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Semisimples in Varieties of Commutative Integral Bounded Residuated Lattices

Abstract. In any variety of bounded integral residuated lattice-ordered commutative monoids (bounded residuated lattices for short) the class of its semisimple members is closed under isomorphic images, subalgebras and products, but it is not closed under homomorphic images, and so it is not a variety. In this paper we study varieties of bounded residuated lattices whose semisimple members form a variety, and we give an equational presentation for them. We also study locally representable varieties whose semisimple members form a variety. Finally, we analyze the relationship with the property "to have radical term", especially for k-radical varieties, and for the hierarchy of varieties $(\mathsf{WL_k})_{k>0}$ defined in Cignoli and Torrens (*Studia Logica* 100:1107–1136, 2012 [7]).

Keywords: Residuated lattices, Semisimple and local residuated lattices, k-Radical varieties, Radical term.

Introduction

The aim of this paper is to present an approach to varieties of bounded integral residuated lattice-ordered commutative monoids (bounded residuated lattices for short) whose semisimple members form a variety; and their relationship with the property "to admit radical term". This property tells us that there exists a unary term $t(x)$ such that the (maximal) radical of any member of the variety is the set all elements satisfying the equation $t(x) \approx \top$.

Our interest in these varieties arises from the results about varieties of bounded residuated lattices having boolean retraction term obtained in [7]. In these varieties the class of semisimple members is the class of boolean algebras, and so a variety. Moreover, the boolean retraction term is also radical term. Something similar happens to the variety of n-contractive (n-potent) bounded residuated lattices for any $n > 0$, because their semisimple members form a variety and it admits radical term.

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The first goal of our work is to analyze varieties of bounded residuated lattices whose semisimple members form a variety. This is done in Sect. 3, in which we give some basic results on the class of semisimple algebras of a variety of bounded residuated lattices, and we obtain an equational characterization of these varieties (Theorem 3.4). For this purpose we use some results of semisimple varieties given by T. Kowalski in [12] (see also [8]).

In Sect. 4, we analyze varieties whose members are isomorphic to a subdirect product of local bounded residuated lattices. For each $m > 0$, we define m-locally representable varieties and we show that their semisimple members form a variety (Corollary 4.9).

In Sect. 5, we study varieties admitting radical term. A simple argument shows that in these varieties the class of semisimple members form a quasivariety. Moreover, if the variety is k-radical, then the converse also holds, and so the class of semisimple members form a quasivariety if and only if the variety admits radical term. This result can be improved for subvarieties of the variety $\mathsf{WL_k}$, defined by the equation $k.x \vee k.\neg x \approx \top$, $k > 0$. We prove in Theorem 5.7 that the subvarieties of $\mathsf{WL_k}$ whose semisimple members form a variety are its radical term subvarieties.

We also include two sections containing the main results and properties of bounded residuated lattices needed throughout the paper.

In Sect. 1, after giving basic definitions, we introduce elementary terms and recall their arithmetical properties. The proof of these properties can be found in detail in [8] (see also [7,14]). Moreover, we also introduce two hierarchies of varieties of bounded residuated lattices, namely the well-known hierarchy of contractive bounded residuated lattices $(\mathsf{E_m})_{m>0}$ and the hierarchy $(\mathsf{WL_k})_{k>0}$ introduced in [7]; both hierarchies are used through the paper.

In Sect. 2, we recall some notions of the filter theory of bounded residuated lattices. In particular the isomorphism between implicative filters and congruence relations. We also give the definition and several properties of maximal implicative filters and the radical. In Lemma 2.8 we show that, in general, product of radicals is not equal to the radical of the product, in contrast to [9, Proposition 23]. We also define k-radical variety, and we show that $\mathsf{WL_k}$ and $\mathsf{E_k}$ are k-radical, for any $k > 0$.

Finally, at the end of the paper, we present an example of an n-contractive bounded residuated which is not locally representable. This example also clarifies the relationship between the considered hierarchies of varieties.

We assume that the reader is familiar with basic notions of residuated lattices and Universal Algebra. The results needed to understand the paper can be found in [8,14] for residuated lattices, and in [1,4] for Universal Algebra.

1. Preliminaries

Throughout this paper BRL denotes the class of bounded residuated lattices, that is, the class of algebras $\boldsymbol{A} = \langle A; \cdot, \to, \vee, \wedge, \top, \bot\rangle$ of type $(2, 2, 2, 2, 0, 0)$ such that:

- $\langle A; \cdot, \top\rangle$ is a commutative monoid,
- $\langle A; \vee, \wedge, \bot, \top\rangle$ is a bounded lattice with smallest element $\bot$, and greatest element $\top$,
- for any $a, b, c \in A$ $a \cdot b \leqslant c$ if and only if $a \leqslant b \to c$, where $\leqslant$ is the partial order given by the lattice structure.

It is well known that bounded residuated lattices admit an equational presentation, and so BRL is a variety (see for example [7,8,14]), that is, if H, S and P respectively represent the operators homomorphic images, isomorphic images of subalgebras and isomorphic images of direct products, then $HSP(\mathsf{BRL}) \subseteq \mathsf{BRL}$.

In the next lemmata we list, for further reference, some well known properties of bounded residuated lattices.

Lemma 1.1. *Let $\boldsymbol{A}$ be a bounded residuated lattice. Then for any $a, b, c \in A$ it satisfies*

(*a*) $a \leqslant b$ *if and only if* $a \to b = \top$,

(*b*) $\top \to a = a$,

(*c*) $(a \to b) \to ((b \to c) \to (a \to c)) = \top$,

(*d*) $(a \cdot b) \to c = a \to (b \to c)$,

(*e*) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$,

(*f*) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

On every bounded residuated lattice $\boldsymbol{A}$ we consider the unary operation:

$$\neg x := x \to \bot, \text{ for all } x \in A.$$

By taking into account that the $\{\to, \bot, \top\}$-reduct of a bounded residuated lattice is a bounded BCK-algebra we have ([7,10]):

Lemma 1.2. *If $\boldsymbol{A}$ is a bounded residuated lattice, then for any $a, b \in A$ the following properties hold true:*

(*a*) $a \leqslant b \Rightarrow \neg b \leqslant \neg a$,

(*b*) $a \leqslant \neg\neg a$,

(*c*) $\neg a = \neg\neg\neg a$,

(*d*) $a \to \neg b = b \to \neg a$,

(*e*) $a \to \neg b = \neg\neg a \to \neg b$,

(*f*) $\neg\neg(a \to \neg b) = a \to \neg b$,

(*g*) $\neg a \to (a \to b) = \top$.

If we consider the binary term operation

$$x + y := \neg(\neg x \cdot \neg y),$$

then for every $\boldsymbol{A} \in \mathsf{BRL}$ $\langle A; + \rangle$ is a commutative semigroup. For any non negative integer n we define terms x^n and $n.x$ recursively:

- $x^0 = \top$ and $x^{n+1} = x \cdot x^n$,
- $0.x = \bot$ and $(n+1).x = x + n.x$.

We write $\neg\, x^n$ instead of $\neg(x^n) = x^n \to \bot$. Then it is straightforward to show the properties listed in the next lemma.

LEMMA 1.3. ([7, Lemma 1.3]) *Let $\boldsymbol{A}$ be a bounded residuated lattice. If $a, b \in A$ and n, m are non negative integers, then*

(*a*) $a + b = \neg a \to \neg\neg b$,

(*b*) $1.a = \bot + a = \neg\neg a$,

(*c*) $n.a = \neg(\neg a)^n$,

(*d*) $n.a = \neg\neg(n.a) = n.(\neg\neg a)$,

(*e*) $(n+m).a = n.a + m.a$,

(*f*) $n.(a+b) = n.a + n.b$,

(*g*) $(mn).a = m.(n.a)$,

(*h*) *if* $n \leqslant m$, *then* $n.a \leqslant m.a$ *and* $a^m \leqslant a^n$,

(*i*) $\neg(n.a)^m = m.(\neg a)^n$,

(*j*) *if* $a \leqslant b$, *then* $n.a^m \leqslant n.b^m$.

Through the paper we consider several hierarchies of varieties of bounded residuated lattices.

For any integer $m > 0$, let $\mathsf{E_m}$ denote the subvariety of BRL determined by the identity

(Em) $x^m \approx x^{m+1}$.

The members of $\mathsf{E_m}$ are called *m-contractive* or *m-potent* bounded residuated lattices. Moreover, it is easy to check that for any $m > 0$, $\mathsf{E_m} \subsetneq \mathsf{E_{m+1}}$. Algebras in $\mathsf{E_1}$ are also called *Heyting algebras.*

For each integer $k > 0$, $\mathsf{WL_k}$ represents the subvariety of BRL given by the identity (see [7])

(WLk) $k.x \vee k.\neg x \approx \top$.

The variety $\mathsf{WL_2}$ contains the class MTL of MTL-*algebras*, bounded residuated lattices representable as subdirect product of totally ordered bounded residuated lattices. MTL may be defined as the subvariety of BRL given by the identity $(x \to y) \vee (y \to x) \approx \top$.

From the results given in [7] we deduce:

LEMMA 1.4. *The following properties hold true:*

(a) $\mathsf{WL_1}$ *is the variety of stonean residuated lattices, i.e., the subvariety of* BRL *given by the equation* $\neg\neg x \vee \neg x \approx \top$.

(b) *For any* $k > 0$, $\mathsf{WL_k} \subsetneq \mathsf{WL_{k+1}}$.

(c) *For any* $k > 0$, $\mathsf{WL_k} \cap \mathsf{PRL} = \mathsf{WL_1}$, *where* PRL *denotes the variety of pseudocomplemented residuated lattices, i.e., the subvariety of* BRL *given by the identity* $x \wedge \neg x \approx \bot$.

2. Implicative Filters, Congruence Relations and Radical

An *implicative filter* (*i-filter* for short) of a bounded residuated lattice $\boldsymbol{A}$ is a subset F of A satisfying the following conditions:

(F1) $\top \in F$.

(F2) For all $a, b \in A$, if $b \in F$ and $a \leqslant b$, then $b \in F$.

(F3) If a, b are in F, then $a \cdot b \in F$.

Alternatively, i-filters may be defined as subsets F of A satisfying **(F1)** and

(F4) if $a, a \to b \in F$, then $b \in F$.

It is easy to prove that for each non empty set $B \subseteq A$,

$$\langle B \rangle = \{a \in A : x_1^{n_1} \cdot x^{n_2} \cdots x_k^{n_k} \leqslant a,\ 1 \leqslant k, n_1, \ldots, n_k,\ x_1, \ldots, x_k \in B\}$$

is the *smallest i-filter containing* B, and it is equal to the intersection of all i-filters containing B. For each $a \in A$, we shall write $\langle a \rangle$ instead of $\langle \{a\} \rangle$.

The properties listed in the next lemma are easy to prove.

LEMMA 2.1. *If* $\boldsymbol{A}$ *is a bounded residuated lattice, then*

(a) $\{\top\}$ *is the least i-filter, and* $\langle \bot \rangle = A$,

(b) *for any* $B \subseteq A$ *and any* $a \in A$,

$$\langle B \cup \{a\} \rangle = \{b \in A : a^n \to b \in \langle B \rangle \text{ for some } n \geqslant 0\},$$

(c) *for any* $a \in A$, $\langle a \rangle = \{b \in B : a^n \to b = \top \text{ for some } n \geqslant 0\}$.

Given an i-filter F of a bounded residuated lattice $\boldsymbol{A}$, the binary relation

$$\theta(F) := \{(x, y) \in A \times A : x \to y \in F \text{ and } y \to x \in F\}$$

is a congruence on $\boldsymbol{A}$ such that $F = \top/\theta(F)$, the equivalence class of $\top$. Actually, the correspondence $F \mapsto \theta(F)$ is an order isomorphism from the set of i-filters of $\boldsymbol{A}$ onto the set of congruences of $\boldsymbol{A}$, both sets ordered by inclusion. Its inverse is given by the map $\theta \mapsto \top/\theta$. We write $\boldsymbol{A}/F$ instead of $\boldsymbol{A}/\theta(F)$, and a/F instead of $a/\theta(F)$, the equivalence class of a modulo $\theta(F)$. Notice that in $\boldsymbol{A}$, $\theta_{\{\top\}}$ is the identity and $\theta_{\langle \bot \rangle} = A \times A$ the universal equivalence relation.

An i-filter F of a non trivial bounded residuated lattice $\boldsymbol{A}$ is *proper* provided $F \neq A$, that is $\bot \notin A$. A *maximal i-filter* is a proper i-filter M of $\boldsymbol{A}$ such that for each $a \in A \setminus M$, $\langle M \cup \{a\} \rangle = A$.

REMARK 2.2. Since in any bounded residuated lattice the set of its i-filters is closed under upward directed families, by Zorn's Lemma, an i-filter is proper if and only if it is contained in a maximal i-filter.

The radical of a bounded residuated lattice $\boldsymbol{A}$, represented by $Rad(\boldsymbol{A})$, is the intersection of its maximal i-filters, that is,

- $a \in Rad(\boldsymbol{A})$ if and only if $a \in M$ for each maximal i-filter M of $\boldsymbol{A}$.

The following two results are well known and they can be found in the literature. We include simple proofs of them (cf. [11,14]).

LEMMA 2.3. *Let* F *be an i-filter of a bounded residuated lattice* $\boldsymbol{A}$. *Then* F *is maximal if and only if*

(Mx) *for any* $a \in A$, $a \notin F$ *if and only if* $\exists\, n > 0$ *such that* $\neg\, a^n \in F$.

PROOF. Assume that F is maximal. Then $a \in A \setminus F$ if and only if $\langle F \cup \{a\} \rangle = A$ if and only if $\bot \in \langle F \cup \{a\} \rangle$ if and only if there exists $n > 0$ such that

$\neg a^n = a^n \to \bot \in F$. Conversely, assume that F is a proper i-filter satisfying $a \notin F$ if and only if $\exists n > 1$ such that $\neg a^n \in F$. Then for any $a \notin F$, $\bot \in \langle F \cup \{a\}\rangle$, and so $\langle F \cup \{a\}\rangle = A$. Thus F is maximal. ■

LEMMA 2.4. *For every bounded residuated lattice* $\boldsymbol{A}$ *and for any* $a \in A$*, the following are equivalent:*

(i) $a \in Rad(\boldsymbol{A})$.

(ii) *For all* $n > 0$ $\langle \neg a^n \rangle = A$.

(iii) *For any* $n > 0$ *there is* $k > 0$ *such that* $k.a^n = \top$.

PROOF. $(i) \Leftrightarrow (ii)$: Assume $a \in Rad(\boldsymbol{A})$. Then for any maximal i-filter M and for any $n > 0$, $\neg a^n \notin M$, and by Remark 2.2, we have that for any $n > 0$ the i-filter $\langle \neg a^n \rangle$ is not proper. Conversely, if for any $n > 0$, $\langle \neg a^n \rangle = A$, then $\neg a^n$ does not belong to any maximal i-filter, hence by Lemma 2.3, a belongs to every maximal i-filter and so $a \in Rad(\boldsymbol{A})$.

$(ii) \Leftrightarrow (iii)$ follows from the fact that for any $a \in A$, and any $n > 0$,

$$\langle \neg a^n \rangle = A \text{ iff } \bot \in \langle \neg a^n \rangle \text{ iff } \exists k > 0 \text{ s.t.} \top = (\neg a^n)^k \to \bot = k.a^n.$$ ■

COROLLARY 2.5. *For each bounded residuated lattice* $\boldsymbol{A}$ *we have*

$$Rad(\boldsymbol{A}) = \{a \in A : \forall n > 0, \exists k > 0 \textit{ such that } k.a^n = \top\}.$$

The following result is a consequence of above

LEMMA 2.6. *The following properties hold true:*

(a) *If* $\boldsymbol{B}$ *is a subalgebra of a bounded residuated lattice* $\boldsymbol{A}$*, then* $Rad(\boldsymbol{B}) \subseteq Rad(\boldsymbol{A})$.

(b) *If* $(\boldsymbol{A}_i)_{i\in I}$ *is a family of bounded residuated lattices, then* $Rad(\prod_{i\in I} \boldsymbol{A}_i) \subseteq \prod_{i\in I} Rad(\boldsymbol{A}_i)$.

(c) *If* $h\colon \boldsymbol{A} \to \boldsymbol{B}$ *is an homomorphism of bounded residuated lattices, then* $h[Rad(\boldsymbol{A})] \subseteq Rad(\boldsymbol{B})$.

(d) *For any i-filter* F *of a bounded residuated lattice* $\boldsymbol{A}$*,* $Rad(\boldsymbol{A})/F \subseteq Rad(\boldsymbol{A}/F)$.

Some authors claim without proof, that in item (b) of above lemma the reverse inclusion is also satisfied, and in fact one has an equality (see [9, Proposition 2.3] for example). However this claim is not true, as the next lemma shows.

LEMMA 2.7. *There is a family* $(\boldsymbol{A}_n)_{n>1}$ *of bounded residuated lattices such that* $\prod_{n>1} Rad(\boldsymbol{A}_n) \nsubseteq Rad(\prod_{n>1} \boldsymbol{A}_n)$.

Proof. For $n > 1$, we consider the bounded residuated lattice $\widehat{\boldsymbol{L_{n+1}}}$ described in [7, Sect. 5.1, p. 1131]. From the results given in [7], one deduces:

- For each $n > 1$ we can choose $a_n \in Rad(\widehat{\boldsymbol{L_{n+1}}})$ such that $n.a_n \neq \top$.

Then $\overline{a} = (a_n)_{n>1} \in \prod_{n>1} Rad(\widehat{\boldsymbol{L_{n+1}}})$. Moreover, for any $k > 1$, $k.a_k \neq \top^{\widehat{\boldsymbol{L_{k+1}}}}$, and so for any $k > 0$, $k{\cdot}\overline{a} \neq \top^{\prod_{n>1}\widehat{\boldsymbol{L_{n+1}}}}$. Thus $\overline{a} \notin Rad(\prod_{n>1}\widehat{\boldsymbol{L_{n+1}}})$. This closes the proof. ■

Given an integer $k > 0$, we say that a bounded residuated lattice $\boldsymbol{A}$ is *k-radical* provided that

$$Rad(\boldsymbol{A}) = \{a \in A : \forall n > 0,\ k.a^n = \top\}. \tag{2.1}$$

A variety V is called *k-radical* whenever all its members are k-radical.

For any $m > 0$, $\mathsf{E_m}$ is m-radical because for $\boldsymbol{A} \in \mathsf{E_m}$, $Rad(\boldsymbol{A}) = \{a \in A : m.a^m = \top\}$; moreover, it is shown in [7, Lemma 1.8] that for each $k > 0$, $\mathsf{WL_k}$ is a k-radical variety. In particular since $\mathsf{MTL} \subseteq \mathsf{WL_2}$, then MTL is a 2-radical variety.

For k-radical varieties we can improve item (b) of Lemma 2.6.

Lemma 2.8. *Let k be a positive integer. If $(\boldsymbol{A}_i)_{i\in I}$ is a family of k-radical bounded residuated lattices, then $Rad(\prod_{i\in I}\boldsymbol{A}_i) = \prod_{i\in I} Rad(\boldsymbol{A}_i)$.*

Proof. Let $a = (a_i)_{i\in I} \in \prod_{i\in I} A_i$, then since the operations in $\prod_{i\in I}\boldsymbol{A}_i$ are defined componentwise, we have that for any $n > 0$, $k.a^n = \top$ if and only if for all $i \in I$ $k.a_i^n = \top_i$, hence $a \in Rad(\prod_{i\in I}\boldsymbol{A}_i)$ if and only if $a_i \in Rad(\boldsymbol{A}_i)$ for all $i \in I$. Thus $\prod_{i\in I} Rad(\boldsymbol{A}_i) = Rad(\prod_{i\in I}\boldsymbol{A}_i)$. ■

3. Simple and Semisimple Bounded Residuated Lattices

Given a class K of algebras we represent by $\mathsf{K_{FSI}}$ the class of its finitely subdirectly irreducible members, and we represent by $\mathsf{K_{SI}}$ the class of its subdirectly irreducible members. Clearly $\mathsf{K_{SI}} \subseteq \mathsf{K_{FSI}}$. Every variety is generated by its (finitely) subdirectly irreducible members. After [14, Proposition 1.4], we know that $\mathsf{BRL_{FSI}}$ *is the class of bounded residuated lattices with* $\top$ *join irreducible.*

A special case of subdirectly irreducible algebras are simple algebras. An algebra is called *simple* provided that it only has trivial congruence relations, namely the identity and the universal. Hence a bounded residuated lattice is simple if and only if $\{\top\}$ is its unique proper i-filter, or equivalently, $\{\top\}$

is maximal i-filter. Therefore a proper i-filter F of a bounded residuated lattice $\boldsymbol{A}$ is maximal if and only if the quotient algebra $\boldsymbol{A}/F$ is simple.

Algebras representable as subdirect product of simple algebras are called *semisimples*. Then an algebra is semisimple if and only if the intersection of its proper maximal congruences is the identity. Therefore a bounded residuated lattice $\boldsymbol{A}$ is semisimple if and only if $Rad(\boldsymbol{A}) = \{\top\}$. Notice that for any bounded residuated lattice $\boldsymbol{A}$, the quotient algebra $\boldsymbol{A}/Rad(\boldsymbol{A})$ is always semisimple.

From now on V represents a variety of bounded residuated lattices, $\mathsf{V_S}$ represents the class of its simple members, and $\mathsf{V_{SS}}$ represents the class of its semisimple members. It follows from Lemma 2.6 that bounded residuated lattices are hereditarily semisimple, then $\mathsf{V_{SS}}$ is closed under isomorphic images, subalgebras and products. Moreover, since $\mathsf{V_{SS}} \subseteq SP(\mathsf{V_S})$ and $\mathsf{V_S} \subseteq \mathsf{V_{SS}}$, we have that $\mathsf{V_S}$ and $\mathsf{V_{SS}}$ generate the same variety $HSP(\mathsf{V_S})$. Notice that $HSP(\mathsf{V_S})_{\mathsf{S}} = \mathsf{V_S}$, and $HSP(\mathsf{V_S})_{\mathsf{SS}} = \mathsf{V_{SS}}$.

It is shown in [13] that BRL, as a variety, is generated by its finite simple members (see also [8]), but $\mathsf{BRL_{SS}} \neq \mathsf{BRL}$. On the other hand, for any $m > 0$ it is straightforward to see that $\mathsf{E_{mSS}}$ is the variety $\mathsf{EM_m}$ of bounded residuated lattices given by the equation:

(MEm) $x \vee \neg x^m \approx \top$.

Notice that $\mathsf{EM_1}$ is the variety of Boolean algebras.

We say that a variety V *is semisimple* provided that all its members are semisimple, that is, $\mathsf{V} = \mathsf{V_{SS}}$. The next result follows from those given in [8,12] (cf. [16]).

THEOREM 3.1. *Each variety of bounded residuated lattices* V *satisfies:*

(a) *if* V *is semisimple, then* $\mathsf{V} \subseteq \mathsf{E_m}$ *for some* $m > 0$*, and*

(b) V *is semisimple if and only* $\mathsf{V} \subseteq \mathsf{EM_m}$*, for some* $m > 0$.

As a consequence of the above theorem, we obtain the following corollary

COROLLARY 3.2. *For every variety* V *of bounded residuated lattices, the following are equivalent:*

(i) $\mathsf{V_{SS}}$ *is a variety.*

(ii) $\mathsf{V_S} \subseteq \mathsf{E_m}$*, for some* $m > 0$.

(iii) $\mathsf{V_{SS}} \subseteq \mathsf{E_m}$*, for some* $m > 0$.

(iv) $\mathsf{V_{SS}} \subseteq \mathsf{EM_m}$*, for some* $m > 0$.

(v) $\mathsf{V_{SS}} = HSP(\mathsf{V_S})$.

Proof. $(i) \Rightarrow (ii)$, $(iii) \Rightarrow (iv)$ follow from Theorem 3.1, $(ii) \Rightarrow (iii)$ is true because $\mathsf{V_{SS}} \subseteq SP(\mathsf{V_S})$, and $(v) \Rightarrow (i)$ is trivial.
$(iv) \Rightarrow (v)$: Assume that there is $m > 0$ such that $\mathsf{V_{SS}} \subseteq \mathsf{EM_m}$. Then $HSP(\mathsf{V_S}) = HSP(\mathsf{V_{SS}}) \subseteq \mathsf{EM_m}$. Hence every algebra in $HSP(\mathsf{V_S})$ is semisimple, and so $HSP(\mathsf{V_S}) \subseteq \mathsf{V_{SS}}$. The other inclusion always holds. ■

Our next task is to analyze varieties of bounded residuated lattices whose semisimple algebras form a variety. For this purpose we need some results about free algebras. For every set of variables X, $\boldsymbol{F}_{\mathsf{V}}(\overline{X})$ denotes the $|X|$-free algebra in V with set of free generators $\overline{X} = \{\overline{x} : x \in X\}$. The next lemma follows from Lemma 4.1 and Theorem 4.3 of [5]. We include a direct proof of item (b) obtained from the one given for Theorem 4.3 in [5], by taking $\boldsymbol{A} \in \mathsf{V_{SS}}$ in place of $\boldsymbol{S} \in \mathsf{V_S}$.

Lemma 3.3. *For each non-empty set of variables X, the following properties hold true:*

(a) *$\theta(Rad(\boldsymbol{F}_{\mathsf{V}}(\overline{X})))$ is a fully invariant congruence on $\boldsymbol{F}_{\mathsf{V}}(\overline{X})$, and $\boldsymbol{F}_{\mathsf{V}}(\overline{X})/Rad(\boldsymbol{F}_{\mathsf{V}}(\overline{X})) \in \mathsf{V_{SS}}$;*

(b) *$\boldsymbol{F}_{\mathsf{V}}(\overline{X})/Rad(\boldsymbol{F}_{\mathsf{V}}(\overline{X}))$ is the $|X|$-free algebra in $\mathsf{V_{SS}}$, with $\overline{X}/Rad(\boldsymbol{F}_{\mathsf{V}}(\overline{X})) = \{\overline{x}/Rad(\boldsymbol{F}_{\mathsf{V}}(\overline{X})) : x \in X\}$ as set of free generators;*

(c) *$\boldsymbol{F}_{\mathsf{V}}(\overline{X})/Rad(\boldsymbol{F}_{\mathsf{V}}(\overline{X}))$ is the $|X|$-free algebra in $HSP(\mathsf{V_S})$, with $\overline{X}/Rad(\boldsymbol{A})$ as set of free generators.*

Proof. (b): Let X be a set of variables. Then $\boldsymbol{F}_{\mathsf{V}}(\overline{X})/Rad(\boldsymbol{F}_{\mathsf{V}}(\overline{X}))$ is semisimple and it belongs to $\mathsf{V_{SS}}$. Let $\boldsymbol{A} \in \mathsf{V_{SS}}$. Given a mapping $h : \overline{X}/Rad(\boldsymbol{F}_{\mathsf{V}}(\overline{X})) \to A$, we consider $h_0 : \overline{X} \to A$ defined by $h_0(\overline{x}) := h\left(\overline{x}/Rad(\boldsymbol{F}_{\mathsf{V}}(\overline{X}))\right)$. Let $H_0 \colon \boldsymbol{F}_{\mathsf{V}}(\overline{X}) \to \boldsymbol{A}$ be the homomorphism extending h_0. Since $Rad(\boldsymbol{A}) = \{\top\}$, by item (c) of Lemma 2.6, $Rad(\boldsymbol{F}_{\mathsf{V}}(\overline{X})) \subseteq \top/(\ker H_0)$. Hence there is a homomorphism H from $\boldsymbol{F}_{\mathsf{V}}(\overline{X})/Rad(\boldsymbol{F}_{\mathsf{V}}(\overline{X}))$ into $\boldsymbol{A}$ such that for any $\overline{t} \in \boldsymbol{F}_{\mathsf{V}}(\overline{X})$, $H(\overline{t}/Rad(\boldsymbol{F}_{\mathsf{V}}(\overline{X})) = H_0(\overline{t})$ and clearly this homomorphism extends h. Thus $\boldsymbol{F}_{\mathsf{V}}(\overline{X})/Rad(\boldsymbol{F}_{\mathsf{V}}(\overline{X}))$ is, up isomorphism, the $|X|$-free algebra in $\mathsf{V_{SS}}$. ■

Now we can give the main result of this section.

Theorem 3.4. *Let V be a variety of bounded residuated lattices. Then $\mathsf{V_{SS}}$ is a variety if and only if there exists $m > 0$ such that*

(SS) *for each $n > 0$, there is $k_n > 0$ such that $\mathsf{V} \models k_n.(x \vee \neg x^m)^n \approx \top$.*

Proof. Assume that $\mathsf{V_{SS}}$ is a variety. Then there exists $m > 0$ such that $\mathsf{V_{SS}} \models x \vee \neg x^m \approx \top$. Let $\boldsymbol{F}$ be a free algebra in V with at least one free generator $\overline{x}$. Since $\boldsymbol{F}/Rad(\boldsymbol{F}) \in \mathsf{V_{SS}}$, then $\boldsymbol{F}/Rad(\boldsymbol{F}) \models x \vee \neg x^m \approx \top$. Hence $(\overline{x} \vee \neg \overline{x}^m)/Rad(\boldsymbol{F}) = \top/Rad(\boldsymbol{F})$, and so $\overline{x} \vee \neg \overline{x}^m \in Rad(\boldsymbol{F})$, hence:

- for any $n > 0$ there is $k_n > 0$ such that $k_n.(\overline{x} \vee \neg \overline{x}^m)^n = \top$.

Since $\overline{x}$ is a free generator we have that (SS) holds.
Conversely, assume that (SS) holds. Let $\boldsymbol{F}$ be a free algebra in V, with set of free generators $\overline{X} \neq \emptyset$. Then, by Lemma 3.3, $\boldsymbol{F}/Rad(\boldsymbol{F})$ is a free algebra of the variety generated by $\mathsf{V_{SS}}$, and if $\overline{x} \in \overline{X}$. Then $\overline{x}/Rad(\boldsymbol{F})$ is a free generator of $\boldsymbol{F}/Rad(\boldsymbol{F})$. By (SS), $\overline{x} \vee \neg \overline{x}^m \in Rad(\boldsymbol{F})$, and so $\overline{x}/Rad(\boldsymbol{F}) \vee \neg (\overline{x}/Rad(\boldsymbol{F}))^m = \top/Rad(\boldsymbol{F})$. Hence $\mathsf{V}_{SS} \models x \vee \neg x^m \approx \top$ and then $\mathsf{V_{SS}} \subseteq \mathsf{EM_m}$. Thus, by Corollary 3.2, $\mathsf{V_{SS}}$ is a variety. ■

Observe that, by (h) and (j) of Lemma 1.3, in (SS) the sequence $\mathsf{k} = (k_n)_{n>0}$ can be taken increasing.

Theorem 3.4 shows how we can give an equational presentation of varieties of bounded residuated lattices whose semisimple members form a variety. However, these presentations depend of infinitely many parameters. For any $m > 0$ and any increasing sequence of positive integers $\mathsf{k} = (k_n)_{n>0}$ we consider the variety

$$\mathsf{V_{m,k}} = \{\boldsymbol{A} \in \mathsf{BRL} : \forall n > 0, \boldsymbol{A} \models k_n.(x \vee \neg x^m)^n \approx \top\}.$$

Then Theorem 3.4 can be reformulated as follows

Corollary 3.5. *Let V be a variety of bounded residuated lattices. Then $\mathsf{V_{SS}}$ is a variety if and only if there are $m > 0$ and a increasing sequence of positive integers k such that $\mathsf{V} \subseteq \mathsf{V_{m,k}}$. In this case $\mathsf{V_{SS}} \subseteq \mathsf{EM_m}$.*

In particular for k-radical varieties we have:

Corollary 3.6. *Let V be a k-radical variety of bounded residuated lattices, then $\mathsf{V_{SS}}$ is a variety if and only if there is $m > 0$ such that for any $n > 0$*

$$\mathsf{V} \models k.(x \vee \neg x^m)^n \approx \top.$$

4. Local and Locally Representable Bounded Residuated Lattices

A bounded residuated lattice $\boldsymbol{A}$ is called *local* provided that it has only one maximal i-filter, that is $Rad(\boldsymbol{A})$ is (the unique) maximal i-filter. Observe this is equivalent to $\boldsymbol{A}/Rad(\boldsymbol{A})$ being simple. Then we have (cf. [9, Proposition 2.4.]):

Theorem 4.1. *A bounded residuated lattice $\boldsymbol{A}$ is local if and only if*

(♠) *for any* $a \in A$, $a \notin Rad(\boldsymbol{A})$ *if and only if there is* $n > 0$ *such that* $a^n = \bot$.

Proof. Assume that $\boldsymbol{A}$ is local. Since $Rad(\boldsymbol{A})$ is the unique maximal i-filter of $\boldsymbol{A}$, then for any $a \in A$: $a \notin Rad(\boldsymbol{A})$ iff $\langle a \rangle = A$ iff $\bot \in \langle a \rangle$ iff there is $n > 0$ such that $a^n = \bot$.
Conversely, assume $\boldsymbol{A}$ satisfies (♠). Then for any $a \in A$, if $a \notin Rad(\boldsymbol{A})$, then $\bot \in \langle Rad(\boldsymbol{A}) \cup \{a\} \rangle$. Thus $Rad(\boldsymbol{A})$ is maximal i-filter. ■

As immediate consequences we have:

Corollary 4.2. *Let* F *be a proper i-filter of a bounded residuated lattice* $\boldsymbol{A}$. *Then* $\boldsymbol{A}/F$ *is local if and only if* F *is contained in only one maximal i-filter.*

Corollary 4.3. *Every homomorphic image of a local bounded residuated lattice is also local.*

We say that an algebra $\boldsymbol{A}$ in BRL is *locally representable* provided that $\boldsymbol{A}$ is isomorphic to a subdirect product of local algebras. Then, a variety is called *locally representable* if all its members are locally representable. From subdirect product representation in varieties we deduce:

Lemma 4.4. *A variety* V *of bounded residuated lattices is locally representable if and only if any algebra in* V_{SI} *is a local algebra.*

For k-radical varieties we have:

Theorem 4.5. *Let* $\boldsymbol{A} \in \mathsf{BRL}_{\mathsf{FSI}}$ *be* k*-radical. Then* $\boldsymbol{A}$ *is local if and only if*

(♣) *for any* $a \in A$ *there is* $n_a > 0$ *such that* $k.a \vee \neg a^{n_a} = \top$.

Proof. Assume that $\boldsymbol{A}$ is local. If $a \in A$ is such that $k.a \neq \top$, then $a \notin Rad(\boldsymbol{A})$, and there is n_a such that $a^{n_a} = \bot$, so $\neg a^{n_a} = \top$. Hence for any $a \in A$ there is n_a such that $k.a \vee \neg a^{n_a} = \top$, and (♣) holds.
Conversely, assume (♣). If $a \notin Rad(\boldsymbol{A})$, then there is $m > 0$ such that $k.a^m \neq \top$, hence since $\top$ is join irreducible, by (♣) there is $n = n_{a^m} > 0$ such that $\neg a^{mn} = \neg(a^m)^n = \top$, and so $a^{mn} = \bot$, hence, by Theorem 4.1, $\boldsymbol{A}$ is local. ■

It is well known that the variety MTL is locally representable, because any totally ordered bounded residuated lattice is local. Moreover, we have:

Lemma 4.6. *If* $k > 0$, *then:*

(*a*) *any* $\boldsymbol{A} \in \mathsf{WL}_{\mathsf{k\,FSI}}$ *is local,*

(b) *any algebra in* $\mathsf{WL_k}$ *is locally representable.*

PROOF. (a): By item (i) in Lemma 1.3, (WLk) can be rewritten as $k.x \vee \neg x^k \approx \top$, and so any member of $\mathsf{WL_{k\,FSI}}$ satisfies (♣). Then by Theorem 4.5 $\boldsymbol{A}$ is local.

(b) follows from Lemma 4.4, because $\mathsf{WL_{k\,SI}} \subseteq \mathsf{WL_{k\,FSI}}$. ■

We recall that a proper i-filter P of a bounded residuated lattice $\boldsymbol{A}$ is called *prime* provided that for any $a, b \in P$, $a \vee b \in P$ implies $a \in P$ or $b \in P$. It is well known that a proper i-filter F of $\boldsymbol{A}$ is prime if and only if $\boldsymbol{A}/F$ is a non trivial finitely subdirectly irreducible.

COROLLARY 4.7. *Each prime i-filter of a member of* $\mathsf{WL_k}$ *is contained in only one maximal i-filter.*

Given an integer $m > 0$, we say that a local bounded residuated lattice $\boldsymbol{A}$ is *m-local*, provided that $\boldsymbol{A}/Rad(\boldsymbol{A}) \in \mathsf{EM_m}$, i.e., $\boldsymbol{A}/Rad(\boldsymbol{A}) \in \mathsf{EM_{mS}}$.

THEOREM 4.8. *A bounded residuated lattice* $\boldsymbol{A}$ *is* m*-local if and only if:*

(**m**♠) *for any* $a \in A$, $a \notin Rad(\boldsymbol{A})$ *iff* $\neg a^m \in Rad(\boldsymbol{A})$.

PROOF. If $\boldsymbol{A}$ is m-local, then $\boldsymbol{A}/Rad(\boldsymbol{A}) \in \mathsf{EM_{mS}}$. Thus

$$\begin{aligned} a \notin Rad(\boldsymbol{A}) &\text{ iff } a/Rad(\boldsymbol{A}) \neq \top/Rad(\boldsymbol{A}) \\ &\text{ iff } \neg\,(a/Rad(\boldsymbol{A}))^m = \top/Rad(\boldsymbol{A}) \\ &\text{ iff } \neg\,a^m/Rad(\boldsymbol{A}) = \top/Rad(\boldsymbol{A}) \\ &\text{ iff } \neg\,a^m \in Rad(\boldsymbol{A}). \end{aligned}$$

Conversely, (**m**♠) implies that $Rad(\boldsymbol{A})$ is maximal i-filter, at the same time, that $\boldsymbol{A}/Rad(\boldsymbol{A})$ satisfies $x \vee \neg x^m \approx \top$. ■

Given a positive integer $m > 0$, we say that a bounded residuated lattice $\boldsymbol{A}$ is *m-locally representable* provided that $\boldsymbol{A}$ is isomorphic to a subdirect product of m-local algebras. Then, a variety is called *m-locally representable* if all its members are m-locally representable.

COROLLARY 4.9. *If* V *is an* m*-locally representable variety of bounded residuated lattices, then* $\mathsf{V_{SS}} \subseteq \mathsf{EM_m}$.

Moreover, from subdirect product representation, we have:

LEMMA 4.10. *A variety* V *of bounded residuated lattices has all its members* m*-locally representable if and only if any algebra in* $\mathsf{V_{SI}}$ *is* m*-local.*

Notice that for any $m > 0$, there are algebras in $\mathsf{E_m}$ which are not locally representable, while $\mathsf{E_{m_{SS}}}$ is the variety $\mathsf{EM_m}$. See the example given at the end of the Sect. 5. We also note that for the variety of Heyting algebras $\mathsf{E_1}$, $\mathsf{E_{1_{SS}}}$ is the variety $\mathsf{EM_1}$ of boolean algebras.

5. Radical Terms

In what follows, we shall consider unary $\{\cdot, \to, \wedge, \vee, \bot, \top\}$-terms, which we call simply *unary terms*. Given a unary term t we write $t(x)$ to indicate that the variable which appears in t is x. If $\boldsymbol{A}$ is a bounded residuated lattice, then for any $a \in A$, $t^{\boldsymbol{A}}(a)$ represents the interpretation of t on $\boldsymbol{A}$ given by the assignment $x \mapsto a$.

Let $t(x)$ be a unary term. We say that a variety V of bounded residuated lattices has $t(x)$ as a *radical term*, or that $t(x)$ *is a radical term for* V, whenever any $\boldsymbol{A} \in \mathsf{V}$ satisfies:

$$Rad(\boldsymbol{A}) = \{a \in A : t^{\boldsymbol{A}}(a) = \top\}. \tag{5.1}$$

Notice that for any $m > 0$ $m.x^m$ is a radical term for $\mathsf{E_m}$, besides $\mathsf{E_{m_{SS}}} = \mathsf{EM_m}$. We also know that any subvariety of BRL having $t(x)$ as a boolean retraction term has $t(x)$ as a radical term. In particular, for any $k > 0$ the variety $\mathsf{V^k}$ given by the equation $k.x^k \vee k.(\neg x^k) \approx \top$ has $k.x^k$ as radical term, see Theorem 5.1 of [7] for details. Actually, the variety $\mathsf{V^k}$ is the greatest subvariety of $\mathsf{WL_k}$ admitting a boolean retraction term ([7, Theorem 5.7]). Moreover, if the variety V has boolean retraction term, then $\mathsf{V_{SS}}$ is the variety of boolean algebras $\mathsf{EM_1}$. In these cases the class of semisimple algebras is a variety.

In general, if $t(x)$ is a radical term for V, then $\mathsf{V_{SS}}$ is the *subquasivariety of* V given by the quasiidentity

$$t(x) \approx \top \Rightarrow x \approx \top. \tag{5.2}$$

In similar way to Lemma 2.8 we can prove the following

Lemma 5.1. (cf. [7, Lemma 3.7]) *If $(\boldsymbol{A}_i)_{i\in I}$ is a family of bounded residuated lattices having $t(x)$ as radical term, then $Rad(\prod_{i\in I} \boldsymbol{A}_i) = \prod_{i\in I} Rad(\boldsymbol{A}_i)$.*

Our next task in to analyze k-radical varieties admitting radical term.

Given a class of algebras K, let $\models_{\mathsf{K}}$ denote the equational consequence relation relative to K, or determined by K, (see [3, Chapter 2] and [8, Page 54]). It is well known that if K is a quasivariety, then $\models_{\mathsf{K}}$ is finitary (compact) in the sense of [3, Chapter 2], that is, for any set of equations $\Sigma \cup \{\varphi \approx \psi\}$ in the algebraic language of K, one has

- $\Sigma \models_{\mathsf{K}} \psi \approx \varphi$ if and only if there is a finite subset $\{\psi_i \approx \varphi_i : 0 < i \leqslant m\}$ of Σ such that $\{\psi_i \approx \varphi_i : 0 < i \leqslant m\} \models_{\mathsf{K}} \psi \approx \varphi$.

Recall that for every class K, $\{\psi_i \approx \varphi_i : 0 < i \leqslant m\} \models_{\mathsf{K}} \psi \approx \varphi$ is equivalent to $\mathsf{K} \models (\psi_1 \approx \varphi_1 \,\&\, \dots \,\&\, \psi_m \approx \varphi_m) \Rightarrow \psi \approx \varphi$.

THEOREM 5.2. *For every variety* V *of* k*-radical bounded residuated lattices, the following are equivalent:*

(*i*) $\mathsf{V_{SS}}$ *is a quasivariety.*

(*ii*) *There is* $r > 0$ *such that* $\mathsf{V_{SS}} \models k.x^r \approx \top \Rightarrow x \approx \top$.

(*iii*) *There is* $r > 0$ *such that for all* $\boldsymbol{A} \in \mathsf{V}$, $Rad(\boldsymbol{A}) = \{a \in A : k.a^r = \top\}$.

(*iv*) *There is* $r > 0$ *such that* $\mathsf{V_{SS}} = \{\boldsymbol{A} \in \mathsf{V} : \boldsymbol{A} \models k.x^r \approx \top \Rightarrow x \approx \top\}$.

PROOF. $(i) \Rightarrow (ii)$: Assume that $\mathsf{V_{SS}}$ is a quasivariety. From the definition of k-radical variety it follows that

$$\{k.x^n \approx \top : n \in \omega\} \models_{\mathsf{V_{SS}}} x \approx \top,$$

then since $\mathsf{V_{SS}}$ is a quasivariety, the operator $\models_{\mathsf{V_{SS}}}$ is finitary, and so there are integers $0 < m$, and $0 \leqslant n_1 < \cdots < n_m$, such that

$$\{k.x^{n_1} \approx \top, \dots, k.x^{n_m} \approx \top\} \models_{\mathsf{V_{SS}}} x \approx \top.$$

For any $\boldsymbol{A} \in \mathsf{V_{SS}}$ and any $a \in \boldsymbol{A}$ $k.a^{n_m} \leqslant \cdots \leqslant k.a^{n_1}$, hence if $r = n_m$, we have

$$\{k.x^r \approx \top\} \models_{\mathsf{V_{SS}}} x \approx \top,$$

and so $\mathsf{V_{SS}} \models k.x^r \approx \top \Rightarrow x \approx \top$.
$(ii) \Rightarrow (iii)$: For any $\boldsymbol{A} \in \mathsf{V}$, $Rad(\boldsymbol{A}) \subseteq \{a \in A : k.a^r = \top\}$ always holds. To see the reverse inclusion, we consider $a \in A$ such that $k.a^r = \top$. Then $k.(a/Rad(\boldsymbol{A}))^r = \top/Rad(\boldsymbol{A})$ and since $\boldsymbol{A}/Rad(\boldsymbol{A}) \in \mathsf{V_{SS}}$, by (ii), we have $a/Rad(\boldsymbol{A}) = \top/Rad(\boldsymbol{A})$. Hence $a \in Rad(\boldsymbol{A})$.
$(iii) \Rightarrow (iv)$: Assume that there exists $r > 0$ such that (iii) holds. Then (iv) follows from the fact that $\boldsymbol{A} \in \mathsf{V_{SS}}$ if and only if $\boldsymbol{A} \in \mathsf{V}$ and $Rad(\boldsymbol{A}) = \{a \in A : k.a^r = \top\} = \{\top\}$. Indeed, $\boldsymbol{A} \in \mathsf{V_{SS}}$ if and only if $\boldsymbol{A} \in \mathsf{V}$ and for any $a \in A$, $k.a^r = \top$ implies $a = \top$. This proves (iv).
$(iv) \Rightarrow (i)$ is trivial. ∎

REMARK 5.3. Observe that if the variety V satisfies (ii) of the above theorem, then for any $l, m > 0$,

$$\mathsf{V_{SS}} \models k.x^r \approx \top \Rightarrow l.x^m \approx \top.$$

Moreover, for any $s \geqslant r$, since $\mathsf{V} \models k.x^s \to k.x^r \approx \top$, we have

$$\mathsf{V_{SS}} \models k.x^s \approx \top \Rightarrow x \approx \top,$$

hence if $r \leqslant k$ we can take $r = k$, and we can assume that $r \geqslant k$, because if $k.x^r$ is radical term for V, then for any $s \geqslant r$, $k.x^s$ is also radical term.

Corollary 5.4. *Let* V *be a* k*-radical variety of bounded residuated lattices. Then* V *has a radical term if and only if* $\mathsf{V_{SS}}$ *is a quasivariety. In this case the radical term is given by* $k.x^r$*, for some* $r \geqslant k$*.*

Proof. If V has a radical term, then $\mathsf{V_{SS}}$ is a quasivariety by (5.1), and by Theorem 5.2, $\mathsf{V_{SS}}$ is a quasivariety if and only if there exists $r > 0$ such that $k.x^r$ is a radical term for V. ■

The above results can be reformulated as follows:

Corollary 5.5. *For any* $k > 0$*, and any variety* V *of bounded residuated lattices, the following are equivalent:*

(i) *There is* $r \geqslant k$ *such that* $k.x^r$ *is a radical term for* V*.*

(ii) V *is* k*-radical and* $\mathsf{V_{SS}}$ *is a quasivariety.*

Lemma 5.6. *Let* V *be a* k*-radical variety of bounded residuated lattices. If* $\mathsf{V_{SS}} \subseteq \mathsf{EM_m}$ *then* $k.x^m$ *is a radical term for* V*.*

Proof. Consider $\boldsymbol{A} \in \mathsf{V}$, and $a \in A$. We show that $a \in Rad(\boldsymbol{A})$ if and only if $k.a^m = \top$.

- If $a \in Rad(\boldsymbol{A})$, then $k.a^m = \top$, because V is k-radical.
- If $k.a^m = \top$, then $(\neg a^m)^k \to \bot = \top$, and hence $\langle \neg a^m \rangle = A$. Thus $\neg a^n \notin M$ for any maximal i-filter M as it is proper. By taking into account that $\boldsymbol{A}/M \models x \vee \neg x^m \approx \top$, we have $a \in M$. Therefore $a \in Rad(\boldsymbol{A})$. ■

For subvarieties of $\mathsf{WL_k}$ we can improve Theorem 5.2.

Theorem 5.7. *Assume that* V *is a subvariety of* $\mathsf{WL_k}$*,* $k > 0$*, then the following conditions are equivalent:*

(i) V *admits radical term.*

(ii) $\mathsf{V_{SS}}$ *is a variety.*

Proof. It is enough to see that (i) implies (ii). Assume that $k.x^r$ is a radical term for V. By Corollaries 3.2, 4.9 and Lemma 4.10, to show that $\mathsf{V_{SS}}$ is a variety it suffices to prove that any algebra in $\mathsf{V_{SI}}$ is kr-local. If

$\boldsymbol{A} \in \mathsf{V_{SI}}$ and $a \notin Rad(\boldsymbol{A})$, then $k.a^r \neq \top$ and since $\top$ is join irreducible in $\boldsymbol{A}$, $\neg(a^{rk}) = k.\neg a^r = \top \in Rad(\boldsymbol{A})$, by Theorem 4.8, $\boldsymbol{A}$ is kr-local. ■

REMARK 5.8. It follows from the above theorem that if a variety $\mathsf{V} \subseteq \mathsf{WL_k}$ has $k.x^r$ as radical term, then $\mathsf{V_{SS}} \subseteq \mathsf{EM_{kr}}$. However, it may happen that $\mathsf{V_{SS}} \subseteq \mathsf{EM_m}$ for some $m < kr$.

COROLLARY 5.9. *Let* V *be a subvariety of* $\mathsf{WL_k}$*, then* V *has a radical term if and only if* $\mathsf{V_{SS}}$ *is a variety. In this case the term is given by* $k.x^r$*, for some* $r \geqslant k$*.*

There are well known subvarieties of $\mathsf{WL_2}$ whose semisimple members do not form a variety. The variety of BL-algebras is the subvariety BL of MTL given by the equation $x \cdot (x \to y) \approx y \cdot (y \to x)$, and the variety MV of MV-algebras is the subvariety of BL given by the equation $\neg\neg x \approx x$. It is well known (see [6]) that for any $\boldsymbol{A} \in \mathsf{BL}$, $\boldsymbol{A}/Rad(\boldsymbol{A}) \in \mathsf{MV}$, hence $\mathsf{BL_{SS}} = \mathsf{MV_{SS}}$, however $\mathsf{MV_{SS}} \subsetneq HSP(\mathsf{MV_{SS}}) = \mathsf{MV}$. This means that neither $\mathsf{BL_{SS}}$ or $\mathsf{MV_{SS}}$ is a variety. Hence by Theorem 5.7, both BL and MV are subvarieties of $\mathsf{WL_2}$ that do not admit radical term.

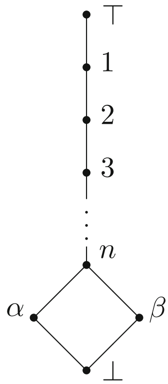
An Example[1]

For $n > 1$, let $A_n = \{r \in \omega : 1 \leqslant r \leqslant n\} \cup \{\bot, \top, \alpha, \beta\}$. Consider the algebra $\boldsymbol{A}_n = \langle A_n; \cdot, \to, \vee, \wedge, \bot, \top\rangle$[2] of type $(2, 2, 2, 2, 0, 0)$ such that $\langle A_n; \vee, \wedge, \bot, \top\rangle$ is the bounded lattice given by the diagram depicted in Figure 1.
Moreover, $\cdot$ is defined by the following prescription (cf. table in Figure 1):

- for any $x, y \in A_n$, $x \cdot y = y \cdot x$, $x \cdot \top = x$, and $x \cdot \bot = \bot$.
- for any $1 \leqslant r, s \leqslant n$, $r \cdot s = \min\{r + s, n\}$,
- for any $1 \leqslant r \leqslant n$ and $x \in \{\alpha, \beta\}$, $r.x = x = x \cdot x$,
- $\alpha \cdot \beta = \bot$.

[1] This example has been obtained with the help of Prover9-Mace4 available in [15].

[2] In fact, this algebra is the ordinal sum of the four element boolean algebra and the $n+1$-element MV-chain, see [2].



$\cdot$	$\bot$	$\top$	1	2	3	$\cdots$	n	α	β
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\cdots$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\top$	1	2	3	$\cdots$	n	α	β
1	$\bot$	1	2	3	4	$\cdots$	n	α	β
2	$\bot$	2	3	4		$\cdots$	n	α	β
$\cdots$			$\cdots$			$\cdots$			
n	$\bot$	n	n	n	n	$\cdots$	n	α	β
α	$\bot$	α	α	α	α	$\cdots$	α	α	$\bot$
β	$\bot$	β	β	β	β	$\cdots$	β	$\bot$	β

Figure 1. The lattice reduct of $\boldsymbol{A}_n$, and table of its monoid operation

It is easy to prove that $\cdot$ distributes $\vee$, and then $\cdot$ admits residual which we represent by $\rightarrow$. One can check that $\rightarrow$ is given by the following table

$\rightarrow$	$\bot$	$\top$	1	2	3	$\cdots$	n	α	β
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\cdots$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\top$	1	2	3	$\cdots$	n	α	β
1	$\bot$	$\top$	$\top$	1	2	$\cdots$	$n-1$	α	β
2	$\bot$	$\top$	$\top$	2	3	$\cdots$	$n-2$	α	β
$\cdots$			$\cdots$			$\cdots$			
n	$\bot$	$\top$	$\top$	$\top$	$\top$	$\cdots$	$\top$	α	β
α	β	$\top$	$\top$	$\top$	$\top$	$\cdots$	$\top$	$\top$	β
β	α	$\top$	$\top$	$\top$	$\top$	$\cdots$	$\top$	α	$\top$

Notice that $\boldsymbol{A}_1$ is a Heyting algebra. It follows from the definition that for any $n > 0$, $\boldsymbol{A}_n \in \mathsf{E_n}$. Moreover:

- if $k > 0$, then $\alpha^k = \alpha > \bot$. Therefore $\boldsymbol{A}_n \notin \mathsf{EM_m}$ for each $m > 0$,
- for any $a \in A_n$, $a^n = a^{n+1}$, however $1^{n-1} = n-1 > 1^n = n$. Thus $\boldsymbol{A}_n \in \mathsf{E_n} \smallsetminus \mathsf{E_{n-1}}$,
- for any $k > 0$, $k.\alpha = \alpha$ and $k.\neg\alpha = k.\beta = \beta$, and so $k.\alpha \vee k.(\neg\alpha) = n \neq \top$. Hence $\boldsymbol{A}_n \notin \bigcup_{k>0} \mathsf{WL_k}$.
- $\boldsymbol{A}_n$ is subdirectly irreducible, but it is not local, because $Rad(\boldsymbol{A})$ is not maximal, and $\mathsf{E_n}$ is not locally representable.

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