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Probabilistic Logics with Independence and Confirmation

Abstract. The main goal of this work is to present the proof-theoretical and model-theoretical approaches to probabilistic logics which allow reasoning about independence and probabilistic support. We extend the existing formalisms [14] to obtain several variants of probabilistic logics by adding the operators for independence and confirmation to the syntax. We axiomatize these logics, provide corresponding semantics, prove that the axiomatizations are sound and strongly complete, and discuss decidability issues.

Keywords: Probabilistic support, Independence, Axiomatization, Completeness theorem, Decidability.

1. Introduction

Independence is one of the main notions in probability theory. Two events, A and B are said to be *independent* w.r.t. a probability measure μ, if occurrence of one of them does not affect the probability of the other, i.e. $\mu(A \cap B) = \mu(A)\mu(B)$ (or, more intuitively, $\mu(A|B) = \mu(A)$). Surprisingly, in spite of extensive development of various probabilistic logics in past decades, the notion of independence received little attention from the logical side.

A strongly related notion is Carnap's notion of *confirmation*, or *probabilistic support*. In his book [2], one of the main tasks is "the explication of certain concepts which are connected with the scientific procedure of confirming or disconfirming hypotheses with the help of observations and which we therefore will briefly call *concepts of confirmation*" (page 19). He distinguished three different semantical concepts of confirmation: the classificatory concept of confirmation, ("a hypothesis is confirmed by an evidence"), the comparative concept ("a hypothesis A is confirmed by an evidence B at least as strongly as C is confirmed by D") and the quantitative concept of confirmation. The third concept is formalized by a numerical function c which maps pairs of sentences to a real number from the unit interval, where $c(A, B)$ is the degree of confirmation of the hypothesis A on the basis of the

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evidence B. Carnap used the concept of degree of confirmation as the basic concept of inductive logic. The notion of confirmation has drown particular attention for the famous discussion on its nature, after Popper and Miller [16] claimed that probabilistic support is deductive, not inductive (see e.g. [13,17]).

There are several nonequivalent proposals for the measure of confirmation $c(A, B)$, all of them agreeing in the qualitative way: $c(A, B) > 0$ iff the posterior probability of A on the evidence B is greater than the prior probability of A [9]. That leads to the natural choice for Carnap's classificatory concept: B confirms A just in case that $\mu(A|B) > \mu(A)$ (or $\mu(A \cap B) > \mu(A)\mu(B)$).

In this paper, we formalize the classificatory notion of confirmation. Whilst in concrete situations people use this intuitive notion in a correct way, in an abstract environment though, they often they misuse it, transferring the properties of conditionals (eg. transitivity) to properties of confirmation [18]. The notion of independence also caused some reasoning mistakes, even among the scholars in probability theory. De Morgan [4], who devoted a significant part of his work to analyzing relationships between logic and probability, discussed uncertain reasoning with necessary valid inferences (he called them arguments) and probable premises (testimonies) by analyzing some examples. Hailperin in [8] noticed some systematical failures in the work of De Morgan and showed that some of his results hold only under the assumption that the premises are independent. Logical formalization of the notions of independence and confirmation might be helpful in avoiding the situations of their incorrect applications.

In this work, we propose several sound, complete and decidable logics for reasoning about independence and confirmation. We start with probabilistic logics that extend classical propositional calculus with modal-like unary operators of the form $P_{\geq r}$, where r range over the set of all rational numbers from the unit interval [14]. The intended meaning of the formula $P_{\geq r}\alpha$ is "the probability of α is at least r". We extend the language with two binary operators, $\perp$ and $\uparrow$. The formula $\alpha \perp \beta$ is read as "α and β are (probabilistically) independent", while $\alpha \uparrow \beta$ is read as "α confirms β". In fact, we find it convenient to introduce the operators of weak confirmation $\nearrow$ and weak disconfirmation $\searrow$ as the primitive operators, and define $\perp$ and $\uparrow$ as abbreviations. We introduce four formal systems: the logic LPP_2^{ind},[1] without nesting of probabilistic operators, LPP_1^{ind} where nesting is allowed, and

[1] Here *ind* stands for "independence".

their restrictions $LPP_2^{Fr(n),ind}$ and $LPP_1^{Fr(n),ind}$ with a fixed finite range for probabilistic measures.

There are not many probabilistic logics in which independence and confirmation are expressible. Reason for that lies in difficulties in combining high expressivity of the language and inference mechanisms. In the most famous paper in the field, Fagin, Halpern and Megiddo [6] axiomatized logic with linear weight formulas (LWFs), i.e. Boolean combinations of the expressions of the form $r_1P(\alpha_1)+\cdots+r_nP(\alpha_n)\geq r_{n+1}$, where r_i are rational numbers and α_i are propositional formulas. In the language, one can express the statement "conditional probability of α given β is at least r", but not linear combinations of conditional probabilities. However, only simple completeness (every consistent formula is satisfiable) is proved, so there are unsatisfiable sets of formulas that are consistent with respect to the given finitary axiomatization. There are two axiomatized logics which extend LWFs, that can capture independency. In [11], LWFs are enriched with an operator that can express independence of sets of propositional letters. In [5], an axiomatization of the language that contains linear combinations of conditional probabilities is presented. Consequently, both independence and support are expressible. The language with polynomial weight formulas (PWFs), which extends all previously mentioned languages, is also considered in [6] (see also [7]), but the authors haven't axiomatized the logic. Instead, they have axiomatized the reasoning about PWFs within the first order logic. A complete axiomatization of PWFs is presented in [15].

This work offers the first axiomatization of the logics without arithmetical operations built into syntax, that can represent independence and confirmation. Those two notions are formalized using binary operators. Also, to the best of our knowledge, none of the previous attempts deal with the nesting of those operators. On the other hand, we offer a formalism in which we can formalize independence and support of both probabilistic and classical formulas. As we discuss in Section 2.3, the key issue for real-valued probabilistic logics is the non-compactness phenomena. As a consequence, for any finitary axiomatization there are unsatisfiable sets of formulas that are consistent w.r.t. the axiomatization [19]. In this paper we present the infinitary axiomatizations, as a way to overcome the problem and prove strong completeness for LPP_1^{ind} and LPP_2^{ind}. Here the term infinitary concerns the meta language only. Object languages are countable and formulas are finite, while only proofs are allowed to be infinite. On the other hand, for the restricted logics $LPP_1^{Fr(n),ind}$ and $LPP_2^{Fr(n),ind}$ we offer finitary axiomatizations.

The contents of this paper are as follows. First we introduce LPP_2^{ind}. In Section 2 we present the syntax and semantics of LPP_2^{ind} in detail. Section 3 contains an axiomatization for the logic. Some useful statements are proved in Section 4. In Section 5 we prove the completeness of our axiomatization. In Section 6 we prove decidability of LPP_2^{ind}-formulas. We provide the syntax, semantics, and a complete axiomatization for the logic LPP_1^{ind} in Section 7, but do not go into the details. We also prove that the logic is decidable in the same section. Finally, in Section 8 we restrict the logics LPP_1^{ind} and LPP_2^{ind}, focusing only on the probabilistic models which have a finite fixed range $\{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$. We propose finitary axiomatization for the logics. We conclude in Section 9.

2. The Logic LPP_2^{ind}: Syntax and Semantics

In this section we present the syntax and semantics of the propositional probabilistic logic without nesting of the probability operators. This logic, that we call LPP_2^{ind}, contains two types of propositional formulas: one with probabilities and one without probabilities.

2.1. Syntax

Let $\mathcal{P}$ be a countable set of propositional letters. The language of LPP_2^{ind} consists of the elements of $\mathcal{P}$, classical propositional connectives $\neg$ and $\wedge$, the binary probability operators $\nearrow$ and $\searrow$ and the list of probability operators of the form $P_{\geq r}$, for every $r \in \mathbb{Q} \cap [0, 1]$. Note that we use negation and conjunction as a complete list of boolean connectives. We use the usual abbreviations for the other classical connectives.

By For_C we will denote the set of all propositional formulas over $\mathcal{P}$. The propositional formulas will be denoted by α, β and γ, possibly with subscripts.

Definition 2.1. (Probabilistic formula) A *basic probabilistic formula* is any formula of the form

- $P_{\geq r}\alpha$ where $\alpha \in For_C$, or
- $\alpha \nearrow \beta$ or $\alpha \searrow \beta$, where $\alpha, \beta \in For_C$.

A *probabilistic formula* is any Boolean combination of basic probabilistic formulas. We use For_P for the set of all probabilistic formulas and denote arbitrary probabilistic formulas by ϕ and ψ (indexed if necessary).

We use the following abbreviations to introduce other types of probability operators:

- $P_{<s}\alpha$ is $\neg P_{\geq s}\alpha$, $P_{\leq s}\alpha$ is $P_{\geq 1-s}\neg\alpha$, $P_{>s}\alpha\neg P_{\leq s}\alpha$, and $P_{=s}\alpha$ is $P_{\geq s}\alpha \wedge P_{\leq s}\alpha$,
- $\alpha \perp \beta$ is $\alpha \nearrow \beta \wedge \alpha \searrow \beta$, $\alpha \uparrow \beta$ is $\neg(\alpha \searrow \beta)$ and $\alpha \downarrow \beta$ is $\neg(\alpha \nearrow \beta)$.

Example 2.2. *Let $\alpha, \beta \in For_C$. Then*

$$(\alpha \perp \beta) \rightarrow P_{\geq \frac{1}{2}}(\alpha \wedge \beta)$$

is a probabilistic formula. Its meaning is that if α and β are independent, then probability of $\alpha \wedge \beta$ is at least one half. Furthermore, the expression

$$(P_{=1}\alpha) \nearrow \beta$$

is not a probabilistic formula of the logic LPP_2^{ind}, since nested probabilistic operators are not allowed.

Definition 2.3. (Formula) $For_{LPP_2^{ind}} = For_C \cup For_P$. We denote arbitrary formulas by ρ and σ (possibly with subscripts).

Obviously, mixing of pure propositional formulas and probability formulas is not allowed. For example, $\alpha \vee P_{\geq \frac{1}{2}}\beta$ is not a formula of the logic LPP_2^{ind}.

In the rest of the paper, we denote both $\alpha \wedge \neg\alpha$ and $\phi \wedge \neg\phi$ by $\perp$ (and similarly for $\top$), letting the context determine the meaning.

2.2. Semantics

The semantics for $For_{LPP_2^{ind}}$ is based on the possible-world approach.

Definition 2.4. (LPP_2^{ind}-structure) An LPP_2^{ind}-*structure* is a tuple $\langle W, H, \mu, v\rangle$, where:

- W is a nonempty set. The elements of W are called *worlds*.
- H is an algebra of subsets of W, i.e., a set of subsets of W with the property:
  - $W \in H$,
  - If $A, B \in H$, then $W \setminus A \in H$ and $A \cup B \in H$.

  The elements of H are called *measurable sets*.
- $\mu : H \longrightarrow [0, 1]$ is a finitely additive measure, i.e.,
  - $\mu(W) = 1$,
  - If $A, B \in H$ and $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

- $v : W \times \mathcal{P} \to \{true, false\}$ provides for each world $w \in W$ a two-valued evaluation $v(w, \cdot)$ of the primitive proposition, which is extended to For_C as usual.

For given $\alpha \in For_C$ and LPP_2^{ind}-structure M, let $[\alpha]_M = \{w \in W \mid v(w, \alpha) = true\}$. We will omit the subscript M when it is clear from context.

Definition 2.5. (Measurable structure) The class of *measurable* structures of the logic LPP_2^{ind}, denoted by $LPP_{2,Meas}^{ind}$, is the class of all LPP_2^{ind}-structures M such that $[\alpha]_M \in H$ for every $\alpha \in For_C$.

Next we define the satisfiability of a formula in a measurable structure ($\models$).

Definition 2.6. (Satisfiability) Let $M \in LPP_{2,Meas}^{ind}$ be a measurable structure. We define the satisfiability relation $\models$ recursively as follows:

- $M \models \alpha$ iff $v(w, \alpha) = true$ for all $w \in W$.
- $M \models \alpha \nearrow \beta$ if $\mu([\alpha \wedge \beta]) \geq \mu([\alpha])\mu([\beta])$,
- $M \models \alpha \searrow \beta$ if $\mu([\alpha \wedge \beta]) \leq \mu([\alpha])\mu([\beta])$,
- $M \models P_{\geq r}\alpha$ if $\mu([\alpha]) \geq r$,
- $M \models \neg\phi$ if $M \not\models \phi$,
- $M \models \phi \wedge \psi$ if $M \models \phi$ and $M \models \psi$.

Note that, by the first item of the previous definition, the formulas from For_C do not behave in the usual way: for some $\alpha \in For_C$ and $M \in LPP_{2,Meas}^{ind}$ we might have both $M \not\models \alpha$ and $M \not\models \neg\alpha$. This corresponds to the intuition that $M \models \alpha$ means that α is known with certainty (in M), implying that $\mu([\alpha]) = \mu(W) = 1$. Thus, $M \not\models \alpha$ and $M \not\models \neg\alpha$ allows nontrivial probability ($\neq 0, 1$) of $[\alpha]$.

Directly from Definition 2.6 we obtain that satisfiability for the operators of independence and confirmation correspond to their intuitive meaning:

- $M \models \alpha \perp \beta$ if $\mu([\alpha \wedge \beta]) = \mu([\alpha])\mu([\beta])$,
- $M \models \alpha \uparrow \beta$ if $\mu([\alpha \wedge \beta]) > \mu([\alpha])\mu([\beta])$.

Definition 2.7. (Model) For a measurable structure M and a set of formulas T, we call M a *model* of T and write $M \models T$ iff $M \models \rho$ for every $\rho \in T$. We also say that T is *satisfiable*, if there is M such that $M \models T$.

Definition 2.8. (Entailment) We say that a set of formulas T *entails* a formula ρ and write $T \models \rho$, if all models of T are models of ρ. Moreover, a formula ρ is *valid* if $\emptyset \models \rho$.

Note that, in spite of nonstandard behavior of the classical formulas, the set of valid formulas from For_C w.r.t. classical propositional satisfiability relation coincide with the set of all formulas that are valid with respect to the semantics presented here.

2.3. Noncompactness

Compactness theorem states that for every set of formulas T, T is satisfiable if and only if every finite subset of T is satisfiable. It is known that Compactness theorem does not hold for real-valued probabilistic logics. The well known example is the following set:

$$T_1 = \{P_{>0}\alpha\} \cup \{P_{\leqslant \frac{1}{n}}\alpha \mid n = 1, 2, 3, \dots\}.$$

The following set of formulas shows that by adding the new operator $\nearrow$ we obtain more examples for noncompactness. Let

$$T_2 = \{\alpha \downarrow \beta\} \cup \{(P_{\geq r}\alpha \wedge P_{\geq s}\beta) \rightarrow P_{\geq rs}(\alpha \wedge \beta) \mid r, s \in [0,1] \cup \mathbb{Q}\}.$$

Let us show that T_2 is not satisfiable. Suppose that M is a model of T_2, and let $a, b \in \mathbb{R}$ be reals such that $a = \mu([\alpha])$ and $b = \mu([\beta])$. Then $M \models P_{\geq r}\alpha$ for all $r \in \mathbb{Q} \cap [0, a)$ and $M \models P_{\geq s}\beta$ for all $s \in \mathbb{Q} \cap [0, b)$. By assumption, $M \models (P_{\geq r}\alpha \wedge P_{\geq s}\beta) \rightarrow P_{\geq rs}(\alpha \wedge \beta)$ for all rational numbers r and s from the unit interval, so $M \models P_{\geq rs}(\alpha \wedge \beta)$ for all $r \in \mathbb{Q} \cap [0, a)$ and $s \in \mathbb{Q} \cap [0, b)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, we obtain $\mu([\alpha \wedge \beta]) \geq ab = \mu([\alpha])\mu([\beta])$. This contradicts our assumption that $M \models \alpha \downarrow \beta$, so we conclude that T_2 is unsatisfiable.

On the other hand, it is easy to show that T_2 is finitely satisfiable. Indeed, for each finite subset of the form $\{\alpha \downarrow \beta, (P_{\geq r_1}\alpha \wedge P_{\geq s_1}\beta) \rightarrow P_{\geq r_1 s_1}(\alpha \wedge \beta), \dots, (P_{\geq r_n}\alpha \wedge P_{\geq s_n}\beta) \rightarrow P_{\geq r_n s_n}(\alpha \wedge \beta)\}$, by density of $\mathbb{Q}$ in $\mathbb{R}$ we can choose a model such that $\mu([\alpha \wedge \beta]) < \mu([\alpha])\mu([\beta])$ and there is no $k \in \{1, \dots, n\}$ such that $\mu([\alpha]) \geq r_k$, $\mu([\beta]) \geq s_k$ and $\mu([\alpha \wedge \beta]) < r_k s_k$.

In [19], an unpleasant consequence of noncompactness is pointed out: for any finitary axiomatization there are unsatisfiable sets of formulas that are consistent w.r.t. the axiomatization. In the following section we present the axiomatization that includes infinitary rules of inference, as a way to overcome the problem and obtain the strong completeness. The so called Archimedean rule is used in [14] to overcome the problem with sets similar to T_1. This rule is also part of our axiomatization. We also introduce two novel infinitary rules to enforce inconsistency of the sets of formulas which are similar to T_2.

3. The Axiomatization $Ax_{LPP_2^{ind}}$

In this section we introduce the axiomatizatic system for the logic LPP_2^{ind}, denoted by $Ax_{LPP_2^{ind}}$.

Axiom Schemes

A1 all instances of propositional theorems for both For_C and For_P

A2 $P_{\geq 0}\alpha$

A3 $P_{\leqslant r}\alpha \rightarrow P_{<s}\alpha$ whenever $r < s$

A4 $P_{<r}\alpha \rightarrow P_{\leqslant r}\alpha$

A5 $(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\geq 1}(\neg\alpha \vee \neg\beta)) \rightarrow P_{\geq \min\{1,r+s\}}(\alpha \vee \beta)$

A6 $(P_{\leqslant r}\alpha \wedge P_{<s}\beta) \rightarrow P_{<r+s}(\alpha \vee \beta)$, whenever $r + s \leqslant 1$

A7 $(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge \alpha \nearrow \beta) \rightarrow P_{\geq rs}(\alpha \wedge \beta)$

A8 $(P_{\leq r}\alpha \wedge P_{\leq s}\beta \wedge \alpha \searrow \beta) \rightarrow P_{\leq rs}(\alpha \wedge \beta)$

Inference rules

R1 (a) From α and $\alpha \rightarrow \beta$ infer β.
(b) From ϕ and $\phi \rightarrow \psi$ infer ψ.

R2 From α infer $P_{\geq 1}\alpha$.

R3 From the set of premises

$$\{\phi \rightarrow P_{\geq r-\frac{1}{k}}\alpha \mid k \in \mathbb{N}, k \geq \frac{1}{r}\}$$

infer $\phi \rightarrow P_{\geq r}\alpha$.

R4 From the set of premises

$$\{\phi \rightarrow ((P_{\geq r}\alpha \wedge P_{\geq s}\beta) \rightarrow P_{\geq rs}(\alpha \wedge \beta)) \mid r, s \in [0,1] \cup \mathbb{Q}\}$$

infer $\phi \rightarrow (\alpha \nearrow \beta)$.

R5 From the set of premises

$$\{\phi \rightarrow ((P_{\leq r}\alpha \wedge P_{\leq s}\beta) \rightarrow P_{\leq rs}(\alpha \wedge \beta)) \mid r, s \in [0,1] \cup \mathbb{Q}\}$$

infer $\phi \rightarrow (\alpha \searrow \beta)$.

The axioms A1-A6 and the inference rules R1 (Modus Ponens, applied to both classical and probabilistic formulas), R2 (probabilistic Necessitation) and R3 (Archimedean rule) form axiomatic system for the logic LPP_2 from [14].

The axioms A7 and A8 and the rules R4 and R5 are added to the system. In combination with other axioms, A7 and R4 characterize the new operator $\nearrow$, while A8 and R5 characterize $\searrow$. The rules R3, R4 and R5 are infinitary rules of inference, and the implicative form of the formulas in the rules allows proof of Deduction theorem. Now we define some basic notions of proof theory:

- $T \vdash_{Ax_{LPP_2^{ind}}} \rho$ ("ρ is *deducible* from a set of formulas T") if there is an at most countable sequence (called *proof*) of formulas $\rho_0, \rho_1, \ldots, \rho$, such that every ρ_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule (we write $\vdash$ instead of $\vdash_{Ax_{LPP_2^{ind}}}$ when it is clear from context).
- $\vdash \rho$ (ρ is a *theorem*) if $\emptyset \vdash \rho$.
- T is *inconsistent* if there a formula $\phi \in For_P$ such that $T \vdash \phi \wedge \neg\phi$, otherwise it is consistent.
- T is *maximally consistent* if it is consistent and every proper superset of T is inconsistent.

Note that the notion of consistency is defined using probabilistic formulas only. Nevertheless, if T is consistent, then there is no α such that $T \vdash \alpha \wedge \neg\alpha$, because otherwise by Necessitation we have $T \vdash P_{=1}\alpha \wedge P_{=1}\neg\alpha$. Also, note that maximal consistency of T implies that[2]

$$\phi \in T \quad \text{or} \quad \neg\phi \in T, \quad \text{whenever} \quad \phi \in For_P,$$

but there might exist $\alpha \in For_C$ such that both $\alpha \notin T$ and $\neg\alpha \notin T$. On the other hand, T is *deductively closed* for all formulas, i.e.

$$T \vdash \rho \quad \text{implies} \quad \rho \in T, \quad \text{for all} \quad \rho \in For_{LPP_2^{ind}}.$$

4. Some Theorems of LPP_2^{ind}

In this section we prove several theorems about our system. We will use some of them later in proving the completeness of the axiomatization. We start with the soundness theorem.

THEOREM 4.1. (Soundness) *The axiomatic system $Ax_{LPP_2^{ind}}$ is sound with respect to the class of measurable structures $LPP_{2,Meas}^{ind}$.*

[2]This will follow immediately from Theorem 4.2.2 (in Section 4).

Proof. We need to show that every instance of an axiom schema holds in every structure, and that the inference rules preserve the validity. For example, let us consider the axiom A7. Suppose that $M \in LPP^{ind}_{2,Meas}$ is a structure such that $M \models P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge \alpha \nearrow \beta$. Then $\mu([\alpha]) \geq r$, $\mu([\beta]) \geq s$ and $\mu([\alpha \wedge \beta]) \geq \mu([\alpha])\mu([\beta])$. This further implies $\mu([\alpha \wedge \beta]) \geq rs$, so $M \models P_{\geq rs}(\alpha \wedge \beta)$.

Let us show that R4 preserves validity. Suppose that $M \models \{\phi \rightarrow ((P_{\geq r}\alpha \wedge P_{\geq s}\beta) \rightarrow P_{\geq rs}(\alpha \wedge \beta)) \mid r, s \in [0,1] \cup \mathbb{Q}\}$. If $M \not\models \phi$, then trivially $M \models \phi \rightarrow (\alpha \nearrow \beta)$. Now assume that $M \models \phi$. Then

$$M \models (P_{\geq r}\alpha \wedge P_{\geq s}\beta) \rightarrow P_{\geq rs}(\alpha \wedge \beta)$$

for all $r, s \in [0,1] \cup \mathbb{Q}$. If r and s are rational numbers from the unit interval such that $r \leqslant \mu([\alpha])$ and $s \leqslant \mu([\beta])$, then $M \models P_{\geq r}\alpha \wedge P_{\geq s}\beta$, so $M \models P_{\geq rs}(\alpha \wedge \beta)$, or, equivalently, $\mu([\alpha \wedge \beta]) \geq rs$. By density of rational numbers, we obtain $\mu([\alpha \wedge \beta]) \geq \mu([\alpha])\mu([\beta])$, so $M \models \alpha \nearrow \beta$. Consequently, $M \models \phi \rightarrow (\alpha \nearrow \beta)$. ∎

Next we prove the deduction theorem. Because of the two types of formulas, there are two versions.

Theorem 4.2. (Deduction theorem) *Let T be a set of formulas. Then:*

1. *$T \cup \{\alpha\} \vdash \beta$ iff $T \vdash \alpha \rightarrow \beta$.*
2. *$T \cup \{\phi\} \vdash \psi$ iff $T \vdash \phi \rightarrow \psi$.*

Proof. First statement is the theorem of classical propositional calculus. Let us prove 2. It is sufficient to prove the direction from left to right, because the other direction follows immediately from R1. So, suppose that $T \cup \{\phi\} \vdash \psi$. We proceed by the length of the inference.

The cases when $\vdash \psi$ and $\psi \in T \cup \{\phi\}$ are the same as in the classical propositional calculus, as well as the case when we apply R1(b). The cases when ψ is obtained from $T \cup \{\phi\}$ by means of the rules R2 or R3 are proved in [14]. Let us prove the case when ψ is obtained from $T \cup \{\phi\}$ by means of the inference rule R4.

Suppose that $T \cup \{\phi\} \vdash \psi$ and that ψ is the formula $\phi_1 \rightarrow (\alpha \nearrow \beta)$, obtained from the set of premises

$$\{\phi_1 \rightarrow ((P_{\geq r}\alpha \wedge P_{\geq s}\beta) \rightarrow P_{\geq rs}(\alpha \wedge \beta)) \mid r, s \in [0,1] \cup \mathbb{Q}\}$$

applying R4. Then, by the induction hypothesis, for every $r, s \in [0,1] \cup \mathbb{Q}$ we have

$$T \vdash \phi \rightarrow (\phi_1 \rightarrow ((P_{\geq r}\alpha \wedge P_{\geq s}\beta) \rightarrow P_{\geq rs}(\alpha \wedge \beta))).$$

By propositional reasoning, we obtain

$$T \vdash (\phi \wedge \phi_1) \to ((P_{\geq r}\alpha \wedge P_{\geq s}\beta) \to P_{\geq rs}(\alpha \wedge \beta)),$$

for every $r, s \in [0,1] \cup \mathbb{Q}$. Then, using the rule R4 we infer

$$T \vdash (\phi \wedge \phi_1) \to (\alpha \nearrow \beta).$$

Finally, using propositional reasoning we obtain $T \vdash \phi \to (\phi_1 \to (\alpha \nearrow \beta))$, i.e., $T \vdash \phi \to \psi$.

In the case when ψ is obtained from $T \cup \{\phi\}$ by means of the inference rule R5, we can reason in the similar way as above. ■

Remark 4.3. In Section 3 we stressed that there exist $\alpha \in For_C$ and a maximal consistent set T such that both $\alpha \notin T$ and $\neg\alpha \notin T$. Note that it does not contradict Deduction theorem. Namely, from $\alpha \notin T$ we can conclude that $T \cup \{\alpha\}$ is inconsistent, i.e.,

$$T \cup \{\alpha\} \vdash \bot, \tag{I}$$

but we cannot deduce $T \vdash \neg\alpha$ using Theorem 4.2.1. The reason is that $\bot$ in (I) is a probabilistic contradiction, i.e. a formula of the form $\phi \wedge \neg\phi$ (recall the definition of inconsistency from Section 3), so Theorem 4.2.1 cannot be applied.

Note that our axiomatic system does not contain axioms and rules that reason solely about independence and confirmation. For example, commutativity is an obvious property of independence operator. We believe that it is important to illustrate the possibility of deriving that type of properties in our system $Ax_{LPP_2^{ind}}$.

Proposition 4.4. *Let $\alpha, \beta \in For_C$. Then*

- $\vdash (\alpha \nearrow \beta) \to (\beta \nearrow \alpha)$
- $\vdash (\alpha \searrow \beta) \to (\beta \searrow \alpha)$

Proof. Let $r, s \in [0,1] \cup \mathbb{Q}$. Using the axiom A7, propositional reasoning (A1 and R1) and Deduction theorem, we obtain $\alpha \nearrow \beta \vdash (P_{\geq r}\alpha \wedge P_{\geq s}\beta) \to P_{\geq rs}(\alpha \wedge \beta)$. By commutativity of both conjunction of formulas and multiplication of rational numbers, we obtain

$$\alpha \nearrow \beta \vdash (P_{\geq s}\beta \wedge P_{\geq r}\alpha) \to P_{\geq sr}(\beta \wedge \alpha)$$

for all $r, s \in [0,1] \cup \mathbb{Q}$. Using R4 (choosing $\phi = \top$), from the set of premises $\{(P_{\geq s}\beta \wedge P_{\geq r}\alpha) \to P_{\geq sr}(\beta \wedge \alpha) \mid r, s \in [0,1] \cup \mathbb{Q}\}$ we can infer $\beta \nearrow \alpha$. Thus, $\alpha \nearrow \beta \vdash \beta \nearrow \alpha$. Using Theorem 4.2.2 we obtain $\vdash (\alpha \nearrow \beta) \to (\beta \nearrow \alpha)$.
The second statement can be proved in the same way, using A8 and R5. ■

Note that commutativity of other binary operators ($\perp, \uparrow$ and $\downarrow$), is a direct consequence of Proposition 4.4.

We conclude this section with a lemma which we use in the construction of maximally consistent extensions of consistent set of formulas.

LEMMA 4.5. *Let T be a consistent set of formulas.*

1. *If the set of formulas $T \cup \{\phi \to P_{\geq r}\alpha\}$ is inconsistent, then there is an integer $k > 0$ such that $r - \frac{1}{k} \geq 0$ and $T \cup \{\phi \to P_{<r-\frac{1}{k}}\alpha\}$ is consistent.*
2. *If the set of formulas $T \cup \{\phi \to (\alpha \nearrow \beta)\}$ is inconsistent, then there are r and s from $\mathbb{Q} \cap [0,1]$ such that $T \cup \{\phi \to (P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{<rs}(\alpha \wedge \beta))\}$ is consistent.*
3. *If the set of formulas $T \cup \{\phi \to (\alpha \searrow \beta)\}$ is inconsistent, then there are r and s from $\mathbb{Q} \cap [0,1]$ such that $T \cup \{\phi \to (P_{\leq r}\alpha \wedge P_{\leq s}\beta \wedge P_{>rs}(\alpha \wedge \beta))\}$ is consistent.*

PROOF. For the proof of the first statement, we refer the reader to [14]. Let us prove the second statement. Suppose that $T \cup \{\phi \to (\alpha \nearrow \beta)\}$ is inconsistent. Then the set $T \cup \{\alpha \nearrow \beta\}$ is inconsistent as well. From Theorem 4.2.2 we obtain $T \vdash \alpha \downarrow \beta$.

Now suppose that the set

$$T \cup \{P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{<rs}(\alpha \wedge \beta)\}$$

is inconsistent for all r and s. By Theorem 4.2.2,

$$T \vdash \neg(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{<rs}(\alpha \wedge \beta))$$

for all r and s. Since the formula $\neg(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{<rs}(\alpha \wedge \beta))$ is equivalent to $\top \to ((P_{\geq r}\alpha \wedge P_{\geq s}\beta) \to P_{\geq rs}(\alpha \wedge \beta))$, from the inference rule R4 we obtain $T \vdash \alpha \nearrow \beta$. This contradicts the fact that $T \cup \{\alpha \nearrow \beta\}$ is inconsistent. This means that there are r and s such that the set $T \cup \{P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{<rs}(\alpha \wedge \beta)\}$ is consistent. Consequently, the set

$$T \cup \{\phi \to (P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{<rs}(\alpha \wedge \beta))\}$$

is also consistent.

Finally, note that the third statement can be proved in the same way as the second statement. ■

5. Completeness of LPP_2^{ind}

In this section we show that the axiomatization $Ax_{LPP_2^{ind}}$ makes LPP_2^{ind} complete for the class of structures $LPP_{2,Meas}^{ind}$. We prove completeness in three steps. First, we extend a consistent set of formulas T to a maximally consistent set T^*. Second, we use T^* to construct a measurable structure M_{T^*}. Finally, we prove that M_{T^*} is a model of T^* (and, consequently, of T).

In the following definition we show how to extend a consistent set T to a maximal consistent set T^*. Here we assume that an enumeration of all formulas is given, say $For_{LPP_2^{ind}} = \{\rho_i \mid i = 0, 1, 2, \ldots\}$. We start with the set T and at each step we consider the next formula in the enumeration. Note that different enumerations may lead to different extensions.

Definition 5.1. (Extension to a maximally consistent set) Let T be a consistent set of formulas and let $\{\rho_i \mid i = 0, 1, 2, \ldots\}$ be an enumeration of all formulas from $For_{LPP_2^{ind}}$. We construct T^* recursively as follows:

1. $T_0 = T$.
2. If the formula ρ_i is consistent with T_i, then $T_{i+1} = T_i \cup \{\rho_i\}$.
3. If the formula ρ_i is not consistent with T_i, then there are three cases:
   (a) If $\rho_i = \phi \to P_{\geq r}\alpha$, then
   $$T_{i+1} = T_i \cup \{\phi \to P_{<r-\frac{1}{k}}\alpha\},$$
   where k is the smallest positive integer such that $r - \frac{1}{k} \geq 0$ and T_{i+1} is consistent.
   (b) If $\rho_i = \phi \to (\alpha \nearrow \beta)$, then
   $$T_{i+1} = T_i \cup \{\phi \to (P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{<rs}(\alpha \wedge \beta))\},$$
   where r and s are two (arbitrarily chosen) rational numbers from the unit interval such that T_{i+1} is consistent.
   (c) If $\rho_i = \phi \to (\alpha \searrow \beta)$, then
   $$T_{i+1} = T_i \cup \{\phi \to (P_{\leq r}\alpha \wedge P_{\leq s}\beta \wedge P_{>rs}(\alpha \wedge \beta))\},$$
   where r and s are two (arbitrarily chosen) rational numbers from the unit interval such that T_{i+1} is consistent.
   (d) Otherwise, $T_{i+1} = T_i$.
4. $T^* = \bigcup_{n=0}^{\infty} T_n$.

The steps 3(a), 3(b) and 3(c) of the construction are added to make sure that the infinitary rules R3, R4 and R5 (respectively) cannot be applied to T^* in order to produce inconsistencies. The step 3(a) is correctly defined, since the existence of such k is provided by Lemma 4.5.1. Similarly, Lemma 4.5.2 provides existence of r and s from the step 3(b), while the existence of r and s from 3(c) is ensured by Lemma 4.5.3.

THEOREM 5.2. (Lindenbaum's lemma) *Every consistent set of formulas can be extended to a maximally consistent set.*

PROOF. Let T be a consistent set and let T^* be its extension constructed in Definition 5.1. We need to show that the set T^* is maximally consistent. Each T_i is consistent by the construction. In order to prove the consistency of T^*, we first prove that it is deductively closed. If the formula ρ is an instance of some axiom, then $\rho \in T^*$ by construction of T^*. Next we prove that T^* is closed under inference rules. The only possible problem is with the infinitary inference rules R3, R4 and R5. We will show that T^* is closed under R4, while the case when we deal with R3 or R5 can be proved in a similar way. First we need to show maximality of T^* for the probabilistic formulas, i.e.,

$$\phi \in T^* \quad \text{or} \quad \neg\phi \in T^*, \quad \text{whenever} \;\; \phi \in For_P.$$

For an arbitrary $\phi \in For_P$, let i and j be the nonnegative integers such that $\rho_i = \phi$ and $\rho_j = \neg\phi$. By Theorem 4.2.2, either ϕ or $\neg\phi$ is consistent with $T_{\max\{i,j\}}$. But then either $\phi \in T_{i+1}$ or $\neg\phi \in T_{j+1}$.

Now we turn to R4. Assume that

$$\phi \to ((P_{\geq r}\alpha \wedge P_{\geq s}\beta) \to P_{\geq rs}(\alpha \wedge \beta)) \in T^*$$

for all $r, s \in [0,1] \cup \mathbb{Q}$. We need to show that $\phi \to (\alpha \nearrow \beta) \in T^*$. So suppose that $\phi \to (\alpha \nearrow \beta) \notin T^*$. By maximality of T^*, $\neg(\phi \to (\alpha \nearrow \beta)) \in T^*$. Consequently, $\phi \in T^*$. Then there is i such that $\phi \in T_i$. Let j be a nonnegative integer such that $\rho_j = \phi \to (\alpha \nearrow \beta)$. By step 3(b) of the construction, there are $r', s' \in [0,1] \cup \mathbb{Q}$ such that

$$\phi \to (P_{\geq r'}\alpha \wedge P_{\geq s'}\beta \wedge P_{<r's'}(\alpha \wedge \beta)) \in T_{j+1}.$$

Let k be the nonnegative integer such that $\rho_k = \phi \to ((P_{\geq r'}\alpha \wedge P_{\geq s'}\beta) \to P_{\geq r's'}(\alpha \wedge \beta))$. Then

$$T_{\max\{i,k+1\}} \vdash (P_{\geq r'}\alpha \wedge P_{\geq s'}\beta) \to P_{\geq r's'}(\alpha \wedge \beta).$$

On the other hand, we have

$$T_{\max\{i,j+1\}} \vdash P_{\geq r'}\alpha \wedge P_{\geq s'}\beta \wedge P_{<r's'}(\alpha \wedge \beta).$$

Consequently, $T_{\max\{i,j+1,k+1\}} \vdash \bot$, a contradiction. Thus, T^* is deductively closed.

Finally, T^* is consistent. Otherwise there is $\phi \in For_P$ such that $T^* \vdash \phi \wedge \neg\phi$. Then there is i such that $\phi \wedge \neg\phi \in T_i$, a contradiction. ■

Next we use T^* to define a measurable structure.

Definition 5.3. Let T^* be a maximally consistent set of formulas. We define a tuple $M_{T^*} = \langle W, H, \mu, v\rangle$ as follows:

- $W = \{w \mid w$ is a classical propositional interpretation that satisfies the set $T^* \cap For_C\}$
- $H = \{[\alpha] \mid \alpha \in For_C\}$, where $[\alpha] = \{w \in W \mid w \models_{prop} \alpha\}$ (here $\models_{prop}$ denotes the satisfiability relation of propositional logic),
- $\mu : H \to [0,1]$ such that $\mu([\alpha]) = \sup\{r \in [0,1] \cup \mathbb{Q} \mid P_{\geq r}\alpha \in T^*\}$,
- for every world w and every propositional letter $p \in \mathcal{P}$, $v(w,p) = true$ iff $w \models_{prop} p$.

The following result says that the tuple defined in Definition 5.3 is a measurable structure.

Lemma 5.4. *For each maximally consistent set of formulas T^*, $M_{T^*} \in LPP^{ind}_{2,Meas}$.*

Proof. The proof that H is an algebra is straightforward. The proof that μ is a finitely additive probability measure is identical to the one presented in [14]. Finally, M^* is measurable by definition. ∎

Next we state an important relationship between the constructed structure M^* and the operators $\nearrow$ and $\searrow$.

Lemma 5.5. *Let T^* be a maximally consistent set of formulas and let M_{T^*} be the corresponding measurable structure. Then*

1. *$\alpha \nearrow \beta \in T^*$ iff $\mu([\alpha \wedge \beta]) \geq \mu([\alpha])\mu([\beta])$,*
2. *$\alpha \searrow \beta \in T^*$ iff $\mu([\alpha \wedge \beta]) \leq \mu([\alpha])\mu([\beta])$.*

Proof. We will only prove the first statement, since the second statement can be proved in a similar way. Suppose that $\alpha \nearrow \beta \in T^*$. Let $\{r_n \mid n \in \mathbb{N}\}$ and $\{s_n \mid n \in \mathbb{N}\}$ be two increasing sequences of numbers ($r_n < r_{n+1}$ and $s_n < s_{n+1}$ for each $n \in \mathbb{N}$) from $[0,1] \cup \mathbb{Q}$, such that $\lim_{n\to\infty} r_n = \mu([\alpha])$ and $\lim_{n\to\infty} s_n = \mu([\beta])$. Let n be any number from $\mathbb{N}$. Then $T^* \vdash P_{\geq r_n}\alpha \wedge P_{\geq s_n}\beta$. Using the assumption $\alpha \nearrow \beta \in T^*$, the axiom A7 and propositional reasoning (A1 and R1(b)) we obtain $T^* \vdash P_{\geq r_n s_n}(\alpha \wedge \beta)$. Finally, by Definition 5.3 we have

$$\mu([\alpha \wedge \beta]) \geq \lim_{n\to\infty} r_n s_n = \mu([\alpha])\mu([\beta]).$$

Now assume that $\mu([\alpha \wedge \beta]) \geq \mu([\alpha])\mu([\beta])$. We will show that

$$T^* \vdash (P_{\geq r}\alpha \wedge P_{\geq s}\beta) \to P_{\geq rs}(\alpha \wedge \beta) \quad \text{for all } r, s \in [0,1] \cup \mathbb{Q}. \qquad \text{(II)}$$

If $r > \mu([\alpha])$ or $s > \mu([\beta])$, then $T^* \nvdash P_{\geq r}\alpha \wedge P_{\geq s}\beta$. By maximality of T^*, $T^* \vdash \neg(P_{\geq r}\alpha \wedge P_{\geq s}\beta)$, and consequently $T^* \vdash (P_{\geq r}\alpha \wedge P_{\geq s}\beta) \rightarrow P_{\geq rs}(\alpha \wedge \beta)$. If $r \leqslant \mu([\alpha])$ and $s \leqslant \mu([\beta])$, then $rs \leqslant \mu([\alpha \wedge \beta])$ by the assumption, so $T^* \vdash P_{\geq rs}(\alpha \wedge \beta)$ by Definition 5.3. Thus we proved (II) and we can apply R4 to obtain $T^* \vdash \alpha \nearrow \beta$. Now the result follows from the fact that T^* is deductively closed. ■

Now we prove the main result of this section.

Theorem 5.6. (Strong completeness theorem) *A set of formulas T is consistent if and only if there is an $M \in LPP^{ind}_{2,Meas}$ such that $M \models T$.*

Proof. By Theorem 5.2, there is a maximally consistent superset T^* of T. Let M_{T^*} be the corresponding measurable structure from Definition 5.3. It is sufficient to show that $\rho \in T^*$ if and only if $M_{T^*} \models \rho$, for every formula $\rho \in For_{LPP_2^{ind}}$. If ρ is a propositional formula α, suppose that $\alpha \in T^*$. Then $w \models_{prop} \alpha$ for every $w \in W$. By Definition 5.3, $v(w,\alpha) = true$ for every $w \in W$, so $M_{T_*} \models \alpha$. On the other hand, if $\alpha \notin T^*$, then $T^* \cup \{\neg\alpha\}$ is consistent by Theorem 4.2.1. By Completeness theorem for propositional logic, there is w such that $w \models_{prop} T^* \cup \{\neg\alpha\}$. Obviously $w \in W$ and $v(w,\alpha) = false$, so $M_{T_*} \not\models \alpha$. If ρ is a probabilistic formula ϕ, we use induction on the complexity of the formulas. The cases when ϕ is a conjunction or a negation are straightforward. If ϕ is $\alpha \nearrow \beta$ or $\alpha \searrow \beta$, the statement follows from Lemma 5.5. The case when $\phi = P_{\geq r}\alpha$ is proved in [14]. ■

Finally, we point out that from the Theorems 4.1 and 5.6 we obtain the usual formulation of completeness: $T \vdash \rho$ iff $T \models \rho$.

6. Decidability of LPP_2^{ind}

It is known that the problem of satisfiability of classical propositional formulas is decidable and that the problem is NP-complete. Now we turn to probabilistic formulas. We will prove that the satisfiability of the probabilistic formulas is decidable in PSPACE. The proof we give is essentially that of Fagin, Halpern and Megiddo [6] for the decidability of PWFs: we first reduce the problem to a problem of satisfiability of an existential sentence of real closed fields, and then we apply Canny's decision procedure [1]. In fact, our complexity result can also be obtained by the result from [6], since it is possible to rewrite any formula of LPP_2^{ind} as a PWF. To make this paper self-contained, we give the proof of the decidability result for LPP_2^{ind}, since we extend the idea (of translating the problem to a system of inequalities) in the proof of decidability of satisfiability problem for LPP_1^{ind} in Section 7.2.

THEOREM 6.1. *The satisfiability of probabilistic formulas is decidable in PSPACE.*

PROOF. For a formula $\phi \in For_P$, by $Basic(\phi)$ we denote the set of all basic probabilistic formulas which are subformulas of ϕ. We will assume that the formula $\phi \in For_P$ is in the complete disjunctive normal form (CDNF),. i.e.,

$$\phi = \bigvee_{i=1}^{m} \phi_i,$$

where every ϕ_i is a conjunction of the formulas from $Basic(\phi)$ or their negations, using all elements of $Basic(\phi)$ (i.e. the number of conjuncts of each ϕ_i is $|Basic(\phi)|$). Since ϕ is satisfiable iff at least one ϕ_i is satisfiable, we can focus on satisfiability of the formulas of the form

$$\bigwedge_{k=1}^{|Basic(\phi)|} \psi_k, \tag{III}$$

where $\psi_k \in \{P_{\geq r}\alpha, \alpha \nearrow \beta, \alpha \searrow \beta, P_{<r}\alpha, \alpha \downarrow \beta, \alpha \uparrow \beta, \mid \alpha, \beta \in For_C\}$. For the given formula (III), we denote the set $\{\psi_k \mid k = 1, \ldots, |Basic(\phi)|\}$ by F.

We also use CDNF for the classical propositional formulas. Thus, if $p_1, \ldots, p_n$ are all of the propositional letters appearing in (III), let $\gamma_1, \ldots, \gamma_{2^n}$ be all of the formulas of the form

$$\pm p_1 \wedge \cdots \wedge \pm p_n,$$

where $+p = p$ and $-p = \neg p$. Clearly, γ_i's are pairwise disjoint and form a partition of $\top$. Furthermore, for each α appearing in (III) there is a unique set of indices $I_\alpha \subseteq \{1, \ldots, 2^n\}$ such that $\alpha \leftrightarrow \bigvee_{i \in I_\alpha} \gamma_i$ is a tautology. We denote by Γ_α the corresponding set $\{\gamma_i \mid i \in I_\alpha\}$.

If ψ_k is a formula from (III) of the form $P_{\geq r}\alpha$, using finite additivity we obtain that $M = \langle W, H, \mu, v \rangle$ satisfies ψ_k iff

$$\sum_{\gamma \in \Gamma_\alpha} \mu([\gamma]) \geq r. \tag{IV}$$

Similarly, if ψ_k is of the form $\alpha \nearrow \beta$, we have the condition

$$\sum_{\gamma_{\alpha\wedge\beta} \in \Gamma_{\alpha\wedge\beta}} \mu\left([\gamma_{\alpha\wedge\beta}]\right) \geq \left(\sum_{\gamma_\alpha \in \Gamma_\alpha} \mu([\gamma_\alpha])\right) \left(\sum_{\gamma_\beta \in \Gamma_\beta} \mu([\gamma_\beta])\right). \tag{V}$$

The conditions for satisfiability of the formulas ψ_k of the form $\alpha \searrow \beta$, $P_{<r}\alpha$, $\alpha \downarrow \beta$ and $\alpha \uparrow \beta$ can be written in a similar way.

For each $i \in \{1, \dots, 2^n\}$ we denote by x_i the value of the formula γ_i in a potential model $M = \langle W, H, \mu, v\rangle$ of the formula (III), i.e., $x_i = \mu([\gamma_i])$. Then the formula is satisfiable iff the following sentence of the language of real closed fields is satisfiable:

$$\begin{aligned}
\exists x_1 \dots \exists x_{2^n} \Bigg(& \bigwedge_{k=1}^{2^n} (x_k \geq 0) \\
& \wedge \sum_{k=1}^{2^n} x_k = 1 \\
& \wedge \bigwedge_{P_{\geq r}\alpha \in F} \left(\sum_{\gamma_i \in \Gamma_\alpha} x_i \geq r \right) \\
& \wedge \bigwedge_{\alpha \nearrow \beta \in F} \sum_{\gamma_i \in \Gamma_{\alpha\wedge\beta}} x_i \geq \left(\sum_{\gamma_i \in \Gamma_\alpha} x_i \right) \left(\sum_{\gamma_i \in \Gamma_\beta} x_i \right) \\
& \wedge \bigwedge_{\alpha \searrow \beta \in F} \sum_{\gamma_i \in \Gamma_{\alpha\wedge\beta}} x_i \leq \left(\sum_{\gamma_i \in \Gamma_\alpha} x_i \right) \left(\sum_{\gamma_i \in \Gamma_\beta} x_i \right) \\
& \wedge \bigwedge_{P_{<r}\alpha \in F} \left(\sum_{\gamma_i \in \Gamma_\alpha} x_i < r \right) \\
& \wedge \bigwedge_{\alpha \downarrow \beta \in F} \sum_{\gamma_i \in \Gamma_{\alpha\wedge\beta}} x_i < \left(\sum_{\gamma_i \in \Gamma_\alpha} x_i \right) \left(\sum_{\gamma_i \in \Gamma_\beta} x_i \right) \\
& \wedge \bigwedge_{\alpha \uparrow \beta \in F} \sum_{\gamma_i \in \Gamma_{\alpha\wedge\beta}} x_i > \left(\sum_{\gamma_i \in \Gamma_\alpha} x_i \right) \left(\sum_{\gamma_i \in \Gamma_\beta} x_i \right) \Bigg).
\end{aligned}$$

The sentence represents a nonlinear system of inequalities. The first line represents non-negativity of probability measures. The second line represents the condition $\mu(W) = \mu([\top]) = 1$. The third and the fourth line represent the conditions (IV) and (V), respectively, while the last four lines represent the similar conditions for the other basic formulas from F.

Since the theory of real closed fields is decidable, our logic is decidable as well. Moreover, note that the above sentence is an existential sentence. Thus, we can use Canny's decision procedure from [1]. Since the procedure decides satisfiability of the formula in PSPACE, we conclude that satisfiability of probabilistic formulas is in PSPACE as well. ∎

7. The Logic LPP_1^{ind}

In this section we present the logic LPP_1^{ind} which extends LPP_2^{ind} in the way that nesting of the probabilistic operators is allowed. The set $For_{LPP_1^{ind}}$ of formulas is the smallest set such that:

- $\mathcal{P} \subset For_{LPP_1^{ind}}$,
- if $\{\alpha, \beta\} \subseteq For_{LPP_1^{ind}}$ and $r \in [0,1] \cap \mathbb{Q}$, then $\{P_{\geq r}\alpha, \alpha \nearrow \beta, \alpha \searrow \beta, \alpha \wedge \beta, \neg\alpha\} \subseteq For_{LPP_1^{ind}}$.

We can also see LPP_1^{ind} as an extension of the logic LPP_2 from [14], obtained by adding the operators $\nearrow$ and $\searrow$. Now we can mix probabilistic and classical propositional knowledge in a single formula, so we don't introduce two different notations – we denote all the formulas from $For_{LPP_1^{ind}}$ by α, β, ...

Example 7.1. *If* $\alpha, \beta \in For_{LPP_1^{ind}}$, *then*

$$(\alpha \perp \beta) \uparrow (\alpha \vee \beta)$$

is also a formula from $For_{LPP_1^{ind}}$. *Its meaning is that the independence of* α *and* β *confirms their disjunction* $\alpha \vee \beta$. *If* $p \in \mathcal{P}$, *the formula*

$$p \rightarrow P_{\geq 1}p$$

also belongs to $For_{LPP_1^{ind}}$, *but not to* $For_{LPP_2^{ind}}$, *since it mixes nonprobabilistic and probabilistic knowledge.*

The semantics for the logic $For_{LPP_1^{ind}}$ is defined in a modal way, and now each world is equipped with a probability space. More formally, a structure is any tuple $M = \langle W, Prob, v\rangle$ where W is a nonempty set of worlds, v assigns to every $w \in W$ a two-valued evaluation of propositional letters, while $Prob(w) = \langle W(w), H(w), \mu(w)\rangle$ is a triple where:

- $W(w)$ is a non empty subset of W,
- $H(w)$ is an algebra of subsets of $W(w)$ and
- $\mu(w) : H(w) \rightarrow [0,1]$ is a finitely additive probability measure.

The definition of $\models$ is also modified, since now it depends on worlds of a model.

- $M, w \models P_{\geq r}\alpha$ if $\mu(w)(\{u \in W(w) \mid M, u \models \alpha\}) \geq r$,
- $M, w \models \alpha \nearrow \beta$ if $\mu(w)(\{u \in W(w) \mid M, u \models \alpha \wedge \beta\}) \geq$ $\mu(w)(\{u \in W(w) \mid M, u \models \alpha\})\mu(w)(\{u \in W(w) \mid M, u \models \beta\})$,

- $M, w \models \alpha \searrow \beta$ if $\mu(w)(\{u \in W(w) \mid M, u \models \alpha \wedge \beta\}) \leq \mu(w)(\{u \in W(w) \mid M, u \models \alpha\})\mu(w)(\{u \in W(w) \mid M, u \models \beta\})$.

Satisfiability of conjunctions and negations is defined as usual.

We denote the set $\{u \in W(w) \mid M, u \models \alpha\}$ by $[\alpha]_{M,w}$ (we omit M from $M, w \models \alpha$ and $[\alpha]_{M,w}$ if it is clear from the context). Like in the logic $For_{LPP_2^{ind}}$, we consider measurable models only. We say that α is satisfiable, if there is $M = \langle W, Prob, v\rangle$ and $w \in W$ such that $M, w \models \alpha$.

7.1. Completeness of LPP_1^{ind}

The axiomatic system for the logic LPP_2^{ind} is also sound and strongly complete for the logic LPP_1^{ind}. Thus, $Ax_{LPP_1^{ind}} = Ax_{LPP_2^{ind}}$. Of course, A1 and R1 now don't have two variants. A proof is still a countable sequence of formulas. The only difference is that Necessitation is now restricted to theorems only. Thus, the set of theorems is the set of all formulas deducible from the empty set, while we derive from T using all elements of T, all theorems and we apply the rules R1, R3, R4 and R5. The restricted use of Necessitation allows easy proof of Deduction theorem.

The proof of completeness theorem is almost identical to the proof for the logic LPP_2^{ind}. The only difference is when we use a maximally consistent set T^* to construct M_{T^*}. Instead, we define a model $M^* = \langle W, Prob, v\rangle$ where W is the set of all maximally consistent formulas of $For_{LPP_1^{ind}}$, $v(w, p) = true$ iff $p \in w$, while $Prob(w) = \langle W(w), H(w), \mu(w)\rangle$ is defined in the following way:

- $W(w) = W$,
- $H(w) = \{\{u \in W \mid \alpha \in u\} \mid \alpha \in For_{LPP_1^{ind}}\}$,
- $\mu(w)(\{u \in W \mid \alpha \in u\}) = \sup\{r \in [0, 1] \cup \mathbb{Q} \mid P_{\geq r}\alpha \in w\}$.

The proof that $\alpha \in w$ iff $w \models \alpha$ is the same as before. Then, if T is a consistent set and if w is a maximally consistent superset of T, we obtain $M, w \models T$.

7.2. Decidability of LPP_1^{ind}

In the proof of decidability of satisfiability problem for LPP_1^{ind}, we use the method of filtration [10]. We show that if α is satisfiable in $M = \langle W, Prob, v\rangle$, then there is a model M^* of α, with at most $2^{|\mathrm{Subf}(\alpha)|}$ worlds, where $\mathrm{Subf}(\alpha)$ denote the set of all subformulas of α. By $\approx$ we denote the equivalence relation over W^2, such that $w \approx u$ iff for every $\beta \in \mathrm{Subf}(\alpha)$, $w \models \beta$ iff $u \models \beta$. Then cardinality of the quotient set $W_{/\approx}$ is at most $2^{|\mathrm{Subf}(\alpha)|}$. From every class C_i we choose an element and denote it w_i. We define the structure $M^* = \langle W^*, Prob^*, v^* \rangle$, where

- $W^* = \{w_i \mid C_i \in W_{/\approx}\}$,
- $Prob^*$ is defined by:
  - $W^*(w_i) = \{w_j \in W^* : (\exists u \in C_{w_j}) u \in W(w_i)\}$
  - $H^*(w_i)$ is the powerset of $W^*(w_i)$,
  - $\mu^*(w_i)$ is a probability measure s.t. $\mu^*(w_i)(\{w_j\}) = \mu(w_i)(C_{w_j})$,
- $v^*(w_i, p) = v(w_i, p)$.

Obviously, $\mu^*(w_i)$ is well defined, and for any $D \in H^*(w_i)$

$$\mu^*(w_i)(D) = \sum_{w_j \in D} \mu^*(w_i)(\{w_j\}).$$

Note that for each $w' \in C_w$, $M, w \models \beta$ iff $M, w' \models \beta$. Now we show that for any $\beta \in \text{Subf}(\alpha)$ and $w \in W^*$, $M, w \models \beta$ iff $M^*, w \models \beta$. We prove the statement using induction on the number n, the sum of numbers of appearances of $\nearrow$ and $\searrow$ in β, and for each n we use induction on the complexity of the formulas. The proof is straightforward and we only prove the case when $\beta = \alpha_1 \nearrow \beta_1$:

$M, w \models \beta$

iff $\mu(w)([\alpha_1 \wedge \beta_1]_{M,w}) \geq \ \mu(w)([\alpha_1]_{M,w})\mu(w)[\beta_1]_{M,w})$

iff $\sum_{C_u : M,u \models \alpha_1 \wedge \beta_1} \mu(w_i)(C_u) \geq$

$\quad (\sum_{C_u : M,u \models \alpha_1} \mu(w_i)(C_u))(\sum_{C_u : M,u \models \beta_1} \mu(w_i)(C_u))$

iff $\sum_{C_u : M^*,u \models \alpha_1 \wedge \beta_1} \mu^*(w_i)(\{u\}) \geq$

$\quad (\sum_{C_u : M^*,u \models \alpha_1} \mu^*(w_i)(\{u\}))(\sum_{C_u : M^*,u \models \beta_1} \mu^*(w_i)(\{u\}))$

iff $\mu^*(w)([\alpha_1 \wedge \beta_1]_{M^*,w}) \geq \ \mu^*(w)([\alpha_1]_{M^*,w})\mu^*(w)[\beta_1]_{M^*,w})$

iff $M^*, w \models \beta$.

Thus we proved that for checking satisfiability of a formula it is enough to check structures with at most $2^{|\text{Subf}(\alpha)|}$ worlds. Note that it does not necessarily imply decidability of the satisfiability problem because there are infinitely many possibilities for probability values.

We proceed by considering filtrated structures defined above and modifying the method from Section 6. Instead of standard DNF, we transform the formula α to disjunction of the formulas of the form

$$\bigwedge_{k=1}^{|\mathrm{Subf}(\alpha)|} \beta_k, \tag{VI}$$

where each β_k belongs to $\mathrm{Subf}(\alpha) \cup \{\neg\beta \mid \beta \in \mathrm{Subf}(\alpha)\}$, and each subformula of α appears exactly once (negated or not). Obviously the conjunction of any two different formulas of the form (VI) is a contradiction, while the disjunction of all such formulas is a tautology. This enables us to translate the problem to a system of inequalities. In each world w of each filtrated model M exactly one formula of the form (VI) holds. We can denote such formula by α_w. For each possible cardinality ℓ of W (i.e. $\ell \leqslant 2^{|\mathrm{Subf}(\alpha)|}$), we consider the ℓ formulas of the form (VI) (here denoted by α_w). Note that those formulas are not necessarily different, but at least one of them must contain α. Now we examine whether there is a structure M with ℓ worlds such that for some world w from the model $w \models \alpha$. For **every** formula α_{w_i}, $i < \ell$, we consider a system of linear equalities and inequalities (similarly as in Section 6) of the form (here we denote by $\xi(\alpha_w)$ the set of all conjuncts β_k from $\alpha_w = \bigwedge_{k=1}^{|\mathrm{Subf}(\alpha)|} \beta_k$):

$\mu(w_i)(\{w_j\}) \geq 0$, for every world w_j

$\sum_{j=1}^{\ell} \mu(w_i)(\{w_j\}) = 1$

$\sum_{w_j : \beta \in \xi(\alpha_{w_j})} \mu(w_i)(\{w_j\}) \geq r$, for every $P_{\geq r}\beta \in \xi(\alpha_{w_j})$

$sum_{w_j : \beta\wedge\gamma \in \xi(\alpha_{w_j})} \mu(w_i)(\{w_j\}) \geq (\sum_{w_j : \beta \in \xi(\alpha_{w_j})} \mu(w_i)(\{w_j\}))$

$(\sum_{w_j : \gamma \in \xi(\alpha_{w_j})} \mu(w_i)(\{w_j\}))$, for every $\beta \nearrow \gamma \in \xi(\alpha_{w_j})$

$\sum_{w_j : \beta\wedge\gamma \in \xi(\alpha_{w_j})} \mu(w_i)(\{w_j\}) \leq (\sum_{w_j : \beta \in \xi(\alpha_{w_j})} \mu(w_i)(\{w_j\}))$

$(\sum_{w_j : \gamma \in \xi(\alpha_{w_j})} \mu(w_i)(\{w_j\}))$, for every $\beta \searrow \gamma \in \xi(\alpha_{w_j})$

$\sum_{w_j : \beta \in \xi(\alpha_{w_j})} \mu(w_i)(\{w_j\}) < r$, for every $\neg P_{\geq r}\beta \in \xi(\alpha_{w_j})$

$\sum_{w_j : \beta\wedge\gamma \in \xi(\alpha_{w_j})} \mu(w_i)(\{w_j\}) < (\sum_{w_j : \beta \in \xi(\alpha_{w_j})} \mu(w_i)(\{w_j\}))$

$(\sum_{w_j : \gamma \in \xi(\alpha_{w_j})} \mu(w_i)(\{w_j\}))$, for every $\neg\beta \nearrow \gamma \in \xi(\alpha_{w_j})$

$\sum_{w_j : \beta\wedge\gamma \in \xi(\alpha_{w_j})} \mu(w_i)(\{w_j\}) > (\sum_{w_j : \beta \in \xi(\alpha_{w_j})} \mu(w_i)(\{w_j\}))$

$(\sum_{w_j : \gamma \in \xi(\alpha_{w_j})} \mu(w_i)(\{w_j\}))$, for every $\neg\beta \searrow \gamma \in \xi(\alpha_{w_j})$

The big system collect the systems for each w_i. We can translate the system to the sentence of RCF, so satisfiabilty of the system is decidable. If for fixed ℓ and α_w (i.e. the fixed formula of the form (VI)) the system is satisfiable, in each of ℓ worlds we can define a probability space, and in at least one world the formula α holds, so α is satisfiable. Since we have finitely many possibilities for the choice of ℓ, and for each ℓ finitely many possibilities for the choice of ℓ formulas α_w, the logic LPP_1^{ind} is decidable.

8. The Logics $LPP_2^{Fr(n),ind}$ and $LPP_1^{Fr(n),ind}$

The logics $LPP_2^{Fr(n)}$ and $LPP_1^{Fr(n)}$, introduced in [14], are similar to LPP_2 and LPP_1 (respectively), but with probability functions restricted to have the range $Range(n) = \{0, 1/n, \ldots, (n-1)/n, 1\}$. So there are really denumerably many different logics of both types, one for each n. It is shown in [14] that the infinitary rule R3 can be replaced by the axiom

A9 $\bigwedge_{k=0}^{n-1} P_{>\frac{k}{n}}\alpha \to P_{\geq\frac{k+1}{n}}\alpha$.

The axiomatic system is finitary and the proofs are defined as finite sequences of formulas. The following theorem of the axiomatization

$$\vdash \bigvee_{k=0}^{n} P_{=\frac{k}{n}}\alpha$$

is used in the proof of the completeness theorem. Since the finitary system is strongly complete, the logic is compact.

In the same way, we can restrict both of our logics, LPP_2^{ind} and LPP_1^{ind}, to obtain the logics $LPP_2^{Fr(n),ind}$ and $LPP_1^{Fr(n),ind}$ whose probability measures have the ranges $Range(n)$. Obviously, those logics are extensions of $LPP_2^{Fr(n)}$ and $LPP_1^{Fr(n)}$.

Now we show that there are finitary axiomatizations for those logics. As in the case of the logics without $\nearrow$ and $\searrow$, we replace R3 with A9. Now we turn to R4 and R5. Note that from the finite set

$$T = \{(P_{>\frac{k}{n}}\alpha \wedge P_{>\frac{\ell}{n}}\beta) \to P_{\geq\frac{k\ell}{n^2}}(\alpha \wedge \beta) \mid k, \ell = 0, \ldots, n\}$$

we can infer all the elements of the set

$$\{(P_{\geq r}\alpha \wedge P_{\geq s}\beta) \to P_{\geq rs}(\alpha \wedge \beta) \mid r, s \in [0,1] \cup \mathbb{Q}\}.$$

Indeed, let $r, s \in (0,1] \cup \mathbb{Q}$ (the case when $rs = 0$ is trivial). If k and ℓ are such that $r \in (\frac{k}{n}, \frac{k+1}{n}] \cap \mathbb{Q}$, and $s \in (\frac{\ell}{n}, \frac{\ell+1}{n}] \cap \mathbb{Q}$, using contraposition of A4 we obtain $\{P_{\geq r}\alpha \wedge P_{\geq s}\beta\} \vdash P_{>\frac{k}{n}}\alpha \wedge P_{>\frac{\ell}{n}}\beta$. From A9 we infer $\{P_{\geq r}\alpha \wedge P_{\geq s}\beta\} \vdash P_{\geq\frac{k+1}{n}}\alpha \wedge P_{\geq\frac{\ell+1}{n}}\beta$. Then

$$T \cup \{P_{\geq r}\alpha \wedge P_{\geq s}\beta\} \vdash P_{\geq\frac{(k+1)(\ell+1)}{n^2}}(\alpha \wedge \beta).$$

Using A4 we obtain $T \cup \{P_{\geq r}\alpha \wedge P_{\geq s}\beta\} \vdash P_{\geq rs}(\alpha \wedge \beta)$. Now the claim follows from Deduction theorem.

Thus, for each α and β we can replace the infinite set of premises from R4 with the finite set T.[3]

So we remove R4 from the axiomatization and introduce the new axiom

A10 $\quad (\bigwedge_{k,\ell=0}^{n}(P_{\geq \frac{k}{n}}\alpha \wedge P_{\geq \frac{\ell}{n}}\beta) \to P_{\geq \frac{k\ell}{n^2}}(\alpha \wedge \beta)) \to \alpha \nearrow \beta$.

Similarly, we can prove that we can replace R5 with the axiom

A11 $\quad (\bigwedge_{k,\ell=0}^{n}(P_{\leq \frac{k}{n}}\alpha \wedge P_{\leq \frac{\ell}{n}}\beta) \to P_{\leq \frac{k\ell}{n^2}}(\alpha \wedge \beta)) \to \alpha \searrow \beta$.

Thus, our logics have finitary axiomatizations.

Moreover, in the similar way as above, we can prove that from the finite set

$$\{(P_{\geq \frac{k}{n}}\alpha \wedge P_{\geq \frac{\ell}{n}}\beta \wedge \alpha \nearrow \beta) \to P_{\geq \frac{k\ell}{n^2}}(\alpha \wedge \beta) \mid k, \ell = 0, \ldots, n\}$$

we can infer every instance of A7. Since the formula $(P_{\geq \frac{k}{n}}\alpha \wedge P_{\geq \frac{\ell}{n}}\beta \wedge \alpha \nearrow \beta) \to P_{\geq \frac{k\ell}{n^2}}(\alpha \wedge \beta)$ is equivalent to $\alpha \nearrow \beta \to ((P_{\geq \frac{k}{n}}\alpha \wedge P_{\geq \frac{\ell}{n}}\beta) \to P_{\geq \frac{k\ell}{n^2}}(\alpha \wedge \beta))$, we can restrict A7 to

A7' $\quad \alpha \nearrow \beta \to \bigwedge_{k,\ell=0}^{n}((P_{\geq \frac{k}{n}}\alpha \wedge P_{\geq \frac{\ell}{n}}\beta) \to P_{\geq \frac{k\ell}{n^2}}(\alpha \wedge \beta))$.

Obviously, we can also restrict A8 to

A8' $\quad \alpha \searrow \beta \to \bigwedge_{k,\ell=0}^{n}((P_{\leq \frac{k}{n}}\alpha \wedge P_{\leq \frac{\ell}{n}}\beta) \to P_{\leq \frac{k\ell}{n^2}}(\alpha \wedge \beta))$.

Finally, note that A7' and A9 together say that the operator $\nearrow$ is definable in the logics $LPP_2^{Fr(n)}$ and $LPP_1^{Fr(n)}$ (i.e., $\alpha \nearrow \beta$ can be introduced as an abbreviation for the formula $\bigwedge_{k,\ell=0}^{n}((P_{\geq \frac{k}{n}}\alpha \wedge P_{\geq \frac{\ell}{n}}\beta) \to P_{\geq \frac{k\ell}{n^2}}(\alpha \wedge \beta)))$, while $\searrow$ is definable by A8' and A9. Thus, for each n, the logic $LPP_2^{Fr(n),ind}$ is a conservative extension of the logic $LPP_2^{Fr(n)}$, and $LPP_1^{Fr(n),ind}$ is a conservative extension of $LPP_1^{Fr(n)}$.

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[3] Actually, the premises in R4 have implicative form ($\phi \to$). Since the form is only introduced for the proof of Deduction theorem and is not essential for characterizing $\nearrow$, we proved a restricted version here. Nevertheless, the proof for the original version is practically the same, since we can use Deduction theorem to move ϕ on the left hand side of $\vdash$.

9. Conclusion

In this paper we present several probabilistic logics in which we can express the notions of independence and confirmation. Our starting point for this work was prior research on axiomatization of propositional logics with unary probabilistic operators [14]. We extend the language of the logics from [14], we show decidability of the obtained logics, and we present axiomatizations which are sound and strongly complete with respect to corresponding semantics. Since the logics use the same language, we give the presentation in complete detail for the version LPP_2^{ind} only and then explain briefly how the others differ from it. Since LPP_2^{ind} does't allow nesting of probability operators, we extend it to LPP_1^{ind}, where nesting is allowed. A disadvantage of both of these logics is noncompactness, and we have to use infinitary rules for complete axiomatizations. However, we achieve compactness using only a finite set of probability values for logics $LPP_1^{Fr(n),ind}$ and $LPP_2^{Fr(n),ind}$, which is still enough for many practical applications. We also show that these two logics are conservative extensions of the logics $LPP_1^{Fr(n)}$ and $LPP_2^{Fr(n)}$.

In the paper, we don't explicitly deal with the notion of mutually independent events. A finite set S is said to be mutually independent iff for every $S' \subseteq S$, $\mu(\bigwedge_{A \in S'}) = \prod_{A \in S'} \mu(A)$. We point out that the notion is also expressible in our logics. For any set of formulas F' with n elements, the corresponding formula is

$$\bigwedge_{k \leq n, \alpha_i \in F', \alpha_i \neq \alpha_j} \alpha_1 \perp (\alpha_2 \wedge \cdots \wedge \alpha_k).$$

This fact follows directly from the satisfiability relation.

Note that although the formulation $\mu(A|B) = \mu(A)$ is more intuitive for understanding independence, the formulation $\mu(A \cap B) = \mu(A)\mu(B)$ is more preferred, as the first expression is not defined when $\mu(B) = 0$. Our operator $\perp$ captures the later definition. (Note that in the case of confirmation both definitions are equivalent, since $\mu(A \cap B) > \mu(A)\mu(B)$ implies $\mu(B) > 0$, so $\mu(A|B) > \mu(A)$ can be used as well. Consequently, our operator $\uparrow$ captures both definitions.) Nevertheless, if we would take conditional probabilities as primitive [3], possibly allowing conditions whose probability value is 0, then the former definition cannot be captured in our logics. For that purpose it might be of interest to extend our formalism considering logics that allow conditional probabilities as primitive operators, which can be an avenue for further research on this topic. The logic $LFOCP$ with formulas of the form $CP_{\geq r}(\alpha, \beta)$ and $CP_{\leq r}(\alpha, \beta)$, with the intended meaning "the conditional probability of α given β is at least R" and "at most r", respectively, is considered in [12]. In the future work, we will try to extend $LFOCP$ with the operators of conditional independence and conditional confirmation, modifying the ideas from this paper. If we denote by $(\alpha \nearrow \beta)|\gamma$ the formula which expresses that α weakly conditionally confirms β, given γ, then $(\alpha \nearrow \beta)|\gamma$ would hold in a model with the measure μ on possible worlds, iff $\mu([\alpha \wedge \beta]|[\gamma]) \geq \mu([\alpha]|[\gamma])\mu([\beta]|[\gamma])$.

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