.), *The Philosophy of Nicholas Rescher: Discussion and Replies*, D. Reidel, 1979, pp. 19–28.

[6] Brady, R. T., *Universal Logic.* Number 109 in CSLI Lecture Notes. Stanford: CSLI Publications, 2006.

[7] Correia, F., Semantics for Analytic Containment, *Studia Logica* 77: 87–104, 2004.

[8] Correia, F., Grounding and Truth-Functions, *Logique Et Analyse* 211: 251–279, 2010.

[9] Correia, F., On the Logic of Factual Equivalence, *The Review of Symbolic Logic* 9(1): 103–122, 2016.

[10] Daniels, C., A Story Semantics for Implication, *Notre Dame Journal of Formal Logic* 27: 221–246, 1986.

[11] Deutsch, H., Paraconsistent Analytic Implication, *Journal of Philosophical Logic* 13: 1–11, 1984.

[12] Dragalin, A., *Mathematical Intuitionism. Introduction to Proof Theory.* Providence: American Mathematical Society, 1988.

[13] Ferguson, T. M., Cut-down operations on bilattices, In *Proceedings of the 45th International Symposium on Multiple-Valued Logic (IEEE 2015), Los Alamitos, CA*, IEEE Computer Society Press, 2015, pp. 163–172.

[14] Ferguson, T. M., Faulty Belnap computers and subsystems of FDE, *Journal of Logic and Computation* 26(5): 1617–1636, 2016.

[15] Fine, K., Analytic Implication, *Notre Dame Journal of Formal Logic* 27(2): 169–179, 1986.

[16] Fine, K., Guide to ground. In F. Correia and B. Schnieder (Eds.), *Metaphysical Grounding*, pp. 37–80. Cambridge University Press, 2012 pp. 37–80.

[17] Fine, K., A Note on Partial Content, *Analysis* 73(3): 413–419, 2013.

[18] Fine, K., Truth-Maker Semantics for Intuitionistic Logic, *Journal of Philosophical Logic* 43: 549–577, 2014.

[19] Fine, K., Angellic Content, *Journal of Philosophical Logic* 45(2): 199–226, 2016.

[20] van Fraassen, B., Facts and Tautological Entailments, *The Journal of Philosophy* 66: 477–487, 1969.

[21] Humberstone, L., False Though Partly True – An Experiment in Logic, *Journal of Philosophical Logic* 32: 613–665, 2003.

[22] Negri, S., and J. von Plato, *Structural Proof Theory*, Cambridge University Press, Cambridge, 2001.

[23] Paoli, F., *Substructural Logics: A Primer*, Kluwer Academic Publishers, 2002.

[24] Parry, W., Ein Axiomensystem für eine neue Art von Implikation (analytische Implikation), *Ergebnisse eines mathematischen Kolloquiums* 4: 5–6, 1933.

[25] Parry, W., Comparison of Entailment Theories, *Journal of Symbolic Logic* 37(2): 441–442, 1972.

[26] Pynko, A. P., Characterizing Belnap's Logic via De Morgan's Laws, *Mathematical Logic Quarterly* 41: 442–454, 1995.

[27] Troelstra, A., and H. Schwichtenberg, *Basic Proof Theory.* Cambridge: Cambridge University Press, 2000.

[28] Yablo, S., *Aboutness.* Princeton: Princeton University Press, 2014.

R. French
Department of Philosophy
School of Philosophical Historical and International Studies
Monash University
Victoria
Australia
rohan.french@gmail.com