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The Lambek Calculus Extended with Intuitionistic Propositional Logic

Abstract. We present sound and complete semantics and a sequent calculus for the Lambek calculus extended with intuitionistic propositional logic.

Keywords: Lambek calculus, Intuitionistic propositional logic, Kripke Semantics, Ternary semantics, Sequent calculus, Decidability.

1. Introduction

Adding propositional connectives to the language of the Lambek calculus is not new. A sequent calculus for the associative Lambek calculus extended with conjunction and disjunction was introduced in [9]. In contrast to intuitionistic and classical propositional logics, in the above extension, conjunction and disjunction are not mutually distributive. Namely, only the sequent

$$(A \wedge C) \vee (B \wedge C) \to (A \vee B) \wedge C \tag{1}$$

is derivable in that calculus. To overcome this deficiency, the converse

$$(A \vee B) \wedge C \to (A \wedge C) \vee (B \wedge C) \tag{2}$$

of (1) was added as an additional axiom in [2,3].[1]

The nonassociative Lambek calculus extended with conjunction, disjunction, negation, $\top$ and $\bot$ (both classical and intuitionistic) was studied in [2,3]. It was shown there that this extension is sound and complete with respect to *distributive lattice-ordered residuated groupoids augmented with boolean negation*. It was also shown there that this *algebraic* semantics possesses the strong finite model property (i.e., it also holds for the consequence

[1] A normalizing natural deduction calculus for the extension of the associative Lambek calculus with conjunction and disjunction, where these connectives are mutually distributive, can be found in [22].

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relation). Thus, the extension of the nonassociative Lambek calculus with intuitionistic propositional logic is strongly decidable (i.e., the consequence relation is decidable).

In this paper we also study the extension of both nonassociative and associative Lambek calculi with intuitionistic propositional logic (denoted ***NLI*** and ***LI***, respectively), presented as a Hilbert-style system,[2] but from a different perspective. We show that the combination of the *relational semantics* (see [4]) and the *Kripke semantics* (see [12]) is sound and complete for these extensions and that the latter are conservative extensions of both the corresponding Lambek calculi and intuitionistic propositional logic, cf. the semantics in [16,17], where completeness is proved only for the logic without negation. Then, using filtrations, we obtain the finite model property of the combined semantics of ***NLI*** that yields an alternative proof of its decidability. This part of the paper is an "intuitionistic counterpart" of similar results from [8] for Lambek calculi extended with classical propositional logic.

We also present sequent calculi without additional axioms for both ***NLI*** and ***LI***. These sequent calculi are based on the ideas from [5] and [15] and are similar to distributive full Lambek calculus (DFL) introduced in [11], but, in addition to conjunction and disjunction of DFL, contain implication and negation. Our sequent calculi are extensions of the sequent calculus for the *logic of bunched implications* ***BI*** [16,17] with the left-sided relevance implication.[3] The sequent calculi combine the substructural nature of the Lambek calculus [14] and the full structural nature of the Gentzen calculus [6] by alternating between the rigid structure of antecedents of Lambek sequents (ranked trees and sequences) and the flexible structure of antecedents of Gentzen sequents (multisets). Both calculi admit cut elimination.

As a corollary of cut elimination we obtain that both ***NLI*** and ***LI*** are decidable. As we have mentioned above, (strong) decidability of ***NLI*** is already known from [2,3]. What is new in our paper is decidability of ***LI***.

This paper is organized as follows. In the next section we introduce the extension of the nonassociative Lambek calculus with intuitionistic propositional logic. Then, in Section 3 we define the relational semantics of the extended nonassociative Lambek calculus and prove the corresponding completeness theorem. Section 4 contains a sequent calculus for ***NLI***. In Section 5 we define the (associative) relational semantics of the extended asso-

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[2] Cf. [2,3], where the extension is presented as a sequent system with additional axioms.

[3] Also, the ternary semantics of Lambek calculus is a similar extension of semantics of relevance implication.

ciative Lambek calculus and prove the corresponding completeness theorem. Finally, in Section 6 we present a sequent calculus for $\boldsymbol{LI}$.

2. Nonassociative Lambek Calculus Extended with Intuitionistic Propositional Logic

The axiomatic extension $\boldsymbol{NLI}$ of the nonassociative Lambek calculus ($\boldsymbol{NL}$) with intuitionistic propositional logic ($\boldsymbol{I}$) is defined as follows. Formulas are constructed from propositional variables (atomic formulas), denoted by P_i, $i = 1, 2, \ldots$, by means of the Lambek connectives $\backslash$, $/$, $\cdot$, and the propositional connectives $\wedge$, $\vee$, $\supset$, and $\neg$ (conjunction, disjunction, implication, and negation, respectively).

Formulas constructed from propositional variables by means of the Lambek connectives only are called *formulas of the pure Lambek calculus* and formulas of the form $A \supset B$, where A and B are formulas of the pure Lambek calculus are called *Lambek implications*.

The rules of inference and the axioms of $\boldsymbol{NLI}$ are the rules of inference of the nonassociative Lambek calculus $\boldsymbol{NL}$[4]

$$\frac{A \cdot B \supset C}{A \supset C/B} \tag{3}$$

$$\frac{A \supset C/B}{A \cdot B \supset C} \tag{4}$$

$$\frac{A \cdot B \supset C}{B \supset A\backslash C} \tag{5}$$

$$\frac{B \supset A\backslash C}{A \cdot B \supset C} \tag{6}$$

modus ponens
$$\frac{A, \quad A \supset B}{B}$$

and the axiom schemata of intuitionistic propositional calculus, see, say, [10, p. 82] for the axioms. Of course, in the above rules and in the axiom schemata A, B, and C range over *all* formulas.

REMARK 1. Examples 2 and 3 in the sequel show that axioms

$$A \supset A \tag{7}$$

[4] We replace the Lambek notation $\rightarrow$ with $\supset$.

and the rule of inference

$$\frac{A \supset B, \quad B \supset C}{A \supset C} \tag{8}$$

of ***NL*** are derivable in ***NLI***, see the definition of derivability below.

A formula A is ***NLI*** *derivable* from a set of formulas (assumptions) Θ, denoted $\Theta \vdash_{\boldsymbol{NLI}} A$,[5] if there exists a sequence of formulas $A_1, A_2, \ldots, A_n = A$, such that for all $i = 1, 2, \ldots, n$ one the following conditions holds.

- A_i is an axiom of ***NLI***; or
- $A_i \in \Theta$; or
- for some $i' < i$, A_i is obtained from $A_{i'}$ by one of the rules of inference (3)–(6); or
- for some $i', i'' < i$, A_i is obtained from $A_{i'}$ and $A_{i''}$ by *modus ponens.*

Let $\Theta(P_1, \ldots, P_n)$ and $F(P_1, \ldots, P_n)$ be a set of formulas and a formula, respectively. If

$$\Theta(P_1, \ldots, P_n) \vdash_{\boldsymbol{NLI}} F(P_1, \ldots, P_n),$$

then for all formulas $A_1, \ldots, A_n$,

$$\Theta(A_1, \ldots, A_n) \vdash_{\boldsymbol{NLI}} F(A_1, \ldots, A_n)$$ [6]

as well: we just substitute A_i for P_i, $i = 1, 2, \ldots, n$ in the derivation of $F(P_1, \ldots, P_n)$ from $\Theta(P_1, \ldots, P_n)$.

If $\Theta(P_1, \ldots, P_n)$ and $F(P_1, \ldots, P_n)$ are a set of formulas and a formula, respectively, and

$$\Theta(P_1, \ldots, P_n) \vdash_{\boldsymbol{I}} F(P_1, \ldots, P_n),$$

we say that $F(A_1, \ldots, A_n)$ *is derivable from* $\Theta(A_1, \ldots, A_n)$ *by means of* ***I*** or just derivable by means of ***I*** if $\Theta(P_1, \ldots, P_n)$ is empty.

EXAMPLE 2. ([8, Example 1]) Axioms (7) of ***NL*** are derivable in ***NLI*** by means of ***I***.

EXAMPLE 3. ([8, Example 2]) The implication $A \supset C$ is derivable from $\{A \supset B, B \supset C\}$ by means of ***I***. That is, the transitivity rule (8) of ***NL*** is derivable in ***NLI***.

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[5] In what follows, $\vdash$ subscripted with some logic denotes the derivability relation in that logic.

[6] Of course, $\Theta(A_1, \ldots, A_n) = \{G(A_1, \ldots, A_n) : G(P_1, \ldots, P_n) \in \Theta(P_1, \ldots, P_n)\}$.

Similarly, if $\Theta(P_1, \ldots, P_n)$ and $F(P_1, \ldots, P_n)$ are a set of Lambek implications and a Lambek implication, respectively, and

$$\Theta(P_1, \ldots, P_n) \vdash_{\boldsymbol{NL}} F(P_1, \ldots, P_n),$$

we say that $F(A_1, \ldots, A_n)$ *is derivable from* $\Theta(A_1, \ldots, A_n)$ *by means of* ***NL***, or just derivable by means of ***NL*** if $\Theta(P_1, \ldots, P_n)$ is empty.

Example 4. The implications $A \cdot C \supset B \cdot C$ and $C \cdot A \supset C \cdot B$ are derivable from $A \supset B$ by means of ***NL***.

Example 5. ([8, Example 3]) The formulas

$$A \cdot (B \wedge C) \supset A \cdot B \tag{9}$$

and

$$(B \wedge C) \cdot A \supset B \cdot A \tag{10}$$

are derivable in ***NLI***.

Example 6. ([8, Example 4]) The formulas

$$A \cdot (B \vee C) \supset A \cdot B \vee A \cdot C \tag{11}$$

and

$$(B \vee C) \cdot A \supset B \cdot A \vee C \cdot A. \tag{12}$$

are derivable in ***NLI***.

Example 7. ([8, Example 6]) If $\Theta \vdash_{\boldsymbol{NLI}} \neg A$, then $\Theta \vdash_{\boldsymbol{NLI}} \neg(A \cdot B)$ and $\Theta \vdash_{\boldsymbol{NLI}} \neg(B \cdot A)$.

3. Semantics of *NLI*

The semantics of ***NLI*** we consider here is a combination of the Došen ternary relational semantics ([4]) and the Kripke semantics ([12]). In this combination the Došen and Kripke relations are not independent, but are related as follows.

Definition 8. (Cf. the *bifunctoriality condition* [17, p. 7].) Let $\leq$ and R be a partial order and a ternary relation, respectively, on a set W. We say that R is *monotone* with respect to $\leq$, if

- $R(u, v, w)$ and $u \leq u'$ imply $R(u', v, w)$;
- $R(u, v, w)$ and $v' \leq v$ imply $R(u, v', w)$; and
- $R(u, v, w)$ and $w' \leq w$ imply $R(u, v, w')$.

The following two trivial examples will be used in the sequel.

Example 9. Every ternary relation is monotone with respect to equality.

Example 10. The empty ternary relation is monotone with respect to any partial order.

An ***NLI***-*interpretation* is a quadruple $\mathfrak{I} = \langle W, \leq, R, V\rangle$, where W is a nonempty set of (possible) worlds, $\leq$ is a partial order on W, R is a ternary relation on W that is monotone with respect to $\leq$, and V is a (valuation) function from W into sets of propositional variables such that $u \leq u'$ implies $V(u) \subseteq V(u')$.

The satisfiability relation $\models$ between worlds and formulas is defined as follows. Let $u \in W$.

- If A is a propositional variable, then $\mathfrak{I}, u \models A$, if $A \in V(u)$;
- $\mathfrak{I}, u \models A \cdot B$, if there are $v, w \in W$ such that $R(u, v, w)$, $\mathfrak{I}, v \models A$ and $\mathfrak{I}, w \models B$;
- $\mathfrak{I}, u \models A/B$, if for all $v, w \in W$ such that $R(w, u, v)$, $\mathfrak{I}, v \models B$ implies $\mathfrak{I}, w \models A$;
- $\mathfrak{I}, u \models B\backslash A$, if for all $v, w \in W$ such that $R(v, w, u)$, $\mathfrak{I}, w \models B$ implies $\mathfrak{I}, v \models A$;
- $\mathfrak{I}, u \models A \vee B$, if $\mathfrak{I}, u \models A$ or $\mathfrak{I}, u \models B$;
- $\mathfrak{I}, u \models A \wedge B$, if $\mathfrak{I}, u \models A$ and $\mathfrak{I}, u \models B$; and
- $\mathfrak{I}, u \models A \supset B$, if for all u' such that $u \leq u'$, $\mathfrak{I}, u' \not\models A$ or $\mathfrak{I}, u' \models B$.
- $\mathfrak{I}, u \models \neg A$, if for all u' such that $u \leq u'$, $\mathfrak{I}, u' \not\models A$.

That is, the semantics as defined is a combination of the ternary semantics for ***NL*** and the intuitionistic semantics for ***I***, extending each to formulas of the other.

Proposition 11. *Let $\mathfrak{I} = \langle W, \leq, R, V\rangle$ be an **NLI**-interpretation and let $u, u' \in W$ be such that $u \leq u'$. Then, for all formulas F, $\mathfrak{I}, u \models F$ implies $\mathfrak{I}, u' \models F$.*

Proof. The proof is by induction on the complexity of F. The basis and the induction step in the case of propositional connectives are treated in a standard manner and is omitted. The case of Lambek connectives does not rely on the induction hypothesis and is as follows.

Assume that F is of the form $A \cdot B$ and let v and w be such that $R(u, v, w)$, $\mathfrak{I}, v \models A$, and $\mathfrak{I}, w \models B$. Since R is monotone with respect to $\leq$, we have $R(u', v, w)$ and $\mathfrak{I}, u' \models A \cdot B$ follows.

Assume that F is of the form A/B and let v and w be such that $\mathfrak{I}, v \models B$ and $R(w, u', v)$. We have to show that $\mathfrak{I}, w \models A$.

Since R is monotone with respect to $\leq$, we have $R(w, u, v)$ which, together with $\mathfrak{I}, u \models A/B$, by the definition of $\models$, implies $\mathfrak{I}, w \models A$.

The case in which F is of the form $A\backslash B$ is similar to that of A/B. ■

A formula F is *satisfiable* if $\mathfrak{I}, u \models F$ for some interpretation $\mathfrak{I} = \langle W, \leq, R, V\rangle$ and some $u \in W$. Also, we say that $\mathfrak{I}$ satisfies a formula F, denoted $\mathfrak{I} \models F$, if $\mathfrak{I}, u \models F$, for all $u \in W$ and we say that $\mathfrak{I}$ satisfies a set of formulas Θ, denoted $\mathfrak{I} \models \Theta$, if $\mathfrak{I} \models F$, for all $F \in \Theta$. Finally, a set of formulas Θ *semantically entails* a formula F, denoted $\Theta \models F$, if for each interpretation $\mathfrak{I}$, $\mathfrak{I} \models \Theta$ implies $\mathfrak{I} \models F$.

It can be readily verified by induction on the derivation length that the above semantics is sound for ***NLI***,[7] i.e., $\Theta \vdash_{\boldsymbol{NLI}} F$ implies $\Theta \models F$. In the rest of this section we show that this semantics is also (strongly) complete.[8]

The proof of the completeness theorem, i.e., that $\Theta \models F$ implies $\Theta \vdash_{\boldsymbol{NLI}} F$, is based on the Thomason construction [20], but is more involved because of the Lambek connectives.

We shall need the following extension of ***NLI***.

Let Θ be a set of formulas. The calculus $\boldsymbol{NLI}_\Theta$ results from ***NLI*** by augmenting its set of axioms with Θ. Derivability in $\boldsymbol{NLI}_\Theta$ is denoted $\vdash_{\boldsymbol{NLI}_\Theta}$. Thus, $\vdash_{\boldsymbol{NLI}_\Theta} F$ if and only if $\Theta \vdash_{\boldsymbol{NLI}} F$.

Remark 12. Obviously, the examples from Section 2 extend to $\boldsymbol{NLI}_\Theta$.

In what follows we write $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset F$ if for some finite subset Ψ of Γ, $\vdash_{\boldsymbol{NLI}_\Theta} \bigwedge \Psi \supset F$.[9] For example, $\Theta \vdash_{\boldsymbol{NLI}} F$ if and only if $\vdash_{\boldsymbol{NLI}_\Theta} \emptyset \supset F$.

Proposition 13. *If* $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \cup \{A\} \supset F$ *and* $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \cup \{B\} \supset F$*, then* $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \cup \{A \vee B\} \supset F$.

Proof. Let Ψ_A and Ψ_B be finite subsets of Γ such that

$$\vdash_{\boldsymbol{NLI}_\Theta} A \wedge \bigwedge \Psi_A \supset F$$

[7] Therefore, ***NLI*** is consistent.

[8] Cf. the semantics in [17, Section 4.2], where completeness is proved only for ***BI*** without $\bot$. However, the ***NLI*** semantics and the proofs in this section modify to the whole ***BI*** in a straightforward manner. Moreover, a similar construction in [8] modifies to the classical ***BI***.

[9] As usual, $\bigwedge \Psi$ is the conjunction of all elements of Ψ. Note that $\Gamma \supset F$ is not a formula, but $\bigwedge \Psi \supset F$ is.

and

$$\vdash_{\boldsymbol{NLI}_\Theta} B \wedge \bigwedge \Psi_B \supset F.$$

Then

$$\vdash_{\boldsymbol{NLI}_\Theta} (A \vee B) \wedge \bigwedge (\Psi_A \cup \Psi_B) \supset F$$

and, by definition, $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \cup \{A \vee B\} \supset F$. ■

DEFINITION 14. A set of formulas Γ is called $\boldsymbol{NLI}_\Theta$-*consistent*, if for no finite subset Ψ of Γ, $\vdash_{\boldsymbol{NLI}_\Theta} \neg \bigwedge \Psi$.

Propositions 15 and 16 below follow from Definition 14, by means of $\boldsymbol{I}$, in a standard manner and are presented without proofs.

PROPOSITION 15. *A set of formulas* Γ *is* $\boldsymbol{NLI}_\Theta$*-consistent if and only if for some formula* F, $\not\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset F$.

PROPOSITION 16. *If* $\not\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset \neg F$, *then* $\Gamma \cup \{F\}$ *is* $\boldsymbol{NLI}_\Theta$*-consistent.*

We shall use the following notation.

For two sets of formulas Γ' and Γ'', we denote the set of formulas

$$\{F' \cdot F'' : F' \in \Gamma' \text{ and } F'' \in \Gamma''\}$$

by $\Gamma' \cdot \Gamma''$ and in what follows we write $F \cdot \Gamma''$ and $\Gamma' \cdot F$ for $\{F\} \cdot \Gamma''$ and $\Gamma' \cdot \{F\}$, respectively.

Also for a set of formulas Γ we define the $\boldsymbol{NLI}_\Theta$-*closure* $[\![\Gamma]\!]_\Theta$ of Γ by

$$[\![\Gamma]\!]_\Theta = \{F : \vdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset F\}$$

and we say that Γ is $\boldsymbol{NLI}_\Theta$-*closed*, if $\Gamma = [\![\Gamma]\!]_\Theta$.

REMARK 17. It immediately follows from the definitions that $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset F$ if and only if $\vdash_{\boldsymbol{NLI}_\Theta} [\![\Gamma]\!]_\Theta \supset F$.

For the proof of the completeness theorem (and Theorem 58 in Section 5) we shall need the following definitions and auxiliary results.

DEFINITION 18. A set of formulas Φ is called $\boldsymbol{NLI}_\Theta$-*conjunctive complete* if for each finite subset Ψ of Φ there is a formula $F \in \Phi$ such that $\vdash_{\boldsymbol{NLI}_\Theta} F \supset \bigwedge \Psi$.

REMARK 19. Each $\boldsymbol{NLI}_\Theta$-closed set of formulas is $\boldsymbol{NLI}_\Theta$-conjunctively complete, because with each its finite subset of formulas it also contains its conjunction.

DEFINITION 20. A set of formulas Γ is called Θ-*saturated* if

- Γ is $\boldsymbol{NLI}_\Theta$-consistent;
- $\Gamma = [\![\Gamma]\!]_\Theta$;[10] and
- $A \vee B \in \Gamma$ implies $A \in \Gamma$ or $B \in \Gamma$.

Example 21. It follows from Proposition 13 that maximal (with respect to inclusion) $\boldsymbol{NLI}_\Theta$-consistent sets of formulas are Θ-saturated.

Example 22. (Cf. [8, Example 8].) Let Θ be a set of formulas, $\mathfrak{I} = \langle W, \leq, R, V\rangle$ be an $\boldsymbol{NLI}$-interpretation satisfying Θ, and let $u \in W$. Then the set of formulas

$$[\![u]\!]_{\mathfrak{I}} = \{A : \mathfrak{I}, u \models A\} \tag{13}$$

is Θ-saturated.

Proposition 23. *If $\not\vdash_{\boldsymbol{NLI}_\Theta} \Phi \supset F$, then there is a Θ-saturated set of formulas Γ including Φ such that $\not\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset F$.*

Proof. Let Γ be a maximal set of formulas including Φ such that $\not\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset F$. We contend that Γ is Θ-saturated.

Since $\not\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset F$, by Proposition 15, Γ is $\boldsymbol{NLI}_\Theta$-consistent and, since Γ is maximal, by Remark 17, it is $\boldsymbol{NLI}_\Theta$-closed. To show that $A \vee B \in \Gamma$ implies $A \in \Gamma$ or $B \in \Gamma$, assume to the contrary that $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \cup \{A\} \supset F$ and $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \cup \{B\} \supset F$. Since $A \vee B \in \Gamma$, by Proposition 13, $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset F$, in contradiction with $\not\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset F$. ■

Proposition 24. *If $\not\vdash_{\boldsymbol{NLI}_\Theta} \Phi \supset \neg F$, then there is a Θ-saturated set of formulas Γ including $\Phi \cup \{F\}$.*

Proof. The proof if similar to that of Proposition 23. By Proposition 16, $\Phi \cup \{F\}$ is $\boldsymbol{NLI}_\Theta$-consistent. Let Γ be a maximal $\boldsymbol{NLI}_\Theta$-consistent set of formulas including $\Phi \cup \{F\}$. We contend that Γ is Θ-saturated.

By definition, Γ is $\boldsymbol{NLI}_\Theta$-consistent and, since Γ is maximal, by Remark 17, it is $\boldsymbol{NLI}_\Theta$-closed. To show that $A \vee B \in \Gamma$ implies $A \in \Gamma$ or $B \in \Gamma$, assume to the contrary that both $\Gamma \cup \{A\}$ and $\Gamma \cup \{B\}$ are $\boldsymbol{NLI}_\Theta$-inconsistent. Then, by the contraposition of the "only if" direction of Proposition 15, $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \cup \{A\} \supset \neg A$ and $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \cup \{B\} \supset \neg B$, implying $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset \neg A$ and $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset \neg B$. Thus, $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset \neg(A \vee B)$ in contradiction with $\boldsymbol{NLI}_\Theta$-consistency of Γ and $A \vee B \in \Gamma$. ■

Proposition 25. (Cf. [8, Proposition 10].) *Let Γ be a Θ-saturated set of formulas, Φ' be a $\boldsymbol{NLI}_\Theta$-conjunctively complete set of formulas, and let Φ'' be a set of formulas such that for all finite subsets Ψ of Φ'', $\Phi' \cdot \bigwedge \Psi \subseteq \Gamma$.*

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[10] That is, Γ is $\boldsymbol{NLI}_\Theta$-*closed*.

Then, there exists a Θ-saturated set of formulas Γ'' including Φ'' such that $\Phi' \cdot \Gamma'' \subseteq \Gamma$.

The proof of Proposition 25 is based on the following two lemmas.

LEMMA 26. (Cf. [8, Lemma 11]) *Let Γ be a Θ-saturated set of formulas, Φ' be a $\boldsymbol{NLI}_\Theta$-conjunctively complete set of formulas, and let Φ'' be a set of formulas such that for all finite subsets Ψ of Φ'', $\Phi' \cdot \bigwedge \Psi \subseteq \Gamma$. Then, for each formula in Φ'' of the form $A \vee B$ one of the following holds.*
- *For all finite subsets Ψ of Φ'', $\Phi' \cdot (A \wedge \bigwedge \Psi) \subseteq \Gamma$*

or
- *for all finite subsets Ψ of Φ'', $\Phi' \cdot (B \wedge \bigwedge \Psi) \subseteq \Gamma$.*

PROOF. Assume to the contrary that there are finite subsets Ψ_1 and Ψ_2 of Φ such that

$$\Phi' \cdot (A \wedge \bigwedge \Psi_1) \not\subseteq \Gamma$$

and

$$\Phi' \cdot (B \wedge \bigwedge \Psi_2) \not\subseteq \Gamma,$$

implying, by (9), that

$$\Phi' \cdot (A \wedge \bigwedge \Psi_1 \wedge \bigwedge \Psi_2) \not\subseteq \Gamma$$

and

$$\Phi' \cdot (B \wedge \bigwedge \Psi_1 \wedge \bigwedge \Psi_2) \not\subseteq \Gamma,$$

because Γ is saturated (and, therefore, is $\boldsymbol{NLI}_\Theta$-closed).

Therefore, there are formulas $F_1, F_2 \in \Phi'$ such that

$$F_1 \cdot (A \wedge \bigwedge \Psi_1 \wedge \bigwedge \Psi_2) \notin \Gamma$$

and

$$F_2 \cdot (B \wedge \bigwedge \Psi_1 \wedge \bigwedge \Psi_2) \notin \Gamma.$$

Then, by (10),

$$(F_1 \wedge F_2) \cdot (A \wedge \bigwedge \Psi_1 \wedge \bigwedge \Psi_2) \notin \Gamma$$

and

$$(F_1 \wedge F_2) \cdot (B \wedge \bigwedge \Psi_1 \wedge \bigwedge \Psi_2) \notin \Gamma,$$

because Γ is saturated (and, therefore, is $\boldsymbol{NLI}_\Theta$-closed).

Since Φ' is $\boldsymbol{NLI}_\Theta$-conjunctively complete, for some formula $F \in \Phi'$, $\vdash_{\boldsymbol{NLI}_\Theta} F \supset F_1 \wedge F_2$, implying, by Example 4,

$$F \cdot (A \wedge \bigwedge \Psi_1 \wedge \bigwedge \Psi_2) \notin \Gamma \tag{14}$$

and

$$F \cdot (B \wedge \bigwedge \Psi_1 \wedge \bigwedge \Psi_2) \notin \Gamma. \tag{15}$$

On the other hand,

$$F \cdot ((A \vee B) \wedge \bigwedge \Psi_1 \wedge \bigwedge \Psi_2) \in \Gamma,$$

because $A \vee B \in \Psi''$ and for all finite subsets Ψ of Φ'', $F \cdot \bigwedge \Psi \subseteq \Gamma$. Therefore, by (11) (and *modus ponens*, of course),

$$F \cdot (A \wedge \bigwedge \Psi_1 \wedge \bigwedge \Psi_2) \vee F \cdot (B \wedge \bigwedge \Psi_1 \wedge \bigwedge \Psi_2) \in \Gamma.$$

Since Γ is saturated,

$$F \cdot (A \wedge \bigwedge \Psi_1 \wedge \bigwedge \Psi_2) \in \Gamma$$

or

$$F \cdot (B \wedge \bigwedge \Psi_1 \wedge \bigwedge \Psi_2) \in \Gamma.$$

However, the former containment contradicts (14) and the latter one contradicts (15). ■

LEMMA 27. *Let Γ be a Θ-saturated set of formulas and let Φ' and Φ'' be sets of formulas such that for all finite subsets Ψ of Φ'', $\Phi' \cdot \bigwedge \Psi \subseteq \Gamma$. Then $\Phi' \cdot [\![\Phi'']\!]_\Theta \subseteq \Gamma$.*

PROOF. Let $F \in [\![\Phi'']\!]_\Theta$ and let Ψ be a finite subset of Φ'' such that $\vdash_{\boldsymbol{NLI}_\Theta} \bigwedge \Psi \supset F$. By Example 4, for all formulas $F' \in \Phi'$,

$$\vdash_{\boldsymbol{NLI}_\Theta} \left(F' \cdot \bigwedge \Psi\right) \supset F' \cdot F,$$

which, together with $F' \cdot \bigwedge \Psi \in \Gamma$ and the Θ-saturation of Γ, implies $F' \cdot F \in \Gamma$. ■

PROOF OF PROPOSITION 25. Let Γ'' be a maximal $\boldsymbol{NLI}_\Theta$-closed set of formulas including Φ'' such that $\Phi' \cdot \Gamma'' \subseteq \Gamma$. Such a set exists, by Lemma 27 (and Zorn's lemma). We contend that Γ'' is Θ-saturated.

Since Γ is $\boldsymbol{NLI}_\Theta$-consistent, by Example 7, Γ'' is $\boldsymbol{NLI}_\Theta$-consistent as well and, by definition, it is also $\boldsymbol{NLI}_\Theta$-closed. Thus, it suffices to show that for each formula of the form $A \vee B$ in Γ'', $A \in \Gamma''$ or $B \in \Gamma''$. Since, by Remark 19, Γ'' is $\boldsymbol{NLI}_\Theta$-conjunctively complete, one of the containments follows from Lemma 26 and maximality of Γ''. ■

Corollary 28. (Cf. [8, Corollary 12].) *Let* Γ *be a* Θ*-saturated set of formulas and let* A *and* B *be formulas such that* $A \cdot B \in \Gamma$*. Then, there exist* Θ*-saturated sets of formulas* Γ' *containing* A *and* Γ'' *containing* B *such that* $\Gamma' \cdot \Gamma'' \subseteq \Gamma$*.*

Proof. Since the set of formulas consisting of B only is $\boldsymbol{NLI}_\Theta$-conjunctively complete, similarly to the proof of Proposition 25, one can show that there exists a Θ-saturated set of formulas Γ' containing A such that $\Gamma' \cdot B \subseteq \Gamma$. Then, since, by Remark 19, Θ-saturated sets of formulas are $\boldsymbol{NLI}_\Theta$-conjunctively complete, by Proposition 25, there exists a Θ-saturated set of formulas Γ'' containing B such that $\Gamma' \cdot \Gamma'' \subseteq \Gamma$. ■

Definition 29. Let Θ be a set of ***NLI*** formulas. The Θ-*canonical* ***NLI***-interpretation $\mathfrak{I}_\Theta = \langle W_\Theta, \leq_\Theta, R_\Theta, V_\Theta \rangle$ is defined as follows.

- W_Θ consists of all Θ-saturated sets of formulas,
- $\leq_\Theta$ is $\subseteq$,[11]
- $R_\Theta = \{(\Gamma, \Gamma', \Gamma'') \in W_\Theta^3 : \Gamma' \cdot \Gamma'' \subseteq \Gamma\}$,[12] and
- $V_\Theta(\Gamma) = \Gamma \cap \mathcal{P}$, where $\mathcal{P}$ is the set of all propositional variables (atomic formulas).

Theorem 30. *Let* $\Gamma \in W_\Theta$*. Then, for each formula* F*,* $\mathfrak{I}_\Theta, \Gamma \models F$ *if and only if* $F \in \Gamma$*.*

In fact, there is a stronger correspondence that also implies the finite model property and, consequently, implies strong decidability of ***NLI***. For this stronger correspondence we need the following notation.

For a nonempty *subformula closed* set of formulas Φ[13] including Θ, we define the restriction

$$\mathfrak{I}_\Theta|_\Phi = \langle W_\Theta|_\Phi, \leq_\Theta |_\Phi, R_\Theta|_\Phi, V_\Theta|_\Phi \rangle$$

of $\mathfrak{I}_\Theta$ to Φ by

- $W_\Theta|_\Phi = \{\Gamma \cap \Phi : \Gamma \in W_\Theta\}$,
- $\leq_\Theta |_\Phi$ is $\subseteq$,
- $R_\Theta|_\Phi = \{(\Gamma, \Gamma', \Gamma'') \in (W_\Theta|_\Phi)^3 : [\![(\Gamma' \cdot \Gamma'')]\!]_\Theta \cap \Phi \subseteq \Gamma\}$,[14] and

[11] That is, $\Gamma \leq_\Theta \Gamma'$, if $\Gamma \subseteq \Gamma'$.

[12] It immediately follows from the definition of R_Θ that it is $\subseteq$-monotone.

[13] That is, if $A \in \Phi$ and B is a subformula of A, then also $B \in \Phi$.

[14] Like in the case of $\mathfrak{I}_\Theta$, it immediately follows from the definition of $R_\Theta|_\Phi$ that it is $\subseteq$-monotone.

- $V_\Theta|_\Phi(\Gamma) = V_\Theta(\Gamma) \cap \Phi$.[15]

THEOREM 31. *Let* $\Gamma \in W_\Theta|_\Phi$. *Then, for each formula* $F \in \Phi$, $\mathfrak{I}_\Theta|_\Phi, \Gamma \models F$ *if and only if* $F \in \Gamma$.

Note that Theorem 30 follows from Theorem 31 with Φ being the set of all formulas.

PROOF OF THEOREM 31. The proof is by a standard induction on the complexity of F. In particular, the cases of the Lambek connectives are treated like in [4] (see also [8, Proof of Theorem 14]) and the cases of a propositional variable or an intuitionistic propositional connectives are treated like in [20, Proof of Theorem 2]. The basis, i.e., the case of a propositional variable, immediately follows from the definition of $\models$, and for the induction step we consider the case of the principal connective of F.

- Let F be of the form $A \cdot B$ and let $\mathfrak{I}_\Theta|_\Phi, \Gamma \models A \cdot B$. That is, there are $\Gamma', \Gamma'' \in W_\Theta|_\Phi$ such that $\mathfrak{I}_\Theta|_\Phi, \Gamma' \models A$, $\mathfrak{I}_\Theta|_\Phi, \Gamma'' \models B$, and
$$[\![\Gamma' \cdot \Gamma'']\!]_\Theta \cap \Phi \subseteq \Gamma. \tag{16}$$
By the induction hypothesis, $A \in \Gamma'$ and $B \in \Gamma''$, which, together with (16) and $A \cdot B \in \Phi$ implies $A \cdot B \in \Gamma$.

  Conversely, let $A \cdot B \in \Gamma$ and let $\widetilde{\Gamma} \in W_\Theta$ be such that $\Gamma = \widetilde{\Gamma} \cap \Phi$. By Corollary 28, there are $\widetilde{\Gamma'}, \widetilde{\Gamma''} \in W_\Theta$ such that $A \in \widetilde{\Gamma'}$, $B \in \widetilde{\Gamma''}$, and $\widetilde{\Gamma'} \cdot \widetilde{\Gamma''} \subseteq \widetilde{\Gamma}$. Let Γ' and Γ'' be $\widetilde{\Gamma'} \cap \Phi$ and $\widetilde{\Gamma''} \cap \Phi$, respectively. As $A, B \in \Phi$, by the induction hypothesis, $\mathfrak{I}_\Theta|_\Phi, \Gamma' \models A$ and $\mathfrak{I}_\Theta|_\Phi, \Gamma'' \models B$. Since
$$[\![\Gamma' \cdot \Gamma'']\!]_\Theta \cap \Phi \subseteq \widetilde{\Gamma} \cap \Phi = \Gamma$$
and $A \cdot B \in \Phi$, by the definition of $\models$, we have $\mathfrak{I}_\Theta|_\Phi, \Gamma \models A \cdot B$.

- Let F be of the form A/B and let $\mathfrak{I}_\Theta|_\Phi, \Gamma \models A/B$. We contend that $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \cdot B \supset A$. To prove this, assume to the contrary that $\not\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \cdot B \supset A$. Then, by Proposition 23, there exists a Θ-saturated set of formulas $\widetilde{\Gamma''}$ including $\Gamma \cdot B$ such that $\not\vdash_{\boldsymbol{NLI}_\Theta} \widetilde{\Gamma''} \supset A$ and, by Proposition 25, there exists a Θ-saturated set of formulas $\widetilde{\Gamma'}$ containing B such that $\Gamma \cdot \widetilde{\Gamma'} \subseteq \widetilde{\Gamma''}$.

  Let $\Gamma' = \widetilde{\Gamma'} \cap \Phi$ and let $\Gamma'' = \widetilde{\Gamma''} \cap \Phi$. Then $[\![\Gamma \cdot \Gamma']\!]_\Theta \cap \Phi \subseteq \Gamma''$. Also, by the induction hypothesis, $\mathfrak{I}_\Theta|_\Phi, \Gamma' \models B$ and $\mathfrak{I}_\Theta|_\Phi, \Gamma'' \not\models A$. All this, however, contradicts $\mathfrak{I}_\Theta|_\Phi, \Gamma \models A/B$, which proves our contention.

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[15] This construction resembles *filtration through* Φ, see [19, Chapter I, Section 7].

Since $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \cdot B \supset A$, by (5), we have $\vdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset A/B$. Let $\widetilde{\Gamma} \in W_\Theta$ be such that $\Gamma = \widetilde{\Gamma} \cap \Phi$. Then $A/B \in \widetilde{\Gamma}$, because $\widetilde{\Gamma}$ is $\boldsymbol{NLI}_\Theta$-closed. This, together with $A/B \in \Phi$, implies $A/B \in \Gamma$.

Conversely, let $A/B \in \Gamma$ and let $\Gamma', \Gamma'' \in W_\Theta|_\Phi$ be such that $[\![\Gamma \cdot \Gamma']\!]_\Theta \cap \Phi \subseteq \Gamma''$ and $\mathfrak{I}_\Theta|_\Phi, \Gamma' \models B$. We have to show that $\mathfrak{I}_\Theta|_\Phi, \Gamma'' \models A$.

By the induction hypothesis, $B \in \Gamma'$ which, together with $A/B \in \Gamma$, implies $(A/B) \cdot B \in \Gamma \cdot \Gamma'$. Since $(A/B) \cdot B \supset A$ is derivable by means of $\boldsymbol{NL}$ and $A \in \Phi$, $A \in [\![\Gamma \cdot \Gamma']\!]_\Theta \cap \Phi$, implying $A \in \Gamma''$. Thus, by the induction hypothesis, $\mathfrak{I}_\Theta|_\Phi, \Gamma'' \models A$.

- The case of $\backslash$ is symmetric to that of $/$ and is omitted.
- Let F be of the form $A \vee B$ and let $\mathfrak{I}_\Theta|_\Phi, \Gamma \models A \vee B$. By the definition of $\models$, $\mathfrak{I}_\Theta|_\Phi, \Gamma \models A$ or $\mathfrak{I}_\Theta|_\Phi, \Gamma \models B$. Thus, by the induction hypothesis, $A \in \Gamma$ or $B \in \Gamma$. Let $\widetilde{\Gamma} \in W_\Theta$ be such that $\Gamma = \widetilde{\Gamma} \cap \Phi$. Then, in both cases, $A \vee B \in \widetilde{\Gamma}$, because $\widetilde{\Gamma}$ is $\boldsymbol{NLI}_\Theta$-closed. Since $A \vee B$ is also in Φ, $A \vee B \in \Gamma$ follows.

  Conversely, assume $A \vee B \in \Gamma$. Let $\widetilde{\Gamma} \in W_\Theta$ be such that $\Gamma = \widetilde{\Gamma} \cap \Phi$. Then $A \in \widetilde{\Gamma}$ or $B \in \widetilde{\Gamma}$, because $\widetilde{\Gamma}$ is Θ-saturated. Since both A and B are in Φ, $A \in \Gamma$ or $B \in \Gamma$. Thus, by the induction hypothesis, $\mathfrak{I}_\Theta|_\Phi, \Gamma \models A$ or $\mathfrak{I}_\Theta|_\Phi, \Gamma \models B$. In both cases, by the definition of $\models$, $\mathfrak{I}_\Theta|_\Phi, \Gamma \models A \vee B$.
- Let F be of the form $A \wedge B$ and let $\mathfrak{I}_\Theta|_\Phi, \Gamma \models A \wedge B$. By the definition of $\models$, $\mathfrak{I}_\Theta|_\Phi, \Gamma \models A$ and $\mathfrak{I}_\Theta|_\Phi, \Gamma \models B$. Thus, by the induction hypothesis, $A, B \in \Gamma$. Let $\widetilde{\Gamma} \in W_\Theta$ be such that $\Gamma = \widetilde{\Gamma} \cap \Phi$. Then $A \wedge B \in \widetilde{\Gamma}$, because $\widetilde{\Gamma}$ is $\boldsymbol{NLI}_\Theta$-closed. Since $A \wedge B$ is also in Φ, $A \wedge B \in \Gamma$ follows.

  Conversely, assume $A \wedge B \in \Gamma$. Let $\widetilde{\Gamma} \in W_\Theta$ be such that $\Gamma = \widetilde{\Gamma} \cap \Phi$. Then $A, B \in \widetilde{\Gamma}$, because $\widetilde{\Gamma}$ is $\boldsymbol{NLI}_\Theta$-closed. Since A and B are also in Φ, they are in Γ as well. Thus, by the induction hypothesis, $\mathfrak{I}_\Theta|_\Phi, \Gamma \models A$ and $\mathfrak{I}_\Theta|_\Phi, \Gamma \models B$, and, by the definition of $\models$, $\mathfrak{I}_\Theta|_\Phi, \Gamma \models A \wedge B$.
- Let F be of the form $A \supset B$ and let $\mathfrak{I}_\Theta|_\Phi, \Gamma \models A \supset B$. Assume to the contrary that $A \supset B \notin \Gamma$. Let $\widetilde{\Gamma} \in W_\Theta$ be such that $\Gamma = \widetilde{\Gamma} \cap \Phi$. Since $A \supset B \in \Phi$, $A \supset B \notin \widetilde{\Gamma}$, implying $\not\vdash_{\boldsymbol{NLI}_\Theta} \widetilde{\Gamma} \cup \{A\} \supset B$, because $\widetilde{\Gamma}$ is $\boldsymbol{NLI}_\Theta$-closed. By Proposition 23, there is a Θ-saturated set of formulas $\widetilde{\Gamma}'$ including $\widetilde{\Gamma} \cup \{A\}$ such that $\not\vdash_{\boldsymbol{NLI}_\Theta} \widetilde{\Gamma}' \supset B$. Therefore, $B \notin \widetilde{\Gamma}'$ and, by the induction hypothesis, $\mathfrak{I}_\Theta|_\Phi, \widetilde{\Gamma}' \cap \Phi \models A$, but $\mathfrak{I}_\Theta|_\Phi, \widetilde{\Gamma}' \cap \Phi \not\models B$. Since

$$\Gamma \subseteq \left(\widetilde{\Gamma} \cup \{A\}\right) \cap \Phi \subseteq \widetilde{\Gamma}' \cap \Phi,$$

$\mathfrak{I}_\Theta|_\Phi, \Gamma \not\models A \supset B$, in contradiction with our assumption.

Conversely, assume $A \supset B \in \Gamma$ and let $\Gamma' \in W_\Theta|_\Phi$ include Γ. We have to show that $\mathfrak{I}_\Theta|_\Phi, \Gamma' \models A$ implies $\mathfrak{I}_\Theta|_\Phi, \Gamma' \models B$. So, let $\mathfrak{I}_\Theta|_\Phi, \Gamma' \models A$ and let $\widetilde{\Gamma'} \in W_\Theta$ be such that $\Gamma' = \widetilde{\Gamma'} \cap \Phi$. By the induction hypothesis, $A \in \Gamma'$, which, together with

$$A \supset B \in \Gamma \subseteq \Gamma' \subseteq \widetilde{\Gamma'}$$

and $\boldsymbol{NLI}_\Theta$-closure of $\widetilde{\Gamma'}$, implies $B \in \widetilde{\Gamma'}$. Since $B \in \Phi$,

$$B \in \Gamma' = \widetilde{\Gamma'} \cap \Phi,$$

and, by the induction hypothesis, $\mathfrak{I}_\Theta|_\Phi, \Gamma' \models B$.

- Let F be of the form $\neg A$ and let $\mathfrak{I}_\Theta|_\Phi, \Gamma \models \neg A$. Assume to the contrary that $\neg A \notin \Gamma$. Let $\widetilde{\Gamma} \in W_\Theta$ be such that $\Gamma = \widetilde{\Gamma} \cap \Phi$. Since $\neg A \in \Phi$, $\neg A \notin \widetilde{\Gamma}$, implying $\nvdash_{\boldsymbol{NLI}_\Theta} \widetilde{\Gamma} \supset \neg A$, because $\widetilde{\Gamma}$ is $\boldsymbol{NLI}_\Theta$-closed. By Proposition 24, there is a Θ-saturated set of formulas $\widetilde{\Gamma'}$ including $\widetilde{\Gamma} \cup \{A\}$. Since $A \in \Phi$, $A \in \widetilde{\Gamma'} \cap \Phi$, and, by the induction hypothesis, $\mathfrak{I}_\Theta|_\Phi, \widetilde{\Gamma'} \cap \Phi \models A$. However, the latter, together with

$$\Gamma \subseteq \left(\widetilde{\Gamma} \cup \{A\}\right) \cap \Phi \subseteq \widetilde{\Gamma'} \cap \Phi,$$

contradicts $\mathfrak{I}_\Theta|_\Phi, \Gamma \models \neg A$.

Conversely, assume $\neg A \in \Gamma$ and let $\Gamma' \in W_\Theta|_\Phi$ include Γ. We have to show that $\mathfrak{I}_\Theta|_\Phi, \Gamma' \not\models A$. Assume to the contrary that $\mathfrak{I}_\Theta|_\Phi, \Gamma' \models A$. Then, by the induction hypothesis, $A \in \Gamma'$, which, together with

$$\neg A \in \Gamma \subseteq \Gamma'$$

contradicts consistency of Γ'. ■

THEOREM 32. (Completeness) *If* $\Theta \models F$, *then* $\Theta \vdash F$.

PROOF. Assume to the contrary that $\Theta \nvdash F$. That is, $\nvdash_{\boldsymbol{NLI}_\Theta} \emptyset \supset F$. By Proposition 23, there is a Θ-saturated set of formulas Γ such that $\nvdash_{\boldsymbol{NLI}_\Theta} \Gamma \supset F$. Then, $F \notin \Gamma$, because Γ is $\boldsymbol{NLI}_\Theta$-closed.

Let Φ be the subformula closure of $\Theta \cup \{F\}$.[16] Then, $F \notin \Gamma \cap \Phi$. By Theorem 31, $\mathfrak{I}_\Theta|_\Phi, \Gamma \cap \Phi \not\models F$, in contradiction with this theorem prerequisite. ■

It was shown in [2,3] that ***NLI*** is strongly decidable. The proof in these papers is based on the finite model property of the algebraic semantics.

[16]That is, Φ is the minimal (with respect to inclusion) subformula closed set of formulas including $\Theta \cup \{F\}$.

Below we present an alternative proof of decidability of ***NLI*** that follows from the finite model property of the relational semantic.

Corollary 33. *Let Θ be a finite set of formulas. If $\Theta \nvdash_{\boldsymbol{NLI}} F$, then there is a finite interpretation satisfying Θ, but not satisfying F.*

Proof. Let Φ be the subformula closure of $\Theta \cup \{F\}$. The interpretation $\mathfrak{I}_\Theta|_\Phi$ satisfies Θ, but does not satisfy F. This interpretation is finite, because Θ (and consequently, Φ) is finite. ■

Corollary 34. ***NLI*** *is strongly decidable.*

Another immediate corollary of the completeness theorem is that the intuitionistic and the Lambek calculi are orthogonal, i.e., ***NLI*** is a conservative extension of both ***NL*** and ***I***.

Corollary 35. (Cf. [8, Corollary 16].) ***NLI*** *is a conservative extension of* ***NL***.[17]

Proof. Assume to the contrary that ***NLI*** is not a conservative extension of ***NL*** and let Θ and F be a finite set of Lambek implications and a Lambek implication, respectively, such that $\Theta \vdash_{\boldsymbol{NLI}} F$, but $\Theta \nvdash_{\boldsymbol{NL}} F$. By the "$\Theta$-extension" of the completeness theorem in [4] (see also, e.g., [13]), there is an ***NL*** interpretation $\mathfrak{I} = \langle W, R, V\rangle$ such that $\mathfrak{I} \models \Theta$, but $\mathfrak{I} \not\models F$. Then the ***NLI***-interpretation $\mathfrak{I}_{\boldsymbol{NLI}} = \langle W, =, R, V\rangle$ (see Example 9) also satisfies Θ, but does not satisfy F. This, however, contradicts the soundness of the relational semantics with respect to ***NLI***. ■

Corollary 36. (Cf. [8, Corollary 17].) ***NLI*** *is a conservative extension of* ***I***.[18]

Proof. Assume to the contrary that ***NLI*** is not a conservative extension of ***I*** and let Θ and F be a finite set of propositional formulas and a propositional formula, respectively, such that $\Theta \vdash_{\boldsymbol{NLI}} F$, but $\Theta \nvdash_{\boldsymbol{I}} F$. By the completeness theorem for ***I***, there is a Kripke interpretation $I = \langle W, \leq, V\rangle$ such that $I \models \Theta$, but $I \not\models F$. Then the ***NLI***-interpretation $\mathfrak{I} = \langle W, \leq, \emptyset, V\rangle$ (see Example 10) satisfies Θ, but does not satisfy F. This, however, contradicts the soundness of the relational semantics with respect to ***NLI***. ■

We conclude this section with the *canonical mapping* of ***NLI***-interpretations satisfying a set of formulas Θ into $\mathfrak{I}_\Theta$.

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[17] Note that he language of ***NL*** does not contain propositional connectives.

[18] Note that he language of ***I*** does not contain Lambek connectives.

Definition 37. Let Θ be a set of formulas and let $\mathfrak{I} = \langle W, \leq, R, V \rangle$ be an ***NLI***-interpretation satisfying Θ. The *canonical mapping* $\iota_{\mathfrak{I}} : W \to W_\Theta$ is defined by $\iota_{\mathfrak{I}}(u) = [\![u]\!]_{\mathfrak{I}}$.[19,20]

Corollary 38. (Cf. [8, Corollary 19].) *Let Θ be a set of formulas and let $\mathfrak{I} = \langle W, \leq, R, V \rangle$ be an interpretation satisfying Θ.*

(*i*) *If $u, v \in W$ are such that $u \leq v$, then $\iota_{\mathfrak{I}}(u) \subseteq \iota_{\mathfrak{I}}(v)$.*

(*ii*) *If $u, v, w \in W$ are such that $R(u, v, w)$, then $\iota_{\mathfrak{I}}(v) \cdot \iota_{\mathfrak{I}}(w) \subseteq \iota_{\mathfrak{I}}(u)$.*

(*iii*) *For all formulas F and all $u \in W$, $\mathfrak{I}, u \models F$ if and only if $\mathfrak{I}_\Theta, \iota_{\mathfrak{I}}(u) \models F$.*

Proof. For the proof of clause (*i*) of the corollary, let $u \leq v$ and let $\mathfrak{I}, u \models F$. Then, by Proposition 11, $\mathfrak{I}, v \models F$ and the result follows from the definition of $\iota_{\mathfrak{I}}$.

For the proof of clause (*ii*), let $R(u, v, w)$ and let $A \in [\![v]\!]_{\mathfrak{I}}$ and $B \in [\![w]\!]_{\mathfrak{I}}$. We have to show that $A \cdot B \in [\![u]\!]_{\mathfrak{I}}$. By the definition of $[\![\ ]\!]_{\mathfrak{I}}$, $\mathfrak{I}, v \models A$ and $\mathfrak{I}, w \models B$, which, together with $R(u, v, w)$, implies $\mathfrak{I}, u \models A \cdot B$. Thus, by the definition of $\iota_{\mathfrak{I}}$, we have the desired containment $A \cdot B \in [\![u]\!]_{\mathfrak{I}}$.

The proof of clause (*iii*) is equally easy. By Theorem 30, $\mathfrak{I}_\Theta, \iota_{\mathfrak{I}}(u) \models F$ if and only if $F \in \iota_{\mathfrak{I}}(u)$, and, by the definition of $\iota_{\mathfrak{I}}$, $F \in \iota_{\mathfrak{I}}(u)$ if and only if $\mathfrak{I}, u \models F$. ■

4. A Sequent Calculus for *NLI*

The sequent calculus ***SNLI*** employs as sequent antecedents alternatingly-nested structures, called *bunches* of formulas, or just bunches, which are defined as follows, cf. [16–18].

Definition 39.

- Finite (possibly empty) multisets of formulas are bunches, and
- non-empty finite multisets consisting of formulas and ordered pairs of bunches are bunches.

In this paper by an "ordered pair" we always mean an ordered pair of bunches. Also we use the following notation.

- Elements of a bunch are separated by semicolons and the components of an ordered pair are separated by a comma.

[19] See (13) for the definition of $[\![u]\!]_{\mathfrak{I}}$.

[20] Since $\mathfrak{I} \models \Theta$, $\iota_{\mathfrak{I}}$ is well-defined.

- Bunches are denoted by Γ, possibly indexed or primed and for bunches Γ' and Γ'', (Γ', Γ'') denotes the ordered pair of Γ' and Γ''.
- Formulas and ordered pairs are denoted by γ, possibly indexed or primed, and we write $\Gamma'; \Gamma''$ and $\Gamma; \gamma$ for the multiset unions $\Gamma' \cup \Gamma''$ and $\Gamma \cup \{\gamma\}$, respectively.
- Sometimes we write γ for the one element multiset $\{\gamma\}$. It will be always clear from the context when γ denotes a formula/ordered pair or a one-element bunch.
- We also write (γ', γ''), (Γ, γ), and (γ, Γ) for $(\{\gamma'\}, \{\gamma''\})$, $(\Gamma, \{\gamma\})$, and $(\{\gamma\}, \Gamma)$, respectively. Note that none of (γ', γ''), (Γ, γ), and (γ, Γ) is a bunch.
- By $\Gamma[\Gamma']$ we denote a bunch Γ with a designated bunch Γ' occurring in Γ, and, in this context, we denote by $\Gamma[\Gamma'']$ the replacement of that particular occurrence of Γ' in Γ with Γ''.
- By $\Gamma[\gamma']$ we denote a bunch Γ with a designated formula/ordered pair γ' occurring in a bunch occurring in Γ, and, in this context, we denote by $\Gamma[\gamma'']$ the replacement of that particular occurrence of γ' in Γ with γ''. That is, if $\Gamma[\gamma']$ is of the form $\Gamma[\Gamma'; \gamma']$, where γ' occurs in the bunch $\Gamma'; \gamma'$, then $\Gamma[\gamma'']$ is of the form $\Gamma[\Gamma'; \gamma'']$.

  Similarly, we denote by $\Gamma[\Gamma'']$ the replacement of that particular occurrence of γ' in Γ with all elements of Γ''. That is, if $\Gamma[\gamma']$ is of the form $\Gamma[\Gamma'; \gamma']$, where γ' occurs in the bunch $\Gamma'; \gamma'$, then $\Gamma[\Gamma'']$ is of the form $\Gamma[\Gamma'; \Gamma'']$.
- By $\gamma[\Gamma']$ we denote an ordered pair γ with a designated bunch Γ' occurring in γ, and, in this context, we denote by $\gamma[\Gamma'']$ the replacement of that particular occurrence of Γ' in γ with Γ''.
- Finally, by $\gamma[\gamma']$ we denote an ordered pair γ with a designated formula/ordered pair γ' occurring in a bunch occurring in γ, and, in this context, we denote by $\gamma[\gamma'']$ the replacement of that particular occurrence of γ' in γ with γ''.

  Similarly, we denote by $\gamma[\Gamma'']$ the replacement of that particular occurrence of γ' in Γ with all elements of Γ''.[21]

Sequents are expressions of the form $\Gamma \to \Delta$, where Γ is a bunch and Δ either is a formula singleton or is empty and in what follows we just write F for the singleton $\{F\}$.

[21] Cf. the notion of "S-Formel" in [6, § 2.4].

The axioms of ***SNLI*** are sequents of the form $P \to P$, where P is an atomic formula, and the rules of inference (i.e., the introduction rules of a formula into the antecedent and the succeedent of a sequent) are as follows.

$$(/\to)\quad \frac{\Gamma[B] \to \Delta \qquad \Gamma' \to A}{\Gamma[(B/A, \Gamma')] \to \Delta} \qquad\qquad (\to /)\quad \frac{(\Gamma, A) \to B}{\Gamma \to B/A}$$

$$(\backslash\to)\quad \frac{\Gamma[B] \to \Delta \qquad \Gamma' \to A}{\Gamma[(\Gamma', A\backslash B)] \to \Delta} \qquad\qquad (\to \backslash)\quad \frac{(A, \Gamma) \to B}{\Gamma \to A\backslash B}$$

$$(\cdot\to)\quad \frac{\Gamma[(A, B)] \to \Delta}{\Gamma[A \cdot B] \to \Delta} \qquad\qquad (\to \cdot)\quad \frac{\Gamma \to A \qquad \Gamma' \to B}{(\Gamma, \Gamma') \to A \cdot B}$$

$$(\wedge\to)\quad \frac{\Gamma[A] \to \Delta}{\Gamma[A \wedge B] \to \Delta} \qquad \frac{\Gamma[B] \to \Delta}{\Gamma[A \wedge B] \to \Delta} \qquad\qquad (\to \wedge)\quad \frac{\Gamma \to A \qquad \Gamma \to B}{\Gamma \to A \wedge B}$$

$$(\vee\to)\quad \frac{\Gamma[A] \to \Delta \qquad \Gamma[B] \to \Delta}{\Gamma[A \vee B] \to \Delta} \qquad\qquad (\to \vee)\quad \frac{\Gamma \to A}{\Gamma \to A \vee B} \qquad \frac{\Gamma \to B}{\Gamma \to A \vee B}$$

$$(\supset\to)\quad \frac{\Gamma[\Gamma'; B] \to \Delta \qquad \Gamma' \to A}{\Gamma[\Gamma'; A \supset B] \to \Delta} \qquad\qquad (\to\supset)\quad \frac{\Gamma; A \to B}{\Gamma \to A \supset B}$$

$$(\neg\to)\quad \frac{\Gamma \to A}{\Gamma; \neg A \to} \qquad\qquad (\to \neg)\quad \frac{\Gamma; A \to}{\Gamma \to \neg A}$$

There are also six *structural* rules of inference in ***SNLI***:

$$\text{contraction}\quad \frac{\Gamma[\Gamma'; \gamma; \gamma] \to \Delta}{\Gamma[\Gamma'; \gamma] \to \Delta}$$

$$(\text{bunch thinning} \to)\ \frac{\Gamma[\Gamma'] \to \Delta}{\Gamma[\Gamma'; \gamma] \to \Delta} \qquad (\to \text{bunch thinning})\ \frac{\Gamma \to}{\Gamma \to F}$$

$$\begin{array}{c}\text{left pair}\\ \text{thinning}\end{array}\ \frac{\Gamma \to}{(\Gamma', \Gamma) \to} \qquad\qquad \begin{array}{c}\text{right pair}\\ \text{thinning}\end{array}\ \frac{\Gamma \to}{(\Gamma, \Gamma') \to}$$

and

$$\text{cut}\quad \frac{\Gamma[A] \to \Delta \qquad \Gamma' \to A}{\Gamma[\Gamma'] \to \Delta}$$

Remark 40. There is a natural correspondence between bunches occurring in the antecedent of the conclusion and bunches occurring in the antecedent(s) of the premise(s) of a rule of inference. For example, for the rule $(/\to)$, the bunch containing $(B/A, \Gamma')$ corresponds to the bunch containing B, the bunches in Γ' correspond to themselves in the antecedent Γ' of the premise $\Gamma' \to A$, and the bunches in $\Gamma[(B/A, \Gamma')]$, excluding the bunch containing $(B/A, \Gamma')$, correspond to themselves in the antecedent $\Gamma[B]$ of the other premise.

The correspondence in the other rules is similar and is omitted.[22]

Remark 41. A straightforward induction on the formula complexity shows that for all formulas F, $\vdash_{\boldsymbol{SNLI}} F \rightarrow F$.

Definition 42. With a bunch Γ and an ordered pair γ we associate the formulas $\overline{\Gamma}$ and $\overline{\gamma}$ which are defined by the following recursion.

- $\overline{\{F_1; F_2; \ldots; F_n\}}$ is $\bigwedge_{i=1}^{n} F_i$[23];
- $\overline{(\Gamma', \Gamma'')}$ is $\overline{\Gamma'} \cdot \overline{\Gamma''}$; and
- $\overline{\{\gamma_1; \gamma_2; \ldots; \gamma_n\}}$ is $\bigwedge_{i=1}^{n} \overline{\gamma_i}$.

The *translation* of a sequent $\Gamma \rightarrow F$ is the formula $\overline{\Gamma} \supset F$ and the translation of a sequent $\Gamma \rightarrow$ is the formula $\neg\overline{\Gamma}$.

Theorem 43. *If a sequent* $\Gamma \rightarrow F$ (*respectively,* $\Gamma \rightarrow$) *is derivable in* ***SNLI****, then its translation* $\overline{\Gamma} \supset F$ (*respectively,* $\neg\overline{\Gamma}$) *is derivable in* ***NLI****.*

For the proof of Theorem 43 we need the following property of formulas associated with bunches.

Lemma 44. *Let* $\Gamma[\Gamma']$ *and* $\Gamma[\Gamma'']$ *be bunches such that* $\vdash_{\boldsymbol{NLI}} \overline{\Gamma'} \supset \overline{\Gamma''}$*. Then* $\vdash_{\boldsymbol{NLI}} \overline{\Gamma[\Gamma']} \supset \overline{\Gamma[\Gamma'']}$.

Proof. The proof is by induction on the complexity of $\Gamma[-]$, where $-$ is a special atom that occurs in Γ only once and indicates the place for substitution.[24] The basis is immediate, because in this case, $\Gamma[\Gamma']$ and $\Gamma[\Gamma'']$ are Γ' and Γ'', respectively.

The induction step for the case in which $\Gamma[-]$ is of the form $\{\gamma_1[-]; \gamma_2; \ldots; \gamma_n\}$ follows from the induction hypothesis $\vdash_{\boldsymbol{NLI}} \overline{\gamma_1[\Gamma']} \supset \overline{\gamma_1[\Gamma'']}$ by means of ***I***.

The induction step for the case in which $\Gamma[-]$ is of the form $(\Gamma_1[-], \Gamma_2)$ or of the form $(\Gamma_1, \Gamma_2[-])$ follows from the induction hypothesis $\vdash_{\boldsymbol{NLI}} \overline{\Gamma_i[\Gamma']} \supset \overline{\Gamma_i[\Gamma'']}$, $i = 1, 2$, by means of ***NL***. ■

[22] Actually, here we use a recursive definition of correspondence based on the definition of bunches. It is quite obvious and is left to the reader.

[23] As usual, the empty conjunction is a fixed provable formula.

[24] In order to avoid the notational cluttering involved in a precise definition of positions (in a bunch), we use $\Gamma[-]$ as a way to designate such a position, and $\Gamma[\Gamma']$ as a way to fill the designated position by Γ'.

Proof of Theorem 43. The proof is by induction on the length of the derivation of the sequent. The case of an axiom is immediate, and for the induction step, consider the last step in the derivation of $\Gamma \to \Delta$, where Δ is $\{F\}$ or is empty. The cases of introduction of a connective into the succeedent are trivial, the cases of a structural rule immediately follow from Lemma 44 and Example 7, and the cases of introduction of a formula into the antecedent are similar to each other. We treat only the case of the rule ($\supset\to$) with Δ being $\{F\}$ and leave the other rules to the reader.

Assume that the last step in the derivation is

$$\frac{\Gamma[\Gamma'; B] \to F \qquad \Gamma' \to A}{\Gamma[\Gamma'; A \supset B] \to F}\ (\supset\to).$$ [25]

By the induction hypothesis, $\vdash_{\boldsymbol{NLI}} \overline{\Gamma[\Gamma'; B]} \supset F$ and $\vdash_{\boldsymbol{NLI}} \overline{\Gamma'} \supset A$. Therefore, it suffices to show

$$\overline{\Gamma[\Gamma'; B]} \supset F; \overline{\Gamma'} \supset A \vdash_{\boldsymbol{NLI}} \overline{\Gamma[\Gamma'; A \supset B]} \supset F. \tag{17}$$

The proof of (17) is by induction on the complexity of $\Gamma[-]$. For the basis, $\Gamma[\Gamma'; B]$ is $\Gamma'; B$, and, by means of $\boldsymbol{I}$, we obtain

$$(\overline{\Gamma'} \wedge B) \supset F;\ \overline{\Gamma'} \supset A \vdash_{\boldsymbol{NLI}} (\overline{\Gamma'} \wedge (A \supset B)) \supset F.$$

For the induction step, $\Gamma[-]$ is in one of the following forms.

1. $\{\gamma_1[-]; \gamma_2; \ldots; \gamma_n\}$, $m \geq 1$;
2. $(\Gamma_1, \Gamma_2[-])$; or
3. $(\Gamma_1[-], \Gamma_2)$.

In case 1,

1. $\left(\overline{\gamma_1[\Gamma'; B]} \wedge \bigwedge_{i=2}^{n} \overline{\gamma_i}\right) \supset F$ — assumption
2. $\overline{\Gamma'} \supset A$ — assumption
3. $\overline{\gamma_1[\Gamma'; B]} \supset \left(\bigwedge_{i=2}^{n} \overline{\gamma_i} \supset F\right)$ — follows from 1 by means of $\boldsymbol{I}$
4. $\overline{\gamma_1[\Gamma'; A \supset B]} \supset \left(\bigwedge_{i=2}^{n} \overline{\gamma_i} \supset F\right)$ — follows from 3 and 2 by the induction hypothesis
5. $\left(\overline{\gamma_1[\Gamma'; A \supset B]} \wedge \bigwedge_{i=2}^{n} \overline{\gamma_i}\right) \supset F$ — follows from 4 by means of $\boldsymbol{I}$
6. $\overline{\Gamma[\Gamma'; A \supset B]} \supset F$ — is 5 by Definition 42

[25] That is, the sequent under consideration is $\Gamma[\Gamma'; A \supset B] \to F$.

In case 2,

1.	$\overline{\Gamma_1} \cdot \overline{\Gamma_2[\Gamma'; B]} \supset F$	assumption
2.	$\overline{\Gamma'} \supset A$	assumption
3.	$\overline{\Gamma_2[\Gamma'; B]} \supset \overline{\Gamma_1} \backslash F$	follows from 1 by (5)
4.	$\overline{\Gamma_2[\Gamma'; A \supset B]} \supset \overline{\Gamma_1} \backslash F$	follows from 3 and 2 by the induction hypothesis
5.	$\overline{\Gamma_1} \cdot \overline{\Gamma_2[\Gamma'; A \supset B]} \supset F$	follows from 4 by (6)
6.	$\overline{\Gamma[\Gamma'; A \supset B]} \supset F$	is 5 by Definition 42

Case 3 is similar to case 2 and is omitted. ■

Next we pass to the converse of Theorem 43.

Theorem 45. *If* $\vdash_{\boldsymbol{NLI}} F$, *then* $\vdash_{\boldsymbol{SNLI}} \rightarrow F$.[26]

The proof of Theorem 45 is based on Proposition 46 below (that is also of interest in its own right).

Proposition 46. (Inversion lemma, cf. [21, Proposition 3.5.4, p. 79] and [14, Section 8].) *We have*

(*i*) *If* $\vdash_{\boldsymbol{SNLI}} \Gamma[A \cdot B] \rightarrow \Delta$, *then* $\vdash_{\boldsymbol{SNLI}} \Gamma[(A, B)] \rightarrow \Delta$.

(*ii*) *If* $\vdash_{\boldsymbol{SNLI}} \Gamma \rightarrow B/A$, *then* $\vdash_{\boldsymbol{SNLI}} (\Gamma, A) \rightarrow B$.

(*iii*) *If* $\vdash_{\boldsymbol{SNLI}} \Gamma \rightarrow A \backslash B$, *then* $\vdash_{\boldsymbol{SNLI}} (A, \Gamma) \rightarrow B$.

(*iv*) *If* $\vdash_{\boldsymbol{SNLI}} \Gamma \rightarrow A \supset B$, *then* $\vdash_{\boldsymbol{SNLI}} \Gamma; A \rightarrow B$.

Proof. The proofs of all items are rather standard and we prove item (*ii*) only.[27] Assume $\vdash_{\boldsymbol{SNLI}} \Gamma \rightarrow B/A$. Then

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma \rightarrow B/A} \qquad A \rightarrow A}{(\Gamma, A) \rightarrow (B/A) \cdot A}\,(/ \rightarrow) \qquad \dfrac{\dfrac{A \rightarrow A \qquad B \rightarrow B}{(B/A, A) \rightarrow B}\,(\backslash \rightarrow)}{(B/A) \cdot A \rightarrow B}\,(\cdot \rightarrow)}{(\Gamma, A) \rightarrow B}\ \text{cut.}$$

■

Proof of Theorem 45. The proof is by induction on the derivation length of F. The cases of axioms and a derivation step by *modus ponens* are treated exactly like in the case of sequent calculus for $\boldsymbol{I}$,[28] and for the cases of the Lambek rules of inference (5)–(6) we proceed as follows.

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[26] Of course, Theorem 45 is not exactly the converse of Theorem 43, because it does not mention Γ. There is no Γ in ***NLI***, as it is a Hilbert type calculus.

[27] The proof below was suggested by the referee, instead of the authors' inductive one.

[28] Here we need Remark 41.

Assume that the last derivation step is by (5):

$$\frac{A \cdot B \supset C}{B \supset A\backslash C}.$$[29]

Then

1. $\to A \cdot B \supset C$ induction hypothesis
2. $A \cdot B \to C$ follows from 1 by Proposition 46(*iv*)
3. $(A, B) \to C$ follows from 2 by Proposition 46(*i*)
4. $B \to A\backslash C$ follows from 3 by $(\to /)$
5. $\to B \supset A\backslash C$ follows from 4 by $(\to\supset)$

Assume that the last derivation step is by (6):

$$\frac{B \supset A\backslash C}{A \cdot B \supset C}.$$[30]

Then

1. $\to B \supset A\backslash C$ induction hypothesis
2. $B \to A\backslash C$ follows from 1 by Proposition 46(*iv*)
3. $(A, B) \to C$ follows from 2 by Proposition 46(*iii*)
4. $A \cdot B \to C$ follows from 3 by $(\cdot \to)$
5. $\to A \cdot B \supset C$ follows from 4 by $(\to\supset)$

The case of the rules (3) and (4) are similar and are omitted. ■

Theorem 47. (Cf. [17, Theorem 6.2].) *If a sequent is derivable in **SNLI**, then it is derivable without applications of the cut rule.*

For the proof of Theorem 47 we need the following extension of thinnings.

Lemma 48. *If $\vdash_{\boldsymbol{SNLI}} \Gamma' \to$, then for all $\Gamma[-]$, $\vdash_{\boldsymbol{SNLI}} \Gamma[\Gamma'] \to$.*

Proof. The proof is by induction on the complexity of $\Gamma[-]$ and is similar to that of Lemma 44. The basis is immediate, because in this case, $\Gamma[\Gamma']$ is Γ'.

For the induction step for the case in which $\Gamma[-]$ is of the form $\{\gamma_1[-]; \gamma_2; \ldots; \gamma_n\}$, by the induction hypothesis, $\vdash_{\boldsymbol{SNLI}} \gamma_1[\Gamma'] \to$ and then

$$\frac{\gamma_1[\Gamma'] \to}{\gamma_1[\Gamma']; \gamma_2; \ldots; \gamma_n \to} \quad n-1 \text{ (bunch thinning } \to\text{)s.}$$

For the induction step for the case in which $\Gamma[-]$ is of the form $(\Gamma_1[-], \Gamma_2)$, by the induction hypothesis, $\vdash_{\boldsymbol{SNLI}} \Gamma_1[\Gamma'] \to$ and then

$$\frac{\Gamma_1[\Gamma'] \to}{(\Gamma_1[\Gamma'], \Gamma_2) \to} \begin{array}{l}\text{right pair}\\ \text{thinning}\end{array}.$$

---

[29]That is, F is $B \supset A\backslash C$.

[30]That is, F is $A \cdot B \supset C$.

The case in which $\Gamma[-]$ is of the form $(\Gamma_1, \Gamma_2[-])$ is similar. ■

Proof of Theorem 47. The proof is a straightforward combination of proofs in [14, Section 9] and [21, Section 4.1]. Namely, by the outer induction on the number of structural rules in the derivation,[31] the middle induction on the derivation length, and the inner induction on the complexity of the cut formula, we eliminate the first cut in the derivation. We start with the outer induction (structural rules), skip the middle induction (which is a standard switching of the order of rules of inference), and, for the inner induction, we consider only the cases of the principal connectives $\backslash$ and $\supset$, and leave the rest to the reader.

In the case of contraction, we replace the derivation

$$\dfrac{\dfrac{\Gamma[\Gamma'; \gamma[A]; \gamma[A]] \to \Delta}{\Gamma[\Gamma'; \gamma[A]] \to \Delta}\ \text{contraction} \qquad \Gamma'' \to A}{\Gamma[\Gamma'; \gamma[\Gamma'']] \to \Delta}\ \text{cut}$$

with

$$\dfrac{\dfrac{\dfrac{\Gamma[\Gamma'; \gamma[A]; \gamma[A]] \to \Delta \qquad \Gamma'' \to A}{\Gamma[\Gamma'; \gamma[\Gamma'']; \gamma[A]] \to \Delta}\ \text{cut} \qquad \Gamma'' \to A}{\Gamma[\Gamma'; \gamma[\Gamma'']; \gamma[\Gamma'']] \to \Delta}\ \text{cut.}}{\Gamma[\Gamma'; \gamma[\Gamma'']] \to \Delta}\ \text{contraction}$$

In the case of (bunch thinning $\to$), we replace the derivation

$$\dfrac{\dfrac{\Gamma[\Gamma'] \to \Delta}{\Gamma[\Gamma'; \gamma[A]] \to \Delta}\ (\text{bunch thinning} \to) \qquad \Gamma'' \to A}{\Gamma[\Gamma'; \gamma[\Gamma'']] \to \Delta}\ \text{cut}$$

with

$$\dfrac{\Gamma[\Gamma'] \to \Delta}{\Gamma[\Gamma'; \gamma[\Gamma'']] \to \Delta}\ (\text{bunch thinning} \to).$$

In the case of ($\to$ bunch thinning), we replace the derivation

$$\dfrac{\Gamma[A] \to \Delta \qquad \dfrac{\Gamma' \to}{\Gamma' \to A}\ (\to \text{bunch thinning})}{\Gamma[\Gamma'] \to \Delta}\ \text{cut}$$

with

[31] Actually, the case of thinnings does not rely on the induction hypothesis.

$$\cfrac{\cfrac{\Gamma' \rightarrow}{\Gamma[\Gamma'] \rightarrow}\ \text{Lemma 48}}{\Gamma[\Gamma'] \rightarrow \Delta}\ (\rightarrow \text{bunch thinning})\ ,$$

if $\Delta \neq \emptyset$, and with

$$\cfrac{\Gamma' \rightarrow}{\Gamma[\Gamma'] \rightarrow}\ \text{Lemma 48},$$

otherwise.

In the case of left pair thinning, we replace the derivation

$$\cfrac{\cfrac{\Gamma \rightarrow}{(\Gamma'[A], \Gamma) \rightarrow}\ \begin{array}{l}\text{left pair}\\ \text{thinning}\end{array} \qquad \Gamma'' \rightarrow A}{(\Gamma'[\Gamma''], \Gamma) \rightarrow}\ \text{cut}$$

with

$$\cfrac{\Gamma \rightarrow}{(\Gamma'[\Gamma''], \Gamma) \rightarrow}\ \begin{array}{l}\text{left pair}\\ \text{thinning}\end{array}.$$

Naturally, the case of right pair thinning, is similar.

In the case of $\backslash$, we replace the derivation

$$\cfrac{\cfrac{\Gamma[B] \rightarrow \Delta \qquad \Gamma' \rightarrow A}{\Gamma[(\Gamma', A\backslash B)] \rightarrow \Delta}\ (/ \rightarrow) \qquad \cfrac{(A, \Gamma'') \rightarrow B}{\Gamma'' \rightarrow A\backslash B}\ (\rightarrow /)}{\Gamma[(\Gamma', \Gamma'')] \rightarrow \Delta}\ \text{cut}$$

with

$$\cfrac{\Gamma[B] \rightarrow \Delta \qquad \cfrac{(A, \Gamma'') \rightarrow B \qquad \Gamma' \rightarrow A}{(\Gamma', \Gamma'') \rightarrow B}\ \text{cut}}{\Gamma[(\Gamma', \Gamma'')] \rightarrow \Delta}\ \text{cut}.$$

In the case of $\supset$, we replace the derivation

$$\cfrac{\cfrac{\Gamma[\Gamma'; B] \rightarrow \Delta \qquad \Gamma' \rightarrow A}{\Gamma[\Gamma'; A \supset B] \rightarrow \Delta}\ (\supset\rightarrow) \qquad \cfrac{\Gamma''; A \rightarrow B}{\Gamma'' \rightarrow A \supset B}\ (\rightarrow\supset)}{\Gamma[\Gamma'; \Gamma''] \rightarrow \Delta}\ \text{cut}$$

with

$$\cfrac{\cfrac{\Gamma[\Gamma'; B] \rightarrow \Delta \qquad \cfrac{\Gamma''; A \rightarrow B \qquad \Gamma' \rightarrow A}{\Gamma'; \Gamma'' \rightarrow B}\ \text{cut}}{\Gamma[\Gamma'; \Gamma'; \Gamma''] \rightarrow \Delta}\ \text{cut.}}{\Gamma[\Gamma'; \Gamma''] \rightarrow \Delta}\ \text{contractions}$$

■

Next, we list some immediate corollaries of Theorem 47.

Corollary 49. ***SNLI*** *possesses the subformula property.*

To state Corollary 50 below we need the following notation.

For a set of connectives c (both Lambek and propositional), we denote the restriction of ***SNLI*** to c by $\boldsymbol{SNLI}_c$.

Corollary 50. (Cf. Corollaries 35 and 36.) *For any set of connectives c,* ***SNLI*** *is a conservative extension of* $\boldsymbol{SNLI}_c$.[32]

We conclude this section with the most important consequence of Theorem 47.

Proposition 51. (Cf. Corollary 34.) *For a formula F it is decidable whether* $\vdash_{\boldsymbol{NLI}} F$.

The proof of Proposition 51 is similar to that in [7, § 1]. First we estimate the *depth* of bunches which appear in cut-free ***SNLI***-derivations. For this we need the following definition and notation.

Definition 52. (Cf. Definition 39.) The *depth* of a bunch Γ, denoted $d(\Gamma)$ is defined by the following recursion.

- If Γ is a finite multiset of formulas, then $d(\Gamma) = 0$, and
- if

$$\Gamma = \{F_1; F_2; \ldots, F_m; (\Gamma'_1, \Gamma''_1); (\Gamma'_2, \Gamma''_2); \ldots; (\Gamma'_n, \Gamma''_n)\},$$

  $m = 0, 1, \ldots$ and $n = 1, 2, \ldots$, then

$$d(\Gamma) = \max\{d(\Gamma'_1), d(\Gamma''_1), d(\Gamma'_2), d(\Gamma''_2), \ldots, d(\Gamma'_n), d(\Gamma''_n)\} + 1.$$[33]

We also denote the number of Lambek connectives occurring in a formula F by $\#_{\boldsymbol{NL}}(F)$.

Lemma 53. *Let Γ be a bunch that occurs in a cut-free derivation of a sequent $\rightarrow F$. Then $d(\Gamma) \leq \#_{\boldsymbol{NL}}(F)$.*

Proof. The depth of a bunch can be decreased only by the rules $(\cdot \rightarrow)$, $(\rightarrow /)$, or $(\rightarrow \backslash)$, each introducing the corresponding Lambek connective. Since the derivation is cut-free, all this connectives must occur in F. ■

[32] In particular, it follows that ***NLI*** is a conservative extension of both ***I*** and ***NL***.

[33] In other words, $d(\Gamma)$ is the maximal number of nested parentheses appearing in Γ.

Definition 54. A bunch Γ is *reduced* if it is not of the form $\Gamma[\Gamma'; \gamma; \gamma; \gamma; \gamma]$. [34]

A sequent $\Gamma \to \Delta$ is *reduced*, if Γ is reduced.

A reduced sequent S is a *reduction* of a sequent S' if S can be derived from S' by contractions and bunch thinnings, only.

Finally, a derivation of a sequent is *reduced* if all sequents in it are reduced.

Remark 55. It follows from the definition that, for any two reductions of the same sequent, one can be derived from the other by a reduced derivation consisting of a sequence of contractions followed by a sequence of bunch thinnings.

For the proof of Proposition 51, we show first that each reduction of a derivable sequent can be derived by a cut-free reduced derivation (Lemma 56) and then, using Lemma 53, we establish an upper bound on the length of such a derivation in terms of the derived sequent.

Lemma 56. *If a sequent is derivable in* ***SNLI****, then all its reductions are derivable by a cut-free reduced derivation.*

Proof. By Theorem 47, we may assume that the sequent derivation is cut-free. Then we proceed by induction on the derivation length of the sequent. In the basis case, the sequent is an axiom and therefore, is already reduced.

For the induction step, consider the last step in the derivation

$$\frac{\{S_i\}_{i \in I}}{S},$$

where $I = \{1\}$ or $I = \{1, 2\}$

We "contract" the conclusion and the premises of this step as follows.

Let for a multiset X and $x \in X$, $\#_x(X)$ denote the number of occurrences of x in X. Then, for each bunch Γ occurring in the antecedent of S and each $\gamma \in \Gamma$, if $\#_\gamma(\Gamma) > 2$, we delete $\#_\gamma(\Gamma) - 2$ copies of γ from Γ and from the corresponding bunches occurring in the antecedents of the premises (see Remark 40).[35]

Let S' and S'_i, $i \in I$, be the sequents obtained after the above transformation of the antecedents of S and S_i, respectively. By definition, S' is reduced and

$$\frac{\{S'_i\}_{i \in I}}{S'}. \tag{18}$$

[34] That is, Γ is reduced if no formula or ordered pair appears (as an element) in Γ or in a component of an ordered pair occurring in Γ more than three times.

[35] The deletion process naturally starts with bunches of maximum depth and goes top-down to multisets of formulas.

Since the corresponding bunches in a rule of inference differ from each other by at most one formula/ordered pair, the sequents S'_i, $i \in I$, are also reduced. By the induction hypothesis, there are reduced derivations of S'_i, $i \in I$, which, in combination with (18), result in a reduced derivation of S'. Now, the proof follows from Remark 55. ■

Now we are ready for the proof of Proposition 51.

Proof of Proposition 51. By Theorems 43 and 45, $\vdash_{\boldsymbol{NLI}} F$ if and only if $\vdash_{\boldsymbol{SNLI}} \rightarrow F$, and derivability of the sequent $\rightarrow F$ in ***SNLI*** can be decided as follows.

The sequent $\rightarrow F$ is reduced. Thus, if it is derivable, then, by Lemma 56, it is derivable by a cut-free reduced derivation. By Lemma 53, the depth of bunches occurring in a cut-free derivation of F does not exceed $\#_{\boldsymbol{NL}}(F)$ and, by Corollary 49, all formulas occurring in these bunches are subformulas of F.

Let $\boldsymbol{S}_F$ be the set of all sequents whose antecedent is a reduced bunch of depth not exceeding $\#_{\boldsymbol{NL}}(F)$ such that all formulas occurring in it are subformulas of F and whose succeedent is a subformula of F. Since, obviously, $\boldsymbol{S}_F$ is finite and constructible, derivability of $\rightarrow F$ can be decided by checking all derivations of length not exceeding the number of elements of $\boldsymbol{S}_F$ and consisting of elements of $\boldsymbol{S}_F$ only. ■

5. Associative Lambek Calculus Extended with Intuitionistic Propositional Logic

Associative Lambek calculus ***L*** and its extension with intuitionistic propositional logic ***LI*** result in adding to ***NL*** and ***NLI***, respectively, the axioms

$$(A \cdot B) \cdot C \supset A \cdot (B \cdot C) \tag{19}$$

and

$$A \cdot (B \cdot C) \supset (A \cdot B) \cdot C. \tag{20}$$

It was shown in [4] that the *associative* ternary semantics is sound and complete for ***L***. In this section we present the *associative* ***NLI***-interpretations which are sound and complete for ***LI***.

Definition 57. A ternary relation R on a set W is *associative*, if for all $u, v, w, x \in W$ the following holds.

- There exists y such that $R(y, v, w)$ and $R(u, y, x)$ if and only if there exists z such that $R(z, w, x)$ and $R(u, v, z)$.

An ***NLI***-interpretation $\mathfrak{I} = \langle W, \leq, R, V\rangle$ is *associative*, if R is associative.

By [4, Proposition 2], the axioms (19) and (20) are satisfied by associative interpretations. Thus, the latter is sound for ***LI***.

The notion of Θ-canonical interpretation $\mathfrak{I}_\Theta$ (Definition 29) extends to ***LI*** in a natural manner.[36] Thus, for the proof of completeness of the associative relational semantics with respect to ***LI***, it suffices to show that the canonical ***LI*** interpretation $\mathfrak{I}_\Theta$ is associative.

Theorem 58. *The interpretation* $\mathfrak{I}_\Theta$ *is associative.*

The proof of Theorem 58 is exactly as that of [8, Theorem 23] and is omitted.

Now, exactly like in the proof of Corollaries 35 and 36, it can be shown that ***LI*** is a conservative extension of ***L*** and ***I***, respectively.

It is known from [1] that, in contrast with ***NL***, the general problem of derivability from a *set* of assumptions in ***L*** is undecidable. Therefore, the same problem for ***LI*** is undecidable either.[37]

However, the problem of derivability of a formula in ***L*** is decidable, because there is a sequent calculus for ***L*** that admits cut elimination, see [14], and in the next section we prove that derivability in ***LI*** is decidable as well.

6. A Sequent Calculus for *LI*

The sequent calculus ***SLI*** for ***LI*** is obtained from ***SNLI*** by adding two structural rules which correspond to axioms (19) and (20), respectively, cf. [18, Section 7.1]:

$$\text{restructuring}_{(19)}\ \ \frac{\Gamma[(\Gamma_1, (\Gamma_2, \Gamma_3))] \to \Delta}{\Gamma[((\Gamma_1, \Gamma_2), \Gamma_3)] \to \Delta}$$

and

$$\text{restructuring}_{(20)}\ \ \frac{\Gamma[((\Gamma_1, \Gamma_2), \Gamma_3)] \to \Delta}{\Gamma[(\Gamma_1, (\Gamma_2, \Gamma_3))] \to \Delta}.$$

The proofs of the statements below are similar to the proofs of their nonassociative counterparts and, therefore, are omitted.

Theorem 59. (Cf. Theorem 43.) *If a sequent is derivable in **SLI**, then its translation is derivable in **LI**.*

---

[36] Alternatively, we may assume that Θ includes both (19) and (20).

[37] In particular, filtration does not preserve associativity.

Theorem 60. (Cf. Theorem 45.) *If* $\vdash_{\boldsymbol{LI}} F$, *then* $\vdash_{\boldsymbol{SLI}} \rightarrow F$.

Theorem 61. (Cf. Theorem 47.) *If a sequent is derivable in* ***SLI***, *then it is derivable without applications of the cut rule.*

Proof. The proof follows that of Theorem 47. The only addition is the case of restructuring in which we only consider one of the three possible cases for the cut formula A in restructuring$_{(19)}$. In that case, we replace the derivation

$$\dfrac{\dfrac{\Gamma[(\Gamma_1,(\Gamma_2,A))] \rightarrow \Delta}{\Gamma[((\Gamma_1,\Gamma_2),A)] \rightarrow \Delta}\ \text{restructuring}_{(19)} \qquad \Gamma_3 \rightarrow A}{\Gamma[((\Gamma_1,\Gamma_2),\Gamma_3)] \rightarrow \Delta}\ \text{cut}$$

with

$$\dfrac{\dfrac{\Gamma[(\Gamma_1,(\Gamma_2,A))] \rightarrow \Delta \qquad \Gamma_3 \rightarrow A}{\Gamma[(\Gamma_1,(\Gamma_2,\Gamma_3))] \rightarrow \Delta}\ \text{cut}}{\Gamma[((\Gamma_1,\Gamma_2),\Gamma_3)] \rightarrow \Delta}\ \text{restructuring}_{(19)}.$$

■

Corollary 62. (Cf. Corollary 49.) ***LI*** *possesses the subformula property.*

Corollary 63. (Cf. Corollary 50.) *For any set of connectives c,* ***LI*** *is a conservative extension of* $\boldsymbol{LI}_c$.[38]

Finally, we answer the question of decidability of ***LI*** left open in [2,3].

Proposition 64. (Cf. Proposition 51.) *For a formula F it is decidable whether* $\vdash_{\boldsymbol{LI}} F$.

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