Arun Kumar
Mohua Banerjee

Kleene Algebras and Logic: Boolean and Rough Set Representations, 3-Valued, Rough Set and Perp Semantics

Abstract. A structural theorem for Kleene algebras is proved, showing that an element of a Kleene algebra can be looked upon as an ordered pair of sets, and that negation with the Kleene property (called the 'Kleene negation') is describable by the set-theoretic complement. The propositional logic $\mathcal{L}_K$ of Kleene algebras is shown to be sound and complete with respect to a 3-valued and a rough set semantics. It is also established that Kleene negation can be considered as a modal operator, due to a perp semantics of $\mathcal{L}_K$. Moreover, another representation of Kleene algebras is obtained in the class of complex algebras of compatibility frames. One concludes with the observation that $\mathcal{L}_K$ can be imparted semantics from different perspectives.

Keywords: Kleene algebras, 3-Valued logic, Rough sets, Perp semantics.

1. Introduction

Algebraists, since the beginning of work on lattice theory, have been keenly interested in representing lattice-based algebras as algebras based on *set* lattices. Some such well-known representations are the Birkhoff representation for finite lattices, Stone representation for Boolean algebras, or Priestley representation for distributive lattices. It is also well-known that such representation theorems for classes of lattice-based algebras play a key role in studying set-based semantics of logics 'corresponding' to the classes. In this paper, we pursue this line of investigation, and focus on *Kleene algebras* and their representations. We then move to the corresponding propositional logic, denoted $\mathcal{L}_K$, and define a 3-valued, rough set and perp semantics for it. Through the work here, one is able to establish that $\mathcal{L}_K$ and the 3-element Kleene algebra **3** (cf. Figure 1, Section 2) play the same fundamental role among the Kleene algebras that classical propositional logic and the 2-element Boolean algebra **2** play among the Boolean algebras.

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Kleene algebras were introduced by Kalman [32] and have been studied under different names such as normal i-lattices, Kleene lattices and normal quasi-Boolean algebras, e.g. cf. [12,13]. Let us define these algebras.

Definition 1. An algebra $\mathcal{K} = (K, \vee, \wedge, \sim, 0, 1)$ is called a Kleene algebra if the following hold.

1. $\mathcal{K} = (K, \vee, \wedge, \sim, 0, 1)$ is a De Morgan algebra, i.e.,
   (i) $(K, \vee, \wedge, 0, 1)$ is a bounded distributive lattice, and for all $a, b \in K$,
   (ii) $\sim (a \wedge b) = \sim a \vee \sim b$ (De Morgan property),
   (iii) $\sim\sim a = a$ (involution).
2. $a \wedge \sim a \leq b \vee \sim b$, for all $a, b \in K$ (Kleene property).

There is a lot of work on the algebraic [1,3,10,12,13,32] and logical [16, 23,28,29] aspects of Kleene algebras. The structures have also been studied as reducts of algebras such as 3-valued Łukasiewicz–Moisil (LM) algebras or MV-algebras, e.g. [1,10]. Our interest lies in obtaining a representation result for the algebras. For such an investigation, it would be natural to first turn to the known representation results for De Morgan algebras, as Kleene algebras are based on them. One finds the following, in terms of sets.

– Rasiowa [40] represented De Morgan algebras as set-based De Morgan algebras, where De Morgan negation is defined by an involution function.
– In Dunn's [19,23] representation, each element of a De Morgan algebra can be identified with an ordered pair of sets, where De Morgan negation is defined as reversing the order in the pair.

On the other hand, we also find that there are algebras based on Kleene algebras (as mentioned above) which can be represented by ordered pairs of sets, and where negations are described by set-theoretic complements. Consider the set $B^{[2]} := \{(a, b) : a \leq b, a, b \in B\}$, for any partially ordered set $(B, \leq)$.

– (Moisil (cf. [14])) For each 3-valued LM algebra $\mathcal{A}$, there exists a Boolean algebra B such that $\mathcal{A}$ can be embedded into $B^{[2]}$.
– (Katriňák [33], cf. [10]) Every regular double Stone algebra can be embedded into $B^{[2]}$ for some Boolean algebra B.

Rough set theory [37,38] also provides a way to represent algebras as pairs of sets. In rough set terminology (that will be elaborated on in Section 4), we have the following results for algebraic structures based on Kleene algebras.

– (Comer [17]) Every regular double Stone algebra is isomorphic to an algebra of rough sets in a Pawlak approximation space.
– (Järvinen [31]) Every Nelson algebra defined over an algebraic lattice is isomorphic to an algebra of rough sets in an approximation space based on a quasi order.

It must be mentioned here that there are similar representation results in rough set theory for other algebraic structures as well, e.g. for the class of rough algebras [5], or finite semi-simple Nelson algebras [36]. There also have been studies related to rough sets on other algebras that have Kleene algebras as reducts, for instance, NM-algebras [7] and $BZNM$-algebras [43].

In this article, the following representation results are established for Kleene algebras.

Theorem 1. (i) *Given a Kleene algebra $\mathcal{K}$, there exists a Boolean algebra $\mathcal{B}_{\mathcal{K}}$ such that $\mathcal{K}$ can be embedded into ${\mathcal{B}_{\mathcal{K}}}^{[2]}$.*

(ii) *Every Kleene algebra is isomorphic to an algebra of rough sets in a Pawlak approximation space.*

The De Morgan negation operator with the Kleene property (cf. Definition 1), is referred to as the *Kleene negation*. In literature, one finds various generalizations of the classical (Boolean) negation, including the De Morgan and Kleene negations. It is natural to ask the following question: do these generalized negations arise from (or can be described by) the Boolean negation? The representation result above (Theorem 1) for Kleene algebras shows that Kleene algebras always arise from Boolean algebras, thus answering the above question in the affirmative for the Kleene negation.

Representation of lattice-based algebras as algebras in which objects are pairs of sets, has proved to be of significance in the study of semantics for the logic corresponding to the class of algebras. For instance, such a representation of De Morgan algebras leads to Dunn's well-known 4-valued semantics of De Morgan logic. In a similar way, the above representation results for Kleene algebras help us in the study of semantics for the logic $\mathcal{L}_K$ corresponding to the class of Kleene algebras. $\mathcal{L}_K$ is the De Morgan consequence system [23] with the negation operator satisfying the *Kleene axiom*: $\alpha \wedge \sim \alpha \vdash \beta \vee \sim \beta$. We show that $\mathcal{L}_K$ is sound and complete with respect to a 3-valued as well as a rough set semantics, making use of the representation results.

Furthermore, it is shown that the logic $\mathcal{L}_K$ can be imparted a *perp semantics* [25] as well. Perp semantics provides a framework to study various negations from the points of view of logic as well as algebra. In particular, De Morgan logic is sound and complete with respect to a class of perp (*compatibility*) frames, whence the algebraic semantics, 4-valued semantics and perp semantics for De Morgan logic become equivalent. In this case, $\mathcal{L}_K$ is sound and complete with respect to a class of compatibility frames that we call *Kleene frames*. Thus we obtain that the algebraic, 3-valued, rough set and perp semantics for $\mathcal{L}_K$ are all equivalent. Finally, one obtains another representation of Kleene algebras, in the class of complex algebras of Kleene frames.

The paper is organized as follows. In Section 2, we prove (i) of Theorem 1. The logic $\mathcal{L}_K$ and its 3-valued semantics are introduced in Section 3, and soundness and completeness results are proved. In Section 4, we establish a rough set representation of Kleene algebras, that is, (ii) of Theorem 1, relate rough sets with the 3-valued semantics considered in this work, and then present completeness of $\mathcal{L}_K$ with respect to the rough set semantics. In Section 5, the perp semantics for $\mathcal{L}_K$ is discussed, and the Kleene property in compatibility frames is investigated. Section 5 includes the observations that all the semantics defined for $\mathcal{L}_K$ are equivalent (Theorem 19), and that Kleene algebras are also representable in the class of complex algebras of Kleene frames (Theorem 20). We conclude the article in Section 6.

The lattice theoretic results used in this article are taken from [18]. We use the convention of representing a set $\{x, y, z, \ldots\}$ by $xyz \ldots$.

2. Boolean Representation of Kleene Algebras

Construction of new types of algebras from a given algebra has been of prime interest for algebraists, especially in the context of algebraic logic. Some well known examples of such construction are:

- Nelson algebra from a given Heyting algebra (Vakarelov [45], Fidel [27]).
- Kleene algebras from distributive lattices (Kalman [32]).
- 3-Valued Łukasiewicz–Moisil (LM) algebra from a given Boolean algebra (Moisil, cf. [14]).
- Regular double Stone algebra from a Boolean algebra (Katriňák [33], cf. [10]).

Our work is based on Moisil's construction of a 3-valued LM algebra (which is, in particular, a Kleene algebra). Let us present this construction.
Let $\mathcal{B} := (B, \vee, \wedge, ^c, 0, 1)$ be a Boolean algebra. Consider again, the set

$$B^{[2]} := \{(a, b) : a \leq b, a, b \in B\}.$$

PROPOSITION 1. [10] *$\mathcal{B}^{[2]} := (B^{[2]}, \vee, \wedge, \sim, (0,0), (1,1))$ is a Kleene algebra, where, for $(a,b),(c,d) \in B^{[2]}$,*

$$(a,b) \vee (c,d) := (a \vee c, b \vee d),$$
$$(a,b) \wedge (c,d) := (a \wedge c, b \wedge d),$$
$$\sim (a,b) := (b^c, a^c).$$

PROOF. Let us only demonstrate the Kleene property for $\sim$.

$$(a,b) \wedge \sim (a,b) = (a,b) \wedge (b^c, a^c) = (a \wedge b^c, b \wedge a^c) = (0, b \wedge a^c).$$
$$(c,d) \vee \sim (c,d) = (c,d) \vee (d^c, c^c) = (c \vee d^c, d \vee c^c) = (c \vee d^c, 1).$$

Hence $(a,b) \wedge \sim (a,b) \leq (c,d) \vee \sim (c,d)$. ∎

It may be noted that for a Boolean algebra $\mathcal{B}$, $\mathcal{B}^{[2]}$ is isomorphic to the Kleene algebra formed by the set $B(0) := \{(a,b) \in B \times B : a \wedge b = 0\}$. In fact, $B(0)$ forms a Kleene algebra if $\mathcal{B}$ is a distributive lattice with least element 0, and, even more generally, for any element a in a distributive lattice $\mathcal{B}$, the set $B(a) := \{(x,y) \in B \times B : x \wedge y \leq a \leq x \vee y\}$ forms a Kleene algebra [32]. Pagliani works with similar ordered pairs (X,Y) of certain sets (i.e. satisfying $X \cap Y = \emptyset$, and certain other properties) in [35], and also later in [36] where he obtains a representation of finite semi-simple Nelson algebras in terms of rough sets.

In this section, we prove the representation result stated in Theorem 1(i). Using Stone's representation, each Boolean algebra is embeddable in a power set algebra, so that $B^{[2]}$, for any Boolean algebra B, is embeddable in the Kleene algebra formed by $\mathcal{P}(U)^{[2]}$ for some set U. Thus, because of Theorem 1(i), one can say that each element of a Kleene algebra can also be looked upon as a pair of sets.

Now observe that we already have the following well-known representation theorem, due to the fact that **1, 2** and **3** (Figure 1) are the only subdirectly irreducible (Kleene) algebras in the variety of Kleene algebras.

THEOREM 2. [3] *Let $\mathcal{K}$ be a Kleene algebra. There exists a (index) set I such that $\mathcal{K}$ can be embedded into $\mathbf{3}^I$.*

So, to prove Theorem 1(i), we establish the following.

THEOREM 3. *For the Kleene algebra $\mathbf{3}^I$ correponding to any index set I, there exists a Boolean algebra $B_{\mathbf{3}^I}$ such that $\mathbf{3}^I \cong (\mathcal{B}_{\mathbf{3}^{\mathcal{I}}})^{[2]}$.*

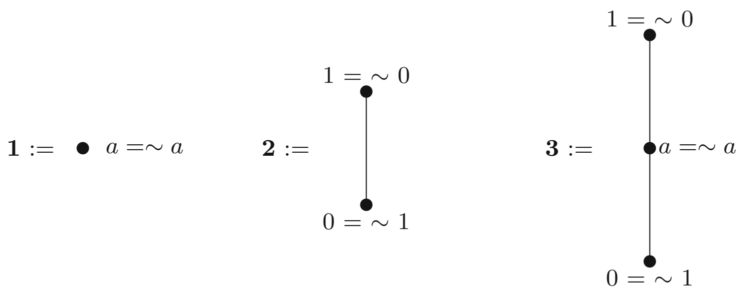

Figure 1. Subdirectly irreducible Kleene algebras

2.1. Completely Join Irreducible Elements of $\mathbf{3}^{I}$ and $(\mathbf{2}^{I})^{[2]}$

Identifying join irreducible elements in lattices can be useful. In [29], for instance, these elements are used to obtain normal forms in the logics corresponding to De Morgan, Kleene and Boolean algebras. Completely join irreducible elements also play a fundamental role in establishing isomorphisms between lattice-based algebras, e.g. cf. [31]. We observe the same in the following.

Let us put the basic definitions and notations in place.

Definition 2. Let $\mathcal{L} := (L, \vee, \wedge, 0, 1)$ be a complete lattice.

(i) An element $a \in L$ is said to be completely join irreducible, if $a = \bigvee S$ implies that $a \in S$, for every subset S of L.

Notation 1. *Let $\mathcal{J}_L$ denote the set of all completely join irreducible elements of L, and $J(x) := \{a \in \mathcal{J}_L : a \leq x\}$, for any $x \in L$.*

(ii) A set S is said to be join dense in $\mathcal{L}$, provided for every element $a \in L$, there is a subset S' of S such that $a = \bigvee S'$.

For a given index set I, let us characterize the sets of completely join irreducible elements of the Kleene algebras $\mathbf{3}^{I}$ and $(\mathbf{2}^{I})^{[2]}$, and prove their join density in the respective lattices.

Let $i, k \in I$. Denote by $f_i^x,\ x \in \{a, 1\}$, the following element in $\mathbf{3}^{I}$.

$$f_i^x(k) := \begin{cases} x & \text{if } k = i \\ 0 & \textit{otherwise} \end{cases}$$

Proposition 2. *The set of completely join irreducible elements of $\mathbf{3}^{I}$ is given by:*

$$\mathcal{J}_{\mathbf{3}^I} = \{f_i^a, f_i^1 : i \in I\}.$$

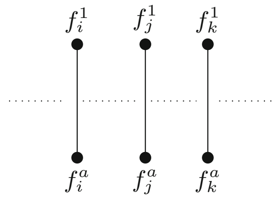

Figure 2. Hasse diagram of $\mathcal{J}_{\mathbf{3}^I}$

Moreover, $\mathcal{J}_{\mathbf{3}^I}$ is join dense in $\mathbf{3}^I$.

PROOF. Let $f_i^a = \vee_{k \in K} f_k,\ K \subseteq I$. This implies that $f_i^a(j) = \vee_{k \in K} f_k(j)$, for each $j \in I$. If $j \neq i$, by the definition of f_i^a, $f_i^a(j) = 0$. So $\vee_{k \in K} f_k(j) = 0$, whence $f_k(j) = 0$, for each $k \in K$. If $j = i$, then $f_i^a(j) = a$, which means $\vee_{k \in K} f_k(j) = a$. But as a is join irreducible in $\mathbf{3}$, there exists a $k' \in K$ such that $f_{k'}(j) = a$. Hence $f_i^a = f_{k'}$. A similar argument works for f_i^1.
Now let $f \in \mathbf{3}^I$. Take $K := \{j \in I : f(j) \neq 0\}$, and for each $j \in K$, define the element f_j of $\mathbf{3}^I$ as

$$f_j(k) := \begin{cases} f(j) & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

Clearly, we have $f = \vee_{j \in K} f_j$, where $f_j \in \mathcal{J}_{\mathbf{3}^I}$. ■

Let us note that for each $i, j \in I$, $f_i^a \leq f_i^1$, and if $i \neq j$, neither $f_i^x \leq f_j^y$ nor $f_j^x \leq f_i^y$ holds for $x, y \in \{a, 1\}$. The order structure of $\mathcal{J}_{\mathbf{3}^I}$ can be visualized by Figure 2.

EXAMPLE 1. Let us consider the Kleene algebra $\mathbf{3}^3$. The set $\mathcal{J}_{\mathbf{3}^3}$ of completely join irreducible elements of $\mathbf{3}^3$ is then given by

$$\begin{aligned} \mathcal{J}_{\mathbf{3}^3} = \{ & f_1^a := (a,0,0), f_1^1 := (1,0,0), f_2^a := (0,a,0), \\ & f_2^1 := (0,1,0), f_3^a := (0,0,a), f_3^1 := (0,0,1)\}. \end{aligned}$$

Let $f := (0, a, 1) \in \mathbf{3}^3$. Then $f = f_2 \vee f_3$, where $f_2 = (0, a, 0)$ and $f_3 = (0, 0, 1)$.

As any complete atomic Boolean algebra is isomorphic to $\mathbf{2}^I$ for some index set I, henceforth, we shall identify any complete atomic Boolean algebra B with $\mathbf{2}^I$. Now, for any such algebra, $B^{[2]}$ is a Kleene algebra (cf. Proposition 1); in fact, it is a completely distributive Kleene algebra.

PROPOSITION 3. *Let B be a complete atomic Boolean algebra. The set of completely join irreducible elements of $B^{[2]}$ is given by*

$$\mathcal{J}_{B^{[2]}} = \{(0, a), (a, a) : a \in \mathcal{J}_B\}.$$

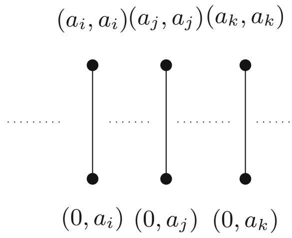

Figure 3. Hasse diagram of $\mathcal{J}_{B^{[2]}}$

Moreover, $\mathcal{J}_{B^{[2]}}$ is join dense in $B^{[2]}$.

Proof. Let $a \in \mathcal{J}_B$ and let $(a, a) = \vee_{k\in K}(x_k, y_k)$, $K \subseteq I$, where $(x_k, y_k) \in B^{[2]}$ for each $k \in K$. $(a, a) = \vee_{k\in K}(x_k, y_k)$ implies $a = \vee_k x_k$. As $a \in \mathcal{J}_B$, $a = x_{k'}$ for some $k' \in K$. We already have $x_{k'} \leq y_{k'} \leq a$, hence combining with $a = x_{k'}$, we get $(a, a) = (x_{k'}, y_{k'})$. With similar arguments one can show that for each $a \in \mathcal{J}_B$, $(0, a)$ is completely join irreducible.

Now, let $(x, y) \in B^{[2]}$. Consider the sets $J(x)$, $J(y)$ (cf. Notation 1, Definition 2). Then $(x, y) = \vee_{a\in J(x)}(a, a) \ \vee \ \vee_{b\in J(y)}(0, b)$. Hence $\mathcal{J}_{B^{[2]}}$ is join dense in $B^{[2]}$. ■

For $a, b \in \mathcal{J}_B$, $(0, a) \leq (a, a)$, and if $a \neq b$, $x, y \in \{a, b\}$ with $x \neq y$, neither $(0, x) \leq (0, y), (y, y)$ nor $(x, x) \leq (0, y), (y, y)$ holds. Then, similar to the case of $\mathbf{3}^I$, the completely join irreducible elements of $B^{[2]}$ can be visualized by Figure 3.

Example 2. Consider the Boolean algebra $\mathbf{4}$ of four elements with atoms a and b. The set of completely join irreducible elements of $\mathbf{4}^{[2]}$ is given by $\mathcal{J}_{\mathbf{4}^{[2]}} = \{(0, a), (a, a), (0, b), (b, b)\}$.

Let $(a, 1) \in \mathbf{4}^{[2]}$. Then $J(a) = \{a\}$ and $J(1) = \{a, b\}$. Hence $(a, 1) = (a, a) \vee (0, a) \vee (0, b)$.

2.2. Structural Theorem for Kleene Algebras

Let us first present the basic lattice-theoretic definitions and results that will be required to arrive at the proof of Theorem 3.

Definition 3.

1. (a) A complete lattice of sets is a family $\mathcal{F}$ such that $\bigcup \mathcal{H}$ and $\bigcap \mathcal{H}$ belong to $\mathcal{F}$ for any $\mathcal{H} \subseteq \mathcal{F}$.
2. Let L be a complete lattice.

(a) L is said to be algebraic if any element $x \in L$ is the join of a set of compact elements of L.

(b) L is said to satisfy the Join-Infinite Distributive Law, if for any subset $\{y_j\}_{j\in J}$ of L and any $x \in L$,

$$(JID)\ x \wedge \bigvee_{j\in J} y_j = \bigvee_{j\in J} x \wedge y_j.$$

THEOREM 4. [18] *Let L be a lattice. The following are equivalent.*

(i) *L is complete, satisfies (JID) and the set of completely join irreducible elements is join dense in L.*

(ii) *L is completely distributive and algebraic.*

It can easily be seen that both the lattices $\mathbf{3}^I$ and $(\mathbf{2}^I)^{[2]}$ are complete and satisfy (JID). We have already observed from Section 2.1 that the sets of completely join irreducible elements of $\mathbf{3}^I$ and $(\mathbf{2}^I)^{[2]}$ are join dense in the respective lattices. So Theorem 4(i) holds for $\mathbf{3}^I$ and $(\mathbf{2}^I)^{[2]}$, and therefore, $\mathbf{3}^I$ and $(\mathbf{2}^I)^{[2]}$ are completely distributive and algebraic lattices.

For the remaining study, let us fix an index set I. Now we can write $\mathcal{J}_{\mathbf{3}^I} = \{f_i^a, f_i^1 : i \in I\}$ and $\mathcal{J}_{(\mathbf{2}^I)^{[2]}} = \{(0, g_i^1), (g_i^1, g_i^1) : i \in I\}$, where g_i^1's are the atoms or completely join irreducible elements of the Boolean algebra $\mathbf{2}^I$, defined as f_i^1 with domain restricted to $\mathbf{2}$. In other words,

$$g_i^1(k) := \begin{cases} 1 & \text{if } k = i \\ 0 & \text{otherwise} \end{cases}$$

THEOREM 5. *The sets of completely join irreducible elements of $\mathbf{3}^I$ and $(\mathbf{2}^I)^{[2]}$ are order isomorphic.*

PROOF. We define the map $\phi : \mathcal{J}_{\mathbf{3}^I} \to \mathcal{J}_{(\mathbf{2}^I)^{[2]}}$ as follows. For $i \in I$,

$$\phi(f_i^a) := (0, g_i^1),$$
$$\phi(f_i^1) := (g_i^1, g_i^1).$$

One can show that ϕ is an order isomorphism due to the following.

– $f_i^x \le f_j^y$ if and only if $i = j$ and $x, y = a$ or $x, y = 1$, or $x = a, y = 1$. In any case, by definition of ϕ, $\phi(f_i^x) \le \phi(f_j^y)$.

– Let $\phi(f_i^x) \le \phi(f_j^y)$ and assume $\phi(f_i^x) = (g_k^1, g_l^1)$ and $\phi(f_j^y) = (g_m^1, g_n^1)$. But then again: $k = l = m = n$ or $g_k^1 = g_m^1 = 0, l = n$ or $g_k^1 = 0, l = m = n$. Again, following the definition of ϕ, we have $f_i^x \le f_j^y$.

– If $(0, g_i^1) \in \mathcal{J}_{(\mathbf{2}^I)^{[2]}}$, then $\phi(f_i^a) = (0, g_i^1)$. Similarly for (g_i^1, g_i^1). Hence ϕ is onto. ■

Lemma 1. [8] *Let L and K be two completely distributive lattices. Further, let $\mathcal{J}_L$ and $\mathcal{J}_K$ be join dense in L and K respectively. Let $\phi : \mathcal{J}_L \rightarrow \mathcal{J}_K$ be an order isomorphism. Then the extension map $\Phi : L \rightarrow K$ given by*

$$\Phi(x) := \bigvee(\phi(J(x))) \quad (\textit{where}\, J(x) := \{a \in \mathcal{J}_L : a \leq x\}), x \in L,$$

is a lattice isomorphism.

Using Theorem 5 and Lemma 1, we have

Theorem 6. *The algebras $\mathbf{3}^I$ and $(\mathbf{2}^I)^{[2]}$ are lattice isomorphic.*

In order to obtain Theorem 3, one would like to extend the above lattice isomorphism to a Kleene isomorphism. We use the technique of Järvinen in [31]. Let us present the preliminaries.

Let $\mathbb{K} := (K, \vee, \wedge, \sim, 0, 1)$ be a completely distributive De Morgan algebra. Define for any $j \in \mathcal{J}_K$,

$$j^* := \bigwedge\{x \in K : x \not\leq \sim j\}.$$

Then $j^* \in \mathcal{J}_K$. For complete details on j^*, one may refer to [31]. Further, it is shown that Lemma 1 can be extended to De Morgan algebras defined over algebraic lattices.

Theorem 7. *Let $\mathbb{L} := (L, \vee, \wedge, \sim, 0, 1)$ and $\mathbb{K} := (K, \vee, \wedge, \sim, 0, 1)$ be two De Morgan algebras defined on algebraic lattices. If $\phi : \mathcal{J}_L \rightarrow \mathcal{J}_K$ is an order isomorphism such that*

$$\phi(j^*) = \phi(j)^*,$$

for all $j \in \mathcal{J}_L$, then Φ is an isomorphism between the algebras $\mathbb{L}$ and $\mathbb{K}$.

Now, let $f_i^a \in \mathcal{J}_{\mathbf{3}^I}$. By definition, $(f_i^a)^* = \bigwedge\{f \in \mathbf{3}^I : f \not\leq \sim (f_i^a)\}$, where for each $i \in I$,

$$\sim (f_i^a)(k) = \begin{cases} a & \text{if } k = i \\ 1 & \text{otherwise} \end{cases}$$

Clearly, we have $f_i^1 \not\leq \sim (f_i^a)$. Now let $f \not\leq \sim (f_i^a)$. Then what does f look like? If $k \neq i$, $f(k) \leq \sim (f_i^a)(k) = 1$. So, for $f \not\leq \sim (f_i^a)$, $f(i)$ has to be 1 (otherwise $f(i) = 0$ or a will lead to $f \leq \sim (f_i^a)$). Hence, $f_i^1 \leq f$ and $(f_i^a)^* = f_i^1$.

Similarly, one can easily show that $(f_i^1)^* = f_i^a$.

On the other hand, let us consider $(0, g_i^1) \in \mathcal{J}_{(\mathbf{2}^I)^{[2]}}$. Then, $(0, g_i^1)^* = \bigwedge\{(g, g') \in (\mathbf{2}^I)^{[2]} : (g, g') \not\leq \sim (0, g_i^1)\}$. By definition of $\sim$, we have $\sim (0, g_i^1) = ((g_i^1)^c, 0^c) = ((g_i^1)^c, 1)$. Observe that $(g_i^1, g_i^1) \not\leq ((g_i^1)^c, 1)$, as, $g_i^1 \not\leq$

$(g_i^1)^c$ is true in a Boolean algebra. Now, let $(g, g') \in \mathcal{J}_{(\mathbf{2}^I)^{[2]}}$ be such that $(g, g') \not\lesssim\sim (0, g_i^1) = ((g_i^1)^c, 1)$. But we have $g' \leq 1$, so for $(g, g') \not\lesssim\sim (0, g_i^1)$ to hold, we must have $g \not\leq (g_i^1)^c$. g_i^1 is an atom of $\mathbf{2}^I$ and $g \not\leq (g_i^1)^c$ imply $g_i^1 \leq g$. Hence $(g_i^1, g_i^1) \leq (g, g')$, and we get $(0, g_i^1)^* = (g_i^1, g_i^1)$. Similarly, we have $(g_i^1, g_i^1)^* = (0, g_i^1)$. Let us summarize these observations in the following lemma.

Lemma 2. *The completely distributive De Morgan algebra $\mathbf{3}^I$ has the following properties. For each $i \in I$,*

1. $(f_i^a)^* = f_i^1$, $(0, g_i^1)^* = (g_i^1, g_i^1)$.
2. $(f_i^1)^* = f_i^a$, $(g_i^1, g_i^1)^* = (0, g_i^1)$.

We return to Theorem 3.

Proof of Theorem 3. Let the Kleene algebra $\mathbf{3}^I$ be given. Consider $\mathbf{2}^I$ as a Boolean subalgebra of $\mathbf{3}^I$. Using the definition of ϕ (cf. Theorem 5) and its extension (cf. Lemma 1), and Lemma 2 we have, for each $i \in I$,

$$\phi((f_i^a)^*) = \phi(f_i^1) = (g_i^1, g_i^1) = \phi(f_i^a)^*,$$
$$\phi((f_i^1)^*) = \phi(f_i^a) = (0, g_i^1) = \phi(f_i^1)^*.$$

By Theorem 6, ϕ is an order isomorphism between $\mathcal{J}_{\mathbf{3}^I}$ and $\mathcal{J}_{(\mathbf{2}^I)^{[2]}}$. Hence using Theorem 7, Φ is an isomorphism between the De Morgan algebras $\mathbf{3}^I$ and $(\mathbf{2}^I)^{[2]}$. As both the algebras are Kleene algebras which are also equational algebras defined over De Morgan algebras, the De Morgan isomorphism Φ extends to Kleene isomorphism. ■

Let us illustrate the above theorem through examples.

Example 3. Consider the Kleene algebra $\mathbf{3} := \{0, a, 1\}$. Then $\mathcal{J}_{\mathbf{3}} = \{a, 1\}$. For $\mathbf{2} := \{0, 1\}$, $\mathbf{2}^{[2]} = \{(0,0), (0,1), (1,1)\}$ and $\mathcal{J}_{\mathbf{2}^{[2]}} = \{(0,1), (1,1)\}$. Further, $a^* = 1$, $1^* = a$ and $(0,1)^* = (1,1)$ and $(1,1)^* = (0,1)$.
Then $\phi : \mathcal{J}_{\mathbf{3}} \to \mathcal{J}_{\mathbf{2}^{[2]}}$ is defined as

$$\phi(a) := (0,1),$$
$$\phi(1) := (1,1).$$

Hence the extension map $\Phi : \mathbf{3} \to \mathbf{2}^{[2]}$ is given as

$$\Phi(a) := (0,1),$$
$$\Phi(1) := (1,1),$$
$$\Phi(0) := (0,0).$$

The diagrammatic illustration of this example is given in Figure 4.

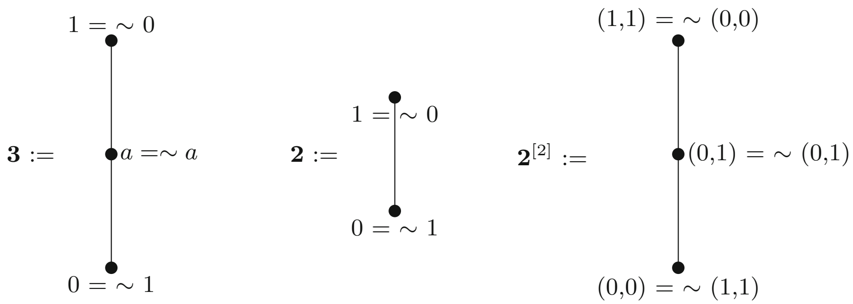

Figure 4. $\mathbf{3} \cong \mathbf{2}^{[2]}$

Example 4. Let us consider the Kleene algebra $\mathbf{3} \times \mathbf{3}$.

$$\mathbf{3} \times \mathbf{3} := \{(0,0),(0,a),(0,1),(a,1),(1,1),(a,0),(1,0),(1,a),(a,a)\}.$$
$$\mathcal{J}_{\mathbf{3}\times\mathbf{3}} = \{(0,a),(0,1),(a,0),(1,0)\} \text{ and}$$
$$(0,a)^* = (0,1),(0,1)^* = (0,a),(a,0)^* = (1,0),(1,0)^* = (a,0).$$

Take the Boolean subalgebra $\mathbf{2}\times\mathbf{2} := \{(0,0),(0,1),(1,0),(1,1)\}$ of $\mathbf{3}\times\mathbf{3}$. For convenience, let us change the notations. We represent the set $\mathbf{2}\times\mathbf{2}$ and its elements as $\mathbf{2}^2 = \{0,x,y,1\}$, where $(0,0)$ is replaced by 0, $(0,1)$ is replaced by x, $(1,0)$ is replaced by y, and $(1,1)$ is replaced by 1. Then

$$(\mathbf{2}^2)^{[2]} = \{(0,0),(0,x),(0,1),(0,y),(x,x),(x,1),(y,1),(y,y),(1,1)\}, \text{ and}$$
$$\mathcal{J}_{(\mathbf{2}^2)^{[2]}} = \{(0,x),(0,y),(x,x),(y,y)\}.$$

Further, $(0,x)^* = (x,x),(x,x)^* = (0,x)$ and $(0,y)^* = (y,y),(y,y)^* = (0,y)$. The diagrammatic illustration of the isomorphism between $\mathbf{3}\times\mathbf{3}$ and $(\mathbf{2}^2)^{[2]}$ is given in Figure 5.

3. The Logic $\mathcal{L}_K$ for Kleene Algebras and a 3-Valued Semantics

As mentioned earlier, Moisil in 1941 (cf. [14]) proved that $B^{[2]}$ forms a 3-valued LM algebra. So, while discussing the logic corresponding to the structures $B^{[2]}$, one is naturally led to 3-valued Łukasiewicz logic. Varlet (cf. [10]) noted the equivalence between regular double Stone algebras and 3-valued LM algebras, whence $B^{[2]}$ can be given the structure of a regular double Stone algebra as well. Here, due to Proposition 1 and Theorem 1(i), we focus on $B^{[2]}$ as a *Kleene algebra*, and study the (propositional) logic corresponding to the class of Kleene algebras and the structures $B^{[2]}$. We denote this system as $\mathcal{L}_K$, and present it in this section.

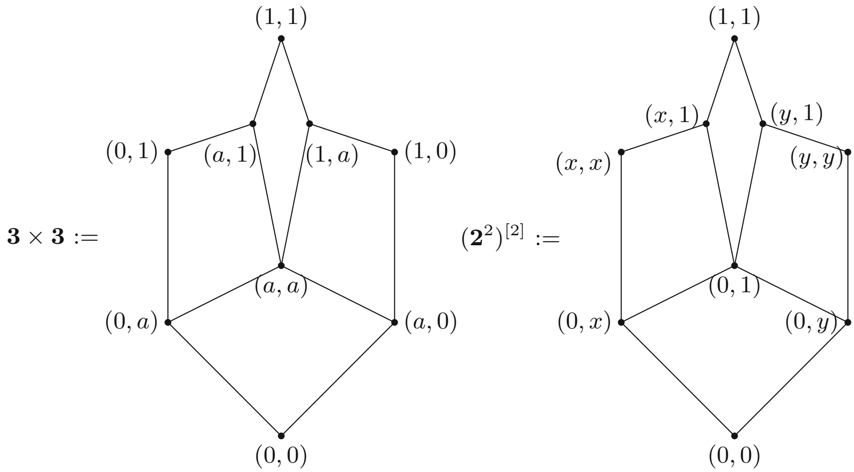

Figure 5. $\mathbf{3} \times \mathbf{3} \cong (\mathbf{2}^2)^{[2]}$

Our approach to the study is motivated by Dunn's 4-valued semantics of the De Morgan consequence system [23]. The 4-valued semantics arises from the fact that each element of a De Morgan algebra can be looked upon as a pair of sets. In our case, we have observed in Section 2 as a consequence of Theorem 1(i), that each element of a Kleene algebra can also be looked upon as a pair of sets. As demonstrated in Example 3 above, the Kleene algebra $\mathbf{3} \cong \mathbf{2}^{[2]}$. We exploit the fact that $\mathbf{3}$, in particular, can be represented as a Kleene algebra of pairs of sets, to get completeness of the logic $\mathcal{L}_K$ with respect to a 3-valued semantics.

The *Kleene* axiom $\alpha \wedge \sim \alpha \vdash \beta \vee \sim \beta$, given by Kalman [32], was studied by Dunn in the context of providing a 3-valued semantics for a fragment of relevance logic [23, 24]. He showed that the De Morgan consequence system coupled with the Kleene axiom (the resulting consequence relation being denoted as $\vdash_{Kalman}$), is sound and complete with respect to a semantic consequence relation (denoted $\models^{\mathbf{3}_R}_{0,1}$) defined on $\mathbf{3}_R$, the *right hand chain* of the De Morgan lattice $\mathbf{4}$ given in Figure 6. $\mathbf{3}_R$ is the side of $\mathbf{4}$ in which the elements are interpreted as t(rue), f(alse) and b(oth), and $\models^{\mathbf{3}_R}_{0,1}$ essentially incorporates truth *and* falsity preservation by valuations in its definition. He called this consequence system, the *Kalman consequence system*.

The completeness result for the Kalman consequence system is obtained considering all 4-valued valuations restricted to $\mathbf{3}_R$: the proof makes explicit reference to valuations on $\mathbf{4}$.

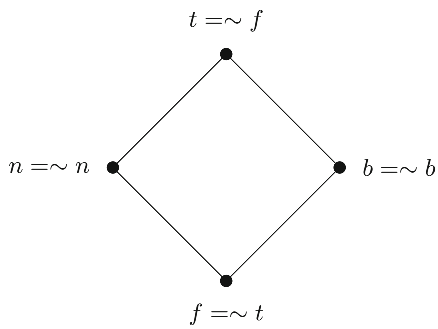

Figure 6. De Morgan lattice **4**

The logic $\mathcal{L}_K$ (K for Kalman and Kleene) that we are considering in our work, is the Kalman consequence system with slight modifications. $\mathcal{L}_K$ is shown to be sound and complete with respect to a 3-valued semantics that is based on the same idea underlying the consequence relation $\models^{\mathbf{3}_R}_{0,1}$, viz. that of truth as well as falsity preservation. However, the definitions and proofs in this case, *do not refer to* **4**.

Let us present $\mathcal{L}_K$. The language consists of

- Propositional variables: $p, q, r, \ldots$.
- Propositional constants: $\top, \bot$.
- Logical connectives: $\vee, \wedge, \sim$.

The well-formed formulae of the logic are defined through the scheme:

$$\top \mid \bot \mid p \mid \alpha \vee \beta \mid \alpha \wedge \beta \mid \sim \alpha.$$

Notation 2. *Denote the set of propositional variables by $\mathcal{P}$, and that of well-formed formulae by $\mathcal{F}$.*

The consequence relation $\vdash_{\mathcal{L}_K}$ is now given through the following postulates and rules, taken from [23] and [25]. These define reflexivity and transitivity of $\vdash$, introduction, elimination principles and the distributive law for the connectives $\wedge$ and $\vee$, contraposition and double negation laws for the negation operator $\sim$, the Kleene property for $\sim$, and some basic requirements from the propositional constants $\top, \bot$. Let $\alpha, \beta, \gamma \in \mathcal{F}$.

Definition 4. ($\mathcal{L}_K$- postulates)

1. $\alpha \vdash \alpha$
2. $\alpha \vdash \beta, \beta \vdash \gamma \; / \; \alpha \vdash \gamma$.

3. $\alpha \wedge \beta \vdash \alpha$, $\alpha \wedge \beta \vdash \beta$.
4. $\alpha \vdash \beta, \alpha \vdash \gamma \;/\; \alpha \vdash \beta \wedge \gamma$.
5. $\alpha \vdash \gamma, \beta \vdash \gamma \;/\; \alpha \vee \beta \vdash \gamma$.
6. $\alpha \vdash \alpha \vee \beta$, $\beta \vdash \alpha \vee \beta$.
7. $\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$ (Distributivity).
8. $\alpha \vdash \beta \;/\; \sim \beta \vdash\sim \alpha$ (Contraposition).
9. $\sim \alpha \wedge \sim \beta \vdash\sim (\alpha \vee \beta)$ ($\vee$-linearity).
10. $\alpha \vdash \top$ (Top).
11. $\bot \vdash \alpha$ (Bottom).
12. $\top \vdash\sim \bot$ (Nor).
13. $\alpha \vdash\sim\sim \alpha$.
14. $\sim\sim \alpha \vdash \alpha$.
15. $\alpha \wedge \sim \alpha \vdash \beta \vee \sim \beta$ (Kalman/Kleene).

Let us now consider any Kleene algebra $(K, \vee, \wedge, \sim, 0, 1)$. We first define valuations on K.

Definition 5. A map $v : \mathcal{F} \rightarrow K$ is called a valuation on K, if it satisfies the following properties for any $\alpha, \beta \in \mathcal{F}$.

1. $v(\alpha \vee \beta) = v(\alpha) \vee v(\beta)$.
2. $v(\alpha \wedge \beta) = v(\alpha) \wedge v(\beta)$.
3. $v(\sim \alpha) =\sim v(\alpha)$.
4. $v(\bot) = 0$.
5. $v(\top) = 1$.

A consequent $\alpha \vdash \beta$ is *valid in* K *under the valuation* v, if $v(\alpha) \leq v(\beta)$. If the consequent is valid under all valuations on K, then it is *valid in* K. Let $\mathcal{A}$ be a class of Kleene algebras. If the consequent $\alpha \vdash \beta$ is valid in each algebra of $\mathcal{A}$, then we say $\alpha \vdash \beta$ *is valid in* $\mathcal{A}$, and denote it as $\alpha \vDash_{\mathcal{A}} \beta$.

Let $\mathcal{A}_K$ denote the class of *all* Kleene algebras. We have, in the classical manner,

Theorem 8. *$\alpha \vdash_{\mathcal{L}_K} \beta$ if and only if $\alpha \vDash_{\mathcal{A}_K} \beta$, for any $\alpha, \beta \in \mathcal{F}$.*

Proof systems for Belnap's 4-valued logic have also been presented in [28,39]. Some abstract algebraic features of the logic obtained by adding the Kalman/Kleene postulate and the corresponding class of algebras are discussed in [28].

We now focus on valuations on the Kleene algebra $B^{[2]}$. For $\alpha \in \mathcal{F}$, $v(\alpha)$ is a pair of the form $(a, b),\ a, b \in B$. Suppose for $\beta \in \mathcal{F}$, $v(\beta) := (c, d),\ c, d \in B$. By definition, the consequent $\alpha \vdash \beta$ is valid in $B^{[2]}$ under v, when $v(\alpha) \leq v(\beta)$, i.e., $(a, b) \leq (c, d)$, or $a \leq c$ and $b \leq d$.

Let $\mathcal{A}_{KB^{[2]}}$ denote the class of Kleene algebras formed by the sets $B^{[2]}$, for *all* Boolean algebras B.

Theorem 9. *$\alpha \vDash_{\mathcal{A}_K} \beta$ if and only if $\alpha \vDash_{\mathcal{A}_{KB^{[2]}}} \beta$, for any $\alpha, \beta \in \mathcal{F}$.*

Proof. Let $\alpha \vDash_{\mathcal{A}_{KB^{[2]}}} \beta$. Consider any Kleene algebra $(K, \vee, \wedge, \sim, 0, 1)$, and let v be a valuation on K. By Theorem 1(i), there exists a Boolean algebra B such that K is embedded in $B^{[2]}$. Let ϕ denote the embedding. It is a routine verification that $\phi \circ v$ is a valuation on $B^{[2]}$. The other direction is trivial, as $\mathcal{A}_{KB^{[2]}}$ is a subclass of $\mathcal{A}_K$. ■

On the other hand, as observed earlier, the structure $B^{[2]}$ is embeddable in the Kleene algebra formed by $\mathcal{P}(U)^{[2]}$ for some set U, utilizing Stone's representation. Hence if v is a valuation on $B^{[2]}$, it can be be extended to a valuation on $\mathcal{P}(U)^{[2]}$. Let $\mathcal{A}_{K\mathcal{P}(U)^{[2]}}$ denote the class of Kleene algebras of the form $\mathcal{P}(U)^{[2]}$, for *all* sets U. So, we get from Theorem 9 the following.

Corollary 1. *$\alpha \vDash_{\mathcal{A}_K} \beta$ if and only if $\alpha \vDash_{\mathcal{A}_{K\mathcal{P}(U)^{[2]}}} \beta$, for any $\alpha, \beta \in \mathcal{F}$.*

Following [23], we now consider semantic consequence relations defined by valuations $v : \mathcal{F} \to \mathbf{3}$ on the Kleene algebra $\mathbf{3}$. Let us re-label the elements of $\mathbf{3}$ as $f,\ u,\ t$, having the standard truth value connotations.

Definition 6. Let $\alpha, \beta \in \mathcal{F}$.

$\alpha \vDash_t \beta$ if and only if, if $v(\alpha) = t$ then $v(\beta) = t$ (Truth preservation).
$\alpha \vDash_f \beta$ if and only if, if $v(\beta) = f$ then $v(\alpha) = f$ (Falsity preservation).
$\alpha \vDash_{t,f} \beta$ if and only if, $\alpha \vDash_t \beta$ and $\alpha \vDash_f \beta$.

We adopt $\vDash_{t,f}$ as the semantic consequence relation for the logic $\mathcal{L}_K$. Note that the consequence relation $\vDash_t$ is the consequence relation used in [44] to interpret the strong Kleene logic. In case of Dunn's 4-valued semantics, the consequence relations $\vDash_t$, $\vDash_f$ and $\vDash_{t,f}$ are defined using valuations on $\mathbf{4}$. As shown in [23], all the three turn out to be equivalent. In order to capture the first-degree entailment fragment of relevance logic, Dunn subsequently uses the semantic consequence relation $\models^{\mathbf{3}_R}_{0,1}$, defined by valuations restricted to $\mathbf{3}_R$, the right hand chain of $\mathbf{4}$. Observe that for valuations on $\mathbf{3}$ that are being considered here, the consequence relations $\vDash_t$, $\vDash_f$ and $\vDash_{t,f}$ are not equivalent: $\alpha \wedge \sim \alpha \vDash_t \beta$, but $\alpha \wedge \sim \alpha \nvDash_f \beta$; $\beta \vDash_f \alpha \vee \sim \alpha$, but $\beta \nvDash_t \alpha \vee \sim \alpha$.

Theorem 10. *$\alpha \vDash_{\mathcal{A}_{K\mathcal{P}(U)^{[2]}}} \beta$ if and only if $\alpha \vDash_{t,f} \beta$, for any $\alpha, \beta \in \mathcal{F}$.*

PROOF. Let $\alpha \vDash_{\mathcal{A}_{K\mathcal{P}(U)^{[2]}}} \beta$, and $v : \mathcal{F} \to \mathbf{3}$ be a valuation. By Example 3 and comments above, $\mathbf{3}$ is embeddable in (in fact, isomorphic to) the Kleene algebra of $\mathcal{P}(U)^{[2]}$ for some set U. If the embedding is denoted by ϕ, $\phi \circ v$ is a valuation on $\mathcal{P}(U)^{[2]}$. Then $(\phi \circ v)(\alpha) \leq (\phi \circ v)(\beta)$ implies $v(\alpha) \leq v(\beta)$. Thus if $v(\alpha) = t$, we have $v(\beta) = t$, and if $v(\beta) = f$, then also $v(\alpha) = f$.
Now, let $\alpha \vDash_{t,f} \beta$. Let U be a set, and $\mathcal{P}(U)^{[2]}$ be the corresponding Kleene algebra. Let v be a valuation on $\mathcal{P}(U)^{[2]}$ – we need to show $v(\alpha) \leq v(\beta)$. For any $\gamma \in \mathcal{F}$ with $v(\gamma) := (A, B),\ A, B \subseteq U$, and for each $x \in U$, define a map $v_x : \mathcal{F} \to \mathbf{3}$ as

$$v_x(\gamma) := \begin{cases} t & \text{if } x \in A \\ u & \text{if } x \in B \setminus A \\ f & \text{if } x \notin B. \end{cases}$$

We show that v_x is a valuation.
Consider any $\gamma, \delta \in \mathcal{F}$, with $v(\gamma) := (A, B)$ and $v(\delta) := (C, D),\ A, B, C, D \subseteq U$.

1. $v_x(\gamma \wedge \delta) = v_x(\gamma) \wedge v_x(\delta)$.

   Note that $v(\gamma \wedge \delta) = (A \cap C, B \cap D)$.

   Case 1 $v_x(\gamma) = t$ and $v_x(\delta) = t$: Then $x \in A \cap C$, and we have $v_x(\gamma \wedge \delta) = t = v_x(\gamma) \wedge v_x(\delta)$.

   Case 2 $v_x(\gamma) = t$ and $v_x(\delta) = u$: $x \in A$, $x \in D$ and $x \notin C$, which imply $x \notin A \cap C$ but $x \in B \cap D$. Hence $v_x(\gamma \wedge \delta) = u = v_x(\gamma) \wedge v_x(\delta)$.

   Case 3 $v_x(\gamma) = t$ and $v_x(\delta) = f$: $x \in A$, $x \notin D$, which imply $x \notin B \cap D$. Hence $v_x(\gamma \wedge \delta) = f = v_x(\gamma) \wedge v_x(\delta)$.

   Case 4 $v_x(\gamma) = u$ and $v_x(\delta) = f$: $x \notin A$ but $x \in B$ and $x \notin D$, which imply $x \notin B \cap D$. Hence $v_x(\gamma \wedge \delta) = f = v_x(\gamma) \wedge v_x(\delta)$.

   Case 5 $v_x(\gamma) = u$, $v_x(\delta) = u$: $x \in B$ but $x \notin A$ and $x \in D$ but $x \notin C$. So, $x \in B \cap D$ and $x \notin A \cap C$. Hence $v_x(\gamma \wedge \delta) = u = v_x(\gamma) \wedge v_x(\delta)$.

   Case 6 $v_x(\gamma) = f$, $v_x(\delta) = f$: $x \notin B$ and $x \notin D$. So, $x \notin B \cap D$. Hence $v_x(\gamma \wedge \delta) = f = v_x(\gamma) \wedge v_x(\delta)$.

2. $v_x(\gamma \vee \delta) = v_x(\gamma) \vee v_x(\delta)$.

   Observe that $v(\gamma \vee \delta) = (A \cup C, B \cup D)$.

   Case 1 $v_x(\gamma) = t$ and $v_x(\delta) = t$: Then $x \in A$, $x \in C$, which imply $x \in A \cup C$. Hence $v_x(\gamma \vee \delta) = t = v_x(\gamma) \vee v_x(\delta)$.

   Case 2 $v_x(\gamma) = t$ and $v_x(\delta) = u$: $x \in A$, $x \in D$ and $x \notin C$, in any way $x \in A \cup C$. Hence $v_x(\gamma \vee \delta) = t = v_x(\gamma) \vee v_x(\delta)$.

Case 3 $v_x(\gamma) = t$ and $v_x(\delta) = f$: $x \in A$, $x \notin D$, which imply $x \in A \cup C$. Hence $v_x(\gamma \vee \delta) = t = v_x(\gamma) \vee v_x(\delta)$.

Case 4 $v_x(\gamma) = u$ and $v_x(\delta) = f$: $x \notin A$ but $x \in B$ and $x \notin D$, which imply $x \notin A \cup C$ but $x \in B \cup D$. Hence $v_x(\gamma \vee \delta) = u = v_x(\gamma) \vee v_x(\delta)$.

Case 5 $v_x(\gamma) = u$, $v_x(\delta) = u$: $x \in B$ but $x \notin A$ and $x \in D$ but $x \notin C$. So, $x \in B \cup D$ and $x \notin A \cup C$. Hence $v_x(\gamma \vee \delta) = u = v_x(\gamma) \vee v_x(\delta)$.

Case 6 $v_x(\gamma) = f$, $v_x(\delta) = f$: $x \notin B$ and $x \notin D$. So, $x \notin B \cup D$. Hence $v_x(\gamma \wedge \delta) = f = v_x(\gamma) \wedge v_x(\delta)$.

3. $v_x(\sim \gamma) = \sim v_x(\gamma)$.

   Note that $v(\sim \gamma) = (B^c, A^c)$.

   Case 1 $v_x(\gamma) = t$: Then $x \in A$, i.e. $x \notin A^c$. Hence $v_x(\sim \gamma) = f = \sim v_x(\gamma)$.

   Case 2 $v_x(\gamma) = u$: $x \notin A$ but $x \in B$. So $x \in A^c$ and $x \notin B^c$. Hence $v_x(\sim \gamma) = u = \sim v_x(\gamma)$.

   Case 3 $v_x(\gamma) = f$: $x \notin B$, i.e. $x \in B^c$. So $v_x(\sim \gamma) = t = \sim v_x(\gamma)$.

Hence v_x is a valuation in **3**. Now let us show that $v(\alpha) \leq v(\beta)$. Let $v(\alpha) := (A', B')$, $v(\beta) := (C', D')$, and $x \in A'$. Then $v_x(\alpha) = t$, and as $\alpha \vDash_{t,f} \beta$, by definition, $v_x(\beta) = t$. This implies $x \in C'$, whence $A' \subseteq C'$.

On the other hand, if $x \notin D'$, $v_x(\beta) = f$. Hence $v_x(\alpha) = f$, so that $x \notin B'$, giving $B' \subseteq D'$. ■

Note that the above proof cannot be applied on the Kleene algebra $B^{[2]}$ instead of $\mathcal{P}(U)^{[2]}$, as we have used set representations explicitly.

An immediate consequence of Theorem 8, Corollary 1 and Theorem 10 is

THEOREM 11. *$\alpha \vdash_{\mathcal{L}_K} \beta$ if and only if $\alpha \vDash_{t,f} \beta$, for any $\alpha, \beta \in \mathcal{F}$.*

4. Rough Set Semantics for $\mathcal{L}_K$

Rough set theory, introduced by Pawlak [37] in 1982, deals with a domain U that is the set of objects, and an equivalence (*indiscernibility*) relation R on U. The pair (U, R) is called an (Pawlak) *approximation space*. For any $A \subseteq U$, one defines the *lower* and *upper approximations* of A in the approximation space (U, R), denoted $\mathsf{L}A$ and $\mathsf{U}A$ respectively, as follows.

$$\mathsf{L}A := \bigcup\{[x] : [x] \subseteq X\},$$

$$\mathsf{U}A := \bigcup\{[x] : [x] \cap X \neq \emptyset\}. \qquad (*)$$

As the information about the objects of the domain is available modulo the equivalence classes in U, the description of any concept, represented extensionally as the subset A of U, is inexact. One then 'approximates' the description from within and outside, through the lower and upper approximations respectively. Unions of equivalence classes are termed as *definable* sets, signifying exactly describable concepts in the context of the given information. In particular, sets of the form $\mathsf{L}A$, $\mathsf{U}A$ are definable sets.

Definition 7. Let (U, R) be an approximation space. For each $A \subseteq U$, the ordered pair $(\mathsf{L}A, \mathsf{U}A)$ is called a rough set in (U, R).

Notation 3. $\mathcal{RS} := \{(\mathsf{L}A, \mathsf{U}A) : A \subseteq U\}$.

The ordered pair (D_1, D_2), where $D_1 \subseteq D_2$ and D_1, D_2 are definable sets, is called a generalized rough set in (U, R).

Notation 4. *$\mathcal{D}$ denotes the collection of definable sets and $\mathcal{R}$ that of the generalized rough sets in (U, R).*

In the following, we proceed to establish part (ii) of Theorem 1 (cf. Section 1). In Section 4.2, we formalize the connection of rough sets with the 3-valued semantics being considered in this work. We end the section with a rough set semantics for $\mathcal{L}_K$ (cf. Theorem 16), obtained as a consequence of the representation results of Section 4.1 below.

4.1. Rough Set Representation of Kleene Algebras

Algebraically, the collection $\mathcal{D}$ of definable sets forms a complete atomic Boolean algebra in which atoms are the equivalence classes. The collection $\mathcal{RS}$ forms a distributive lattice – in fact, it forms a Kleene algebra. On the other hand, observe that $\mathcal{R}$ is the set $\mathcal{D}^{[2]}$ and hence forms a Kleene algebra (cf. Proposition 1) as well. $\mathcal{R}$ has earlier been studied, for instance, by Banerjee and Chakraborty in [5], and shown to form *topological quasi-Boolean*, *pre-rough* and *rough* algebras. Note that, for an approximation space (U, R), sets $\mathcal{R}$ and $\mathcal{RS}$ may not be the same. So, it is natural to ask how $\mathcal{R}$ and $\mathcal{RS}$ differ as algebraic structures. The following result mentioned in [5] gives a connection between the two. The proof is not given in [5]; we sketch it here, as it is used in the sequel.

Theorem 12. *For any approximation space (U, R), there exists an approximation space (U', R') such that $\mathcal{R}$ corresponding to (U, R) is order isomorphic to $\mathcal{R}'$ corresponding to (U', R'). Further, $\mathcal{R}' = \mathcal{RS}'$, the latter denoting the collection of rough sets in the approximation space (U', R') .*

PROOF. Let (U, R) be the given approximation space. Consider the set $\mathbf{A} := \{a \in U : |R(a)| = 1\}$, where $R(a)$ denotes the equivalence class of a in U. So $\mathbf{A}$ is the collection of all elements which are R-related only to themselves. Now, let us consider a set $\mathbf{A}'$ which consists of 'dummy' elements from outside U, indexed by the set $\mathbf{A}$, i.e. $\mathbf{A}' := \{a' : a \in \mathbf{A}\}$ such that $\mathbf{A}' \cap U = \emptyset$. Let $U' = U \cup \mathbf{A}'$. Define an equivalence relation R' on U' as follows.

$$\text{If a } \in \text{U then } R'(a) := R(a) \cup \{x' \in \mathbf{A}' : x \in R(a) \cap \mathbf{A}\}.$$
$$\text{If } a' \in \mathbf{A}' \text{ then } R'(a') := R(a)\ (= \{a, a'\}).$$

Note that the number of equivalence classes in both the approximation spaces is the same. Define the map $\phi : \mathcal{R} \to \mathcal{R}'$ as $\phi(D_1, D_2) := (D_1', D_2')$, where $D_1' := D_1 \cup \{x' \in \mathbf{A}' : x \in D_1 \cap \mathbf{A}\}$ and $D_2' := D_2 \cup \{x' \in \mathbf{A}' : x \in D_2 \cap \mathbf{A}\}$. Then ϕ is an order isomorphism. ∎

Since $\mathcal{R}$ and $\mathcal{RS}$ for any approximation space (U, R) form Kleene algebras, Theorem 12 can easily be extended to Kleene algebras as follows.

THEOREM 13. *Let (U, R) be an approximation space. There exists an approximation space (U', R') such that $\mathcal{R}$ corresponding to (U, R) is Kleene isomorphic to $\mathcal{RS}'$ $(= \mathcal{R}')$ corresponding to (U', R').*

PROOF. Consider (U', R') and ϕ as in Theorem 12. ϕ is a lattice isomorphism, as the restriction of ϕ to the completely join irreducible elements of the lattices $\mathcal{D}^{[2]}$ and $\mathcal{D}'^{[2]}$ is an order isomorphism (using Proposition 3 and Lemma 1). Let us now show that $\phi(\sim (D_1, D_2)) = \sim (\phi(D_1, D_2))$. To avoid confusion, we follow these notations: for $X \subseteq U$ we use X^{c_1} for the complement in U and X^{c_2} for the complement in U'.
Now, $\phi(\sim (D_1, D_2)) = \phi(D_2^{c_1}, D_1^{c_1}) = ((D_2^{c_1})', (D_1^{c_1})')$. By definition of ϕ, we have:

$$(D_2^{c_1})' = D_2^{c_1} \cup \{x' \in \mathbf{A}' : x \in D_2^{c_1} \cap \mathbf{A}\}.$$
$$(D_1^{c_1})' = D_1^{c_1} \cup \{x' \in \mathbf{A}' : x \in D_1^{c_1} \cap \mathbf{A}\}.$$

CLAIM. $(D_2^{c_1})' = (D_2')^{c_2}$, and $(D_1^{c_1})' = (D_1')^{c_2}$.

PROOF OF CLAIM:. Let us first prove that $(D_2^{c_1})' = (D_2')^{c_2}$. Note that $(D_2^{c_1})' = D_2^{c_1} \cup \{x' \in \mathbf{A}' : x \in D_2^{c_1} \cap \mathbf{A}\}$, and $(D_2')^{c_2} = (D_2 \cup \{x' : x \in D_2 \cap \mathbf{A}\})^{c_2} = (D_2)^{c_2} \cap (\{x' \in \mathbf{A}' : x \in D_2 \cap \mathbf{A}\})^{c_2}$.
Let $X := \{x' \in \mathbf{A}' : x \in D_2^{c_1} \cap \mathbf{A}\}$ and $Y := \{x' \in \mathbf{A}' : x \in D_2 \cap \mathbf{A}\}$.
Consider $a \in (D_2^{c_1})' = D_2^{c_1} \cup X$.

<u>Case 1</u> $a \in D_2^{c_1}$:
As $D_2 \subseteq U$, $D_2^{c_1} \subseteq D_2^{c_2}$. Hence $a \in D_2^{c_2}$. As $D_2^{c_1} \subseteq U$, $a \notin \mathbf{A}'$, whence $a \in Y^{c_2}$.

So $a \in (D_2')^{c_2}$.
<u>Case 2</u> $a \in X$:
$a = x'$, where $x \in D_2^{c_1} \cap \mathbf{A}$. As, $x' \in R'(x)$ and $D_2^{c_2}$ is the union of equivalence classes, in particular it contains $R'(x)$. So $a = x' \in D_2^{c_2}$.
$x \in D_2^{c_1}$ implies $x \notin D_2$. Hence $a = x' \in Y^{c_2}$. So $a \in (D_2')^{c_2}$.
Conversely, let $a \in (D_2')^{c_2} = (D_2)^{c_2} \cap Y^{c_2}$.
<u>Case 1</u> $a \in U$:
$a \in D_2^{c_2}$ implies that $a \in D_2^{c_1}$. Hence $a \in (D_2^{c_1})'$.
<u>Case 2</u> $a \in \mathbf{A}'$:
$a \in Y^{c_2}$ implies $a \in \{x' \in \mathbf{A}' : x \in D_2^{c_1} \cap \mathbf{A}\}$. Hence $a \in (D_2^{c_1})'$.
Similar arguments as above show that $(D_1^{c_1})' = (D_1')^{c_2}$.

Proof of Theorem 13. $\phi(\sim (D_1, D_2)) = \phi(D_2^{c_1}, D_1^{c_1}) = ((D_2^{c_1})', (D_1^{c_1})') = ((D_2')^{c_2}, (D_1')^{c_2}) =\sim \phi(D_1, D_2)$.
Hence ϕ is a Kleene isomorphism. ■

It is now not hard to see the correspondence between a complete atomic Boolean algebra and rough sets in an approximation space.

Theorem 14. *Let B be a complete atomic Boolean algebra.*

(i) *There exists an approximation space (U, R) such that*
   (a) $B \cong \mathcal{D}$.
   (b) $B^{[2]}$ *is Kleene isomorphic to* $\mathcal{R}$.

(ii) *There exists an approximation space (U', R') such that $B^{[2]}$ is Kleene isomorphic to $\mathcal{RS}'$.*

Proof. Let U denote the collection of all atoms of B, and R the identity relation on U. (U, R) is the required approximation space. ■

Thus we get Theorem 1(ii):
Given a Kleene algebra $\mathcal{K}$, there exists an approximation space (U, R) such that $\mathcal{K}$ can be embedded into $\mathcal{RS}$. In other words, every Kleene algebra is isomorphic to an algebra of rough sets in a Pawlak approximation space.

4.2. Rough Sets and the Kleene Algebra 3

The definitions (*) of lower and upper approximations of a set A in an approximation space (U, R) immediately yield the following interpretations.

1. x *certainly* belongs to A, if $x \in \mathsf{L}A$, i.e. all objects which are indiscernible to x are in A.
2. x *certainly does not* belong to A, if $x \notin \mathsf{U}A$, i.e. all objects which are indiscernible to x are not in A.

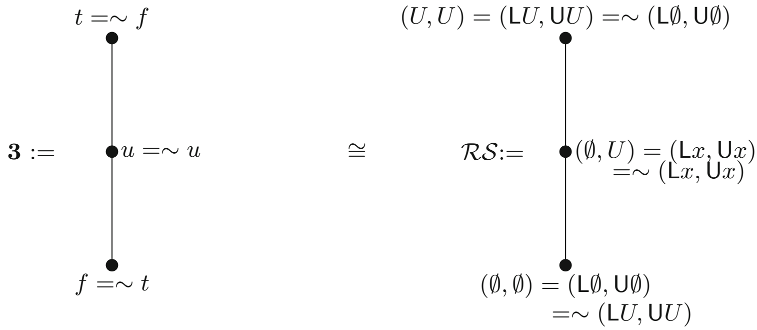

Figure 7. $\mathbf{3} \cong \mathcal{RS}$

3. Belongingness of x to A is *not certain, but possible*, if $x \in \mathsf{U}A$ but $x \notin \mathsf{L}A$. In rough set terminology, this is the case when x is in the *boundary* of A: some objects indiscernible to x are in A, while some others, also indiscernible to x, are in A^c.

These interpretations have led to much work in the study of connections between 3-valued algebras or logics and rough sets, see for instance [2,4, 15,26,30,36]. In particular, in [2], Avron and Konikowska have obtained a non-deterministic logical matrix and studied the 3-valued logic generated by this matrix. A simple predicate language is used, with no quantifiers or connectives, to express membership in rough sets. Connections, in special cases, with 3-valued Kleene, Łukasiewicz and two paraconsistent logics are established. In another direction, from an algebraic motivation, Banerjee and Chakraborty in [4,5] obtained the propositional system of *pre-rough logic* for the class of pre-rough algebras. It was subsequently proved by Banerjee in [4] that 3-valued Łukasiewicz logic and pre-rough logic are equivalent, thereby imparting a rough set semantics to the former.

Let us spell out the natural connections of the Kleene algebra $\mathbf{3}$ with rough sets. Observe that $\mathbf{3}$, being isomorphic to $\mathbf{2}^{[2]}$ (as noted earlier), can also be viewed as a collection of rough sets in an approximation space, due to Theorem 14(ii).

Proposition 4. *There exists an approximation space (U, R) such that $\mathbf{3} \cong \mathcal{RS}$.*

Proof. Let $U := \{x, y\}$ and consider the equivalence relation $R := U \times U$ on U. The correspondence is depicted in Figure 7.

Note that we also have $(\emptyset, U) = (\mathsf{L}y, \mathsf{U}y) = \sim (\mathsf{L}y, \mathsf{U}y)$. ∎

On the other hand, interpretations 1-3 above give rise to a correspondence with the set $\mathbf{3} := \{f, u, t\}$, that assigns to every $x \in U$ and rough set $(\mathsf{L}A, \mathsf{U}A)$ in (U, R), the value t when $x \in \mathsf{L}A$, u when $x \in \mathsf{U}A \setminus \mathsf{L}A$, and f in case $x \notin \mathsf{U}A$. As one can see, this is akin to the valuation defined in the proof of Theorem 10. In fact, using results of the previous sections, we can formally link the 3-valued semantics being considered here, and rough sets.

Let $\mathcal{A}_{K\mathcal{RS}}$ denote the class containing the collections $\mathcal{RS}$ of rough sets over *all* possible approximation spaces (U, R).

THEOREM 15. *For any* $\alpha, \beta \in \mathcal{F}$,

(i) $\alpha \vDash_{\mathcal{A}_K} \beta$ *if and only if* $\alpha \vDash_{\mathcal{A}_{K\mathcal{RS}}} \beta$,

(ii) $\alpha \vDash_{\mathcal{A}_{K\mathcal{RS}}} \beta$ *if and only if* $\alpha \vDash_{t,f} \beta$.

In the process, we have thus obtained a rough set semantics for $\mathcal{L}_K$.

THEOREM 16. *For any* $\alpha, \beta \in \mathcal{F}$, $\alpha \vdash_{\mathcal{L}_K} \beta$ *if and only if* $\alpha \vDash_{\mathcal{A}_{K\mathcal{RS}}} \beta$.

5. Perp Semantics for the Logic $\mathcal{L}_K$

Dunn's framework of perp semantics for negations is of logical, philosophical as well as algebraic importance. On the one hand, it provides relational semantics for various logics with negations (cf. e.g., [23,25]), interpreting the negations as 'impossibility' or 'unnecessity' operators. On the other hand, one can give representations of various algebras as set algebras [20]. In this section, we characterize the Kleene consequent $\alpha \wedge \sim \alpha \vdash \beta \vee \sim \beta$ in Dunn's framework of negations. Further, one obtains a representation of Kleene algebras through duality.

First, we briefly present the basics of perp semantics. For details, one may refer to [20,23–25].

DEFINITION 8. A compatibility frame is a triple $(W, C, \leq)$ with the following properties.

1. $(W, \leq)$ is a partially ordered set.
2. C is a binary relation on W such that for $x, y, x', y' \in W$, if $x' \leq x$, $y' \leq y$ and xCy then $x'Cy'$.

C is called a compatibility relation on W.

A perp frame is a tuple $(W, \perp, \leq)$, where $\perp$, the perp relation on W, is the complement of the compatibility relation C.

As in [25], we do not distinguish between compatibility and perp frames, and present the results in the section in terms of the compatibility relation.

Recall the syntax of the logic $\mathcal{L}_K$ as defined in Section 3.

Definition 9. A relation $\vDash$ between points of W and propositional variables in $\mathcal{P}$ is called an evaluation, if it satisfies the hereditary condition:

– If $x \vDash p$ and $x \leq y$ then $y \vDash p$, for any $x, y \in W$.

Recursively, an evaluation $\vDash$ can be extended to $\mathcal{F}$ as follows. Let $x \in W$.

1. $x \vDash \alpha \wedge \beta$ if and only if $x \vDash \alpha$ and $x \vDash \beta$.
2. $x \vDash \alpha \vee \beta$ if and only if $x \vDash \alpha$ or $x \vDash \beta$.
3. $x \vDash \top$.
4. $x \nvDash \bot$.
5. $x \vDash \sim \alpha$ if and only if for all $y \in W$, xCy implies that $y \nvDash \alpha$.

Then one can easily show that $\vDash$ satisfies the hereditary condition for all formulae in $\mathcal{F}$. Thus, for each formula α in $\mathcal{F}$, an evaluation $\vDash$ gives a subset of W that is *upward closed* in the partially ordered set $(W, \leq)$. ($X \subseteq W$ is upward closed or a *cone*, if $x \in X$ and $x \leq y$, $y \in W$, imply $y \in X$.)

For the compatibility frame $\mathbf{F} := (W, C, \leq)$, the pair $(\mathbf{F}, \vDash)$ for an evaluation $\vDash$ is called a *model*. The notion of validity is introduced next in the following (usual) manner.

– A consequent $\alpha \vdash \beta$ is *valid in a model* $(\mathbf{F}, \vDash)$, denoted as $\alpha \vDash_{(\mathbf{F},\vDash)} \beta$, if and only if, if $x \vDash \alpha$ then $x \vDash \beta$, for each $x \in W$.
– $\alpha \vdash \beta$ is *valid in the compatibility frame* $\mathbf{F}$, denoted as $\alpha \vDash_{\mathbf{F}} \beta$, if and only if $\alpha \vDash_{(\mathbf{F},\vDash)} \beta$ for every model $(\mathbf{F}, \vDash)$.
– Let $\mathbb{F}$ denote a class of compatibility frames. $\alpha \vdash \beta$ is *valid in* $\mathbb{F}$, denoted as $\alpha \vDash_{\mathbb{F}} \beta$, if and only if $\alpha \vDash_{\mathbf{F}} \beta$ for every frame $\mathbf{F}$ belonging to $\mathbb{F}$.

Following [25], let K_i denote the logic whose postulates are 1 to 12 of the logic $\mathcal{L}_K$ (cf. Definition 4). In [25] it has been proved that K_i is the minimal logic which is sound and complete with respect to the class of all compatibility frames. Frame completeness results for various normal logics with negation have been proved using the canonical frames for the logics. Let us give the definitions for the canonical frame [21,25].

Definition 10. A set P of sentences is a prime theory if

(i) $\alpha \vdash \beta$ holds and $\alpha \in P$ imply $\beta \in P$,

(ii) $\alpha, \beta \in P$ imply $\alpha \wedge \beta \in P$,

(iii) $\top \in P$ and $\bot \notin P$,

(iv) $\alpha \vee \beta \in P$ implies $\alpha \in P$ or $\beta \in P$.

Let W_c be the collection of all prime theories of $\mathcal{L}_K$. The *canonical* relation C_c on W_c is defined as: PC_cQ if and only if, for all sentences α, $\sim \alpha \in P$ implies $\alpha \notin Q$. The tuple $(W_c, C_c, \subseteq)$ is the *canonical frame* for $\mathcal{L}_K$.

Note that $\mathcal{L}_K$ contains K_i along with the postulates (13) $\alpha \vdash \sim\sim \alpha$, (14) $\sim\sim \alpha \vdash \alpha$ and (15) $\alpha \wedge \sim \alpha \vdash \beta \vee \sim \beta$ of Definition 4. The consequents $\alpha \vdash \sim\sim \alpha$ and $\sim\sim \alpha \vdash \alpha$ have been characterized by Dunn (e.g. cf. [25]) and Restall [41] respectively as follows.

THEOREM 17. 1. *$\alpha \vdash \sim\sim \alpha$ is valid precisely in the class of all compatibility frames satisfying the following frame condition:*

$$\forall x \forall y (xCy \rightarrow yCx).$$

2. *$\sim\sim \alpha \vdash \alpha$ is valid precisely in the class of all compatibility frames satisfying the frame condition:*

$$\forall x \exists y (xCy \wedge \forall z (yCz \rightarrow z \leq x)).$$

3. *The canonical frame for $\mathcal{L}_K$ satisfies both the above frame conditions.*

It remains for us to characterize the Kleene consequent $\alpha \wedge \sim \alpha \vdash \beta \vee \sim \beta$ with an appropriate frame condition, and prove that the canonical frame for $\mathcal{L}_K$ satisfies the condition.

THEOREM 18. *$\alpha \wedge \sim \alpha \vdash \beta \vee \sim \beta$ is valid in a compatibility frame, if and only if the compatibility relation satisfies the following first order property:*

$$\forall x (xCx \vee \forall y (xCy \rightarrow y \leq x)). \qquad (*)$$

The canonical frame for $\mathcal{L}_K$ satisfies (*).

PROOF. Consider any compatibility frame $(W, C, \leq)$, let (*) hold, and let $x \in W$.
Suppose xCx, then $x \not\models \alpha \wedge \sim \alpha$, and trivially, if $x \models \alpha \wedge \sim \alpha$ then $x \models \beta \vee \sim \beta$.
Now suppose $\forall y(xCy \rightarrow y \leq x)$ is true.

Let $x \not\models \beta$ and xCz. Then $z \leq x$, whence $z \not\models \beta$. So, by definition $x \models \sim \beta$. Hence $x \models \beta \vee \sim \beta$. Hence in any case if $x \models \alpha \wedge \sim \alpha$ then $x \models \beta \vee \sim \beta$.
Let (*) not hold. This implies that there exists x in W such that $not(xCx)$ and there exists y in W such that xCy and $y \not\leq x$. Take such a pair x, y

from W. Let us define, for any $z, w \in W$,

$$z \vDash p \text{ if and only if } x \leq z \text{ and } not(xCz),$$
$$w \vDash q \text{ if and only if } y \leq w.$$

We show that $\vDash$ is an evaluation (cf. Definition 9). Let $z \vDash p$ and $z \leq z'$. Then $x \leq z'$. If xCz', then by the frame condition on C we have

$$x \leq x, z \leq z' xCz' \text{ imply } xCz,$$

which is a contradiction to the fact that $z \vDash p$.

Furthermore, $x \vDash p$, as $x \leq x$ and $not(xCx)$. We also have $x \vDash \sim p$: if xCw for any $w \in W$ then by definition, $w \nvDash p$. Hence, $x \vDash \sim p$ and so $x \vDash p \wedge \sim p$.

On the other hand, $x \nvDash q$ as $y \nleq x$. By the assumption, xCy and $y \vDash q$, hence $x \nvDash \sim q$. So, we have $x \vDash p \wedge \sim p$ but $x \nvDash q \vee \sim q$.

Canonicity :
Let $not(PC_cP)$. Then, by definition of C_c, there exists an $\alpha \in \mathcal{F}$ such that $\alpha, \sim \alpha \in P$. But this implies that $\alpha \wedge \sim \alpha \in P$. Hence for all $\beta \in \mathcal{F}$, $\beta \vee \sim \beta \in P$. So, for all β, either $\beta \in P$ or $\sim \beta \in P$.

Now let PC_cQ and $\beta \in Q$. Then $\sim \beta \notin P$. But as from above, $not(PC_cP)$, we have $\beta \in P$. So, $Q \subseteq P$. ∎

Definition 11. A compatibility frame $(W, C, \leq)$ is called a Kleene frame if it satisfies the following frame conditions.

1. $\forall x \forall y (xCy \rightarrow yCx)$.
2. $\forall x \exists y (xCy \wedge \forall z (yCz \rightarrow z \leq x))$.
3. $\forall x (xCx \vee \forall y (xCy \rightarrow y \leq x))$.

Denote by $\mathbb{F}_K$, the class of all Kleene frames.

We have then arrived at

Theorem 19. *The following are all equivalent, for any $\alpha, \beta \in \mathcal{F}$.*

(a) $\alpha \vdash_{\mathcal{L}_K} \beta$.

(b) $\alpha \vDash_{\mathcal{A}_K} \beta$.

(c) $\alpha \vDash_{\mathcal{A}_{K\mathcal{RS}}} \beta$.

(d) $\alpha \vDash_{t,f} \beta$.

(e) $\alpha \vDash_{\mathbb{F}_K} \beta$.

5.1. Kleene Frames and Complex Algebras

As in the case of Kripke semantics for modal logics, in perp semantics, propositions are interpreted as subsets (that are upward closed) in compatibility frames. Moreover, negations are interpreted using the compatibility relations of the frames. So, negations can be thought of as modal operators. In expected lines therefore, similar to Jónsson-Tarski duality of modal logic, one can establish duality results here, between classes of compatibility frames and classes of various distributive lattices with negation [20,22,25].

Let $(W, C, \leq)$ be a compatibility frame, and let K be the collection of all upward closed subsets of W. Then $(K, \cup, \cap, \emptyset, W)$ is a bounded distributive lattice. A unary operator $\sim$ is defined on K using the compatibility relation C as follows. For any $A \in K$,

$$\sim A := \{x \in W : \text{ for all } y \text{ in } W,\ xCy \text{ implies } y \notin A\}.$$

The algebra $(K, \cup, \cap, \sim, \emptyset, W)$ is called the *complex algebra* of the compatibility frame $(W, C, \leq)$.

Now, let $(K, \vee, \wedge, \sim, 0, 1)$ be a Kleene algebra. Let us consider the set U_K of all prime filters of K. Define a binary relation C_K on U_K as follows. For $P, Q \in U_K$,

$$PC_KQ \text{ if and only if for all } a \in K, \text{ if } \sim a \in P \text{ then } a \notin Q.$$

The structure $(U_K, C_K, \subseteq)$ is called the *canonical frame* for K.

Like the correspondence results presented above, one can easily establish correspondence results between classes of compatibility frames and classes of complex algebras. For the class of Kleene frames, we obtain the following, by mimicking the proofs of Theorems 17 and 18.

Proposition 5. 1. *Let $(W, C, \leq)$ be a compatibility frame. Its complex algebra is a Kleene algebra if and only if $(W, C, \leq)$ is a Kleene frame.*

2. *Let $\mathcal{K}$ be a Kleene algebra. Then its canonical frame is a Kleene frame.*

Using Proposition 5 and employing standard techniques analogous to those in the proofs of Jónsson-Tarski duality results [9], one can prove the following. An embedding between frames is a map that is order and relation preserving in both directions.

Theorem 20. *1. Let $\mathcal{K}$ be a Kleene algebra. There exists a Kleene frame $(W, C, \leq)$ such that $\mathcal{K}$ can be embedded into the complex algebra of $(W, C, \leq)$.*

2. Let $(W, C, \leq)$ be a Kleene frame. Then there exists a Kleene algebra $\mathcal{K}$ such that $(W, C, \leq)$ can be embedded into the canonical frame of $\mathcal{K}$.

Theorem 20, in fact, provides a direct proof of the equivalence of (b) and (e) of Theorem 19.

Theorems 1(i) and 20 are both representation theorems for Kleene algebras. Theorem 1(i) leads us to conclude that the Kleene negation is described by the set-theoretic complement. On the other hand, Theorem 20 tells us that it is described by a compatibility relation.

6. Conclusions

In case of Boolean algebras and classical propositional logic, or De Morgan algebras and De Morgan logic, the algebraic semantics and the 2 or 4-valued semantics (respectively) are equivalent – due to representation theorems for the two classes of algebras. Here, analogously, we have the result for the class of Kleene algebras, that the algebraic semantics and a 3-valued semantics (given by $\vDash_{t,f}$) of the logic $\mathcal{L}_K$ of Kleene algebras are equivalent. This is due to the representation theorem (Theorem 1(i)): any Kleene algebra is embeddable into $B^{[2]}$, for some Boolean algebra B.

Furthermore, the 3-valued semantics of $\mathcal{L}_K$ translates into a rough set semantics for the logic (Theorem 16), because we obtain a representation of Kleene algebras in the class of rough set algebras (Theorem 1(ii)). This part of the work thus adds to the study of connections between 3-valued logics or algebras and rough sets.

$\mathcal{L}_K$ is also shown to be sound and complete with respect to a perp semantics (Theorem 19), and one finds yet another representation of Kleene algebras: any Kleene algebra is embeddable into the complex algebra of a Kleene compatibility frame (Theorem 20). The logic of Kleene algebras can thus be imparted equivalent semantics from different perspectives.

The work presents a hitherto unexplored relationship between rough sets and perp semantics. Theorems 20 and 1(ii) together tell us that the collection of rough sets over any approximation space induces a Kleene compatibility frame, and vice versa: any Kleene compatibility frame induces a Kleene algebra of rough sets over some approximation space. These facts encourage us to investigate the relationship between the two frameworks further. There are different ways in which negations may be defined on rough sets over an approximation space. A study of these negations from the angle of perp semantics has been done in [34]. It may now be worth identifying meaningful compatibility relations on approximation spaces. One can then make a study of negations that may be defined on rough sets using the compatibility relations, and look for relationships if any, with the existing negations.

Extensions of Kleene algebras with modal operators have been studied in literature. One such extension leads to a 3-valued Łukasiewicz algebra [10] or, equivalently, to a pre-rough algebra. Another has been discussed in [11]. Recently, several modal extensions of De Morgan algebras that are weaker than pre-rough algebras but stronger than topological quasi-Boolean algebras [5], have been studied in [42]. It would be interesting to investigate the import of adding the Kleene property to the De Morgan base of these algebras, both from logical and algebraic perspectives.

Study in yet another direction may be worthwhile. In [16], translations of various 3-valued logics into the fragment MEL [6] of the modal logic KD have been obtained. A natural question stemming from there could be about the result of replacing $\mathcal{L}_K$ in our work, with other 3-valued logics. On the algebraic side, one may shift focus to some replacements of Kleene algebras – in particular by algebraic structures relevant to rough sets, such as Post algebras of order three or chain based lattices [36].

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A. Kumar, M. Banerjee
Department of Mathematics and Statistics
Indian Institute of Technology
Kanpur 208016
India
arunk2956@gmail.com

M. Banerjee
mohua@iitk.ac.in