Melvin Fitting

Paraconsistent Logic, Evidence, and Justification

Abstract. In a forthcoming paper, Walter Carnielli and Abilio Rodrigues propose a *Basic Logic of Evidence* (BLE) whose natural deduction rules are thought of as preserving *evidence* instead of truth. BLE turns out to be equivalent to Nelson's paraconsistent logic N4, resulting from adding strong negation to Intuitionistic logic without Intuitionistic negation. The Carnielli/Rodrigues understanding of evidence is informal. Here we provide a formal alternative, using justification logic. First we introduce a modal logic, KX4, in which $\Box X$ can be read as asserting there is implicit evidence for X, where we understand evidence to permit contradictions. We show BLE embeds into KX4 in the same way that Intuitionistic logic embeds into S4. Then we formulate a new justification logic, JX4, in which the implicit evidence motivating KX4 is made explicit. KX4 embeds into JX4 via a realization theorem. Thus BLE has both implicit and explicit possibly contradictory evidence interpretations in a formal sense.

Keywords: Paraconsistent, Justification, Evidence, Modal, Nelson, Strong negation, Belnap.

1. Introduction

In a forthcoming paper [6], two paraconsistent and paracomplete logics are considered. The first is called BLE, for *Basic Logic of Evidence*. The second extends BLE, and is called $\mathsf{LET_J}$, for *Logic of Evidence and Truth*. We do not consider $\mathsf{LET_J}$ in this paper and, without giving too much away, BLE turns out to be equivalent to N4, a well-known logic due to Nelson, though we will continue to call it BLE in this paper. What is of special interest to us in [6] is the motivation provided for BLE/N4. The logic is presented through natural deduction rules, where the underlying idea is that rules should preserve evidence for an assertion, rather than its truth. Further, it is allowed that evidence can be incomplete or contradictory. Evidence is treated informally but plausibly in [6]. It is the purpose of the present paper to fill out the informal ideas motivating BLE with formal evidence based machinery, making use of well-developed ideas coming from *justification logic*.

We will first show that BLE embeds into a modal logic KX4, in which $\Box X$ can be thought of as asserting that there is evidence for X, where this

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evidence is not necessarily correct, that is, factual. We can understand KX4 as being a logic of *implicit* uncertain evidence. We then show KX4 embeds into a justification logic JX4 in which terms represent specific items of uncertain evidence, and there are operations on these pieces of evidence. We can think of JX4 as a logic of *explicit* evidence. Our work parallels that of Artemov [2–4], providing an arithmetic interpretation of intuitionistic logic. Intuitionistic propositional logic embeds into S4, a well-known result of Gödel [9]. Artemov showed that S4 in turn embeds into the first of the justification logics, LP, and that embeds into arithmetic. The notion of evidence represented in S4 implicitly and in LP explicitly is the strongest available, that of *proof*. Evidence as represented in KX4 and JX4 is, of course, weaker, but there are some surprising connections here, as we will see.

2. BLE and Evidence Informally

In [6] BLE is motivated by the idea that it is to be "a paraconsistent formal system capable of expressing the idea of contradictions as conflicting evidence." Further, "*evidence that A is true* is understood as *reasons for believing that A is true*, while *evidence that A is false* means *reasons for believing that A is false*." Thus falsity must be supported by positive evidence, and not simply by lack of evidence for the formula being negated. Writing $\neg A$ for the negation of A, we have the following familiar four-fold division of Belnap–Dunn. Continuing to quote from [6]:

1. No evidence at all: both A and $\neg A$ do not hold;
2. Only evidence that A is true: A holds and $\neg A$ does not hold;
3. Only evidence that A is false: A does not hold and $\neg A$ holds;
4. Conflicting evidence about A: both A and $\neg A$ hold.

Once formulated, the logic BLE turns out to be equivalent to Nelson's N4. As such, much is known about semantics and proof theory [10]. There are both algebraic and possible world semantics, for instance. But our mission here is to make formal the use of evidence for this purpose. We begin by looking at one of the natural deduction rules given for BLE, the rule for conjunction.

$$\frac{A \quad B}{A \wedge B}$$

Don't think of this as saying that if A and B are true, so is $A \wedge B$. Instead, quoting from [6]: "Indeed, if κ and κ' are evidence, respectively, for A and

Table 1. BLE axioms

$A1$	$P \supset (Q \supset P)$
$A2$	$(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$
$A3$	$(P \wedge Q) \supset P$
$A4$	$(P \wedge Q) \supset Q$
$A5$	$(P \supset Q) \supset ((P \supset R) \supset (P \supset (Q \wedge R)))$
$A6$	$P \supset (P \vee Q)$
$A7$	$Q \supset (P \vee Q)$
$A8$	$(P \supset R) \supset ((Q \supset R) \supset ((P \vee Q) \supset R))$
$A9$	$\neg\neg P \equiv P$
$A10$	$\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$
$A11$	$\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$
$A12$	$\neg(P \supset Q) \equiv (P \wedge \neg Q)$

B, κ and κ' together constitute evidence for $A \wedge B$." Similar evidence based motivations are supplied for other connectives. Ignoring negation for the moment, the rules thus motivated for $\wedge$, $\vee$, and $\supset$ are those of intuitionistic logic. The problem we address is the informality of evidence, as used in [6]. What does evidence "together" mean, in the discussion of conjunction? Similar issues arise for disjunction and implication too, of course.

Negation is, so to speak, not treated negatively but positively; "instead of asking about the conditions of assertibility, we ask about the conditions of refutability." An example of a proposed rule is the following.

$$\frac{\neg A}{\neg(A \wedge B)}$$

The motivation supplied is: "If κ is evidence that A is false, κ constitutes evidence that $A \wedge B$ is false." Evidence is understood as something positive. We see that it is not raining, for instance, this is positive evidence that it is false that it is raining, and hence we have positive evidence that it is not both raining and cold. Once again, the use of evidence for motivation is informal. It does, however, lead to Nelsons N4 conditions for negation.

From here on we do not work with natural deduction rules. N4 has well-known axiomatizations, and these will do fine for our purposes. We give the system axiomatically using the schemes in Table 1 for reference purposes. The only rule of inference is *modus ponens*.

3. KX4 and Non-factive Evidence

It is well-known that Gödel introduced the modern axiomatization of the modal logic S4, building on the idea that $\Box$ has the properties one wants

Table 2. KX4 axioms

Classical	All (or enough) tautologies
K	$\Box(X \supset Y) \supset (\Box X \supset \Box Y)$
X	$\Box\Box X \supset \Box X$
4	$\Box X \supset \Box\Box X$

for *provability*. He observed that intuitionistic logic embeds into S4, giving a semi-formal meaning to the idea that intuitionistic truth could be understood by a classical mathematician as a version of provability. Provability is evidence of the strongest kind. A key axiom of S4 is *factivity*, $\Box X \supset X$, which tells us we are talking about evidence that is certain and never mistaken. We want to weaken S4 to a logic of evidence that may be erroneous. Of course we drop the factivity condition. And in its place we add the axiom schema $\Box\Box X \supset \Box X$. Informally, this tells us that evidence for the existence of evidence for X is sufficient for us to assert that we have evidence for X. It is a plausible condition for evidence, even non-factive evidence, to meet. This axiom schema has sometimes been called C4 but more recently X has come into use, and we adopt this name. The logic KX4 is axiomatized using *modus ponens*, *necessitation*, and the schemes shown in Table 2.

In this paper we will think of KX4 as an implicit logic of non-factive evidence, in the same way that Gödel thought of S4 as a logic of provability. We use the term *implicit* for the obvious reason that evidence is not explicitly shown. The necessity symbol indicates the existence of evidence, but does not say what it is. An explicit counterpart to KX4 will be introduced later.

This logic KX4 is complete with respect to frames meeting the conditions of *transitivity* and *denseness*. Denseness says that if $w_1 \mathcal{R} w_2$ then there is a possible world w_3 such that $w_1 \mathcal{R} w_3 \mathcal{R} w_2$. Since $\Box\Box X \supset \Box X$ is a special case of $\Box Y \supset Y$, it follows that every theorem of KX4 is a theorem of S4, that is, KX4 $\subseteq$ S4. The inclusion is strict, as the following shows. Let P be a propositional letter. $\Box P \supset P$ is S4 valid. But consider the KX4 model defined as follows. The set of possible worlds is the half-open real interval, $[0, 1)$. Accessibility is given by the relation $<$. This frame satisfies transitivity and denseness. Build a model on it by taking P to be true at all worlds except 0. Then $\Box P$ evaluates to true at 0, so at 0 the S4 theorem $\Box P \supset P$ theorem fails.

There are even simpler models showing that KX4 is not S4. A one-point model in which P fails at the unique world will do, but the model discussed above also has the *seriality* condition, and thus provides us with a stronger result. We make no further use of possible world semantics in this paper.

Obviously in KX4 we have $\Box\Box X \equiv \Box X$. We also have substitutivity of proved equivalence since KX4 is a normal modal logic. Hence, in any theorem of KX4 any positive number of consecutive $\Box$ symbols can be replaced by one, preserving theoremhood. We now have the following straightforward but important facts.

Theorem 3.1. *The following hold of KX4.*

1. *KX4* $\subsetneq$ *S4*
2. *In any theorem of KX4, the result of replacing any positive number of consecutive* $\Box$ *symbols by a single one yields another KX4 theorem.*

4. A Gödel Embedding into KX4

There are several embeddings of intuitionistic logic into S4. The best-known and most easily motivated embedding amounts to inserting a necessity symbol before every intuitionistic subformula. Remarkably, this also embeds intuitionistic logic into KX4, something that must be well-known, though it was not known to me. This means we can think of intuitionistic logic as being evidence-based, whether we think of that evidence as factive, or not. We will not want intuitionistic negation, but rather Nelson's strong negation instead. For this section we have neither. We will say a few words about intuitionistic negation in Section 5, before dropping it altogether. Strong negation will be introduced in Section 7.

Definition 4.1. (*Positive Intuitionistic Logic*) Positive intuitionistic logic is propositional intuitionistic logic without negation. Only $\wedge$, $\vee$, and $\supset$ are connectives, though we may also allow $\equiv$ as defined in the usual way.

We give a recursive definition of the Gödel embedding in Definition 4.2, writing X^f for the translation of the intuitionistic formula X. Think of X^f as *evidence for* X in an informal sense. We will consider evidence against later on.

Definition 4.2. (*Implicit Evidence For*) P is a propositional letter; X and Y are arbitrary formulas of positive intuitionistic logic.

$$\begin{aligned} P^f &= \Box P \\ (X \wedge Y)^f &= \Box(X^f \wedge Y^f) \\ (X \vee Y)^f &= \Box(X^f \vee Y^f) \\ (X \supset Y)^f &= \Box(X^f \supset Y^f) \end{aligned}$$

The following result is provable, standard, and generally stated with intuitionistic negation present.

THEOREM 4.3. *For a formula X of positive intuitionistic logic, X is an intuitionistic theorem if and only if X^f is a theorem of* S4.

We now show a counterpart to Theorem 4.3. The proof is straightforward, but a bit tedious and unenlightening. There ought to be a proof that gives more insight, but this will do for now.

THEOREM 4.4. *For a formula X of positive intuitionistic logic, X is an intuitionistic theorem if and only if X^f is a theorem of* KX4.

PROOF. One direction is simple. Suppose X^f is a theorem of KX4. Using Theorem 3.1 part 1, X^f is a theorem of S4. Then by Theorem 4.3, X is a theorem of propositional intuitionistic logic.

For the converse direction, it is enough to show that for each line Z of a positive intuitionistic axiomatic proof, Z^f is provable in KX4. This is true if Z is an axiom by the following argument. Lemma 4.5 gives provability in K of formulas corresponding to positive axioms, hence these are also provable in KX4. Then apply Theorem 3.1 part 2 which allows the collapse of multiple occurrences of $\Box$ to a single one. *Modus ponens* is given by Lemma 4.6. ■

LEMMA 4.5. *The following formulas are provable in* K. *In each case, the corresponding intuitionistic axiom or rule is shown.*

1. $\Box\Box P \supset \Box(\Box Q \supset \Box P)$, *corresponding to* $P \supset (Q \supset P)$.
2. $\Box\Box(\Box P \supset \Box(\Box Q \supset \Box R)) \supset \Box(\Box(\Box P \supset \Box\Box Q) \supset \Box(\Box P \supset \Box\Box R))$, *corresponding to* $(P \supset (Q \supset R)) \supset ((P \supset Q) \supset (P \supset R))$.
3. $\Box\Box\Box P \supset \Box(\Box\Box Q \supset \Box(\Box P \wedge \Box Q))$, *corresponding to* $P \supset (Q \supset (P \wedge Q))$.
4. $\Box(\Box P \wedge \Box Q) \supset \Box\Box P$ *and* $\Box(\Box P \wedge \Box Q) \supset \Box\Box Q$, *corresponding to* $(P \wedge Q) \supset P$ *and* $(P \wedge Q) \supset Q$ *respectively.*
5. $\Box\Box P \supset \Box(\Box P \vee \Box Q)$ *and* $\Box\Box Q \supset \Box(\Box P \vee \Box Q)$, *corresponding to* $P \supset (P \vee Q)$ *and* $Q \supset (P \vee Q)$ *respectively.*
6. $\Box\Box\Box(\Box P \supset \Box R) \supset \Box(\Box\Box(\Box Q \supset \Box R) \supset \Box(\Box(\Box P \vee \Box Q) \supset \Box\Box R))$, *corresponding to* $(P \supset R) \supset ((Q \supset R) \supset ((P \vee Q) \supset R))$.

PROOF. Here, as a sample, is an axiomatic proof for 1. The others are more complicated axiomatically and are left to the reader. Actually, tableaus are recommended.

$\Box P \supset (\Box Q \supset \Box P)$ is a tautology. By necessitation, $\Box(\Box P \supset (\Box Q \supset \Box P)$. Then using the K axiom, $\Box\Box P \supset \Box(\Box Q \supset \Box P)$. ■

Lemma 4.6. *If X^f and $(X \supset Y)^f$ are provable in* KX4*, so is Y^f.*

Proof. Every formula W^f begins with $\Box$, and so is of the form $\Box(W^\circ)$ for some particular W°. Now suppose X^f and $(X \supset Y)^f$ are provable in KX4. By the first, $\Box(X^\circ)$ is provable, hence so is $\Box\Box(X^\circ)$ by the 4 axiom. By the second, $\Box(X^f \supset Y^f)$ is provable, that is, $\Box(\Box(X^\circ) \supset \Box(Y^\circ))$ is provable. Then so is $\Box\Box(X^\circ) \supset \Box\Box(Y^\circ)$ by the K axiom. By *modus ponens*, $\Box\Box(Y^\circ)$ is provable, hence so is $\Box(Y^\circ)$ by the X axiom. But this is Y^f. ■

5. Side Remarks on Intuitionistic Negation

We want strong negation, and not intuitionistic negation. Still, it is worth taking a passing look at how intuitionistic negation behaves under an embedding into KX4. We consider it briefly in this section, writing $\sim$ for intuitionistic negation and $\neg$ for the classical version. The symbol $\sim$ will not be used outside this section.

Negation in intuitionistic logic can be taken to be primitive, or defined using $\bot$. The connection is $\sim X \equiv X \supset \bot$. Either can be used when embedding from intuitionistic logic into S4. We can work directly, setting $(\sim X)^f = \Box\neg X^f$ or indirectly, setting $(X \supset \bot)^f = \Box(X^f \supset \bot^f)$. There is a potential problem however. Presumably $\bot$ is treated as if it were a propositional letter, and so $\bot^f = \Box\bot$. Then for a propositional letter P, $(\sim P)^f = \Box\neg P^f = \Box\neg\Box P = \Box(\Box P \supset \bot)$, but $(P \supset \bot)^f = \Box(P^f \supset \bot^f) = \Box(\Box P \supset \Box\bot)$. Fortunately these are equivalent in S4, using the facts that falsehood implies anything, so $\bot \supset \Box\bot$, and we have factivity, so $\Box\bot \supset \bot$.

In KX4 we do not have factivity and the equivalence between the two treatments of negation breaks down. I do not know for certain that the direct approach, using axioms for $\sim$, does not work, but this seems to be the case. The indirect version, via $\bot$, is quite satisfactory since we have the following, continuing Theorem 4.5.

Theorem 5.1. *The following formulas are provable in* K*.*

1. $\Box\bot \supset \Box P$, *corresponding to* $\bot \supset P$.
2. $\Box\Box(\Box P \supset \Box\Box Q) \supset \Box(\Box(\Box P \supset \Box(\Box Q \supset \Box\bot)) \supset \Box(\Box P \supset \Box\Box\bot))$ *corresponding to* $(P \supset Q) \supset ((P \supset \neg Q) \supset \neg P)$, *or properly* $(P \supset Q) \supset ((P \supset (Q \supset \bot)) \supset (P \supset \bot))$.
3. $\Box\Box\Box P \supset \Box(\Box(\Box P \supset \Box\bot) \supset \Box\Box Q)$, *corresponding to* $P \supset (\neg P \supset Q)$, *or properly* $P \supset ((P \supset \bot) \supset Q)$.

As before, we leave this to the reader. Tableaux are easy for this purpose. Now Theorem 4.4 extends to include intuitionistic negation.

THEOREM 5.2. *For a formula X of intuitionistic logic including $\bot$, with $\sim X$ understood as $X \supset \bot$, X is an intuitionistic theorem if and only if X^f is a theorem of* KX4.

This curious result says that proof, represented implicitly by the modal operator of S4, can be replaced by a weaker notion of implicit evidence without factivity, when trying to capture intuitionistic logic.

6. KX4 and Justification Logic

The first justification logic, LP, was introduced by Sergei Artemov, and corresponds to the modal logic S4. A full discussion is not possible here—[4] is a good general reference. The basic idea is that the necessitation operator of S4 is replaced by explicit proof terms (today they are called *justification terms*). These terms can be thought of as representatives of explicit reasons, or proofs. Instead of $\Box X$ one finds $t{:}X$, where t is a justification term. The formula $t{:}X$ can be read: X is so for reason t, or t is evidence for X. The logic LP provides a kind of calculus for these terms. The key items connecting S4 and LP are these.

1. A *forgetful functor* is introduced, mapping each formula X of LP to a formula X° of S4, by replacing each justification term by $\Box$. In effect, justifications are forgotten but it is remembered that they exist.
2. If X is a theorem of LP then X° is a theorem of S4.
3. A *normal realization* of a modal formula is defined to be the result of replacing each occurrence of $\Box$ in the formula by a justification term, which may be different for each occurrence of $\Box$. Negative occurrences of $\Box$ are replaced with distinct justification variables, while positive occurrences are replaced by justification terms that can be more complicated. In effect, negative positions serve as input positions for justifications, positive positions involve justifications that are calculated from these inputs. Note that if Y is a normal realization of modal formula X, then $Y^\circ = X$.
4. Every theorem of S4 has a normal realization that is a theorem of LP.

Items 2 and 4 say that S4 is embeds in LP, meaning that the forgetful functor maps the set of theorems of LP *onto* the set of theorems of S4.

Since the first justification logic, LP, more such logics have been created. It is now known that the family of modal logics having justification logic counterparts is infinite. In [7], and more fully in [8], I give semantic conditions that are sufficient for when this happens. What are sometimes called *Geach logics* meet these conditions; these are logics axiomatized over K using axiom schemes of the form $\Diamond^k\Box^l X \supset \Box^m\Diamond^n X$. The logic KX4 is such a logic, and these general results apply. The details are not reproduced here as they are quite lengthy, but applying them gives us a justification logic we will call JX4, serving as a justification counterpart of KX4 and connected with it via a realization theorem, just as LP and S4 are connected. Here is a formulation. The first four items in the following Definition are from LP; the fifth is new to JX4.

Definition 6.1. (JX4 *Justification Terms*) The family of *justification terms* is built up as follows.

1. There is a set of *justification variables*, x, y, ... Every justification variable is a justification term.
2. There is a set of *justification constants*, a, b, ... Every justification constant is a justification term.
3. There are binary operation symbols, $+$ and $\cdot$. If u and v are justification terms, so are $(u + v)$ and $(u \cdot v)$.
4. There is a unary operation symbol, !. If t is a justification term, so is $!t$.
5. There is a binary function symbol, c. If t and u are justification terms, so is $(t\,\mathsf{c}\,u)$.

Definition 6.2. (JX4 *Formulas*) *Justification formulas* of JX4 are built up from propositional letters using propositional connectives $\wedge$, $\vee$, $\neg$, $\supset$ together with the additional formation rule: if t is a justification term and X is a justification formula, then $t{:}X$ is a justification formula.

Informally a justification term represents a reason why something is so; $t{:}X$ asserts that t is evidence for X. If justification term t has a complex structure we may write $[t]{:}X$, using square brackets, as a visual aid. No formal meaning should be associated with this other than the usual behavior of parentheses.

Justification variables stand for arbitrary justification terms, and can be substituted for under certain circumstances. Justification constants stand for reasons that are not further analyzed—typically they are reasons for axioms. The $\cdot$ operation corresponds to *modus ponens*. If $X \supset Y$ is so for reason s and X is so for reason t, then Y is so for reason $s \cdot t$. (Note that reasons

Table 3. JX4 axioms

Classical	All tautologies (or enough of them)
$\cdot$	$s{:}(X \supset Y) \supset (t{:}X \supset [s \cdot t]{:}Y)$
$+$	$s{:}X \supset [s + t]{:}X$ and $t{:}X \supset [s + t]{:}X$
!	$s{:}X \supset !s{:}s{:}X$
c	$s{:}t{:}X \supset [s \mathsf{c}\, t]{:}X$

are not unique—Y may be true for other reasons too.) The $+$ operation is a kind of weakening. If X is so for either reason s or reason t, then $s + t$ is also a reason for X. The ! operation is a *justification checker* operation. If we have $t{:}X$ then $!t$ verifies this, and we have $!t{:}t{:}X$. Finally, c is a new operation symbol—it is not part of LP. The idea is that a justification for a justification for X can be combined into a simple justification for X. Thus if we have $t{:}u{:}X$ then we should also have $[t \mathsf{c}\, u]{:}X$.

An axiom system is given in Table 3. As usual, what is given are axiom schemes. The only rule is *modus ponens*.

As with all justification logics, JX4 has no necessitation rule. What takes its place is a *constant specification* $\mathcal{C}$. This is a set of formulas of the form $c{:}A$ where A is an axiom and c is a constant symbol. The idea is, axioms are simply assumed and not analyzed further, and so we have constants to justify them. Since classical logic can be axiomatized in many ways, the classical clause in Table 3 is really somewhat ambiguous. We are really giving a scheme for axiomatizations of JX4 rather than a specific one, so the use of constants must be understood as a kind of parameter. All we assume of a constant specification $\mathcal{C}$ here is that each axiom have a constant that justifies it (axiomatically appropriate) and all instances of the same axiom scheme have the same constant justification (schematic). We ignore most of the details in this note.

Let $\mathcal{C}$ be a constant specification meeting our conditions. We write $\vdash_{\mathsf{JX4}(\mathcal{C})} X$ if there is a sequence of formulas of JX4 in which each is either an axiom of JX4, a member of $\mathcal{C}$, or follows from earlier formulas by *modus ponens*.

The following *provable* item is a replacement for the rule of necessitation, and is standard for justification logics. When needed, we will simply cite this fact (which has a constructive proof) rather than giving the details of the justification term involved.

Lemma 6.3. (Internalization) *If* $\vdash_{\mathsf{JX4}(\mathcal{CS})} X$ *then for some justification term* t, $\vdash_{\mathsf{JX4}(\mathcal{CS})} t{:}X$*. Further, this can be done so that* t *contains no justification variables (and no justification function symbols except* $\cdot$ *and* !*, though this is of lesser significance).*

Note: from here on we will suppress mention of constant specifications and simply say that a formula X is provable in JX4, meaning $\vdash_{\mathsf{JX4}(\mathcal{CS})} X$ for some constant specification $\mathcal{C}$ meeting the axiomatically appropriate and schematic conditions.

The central item that we draw from [8] is the following.

THEOREM 6.4. (Embedding) *The forgetful functor embeds JX4 into KX4. In particular, each theorem of KX4 has a normal realization that is provable in JX4.*

Since positive intuitionistic logic embeds into KX4, and by Theorem 6.4 that embeds into JX4, we can understand positive intuitionistic logic as a logic of justification, or evidence, which may be non-factual, uncertain, or contradictory.

We conclude this discussion with a simple example of a normal realization of a KX4 theorem into JX4. The KX4 formula is not the result of embedding a theorem of intuitionistic logic into KX4 since these tend to be too complicated to see easily what is going on, but the example does illustrate the main features of JX4.

EXAMPLE 6.5. $\Box(\Box P \supset \Box Q) \supset (\Box P \supset \Box Q)$ is a theorem of KX4. It is also a theorem of T, where a proof involves factivity. In KX4 we do not have factivity and the proof is quite different. We begin with a sketch of this KX4 proof.

1.	$\Box(\Box P \supset \Box Q) \supset (\Box\Box P \supset \Box\Box Q)$	K axiom
2.	$\Box P \supset \Box\Box P$	4 axiom
3.	$\Box(\Box P \supset \Box Q) \supset (\Box P \supset \Box\Box Q)$	from 1 and 2
4.	$\Box\Box Q \supset \Box Q$	X axiom
5.	$\Box(\Box P \supset \Box Q) \supset (\Box P \supset \Box Q)$	from 3 and 4

Here is a realization in JX4. $v_1{:}(v_2{:}P \supset v_3{:}Q) \supset (v_2{:}P \supset [(v_1 \cdot !v_2) \mathsf{c}\, v_3]{:}Q)$. In it the v_i are justification variables and no constants are used. And here is a proof of this realization in JX4. It amounts to enhancing each line of the KX4 proof above. Not every example works in this simple way, however.

1.	$v_1{:}(v_2{:}P \supset v_3{:}Q) \supset (!v_2{:}v_2{:}P \supset [v_1 \cdot !v_2]{:}v_3{:}Q)$	$\cdot$ axiom
2.	$v_2{:}P \supset !v_2{:}v_2{:}P$	! axiom
3.	$v_1{:}(v_2{:}P \supset v_3{:}Q) \supset (v_2{:}P \supset [v_1 \cdot !v_2]{:}v_3{:}Q)$	from 1 and 2
4.	$[v_1 \cdot !v_2]{:}v_3{:}Q \supset [(v_1 \cdot !v_2) \mathsf{c}\, v_3]{:}Q$	c axiom
5.	$v_1{:}(v_2{:}P \supset v_3{:}Q) \supset (v_2{:}P \supset [(v_1 \cdot !v_2) \mathsf{c}\, v_3]{:}Q)$	from 3 and 4

7. Adding Boolean Negation to BLE

We now add Nelson's strong negation to positive intuitionistic logic to get BLE, the *Basic Logic of Evidence*, equivalently N4. An axiomatization is given in Table 1. We use $\neg$ for strong negation. Evidence that it is not raining is, typically, not based on reasoning but on observation. We are outdoors and we are not wet. $\neg X$ is supposed to represent the existence of direct evidence that X is not so.

Machinery for handling strong negation is, by now, quite standard, and was stated at the beginning of Section 2. One engineers a disconnect between positive and negative evidence—the motivation behind the Belnap/Dunn logic FOUR. Formally, one can think of FOUR as a four-valued lattice, or as two separate two-valued lattices put together. All this is, by now, quite standard, and has already been applied to provide a semantics for strong negation. We do not break any new ground here.

We still want to use the modal logic KX4 to model implicit evidence. In Definition 4.2 we gave an embedding mapping a positive formula X to a modal formula X^f, which we thought of as being in KX4. That is now supplemented with a dual version. In addition to the mapping from X to X^f, we now specify a mapping taking X to X^a, which we can think of as implicitly representing direct evidence *against*. In order to admit incompleteness or inconsistency of evidence we make use of a standard device. For each propositional letter P we assume there is a dual letter, written $\overline{P}$, representing the opposite of P. We assume overlined propositional letters are not part of the original language, and if P and Q are distinct, so are $\overline{P}$ and $\overline{Q}$. Here are the negative embedding conditions.

Definition 7.1. (*Implicit Evidence Against*) P is a propositional letter; X and Y are arbitrary formulas of positive intuitionistic logic plus strong negation.

$$
\begin{aligned}
(X \wedge Y)^a &= \Box(X^a \vee Y^a)\\
(X \vee Y)^a &= \Box(X^a \wedge Y^a)\\
(X \supset Y)^a &= \Box(X^f \wedge Y^a)\\
(\neg X)^a &= X^f\\
(\neg X)^f &= X^a\\
P^a &= \Box\overline{P}
\end{aligned}
$$

Example 7.2. Mapping a formula involving strong negation into KX4.

$$\begin{aligned}(P \supset \neg(Q \supset \neg(R \supset \neg S)))^f &= \Box(P^f \supset (\neg(Q \supset \neg(R \supset \neg S)))^f) \\ &= \Box(\Box P \supset (Q \supset \neg(R \supset \neg S))^a) \\ &= \Box(\Box P \supset \Box(Q^f \wedge (\neg(R \supset \neg S))^a)) \\ &= \Box(\Box P \supset (\Box(\Box Q \wedge (R \supset \neg S)^f))) \\ &= \Box(\Box P \supset (\Box(\Box Q \wedge \Box(R^f \supset (\neg S)^f)))) \\ &= \Box(\Box P \supset (\Box(\Box Q \wedge \Box(\Box R \supset S^a)))) \\ &= \Box(\Box P \supset (\Box(\Box Q \wedge \Box(\Box R \supset \Box\overline{S}))))\end{aligned}$$

While it is natural to have both positive and negative evidence embeddings, it suggests we may have double the amount of work to do in understanding the consequences of our explicit evidence approach. Fortunately, the negative embedding can be eliminated entirely by first reducing formulas to negation normal forms. This, too, is standard material. The following mapping is defined for formulas allowing strong negation, but not containing overlined propositional letters.

Definition 7.3. (*Negation Normal Form*) The *negation normal form* of X, denoted X^N, is given by the following.

$$\begin{aligned}P^N &= P \\ (\neg P)^N &= \overline{P} \\ (X \wedge Y)^N &= X^N \wedge Y^N \\ (\neg(X \wedge Y))^N &= (\neg X)^N \vee (\neg Y)^N \\ (X \vee Y)^N &= X^N \vee Y^N \\ (\neg(X \vee Y))^N &= (\neg X)^N \wedge (\neg Y)^N \\ (X \supset Y)^N &= X^N \supset Y^N \\ (\neg(X \supset Y))^N &= X^N \wedge (\neg Y)^N \\ (\neg\neg X)^N &= X^N\end{aligned}$$

Example 7.4. Consider the formula from Example 7.2 again.

$$\begin{aligned}(P \supset \neg(Q \supset \neg(R \supset \neg S)))^N &= P^N \supset (\neg(Q \supset \neg(R \supset \neg S)))^N \\ &= P \supset (Q^N \wedge (\neg\neg(R \supset \neg S))^N) \\ &= P \supset (Q \wedge (R \supset \neg S)^N)\end{aligned}$$

$$= P \supset (Q \wedge (R^N \supset (\neg S)^N))$$
$$= P \supset (Q \wedge (R \supset \overline{S}))$$

Having computed negation normal form, we now apply the evidence-for mapping, understanding overlined propositional letters simply as a peculiar kind of propositional letter. It is easy to check that $(P \supset (Q \wedge (R \supset \overline{S})))^f = \Box(\Box P \supset \Box(\Box Q \wedge \Box(\Box R \supset \Box\overline{S})))$, which was the outcome of Example 7.2.

This Example generalizes to the following, which has an easy proof by induction on degree. We omit the proof.

Theorem 7.5. *For every formula X of positive intuitionistic logic plus strong negation, but not allowing overlined propositional letters, X^N contains no strong negation occurrences, and $X^f = (X^N)^f$.*

Now we immediately have notions of implicit, and of explicit evidence available to us for BLE, as follows. Suppose X is a formula of BLE, built up using $\wedge$, $\vee$, $\supset$ and $\neg$. That is, we have Nelson's strong negation but not intuitionistic negation. Also assume X does not contain any overlined propositional letters. It is a standard result that X is a theorem of BLE (or N4) iff X^N is a theorem of intuitionistic logic. By our earlier results, X^N is an intuitionistic theorem iff $(X^N)^f$ is a theorem of KX4. But $X^f = (X^N)^f$. We can think of the modal operator of KX4 as asserting the existence of implicit evidence for a proposition so, summarizing, we have the following.

Theorem 7.6. (Implicit Evidence) *Let X be a formula constructed using $\wedge$, $\vee$, $\supset$ and $\neg$ (strong negation) but no overlined propositional letters. X is a theorem of BLE iff X^f is a theorem of KX4.*

This can be carried further. Since X^f does not contain negation our earlier work applies, Theorem 6.4, and we have our version of explicit evidence for BLE.

Theorem 7.7. (Explicit Evidence) *Let X be as in Theorem 7.6. X^f is a theorem of KX4 iff some normal realization of X^f is a theorem of JX4.*

8. Illustrative Examples

We look at two theorems of BLE, providing implicit (in KX4) and explicit (in JX4) evidence analysis for both. It will be seen that an explicit analysis provides significantly more information than the implicit version.

Example 8.1. $\neg(P \supset Q) \supset (Q \supset P)$ is provable in BLE. It embeds into KX4 as follows.

$$\begin{aligned}[\neg(P \supset Q) \supset (Q \supset P)]^f &= \Box[(\neg(P \supset Q))^f \supset (Q \supset P)^f] \\ &= \Box[(P \supset Q)^a \supset (Q \supset P)^f] \\ &= \Box[\Box(P^f \wedge Q^a) \supset \Box(Q^f \supset P^f)] \\ &= \Box[\Box(\Box P \wedge \Box\overline{Q}) \supset \Box(\Box Q \supset \Box P)]\end{aligned}$$

Then our implicit evidence analysis of the BLE theorem $\neg(P \supset Q) \supset (Q \supset P)$ is the KX4 theorem $\Box[\Box(\Box P \wedge \Box\overline{Q}) \supset \Box(\Box Q \supset \Box P)]$, where $\Box$ represents evidence that has not been made explicit. A formula like this is best read, not from left to right, but in a build-up fashion.

$\Box P \wedge \Box\overline{Q}$ there is evidence for P and evidence against Q	$\Box Q \supset \Box P$ evidence for Q entails there is evidence for P
$\Box(\Box P \wedge \Box\overline{Q})$ there is evidence for the situation described above	$\Box(\Box Q \supset \Box P)$ there is evidence for the situation described above

$\Box[\Box(\Box P \wedge \Box\overline{Q}) \supset \Box(\Box Q \supset \Box P)]$
there is evidence that the
left item above entails the right item

Of course none of this tells us anything about what *kind* of evidence we may have. We next turn implicit evidence into explicit evidence by realization. A normal realization is the following, which is provable in JX4.

$$t_2{:}[v_4{:}(v_1{:}P \wedge v_2{:}\overline{Q}) \supset (t_1 \cdot v_4){:}(v_3{:}Q \supset v_1{:}P)]$$

Each v_i is a justification variable. Justification terms t_1 and t_2 are accounted for as follows. The formula $(v_1{:}P \wedge v_2{:}\overline{Q}) \supset (v_3{:}Q \supset v_1{:}P)$ is provable in JX4. The Internalization Lemma 6.3 guarantees the existence of a term t_1 such that $t_1{:}[(v_1{:}P \wedge v_2{:}\overline{Q}) \supset (v_3{:}Q \supset v_1{:}P)]$ is provable in JX4. Likewise the formula $v_4{:}(v_1{:}P \wedge v_2{:}\overline{Q}) \supset (t_1 \cdot v_4){:}(v_3{:}Q \supset v_1{:}P)$ is provable in JX4, and t_2 internalizes a proof of it.

In the justification formula, as is required of a normal realization, justification variables appear in negative positions. We have put these occurrences in boldface to make it easy to see which they are.

$$t_2{:}[\mathbf{v_4}{:}(\mathbf{v_1}{:}P \wedge \mathbf{v_2}{:}\overline{Q}) \supset (t_1 \cdot v_4){:}(\mathbf{v_3}{:}Q \supset v_1{:}P)]$$

Note that v_2 and v_3 have *only* these single negative occurrences. After their introduction, they are never used again, unlike v_1 which occurs elsewhere, and v_4 which occurs as part of $t_1 \cdot v_4$. The variables v_2 and v_3 are introduced, but never used. Their values, reasons for $\overline{Q}$ and for Q, don't matter. Clearly explicit evidence can give us information that is hidden when implicit evidence is used.

EXAMPLE 8.2. $\neg(P \wedge \neg P) \supset ((\neg P \supset P) \supset P)$ is provable in BLE. It embeds into $\mathsf{KX4}$ as the following provable formula.

$$\Box\{\Box(\Box\overline{P} \vee \Box P) \supset \Box(\Box(\Box\overline{P} \supset \Box P) \supset \Box P)\}$$

A realization of this, provable in $\mathsf{JX4}$, is the following.

$$\begin{array}{l} t_4{:}\{v_5{:}(v_2{:}P \vee v_1{:}\overline{P}) \supset [t_3 \cdot v_5]{:}(v_4{:}(v_2{:}\overline{P} \supset v_3{:}P) \supset \\ [((t_2 {\cdot} !v_2 \cdot v_4) \mathsf{c}\, v_3) + (t_1 {\cdot} !v_1 \cdot v_4) \mathsf{c}\, v_1)]{:}P)\} \end{array}$$

Here the Internalization Lemma provides us with the following.

$$\begin{array}{l} t_1 \text{ justifying } v_1{:}P \supset ((v_2{:}\overline{P} \supset v_3{:}P) \supset v_1{:}P) \\ t_2 \text{ justifying } v_2{:}\overline{P} \supset ((v_2{:}\overline{P} \supset v_3{:}P) \supset v_3{:}P) \\ t_3 \text{ justifying } (v_2{:}\overline{P} \vee v_1{:}P) \supset (v_4{:}(v_2{:}\overline{P} \supset v_3{:}P) \supset \\ \qquad\qquad [((t_2 {\cdot} !v_2 \cdot v_4) \mathsf{c}\, v_3) + (t_1 {\cdot} !v_1 \cdot v_4) \mathsf{c}\, v_1)]{:}P) \\ t_4 \text{ justifying } v_5{:}(v_2{:}P \vee v_1{:}\overline{P}) \supset \\ \qquad\qquad [t_3 \cdot v_5]{:}(v_4{:}(v_2{:}\overline{P} \supset v_3{:}P) \supset \\ \qquad\qquad [((t_2 {\cdot} !v_2 \cdot v_4) \mathsf{c}\, v_3) + (t_1 {\cdot} !v_1 \cdot v_4) \mathsf{c}\, v_1)]{:}P) \end{array}$$

As in the previous example, we display the justification formula with variables in negative positions in bold face.

$$\begin{array}{l} t_4{:}\{\mathbf{v_5}{:}(\mathbf{v_2}{:}P \vee \mathbf{v_1}{:}\overline{P}) \supset \\ \quad [t_3 \cdot v_5]{:}(\mathbf{v_4}{:}(\mathbf{v_2}{:}\overline{P} \supset \mathbf{v_3}{:}P) \supset \\ \qquad [((t_2 {\cdot} !v_2 \cdot v_4) \mathsf{c}\, v_3) + (t_1 {\cdot} !v_1 \cdot v_4) \mathsf{c}\, v_1)]{:}P)\} \end{array}$$

Quite unlike the previous example, now all these variables have additional occurrences elsewhere. They are all 'used.' There is no superfluous evidence involved.

9. Conclusion

As promised, we have given a concrete meaning to the motivating idea behind BLE, making evidence usage explicit. There is, however, work that remains to be done. We briefly list three main items, and encourage others to think about them.

1. Carnielli and Rodrigues build a second logic on top of BLE, which they call $\mathsf{LET_J}$. In it there is an operator that, roughly speaking, identifies a formula as having classical behavior. For example, $\neg(P \wedge \neg P) \supset ((\neg P \supset P) \supset P)$ is a theorem of BLE, so if we could express that P behaved classically, presumably that expression would imply $((\neg P \supset P) \supset P)$. An evidence based investigation of $\mathsf{LET_J}$ is still missing. One suspects that there is a natural way of doing this, but it is an open problem.
2. There is a Realization Theorem connecting KX4 and JX4, but, it has a non-constructive proof [8]. For some of the Geach logics there is a constructive proof of Realization available. The first example, connecting S4 and LP, was constructive, for example. Constructive Realization proofs have, so far, made use of cut free proof systems. This has worked for cut free sequent calculi, tableaus, hypersequent calculi, nested sequents, and prefixed tableaus. It is not known yet whether cut free proof systems for KX4 (there are some) will work for this.
3. First-Degree Entailment (FDE) [1], is a logic of fundamental importance, and is the subject of the special issue in which the present paper appears. Quoting from [11], "It is known that Belnap and Dunn's four-valued logic and the $\{\wedge, \vee, \neg\}$-fragment of N4 are the same logic". Then the work in the present paper provides us with an explicit evidence understanding of FDE. But there is also a very natural semantics for FDE [5], in which truth and falsity (each understood essentially classically) are decoupled, and a formula can have one of four truth values: only true, only false, both true and false, and neither true nor false. It is plausible that a special version of justification could be introduced, consisting of a *pair* of justifications in the usual sense. One member of the pair would provide evidence for a proposition and the other, evidence against. The justifications making up the pair would, most likely, be taken from a justification logic counterpart of S5, which corresponds to classical logic under Gödel's mapping in the same way that S4 corresponds to intuitionistic logic. If this can be done, the result may provide some interesting insights into FDE.

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M. Fitting
New York NY
USA
melvin.fitting@gmail.com