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Congruence Lattices of Semilattices with Operators

Abstract. The duality between congruence lattices of semilattices, and algebraic subsets of an algebraic lattice, is extended to include semilattices with operators. For a set G of operators on a semilattice S, we have $\mathrm{Con}(S,+,0,G) \cong^d \mathrm{S_p}(L,H)$, where L is the ideal lattice of S, and H is a corresponding set of adjoint maps on L. This duality is used to find some representations of lattices as congruence lattices of semilattices with operators. It is also shown that these congruence lattices satisfy the Jónsson–Kiefer property.

Keywords: Congruence lattice, Semilattice, Operator, Duality.

An *operator* on a join semilattice $(S,+,0)$ is a function $f : S \to S$ such that $f(x+y) = f(x) + f(y)$ and $f(0) = 0$, i.e., an endomorphism of S. A *semilattice with operators* (SLO) is an algebra $(S,+,0,G)$ where $(S,+,0)$ is a join semilattice and G is a set of operators on S. In this paper we provide a duality for SLOs, and use it to establish representations for certain (distributive, algebraic) lattices as congruence lattices of SLOs.

Our motivation for this study is the theorem of Adaricheva and Nation [1] that for any quasivariety $\mathcal{K}$, the lattice of quasi-equational theories $\mathrm{QTh}(\mathcal{K})$ is isomorphic to the congruence lattice of a semilattice with operators. The semilattice in this case consists of the compact congruences of the free algebra $\mathcal{F}_{\mathcal{K}}(\omega)$, and the operators are derived from the endomorphisms of the free algebra.

Recall that a subset X contained in a complete lattice L is an *algebraic subset* if it contains 1, is closed under arbitrary meets, and is closed under nonempty directed joins. The lattice of all algebraic subsets of L is denoted $\mathrm{S_p}(L)$.

The lattice of quasi-equational theories is dually isomorphic to the lattice $L_q(\mathcal{K})$ of subquasivarieties of $\mathcal{K}$. The characterization theorem of Gorbunov and Tumanov [8] says that $L_q(\mathcal{K})$ is isomorphic to the lattice $\mathrm{S_p}(L,R)$ of all algebraic subsets of an algebraic lattice L that are closed with respect to a distributive binary relation R. The lattice L in this case is again the

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congruence lattice of $\mathcal{F}_{\mathcal{K}}(\omega)$, and the relation is derived from the isomorphism relation, or alternately, the embedding relation. See Section 5.2 of Gorbunov [7]; also cf. Hoehnke [9]. We improve this type of representation by replacing R with a set of adjoint operators which, when regarded as relations, satisfy properties stronger than Gorbunov's.

The basic result, giving the duality between congruence lattices of semilattices (without operators) and lattices of algebraic sets, is Theorem 1 due to Fajtlowicz and Schmidt [4], building on [5,12,13].

Theorem 1. *For any join semilattice S with* 0*,*

$$\mathrm{Con}(S,+,0) \cong^d \mathrm{S_p}(\mathcal{I}(S))$$

where $\mathcal{I}(S)$ denotes the lattice of (nonempty) ideals of S.

The duality for SLOs requires the following definition. An *algebraic operator* on a complete lattice L is a function $h : L \to L$ such that $h(1) = 1$ and h preserves arbitrary meets and directed joins. When H is a set of algebraic operators on L, we say that a subset $X \subseteq L$ is *H-closed* if $h(x) \in X$ for all $h \in H$ and $x \in X$. Let $\mathrm{S_p}(L, H)$ denote the set of all H-closed algebraic subsets of L.

Our main result, Theorem 14, can be stated thusly.

Theorem 2. *The following are equivalent for a lattice K.*

1. *$K \cong \mathrm{Con}(S, +, 0, G)$ for some semilattice with operators.*
2. *$K \cong^d \mathrm{S_p}(L, H)$ for some algebraic lattice L and set H of algebraic operators on L.*

Note that congruence lattices of semilattices are algebraic and meet semidistributive (Papert [11]). These properties are of course inherited by congruence lattices of SLOs.

In the latter sections, we use the duality to obtain representations, focusing on distributive lattices K that are isomorphic to $\mathrm{S_p}(L, H)$ for some chain L. Even this situation presents difficulties in the infinite case. Applications include Theorems 15 and 17, paraphrased below, where $\mathcal{O}(P)$ denotes the lattice of order ideals of the ordered set P.

- If P is a countable ordered set with the property that $\downarrow x$ is finite for every $x \in P$, then there is a semilattice with operators S such that $\mathcal{O}(P) \cong \mathrm{Con}(S, +, 0, G)$.
- For any ordered set P, there is a semilattice with operators such that the linear sum $\mathcal{O}(P) + 1 \cong \mathrm{Con}(S, +, 0, G)$.

On the other hand, while there are plenty of distributive, algebraic lattices that we cannot yet represent, it still may be that every such lattice is isomorphic to the congruence lattice of an SLO.

Another application shows that congruence lattices of SLOs satisfy the Jónsson-Kiefer property (Theorem 21). This demonstrates that some nondistributive, but meet semidistributive, algebraic lattices cannot be represented as congruence lattices of SLOs.

It is interesting to note that finite SLOs themselves are dualizable, as shown in Davey et al. [3].

1. Adjoints: The Finite Case

In this section, we extend the duality

$$\mathrm{Con}(S, +, 0) \cong^d \mathrm{Sub}(S, \wedge, 1)$$

for finite semilattices to SLOs. For simplicity, let us consider a finite semilattice with one operator: $S = \langle S, +, 0, g\rangle$. The extension to semilattices with a monoid of operators is straightforward.

We begin by recalling the general theory of adjoints on finite semilattices. A finite join semilattice with 0 is a lattice, with the naturally induced meet operation. Thus a finite lattice S can be regarded as a semilattice in two ways, either $S = \langle S, +, 0\rangle$ or $S = \langle S, \wedge, 1\rangle$.

For finite semilattices S, T and a $(+, 0)$-homomorphism $g : S \to T$, define the *adjoint* $h : T \to S$ by

$$h(t) = \sum\{s \in S : g(s) \le t\}.$$

Thus $g(s) \le t$ if and only if $s \le h(t)$. The forward direction of this is clear. For the reverse, assume $s_0 \le h(t)$. Then $g(s_0) \le gh(t) = \sum\{g(s) : g(s) \le t\} \le t$, as desired.

As immediate consequences we get, for $s \in S$ and $t \in T$,

- $gh(t) \le t$
- $hg(s) \ge s$.

We can think of $h(t)$ as the largest element that g maps into the ideal $\downarrow t$, and in particular it must be the largest element of its $\ker g$-class.

LEMMA 3. *The adjoint h of an operator satisfies $h(1) = 1$, and h preserves meets: $h(x \wedge y) = h(x) \wedge h(y)$.*

Proof. The map h is order-preserving, so $h(x \wedge y) \leq h(x) \wedge h(y)$. On the other hand, because g preserves order, $g(h(x) \wedge h(y)) \leq gh(x) \leq x$ and $g(h(x) \wedge h(y)) \leq gh(y) \leq y$, whence $g(h(x) \wedge h(y)) \leq x \wedge y$. It follows that $h(x) \wedge h(y) \leq h(x \wedge y)$. ■

Let us denote the adjoint of g by $h = \hat{g}$. In this article, when there is only one operator, we continue to use just g and h.

2. Duality: The Finite Case

Now we recall the duality theorem for congruence lattices of finite semilattices without operators from [5].

Theorem 4. *Let $S = \langle S, +, 0 \rangle$ be a finite join semilattice with 0. Then* $\mathrm{Con}(S, +, 0) \cong^d \mathrm{Sub}(S, \wedge, 1)$ *via the maps*

- $\sigma : \mathrm{Con}(S, +, 0) \to \mathrm{Sub}(S, \wedge, 1)$ *where $\sigma(\theta)$ is the set of all maximal elements of θ-classes,*
- $\rho : \mathrm{Sub}(S, \wedge, 1) \to \mathrm{Con}(S, +, 0)$ *where*

$$\rho(U) = \{(x, y) \in S^2 : \forall u \in U \ (x \leq u \Leftrightarrow y \leq u)\}.$$

It is convenient to use the notation "$x \leq y \mod \theta$" to mean $x + y \ \theta \ y$. Note that $x \, \theta \, y$ if and only if both $x \leq y \mod \theta$ and $y \leq x \mod \theta$ hold.

Lemma 5. *Let $\langle \theta, U \rangle$ be a pair with $\theta = \rho(U)$ and $U = \sigma(\theta)$.*

- *$y \in U$ iff $x \leq y \mod \theta$ implies $x \leq y$.*
- *$x \leq y \mod \theta$ iff $y \leq u$ implies $x \leq u$ for all $u \in U$.*

This lemma provides the central part of the proof of Theorem 4.

Now we turn to SLOs. Of course, $\mathrm{Con}(S, +, 0, g)$ is a sublattice of $\mathrm{Con}(S, +, 0)$, and $\mathrm{Sub}(S, \wedge, 1, h)$ is a sublattice of $\mathrm{Sub}(S, \wedge, 1)$. We just have to be sure we get the appropriate sublattices.

Lemma 6. *Let $\langle \theta, U \rangle$ be a pair with $\theta = \rho(U)$ and $U = \sigma(\theta)$. Let g be an operator on $\langle S, +, 0 \rangle$ and let $h = \hat{g}$ be its adjoint. Then θ respects g if and only if U is closed under h.*

Proof. Assume that θ is a semilattice congruence that respects g, and let $z \in U = \sigma(\theta)$. To see that $h(z) \in U$, assume $s \leq h(z) \mod \theta$. Then $g(s) \leq gh(z) \leq z \mod \theta$. Since $z \in U$, this implies $g(s) \leq z$, whence $s \leq h(z)$, as desired.

Conversely, assume that U is h-closed, and let $x\,\theta\,y$. Then $x \leq y \mod \theta$, so that by Lemma 5, we have $y \leq u$ implies $x \leq u$ whenever $u \in U$. But then $g(y) \leq u$ implies $y \leq h(u) \in U$, wherefore $x \leq h(u)$ and $g(x) \leq u$. Thus $g(x) \leq g(y) \mod \theta$. Symmetrically $g(y) \leq g(x) \mod \theta$, and hence g respects θ. ■

This allows us to conclude:

THEOREM 7. *If S is a finite semilattice with an operator g, and $h = \hat{g}$, then*

$$\mathrm{Con}(S, +, 0, g) \cong^d \mathrm{Sub}(S, \wedge, 1, h).$$

More generally, if G is a monoid of operators on a semilattice S and $H = \widehat{G} = \{\hat{g} : g \in G\}$, then

$$\mathrm{Con}(S, +, 0, G) \cong^d \mathrm{Sub}(S, \wedge, 1, H).$$

3. Application: Representing Finite Distributive Lattices

The duality of Theorem 7 gives us an easy representation of finite distributive lattices. Theorem 8 can also be derived from a result of Tumanov [14], when combined with Adaricheva and Nation [1].

THEOREM 8. *For every finite distributive lattice D, there is a finite semilattice with operators S such that $D \cong \mathrm{Con}(S, +, 0, G)$.*

PROOF. As usual, we view D as the lattice of order ideals of an ordered set P. Let $\sqsubseteq$ be a linear extension of the order $\leq$ on P, so that $x \leq y$ implies $x \sqsubseteq y$, and form the chain $C_0 = \langle P, \sqsubseteq \rangle$. Add a new top element T to C_0, forming $C = C_0 \cup \{T\}$. Since C is a chain, every subset is a meet subsemilattice, and every order-preserving map is meet-preserving.

Now we add a set of operators H to $\langle C, \wedge, T \rangle$ so that the sets $F \cup \{T\}$, with F an order filter of P, will be exactly the H-closed subsets of C. For each pair $a < b$ in P, note that $a \sqsubset b$, and define a function h_{ab} by

$$h_{ab}(x) = \begin{cases} x & \text{if } x \sqsubset a, \\ b & \text{if } x = a, \\ T & \text{if } x \sqsupset a. \end{cases}$$

Let $H = \{h_{ab} : a < b\}$. It is easy to see that this does the trick! For $\mathrm{Sub}(C, \wedge, T, H) \cong \mathcal{F}(P)$, where $\mathcal{F}(P)$ denotes the lattice of order filters of P, and then dually, $\mathrm{Con}(C, +, 0, G) \cong \mathcal{O}(P)$, as desired. ■

4. Adjoints: The General Case

To extend the theory to the general case, where S and T are not-necessarily-finite $(0,+)$-semilattices, we use (nonempty) ideals of S and T. Given a $(+,0)$-homomorphism $g : S \to T$, define the adjoint $h : \mathcal{I}(T) \to \mathcal{I}(S)$ by $h(J) = \{s \in S : g(s) \in J\}$. It is straightforward to confirm that $h(J)$ is an ideal of S. Moreover,

$$g(s) \in J \quad \text{iff} \quad s \in h(J).$$

As an immediate consequence we get $gh(J) \subseteq J$ for $J \in \mathcal{I}(T)$.

Now $g(I)$ for $I \in \mathcal{I}(S)$ need not be an ideal, but it is a directed set, and the union of a directed set of ideals is an ideal. Thus the ideal generated by $g(I)$ is

$$\overline{g}(I) = \{x \in T : x \leq g(z) \text{ for some } z \in I\}.$$

With this minor adjustment, we have

- $\overline{g}h(J) \leq J$ for $J \in \mathcal{I}(T)$,
- $h\overline{g}(I) \geq I$ for $I \in \mathcal{I}(S)$.

Using the above observations, we obtain the analogue of Lemma 3.

Lemma 9. *The adjoint $h : \mathcal{I}(T) \to \mathcal{I}(S)$ satisfies $h(T) = S$, and h preserves both arbitrary intersections and nonempty directed unions.*

5. Duality: The General Case

Now we review the general version of the duality theorem for congruence lattices of semilattices without operators [4]. Recall that if θ is a congruence on S, then an ideal J of S is θ-closed if $x \,\theta\, y$ and $y \in J$ implies $x \in J$.

Theorem 10. *Let $S = \langle S, +, 0\rangle$ be a join semilattice with 0. Then the lattice $\mathrm{Con}(S,+,0)$ is dually isomorphic to $\mathrm{S_p}(\mathcal{I}(S))$ via the maps*

- *$\sigma : \mathrm{Con}(S,+,0) \to \mathrm{S_p}(\mathcal{I}(S))$ such that $\sigma(\theta)$ is the set of all θ-closed ideals of S,*
- *$\rho : \mathrm{S_p}(\mathcal{I}(S)) \to \mathrm{Con}(S,+,0)$ where*

$$\rho(U) = \{(x,y) \in S^2 : \forall J \in U \ (x \in J \Leftrightarrow y \in J)\}.$$

Part of the proof of this theorem, which is used below, is the following analogue of Lemma 5.

LEMMA 11. *Let $\langle \theta, U \rangle$ be a pair with $\theta = \rho(U)$ and $U = \sigma(\theta)$.*

- *$J \in U$ if and only if $x \,\theta\, y$ and $y \in J$ implies $x \in J$.*
- *$x \leq y \mod \theta$ if and only if $y \in K$ and $K \in U$ implies $x \in K$ for all $K \in U$.*

Now we turn to semilattices with operators. $\mathrm{Con}(S, +, 0, g)$ is a sublattice of $\mathrm{Con}(S, +, 0)$, and the h-closed algebraic subsets $\mathrm{S_p}(\mathcal{I}(S), h)$ form a sublattice of $\mathrm{S_p}(\mathcal{I}(S))$. We want to be sure we get the appropriate sublattices. The statement of Lemma 12 is the same as that of Lemma 6, but the interpretation and proof are for the more general situation.

LEMMA 12. *Let $\langle \theta, U \rangle$ be a pair with $\theta = \rho(U)$ and $U = \sigma(\theta)$. Let g be an operator on $\langle S, +, 0 \rangle$ and let $h = \hat{g}$ be its adjoint. Then θ respects g if and only if U is closed under h.*

PROOF. Assume that θ is a semilattice congruence that respects g, and let $J \in U = \sigma(\theta)$. To see that $h(J) \in U$, assume $x \,\theta\, y$ and $y \in h(J)$. Then $g(x) \,\theta\, g(y)$ and $g(y) \in gh(J)$. Since $gh(J) \subseteq J$, we have $g(y) \in J$. By Lemma 11, $g(x) \,\theta\, g(y)$ implies $g(x) \in J$, whence $x \in h(J)$, as desired.

Conversely, assume that U is h-closed, and let $x \,\theta\, y$. Then $x \leq y \mod \theta$, so that by Lemma 11, $y \in J$ implies $x \in J$ whenever $J \in U$. Thus if $g(y) \in J$ with $J \in U$, then $y \in h(J)$, and by assumption $h(J) \in U$. Therefore $x \in h(J)$ and $g(x) \in J$. We have shown that $g(x) \leq g(y) \mod \theta$. Symmetrically $g(y) \leq g(x) \mod \theta$, so that $g(x) \,\theta\, g(y)$, and hence g respects θ. ■

Consequently:

THEOREM 13. *If S is a semilattice with an operator g, and $h = \hat{g}$ is its adjoint, then*

$$\mathrm{Con}(S, +, 0, g) \cong^d \mathrm{S_p}(\mathcal{I}(S), h).$$

More generally, if G is a monoid of operators on S and $H = \widehat{G}$, then

$$\mathrm{Con}(S, +, 0, G) \cong^d \mathrm{S_p}(\mathcal{I}(S), H).$$

The ideals of a semilattice $\langle S, +, 0 \rangle$ form an algebraic lattice $\mathcal{I}(S)$. Moreover, the compact elements of any algebraic lattice L form a join semilattice with 0 such that $L \cong \mathcal{I}(S)$. Restating the previous theorem in these terms yields the first part of our main result.

THEOREM 14. *If S is a semilattice with a monoid G of operators, then there is an algebraic lattice L with a monoid H of algebraic operators such that*

$$(\dagger) \qquad \mathrm{Con}(S, +, 0, G) \cong^d \mathrm{S_p}(L, H).$$

Conversely, if L is an algebraic lattice and H is a monoid of algebraic operators on L, then there is a monoid G of endomorphisms on the semilattice S of compact elements of L such that $H = \widehat{G}$, whence (†) again holds.

Proof. It remains to prove the second part. Assume we are given an algebraic lattice L and a map $h : L \to L$ that preserves arbitrary meets and nonempty directed joins. Letting S denote the semilattice of compact elements of L, we want to define a $(+, 0)$-preserving map $g : S \to S$. Let $g : S \to L$ be given by $g(s) = \bigwedge\{j \in L : s \leq h(j)\}$, so that

$$g(s) \leq j \quad \text{if and only if} \quad s \leq h(j).$$

Again note $s \leq hg(s)$. We claim that if $s \in S$, then $g(s) \in S$, i.e., $g(s)$ is compact.

Suppose $g(s) \leq \bigvee A$ for some $A \subseteq L$. Then

$$s \leq hg(s) \leq h\left(\bigvee A\right) = h\left(\bigvee_{\text{finite } F \subseteq A} \bigvee F\right) = \bigvee_{\text{finite } F \subseteq A} h\left(\bigvee F\right)$$

since the joins of finite subsets form a directed set. Because s is compact, $s \leq h(\bigvee F_0)$ for some finite $F_0 \subseteq A$. This implies $g(s) \leq \bigvee F_0$, as desired.

Now the dual of Lemma 3 shows that h preserves 0 and joins. By construction, g and h are adjoint maps. ■

6. Application: More Distributive Representations

As an application, we can slightly extend Theorem 8.

Theorem 15. *Let P be a countable ordered set with the property that the ideal $\downarrow x$ is finite for every $x \in P$, and let D be the lattice of order ideals $\mathcal{O}(P)$. Then there is a semilattice with operators S such that $D \cong$ Con $(S, +, 0, G)$.*

The proof uses an elementary lemma.

Lemma 16. *An ordered set P has a linear extension to ω if and only if P is countable and has the property that $\downarrow x$ is finite for every $x \in P$.*

Proof of Theorem 15. Let P be an ordered set such that $\downarrow x$ is finite for every $x \in P$. Without loss of generality, P is infinite. Then the order on P has a linear extension $\sqsubseteq$ so that $C_0 = \langle P, \sqsubseteq\rangle$ is isomorphic to ω. Add a new top element T to C_0, forming $C \cong \omega + 1$. Note that every subset of

C containing T is algebraic, and every order-preserving map on C is meet-preserving. Thus we can complete the representation by adding operators exactly as in the finite case, Theorem 8. ∎

When we relax the finiteness condition, an extended argument gives a representation of the linear sum $\mathcal{O}(P)+1$.

THEOREM 17. *For any ordered set P, there is a semilattice with operators such that $\mathcal{O}(P)+1 \cong \mathrm{Con}(S,+,0,G)$.*

PROOF. We prove dually that $1+\mathcal{F}(P)$ is isomorphic to $\mathrm{S_p}(C,H)$ for an algebraic chain C with algebraic operators.

Without loss of generality, P is infinite. Let α be the initial ordinal of cardinality $|P|$. Let L be the set of limit ordinals in α, including 0 as a limit ordinal, and let $N = P \setminus L$ be the set of non-limit ordinals. Add a new top element T to α, forming an algebraic chain C isomorphic to the ordinal $\alpha+1$. Meets are trivial in $\alpha+1$, since every nonempty set has a least element, whereas the join of a set is either 0, or its largest element, or some limit ordinal in L, or T.

The intention is to add operations, so that the H-closed algebraic subsets of C are $\{T\}$ and sets $\{T\} \cup L \cup U$ that are in one-to-one correspondence with the order filters of P. There will be two types of algebraic operations, one for limit ordinals and one for non-limit ordinals.

For every pair of limit ordinals i, $j \in L$ let ℓ_{ij} be defined by

$$\ell_{ij}(x) = \begin{cases} j & \text{if } x \le i, \\ T & \text{if } x > i. \end{cases}$$

In particular, $\ell_{ij}(i) = j$. Note that these operations preserve arbitrary meets and joins in C. If we let $H_0 = \{\ell_{ij} : i, j \in L\}$, then the algebraic subsets $\mathrm{S_p}(C, H_0)$ are $\{T\}$ and all sets of the form $\{T\} \cup L \cup S$ with S an arbitrary subset of N.

For any pair of non-limit ordinals i, $j \in N$ let

$$h_{ij}(x) = \begin{cases} 0 & \text{if } x < i, \\ j & \text{if } x = i, \\ T & \text{if } x > i. \end{cases}$$

In particular, $h_{ij}(i) = j$, and again, in this case because i is a non-limit ordinal, the operations preserve arbitrary meets and joins in C.

It remains only to choose an appropriate subset H_1 of these operations. To do this, set up a bijection $f : N \to P$, and let $H_1 = \{h_{ij} : f(i) \le f(j) \text{ in } P\}$.

Then form $H = H_0 \cup H_1$. By construction, the H-closed subsets of C are $\{T\}$ and $\{T\} \cup L \cup U$ with $f^{-1}(U)$ an order filter in P. ■

Recall that for any ordinal β, $\mathcal{O}(\beta) \cong \beta + 1$ and $\mathcal{O}(\beta^d) \cong 1 + \beta^d$. In particular, β or β^d could be used as the ordered set P.

COROLLARY 18. *For any ordinal β, the lattices $\beta + 2$ and $1 + \beta^d$ can be represented as congruence lattices of semilattices with operators.*

7. The Jónsson–Kiefer Property

The duality of Theorem 14 allows us to prove another property of congruence lattices of SLOs, the dual Jónsson-Kiefer property. A complete lattice L has the *Jónsson-Kiefer property* (JKP) if every element $a \in L$ is the join of elements that are (finitely) join prime *in the ideal* $\downarrow a$. (These elements need not be join prime in the whole lattice.) The JKP property, inspired by [10], was shown by Gorbunov to hold in lattices of quasivarieties [6], and further investigated in Adaricheva et al. [2]. We show that the dual JKP holds for congruence lattices of SLOs by proving that the JKP holds for lattices $\mathrm{S_p}(L, H)$ when L is algebraic.

For an element m in an algebraic lattice L, let $\langle m \rangle$ denote the smallest H-closed algebraic set in $\mathrm{S_p}(L, H)$ containing m.

The proof of Lemma 20 uses a lemma adapted from Gorbunov [7].

LEMMA 19. *For X, $Y \in \mathrm{S_p}(L, H)$, with L algebraic and H a monoid of algebraic operators,*

$$X \vee Y = \{x \wedge y : x \in X,\ y \in Y\}.$$

LEMMA 20. *If m is meet irreducible in L, then $\langle m \rangle$ is join prime in $\mathrm{S_p}(L, H)$.*

PROOF. Let m be meet irreducible in L, and $X, Y \in \mathrm{S_p}(L)$. If $\langle m \rangle \leq X \vee Y$, then $m \in X \vee Y$. By Gorbunov's lemma, $m = x \wedge y$ for some $x \in X$, $y \in Y$. As m is meet irreducible, either $m = x$, whence $m \in X$ and $\langle m \rangle \leq X$, or $m = y$, whence $\langle m \rangle \leq Y$. ■

THEOREM 21. *For any semilattice with operators,* $\mathrm{Con}(S, +, 0, G)$ *satisfies the dual Jónsson–Kiefer property.*

PROOF. It suffices to prove that the least congruence of $(S, +, 0, G)$ is a meet of meet prime congruences, or equivalently, that the largest element of $\mathrm{S_p}(L, H)$, which is L itself, is a join of join prime elements. For a nonzero congruence φ, we just apply the statement to the factor algebra S/φ.

Since L is algebraic, every element x of L is a meet of completely meet irreducible elements, say $x = \bigwedge_{i \in I} m_i$. This implies $x \in \bigvee_{i \in I} \langle m_i \rangle$ with each $\langle m_i \rangle$ join prime, as desired. ■

This theorem can also be derived from Theorem 14 and Gorbunov's result that $\mathrm{S_p}(L, R)$ has the JKP when L is algebraic and R is a distributive relation on L [6].

8. Concluding Remarks

Not every algebraic, meet semidistributive lattice can be represented as the congruence lattice of an SLO. At least two other restrictions are known to apply. One is the dual Jónsson–Kiefer property discussed in the last section. An algebraic, meet semidistributive lattice failing the dual JKP must have uncountably many compact elements and $|L| \geq 2^{\aleph_0}$; see [2]. In that same paper, R. McKenzie constructed such a lattice with no meet prime element, which of course fails the dual JKP.

The second restriction is that every congruence lattice of an SLO supports an *equa-interior operator*, as described in Adaricheva and Nation [1]. Such operators originally arose in the theory of quasivariety lattices; see Gorbunov [7]. This is a strong requirement, in that many finite, meet semidistributive lattices fail to admit an equa-interior operator.

However, it is easy to see that every distributive, algebraic lattice satisfies the dual JKP. Likewise, the identity function serves as an equa-interior operator on these lattices. This raises a natural question.

Problem 22. *Is every distributive, algebraic lattice isomorphic to the congruence lattice of a semilattice with operators?*

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