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Free and Projective Bimodal Symmetric Gödel Algebras

Abstract. Gödel logic (alias Dummett logic) is the extension of intuitionistic logic by the linearity axiom. Symmetric Gödel logic is a logical system, the language of which is an enrichment of the language of Gödel logic with their dual logical connectives. Symmetric Gödel logic is the extension of symmetric intuitionistic logic (L. Esakia, C. Rauszer). The proof-intuitionistic calculus, the language of which is an enrichment of the language of intuitionistic logic by modal operator was investigated by Kuznetsov and Muravitsky. Bimodal symmetric Gödel logic is a logical system, the language of which is an enrichment of the language of Gödel logic with their dual logical connectives and two modal operators. Bimodal symmetric Gödel logic is embedded into an extension of (bimodal) Gödel–Löb logic (provability logic), the language of which contains disjunction, conjunction, negation and two (conjugate) modal operators. The variety of bimodal symmetric Gödel algebras, that represent the algebraic counterparts of bimodal symmetric Gödel logic, is investigated. Description of free algebras and characterization of projective algebras in the variety of bimodal symmetric Gödel algebras is given. All finitely generated projective bimodal symmetric Gödel algebras are infinite, while finitely generated projective symmetric Gödel algebras are finite.

Keywords: Gödel logic, Symmetric intuitionistic logic, Modal logic, Projective algebra, Free algebra.

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1. Introduction

A "symmetric" formulation of the intuitionistic propositional calculus, suggested by various authors (G. Moisil, A. Kuznetsov, C. Rauszer), presupposes that any connective $\&, \vee, \rightharpoonup, \top, \bot$ has its dual $\vee, \&, \neg, \bot, \top$, and the duality principle of the classical logic is restored. The notion of double-Browerian algebras was introduced by McKinsey and Tarski in [26], based on the idea considered by T. Skolem in 1919. In [10] double-Browerian algebras were named Skolem algebras.

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Heyting–Brouwer logic (alias symmetric Intuitionistic logic Int^2) was introduced by C. Rauszer as a Hilbert calculus with an algebraic semantics [30]. Notice that the variety of Skolem (Heyting–Brouwerian) algebras are algebraic models for symmetric Intuitionistic logic Int^2.

The well-known procedure for an embedding of the intuitionistic propositional calculus Int into the classical modal system $S4$ can be extended on the symmetric intuitionistic logic Int^2 [10], which may be embedded into the bimodal (temporal) logical system $S4^2 (= K^2C4T)$ introduced by Segerberg [31]. The language of $S4^2$ consists of $\vee, \wedge, \to, -, \Box_1$ ("it always will be that"), $\Box_2$ ("it always was that"); the temporal connectives $\Diamond_1$ ("it will be the case that"), $\Diamond_2$ ("it was the case that") are introduced in the usual way: $\Diamond_1\varphi = -\Box_1 - \varphi$ and $\Diamond_2\varphi = -\Box_2 - \varphi$. Axioms of $S4^2$: (1) $\Box_1(p \to q) \to (\Box_1 p \to \Box_1 q)$, $\Box_2(p \to q) \to (\Box_2 p \to \Box_2 q)$; (2) $\Box_1 p \to p$, $\Box_2 p \to p$; (3) $\Box_1 p \to \Box_1\Box_1 p$, $\Box_2 p \to \Box_2\Box_2 p$; (4) $\Diamond_2\Box_1 p \to p$, $\Diamond_1\Box_2 p \to p$. Inference rules of $S4^2$: $\varphi, \varphi \to \psi \Rightarrow \psi$, $\varphi \Rightarrow \Box_1\varphi$, $\varphi \Rightarrow \Box_2\varphi$. For every formula α of Int^2 its translation $tr_1(\alpha)$ into $S4^2$ is defined as follows: (1) if α is a propositional variable, then $tr_1(\alpha) = \Box_1\alpha$; (2) $tr_1(\alpha \vee \beta) = tr_1(\alpha) \vee tr_1(\beta)$; (3) $tr_1(\alpha \wedge \beta) = tr_1(\alpha) \wedge tr_1(\beta)$; (4) $tr_1(\alpha \rightharpoonup \beta) = \Box_1(tr_1(-\alpha \vee \beta))$; (5) $tr_1(\alpha \rightharpoondown \beta) = \Diamond_2(tr_1(\alpha \wedge -\beta))$. This translation turns out to be an embedding of Int^2 into $S4^2$ in the sense that $Int^2 \vdash \alpha$ iff $S4^2 \vdash tr_1(\alpha)$ [11,30]. Notice also that we have an embedding of Int into Grz [6].

The proof-intuitionistic calculus, the language of which is an enrichment of the language of intuitionistic logic by modal operator was investigated by Kuznetsov and Muravitsky [20] and Muravitsky [29]. A modal operator on Heyting algebras was introduced in [28] and discussed in [21] (see also [20]) to give an intuitionistic version of the provability logic GL, which formalizes the concept of provability in Peano Arithmetic. This operator was also studied by Caicedo and Cignoli in [17] and Esakia in [14].

The provability Gödel–Löb logic GL can be defined in the following way [5,32]. The language of GL coincides with the language of $S4$. In turn, a Kripke-frame for GL is a Kripke-frame $(W; R)$ (with W a nonempty set of so-called *worlds* or *nodes* and R a binary relation, the so-called *accessibility relation*) with R a transitive relation such that the converse of R is well-founded (Noetherian, in other terms). Such kind Kripke frames are called GL-frames. Then the provability Gödel–Löb logic GL is defined as the set of all formulas that are valid in all GL-frames. Of course, a finite transitive frame is conversely well-founded iff it is irreflexive. Now we can define symmetric Gödel–Löb logic GL^2 in the following way. The language of GL^2 consists of (as in $S4^2$) $\vee, \wedge, \to, -, \Box_1, \Box_2$. GL^2 is the set of all formulas that are valid in all Kripke frames $(W; R)$ with a transitive relation

R such that the R and its converse $\widetilde{R}$ are well-founded. The well-known procedure of an embedding of the intuitionistic propositional calculus Int into Gödel–Löb logic GL [6] also can be extended to the symmetric intuitionistic logic Int^2 into GL^2 in the following way. Define the new operators: $\Box_1^\rho p = \Box_1 p \wedge p$, $\Diamond_2^\rho p = \Diamond_2 p \vee p$. For every formula α of Int^2 its translation $tr_2(\alpha)$ into GL^2 is defined as follows: (1) if α is a propositional variable, then $tr_2(\alpha) = \Box_1^\rho \alpha$; (2) $tr_2(\alpha \vee \beta) = tr_2(\alpha) \vee tr_2(\beta)$; (3) $tr_2(\alpha \wedge \beta) = tr_2(\alpha) \wedge tr_2(\beta)$; (4) $tr_2(\alpha \rightharpoonup \beta) = \Box_1^\rho(tr_2(-\alpha \vee \beta))$; (5) $tr_2(\alpha \rightharpoondown \beta) = \Diamond_2^\rho(tr_2(\alpha \wedge -\beta))$. Items (1)–(4) are defined as in [6] for Int and GL, and item (5) as in [11] for Int^2 and $S4^2$. This translation turns out to be an embedding of the Int^2 into GL^2 in the sense that $Int^2 \vdash \alpha$ iff $GL^2 \vdash tr_2(\alpha)$.

Recall that Gödel logic G, that is also known as Dummett logic LC [8], is an extension of intuitionistic logic Int by the linearity axiom

$$(p \rightharpoonup q) \vee (q \rightharpoonup p).$$

Denote by G^2 the extension of the symmetric Intuitionistic logic Int^2 by Gödel (linearity) axiom and the dual Gödel axiom: $(p \rightharpoonup p) \rightharpoondown ((p \rightharpoondown q) \wedge (q \rightharpoondown p))$. The language of G^2 consists of disjunction $\vee$, conjunction $\wedge$, implication $\rightharpoonup$, co-implication $\rightharpoondown$ (an algebraic interpretation of $\rightharpoondown$ is called pseudo-difference) and constants $\bot, \top$. In other words G^2 is a logic corresponding to Skolem algebras with linearity conditions [11]. We can define the logic G^2 as the set of all formulas which are valid in all finite Kripke frames (W, R) [11], where R is reflexive, transitive, anti-symmetric and such that for every $x \in W$ $R(x)$ (the R-image of x) and $R^{-1}(x)$ (the R-inverse image of x) are chains. The translation of Int^2 into GL^2 can be specified for G^2 into $LinGL^2$, where $LinGL^2$ is the extension of GL^2 by two extra axioms: $tr_2((p \rightharpoonup q) \vee (q \rightharpoonup p))$ and $tr_2((p \rightharpoonup p) \rightharpoondown ((p \rightharpoondown q) \wedge (q \rightharpoondown p)))$, or $(\Diamond_2 \Box_1 p \rightharpoonup p) \wedge (p \rightharpoonup \Diamond_2 \Box_1 p)$ and $(\Box_1 \Diamond_2 p \rightharpoonup p) \wedge (p \rightharpoonup \Box_1 \Diamond_2 p)$. In other words it is a restriction of the translation Int^2 into GL^2 on G^2 into $LinGL^2$.

In this paper we investigate symmetric Gödel logic G^2 enriched by two modalities $\Box$ $(=\Box_1)$ (considered as *Prov* (the formalized provability predicate for Peano Arithmetic modality) and $\Diamond$ $(= -\Box_2-)$. We call this logic a *bimodal symmetric Gödel logic*, denoted by MG^2. Bimodal symmetric Gödel logic MG^2, which is an extended version of prof-intuitionistic logic containing with any logical connective its dual, is introduced for the first time. Moreover, we can regard this logic as a temporal logic with modalities "it always will be that" and "it always was that" that possesses rich expressive possibilities. To be more precise we investigate the variety of algebras, named here $MG^2 - algebras$ (the precise definition of MG^2-algebras will be given below), corresponding to the logic MG^2. Notice that the signature of MG^2-algebra is an extension of the signature of KM-algebra (Δ-pseudo-Boolean algebra in other terminology [20]) by dual operations. Semantically the logic MG^2 is defined in the following way: MG^2 is the set of all formulas which are valid in all finite Kripke frames $(W; R)$ with a transitive relation R such that the R and its converse are well-founded, while the reflexive closure R_ρ of R is a total order. We can extend the embedding of G^2 into $LinGL^2$ to the one of MG^2 into $LinGL^2$ in the same manner as the embedding Int^2 into $S4^2$, and Int^2 into GL^2 transferring the translation of $\Box$ into $\Box_1$ and $\Diamond$ into $\Box_2$ in trivial way: (1) if α is a propositional variable, then $tr(\alpha) = \Box_1^\rho \alpha$; (2) $tr(\alpha \vee \beta) = tr(\alpha) \vee tr(\beta)$; (3) $tr(\alpha \wedge \beta) = tr(\alpha) \wedge tr(\beta)$; (4) $tr(\alpha \rightharpoonup \beta) = \Box_1^\rho(tr(-\alpha \vee \beta))$; (5) $tr(\alpha \rightharpoondown \beta) = \Diamond_2^\rho(tr(\alpha \wedge -\beta))$, $tr(\Box\alpha) = \Box_1\alpha$, $tr(\Diamond\alpha) = \Diamond_2\alpha$.

We give a description of finitely generated free algebras in the variety of algebras corresponding to the bimodal symmetric Gödel logic, which is equivalent to the description of non-equivalent formulas (with a fixed number of variables) in this logic, and the characterization of finitely generated projective algebras, which play an important role in the unification problem for the bimodal symmetric Gödel logic. We will show that in contrast to the variety of all G^2-algebras (and G-algebras as well), the finitely generated free algebras of which are finite, the finitely generated free MG^2-algebras are infinite (this follows also from the fact that 0-generated free KM-algebra is infinite [20]); any finite G^2-algebra (and G-algebra as well) is projective, while every projective MG^2-algebra is not finite.

2. Preliminaries

An algebra $(T, \vee, \wedge, \rightharpoonup, \rightharpoondown, 0, 1)$ is a Skolem algebra [10,30] (or Heyting-Browerian algebra), if $(T, \vee, \wedge, 0, 1)$ is a bounded distributive lattice, $\rightharpoonup$ is an implication (relative pseudo-complement), $\rightharpoondown$ is co-implication (relative pseudo-difference) on T. Notice that in [30] C. Rauszer defines $x \rightharpoondown y$ as $x \dot{-} y$ called *pseudo-difference*: $x \rightharpoondown y (= x \dot{-} y)$ is the least element z such that $y \vee z \geq x$.

An algebra $(T, \vee, \wedge, \rightharpoonup, \rightharpoondown, 0, 1)$ is said to be a G^2-*algebra*, if (i) $(T, \vee, \wedge, \rightharpoonup, 0, 1)$ is a G-algebra, i.e. Heyting algebra that satisfies the linearity axiom, corresponding to Gödel logic; (ii) $(T, \vee, \wedge, \rightharpoondown, 0, 1)$ is a dual G-algebra (alias Brouwerian algebra with linearity condition: $(p \rightharpoondown q) \wedge (q \rightharpoondown p) = 0$).

G^2-algebras, which are algebraic models of the logical system G^2, represent a proper subclass of Skolem algebras.

The *MG^2-algebra* is an algebra $(T, \vee, \wedge, \rightharpoonup, \rightharpoondown, \Box, \Diamond, 0, 1)$, where $(T, \vee, \wedge, \rightharpoonup, \rightharpoondown, 0, 1)$ is the G^2-algebra and the operators $\Box, \Diamond$ satisfy the following conditions:

$$x \leq \Box x,\ \Box x \leq y \vee (y \rightharpoonup x),\ \Box x \rightharpoonup x = x,\ \Box(x \rightharpoonup y) \leq (\Box x \rightharpoonup \Box y),$$
$$\Diamond x \leq x,\ x \rightharpoondown \Diamond x = x,\ \Diamond(x \vee y) = \Diamond x \vee \Diamond y,$$
$$x \leq \Box\Diamond x, \quad \Diamond\Box x \leq x,$$
$$\Diamond(x \rightharpoonup y) \leq \Diamond x \rightharpoonup \Diamond y.$$

We give some comments about these axioms. First of all this set of axioms have no pretensions of economy. The first three axioms represent the algebraic version of the logical axioms of the proof-intuitionistic logic KM [20,21,27,28]. The fourth axiom $\Box(x \rightharpoonup y) \leq (\Box x \rightharpoonup \Box y)$ is deducible from the first three axioms (see [20]) but we add this one for easing proof of some assertion. The next three axioms are intuitively clear. The last three axioms are the algebraic counterparts of the logical axioms given in [11,31] for $K^2C4T(= S4^2)$.

Let us consider simple examples of MG^2-algebra. Let $C(= \{1, 1/2, 0\})$ be 3-element G^2-algebra which we convert into MG^2-algebra defining the operations $\Box$ and $\Diamond$ as follows: $\Box 0 = 1/2$, $\Box 1/2 = 1$, $\Box 1 = 1$, $\Diamond 1 = 1/2$, $\Diamond 1/2 = 0$, $\Diamond 0 = 0$. It is routine to check that all the axioms of MG^2-algebra hold in $C_3 = (\{1, 1/2, 0\}, \vee, \wedge, \rightharpoonup, \rightharpoondown, \Box, \Diamond, 0, 1)$.

Now we show that the operations $\Box$ and $\Diamond$ is defined in a unique way in any MG^2-algebra. Indeed, let us suppose that there exist two box operations $\Box_1$ and $\Box_2$. Then $\Box_1 x \leq \Box_2 x \vee (\Box_2 x \rightharpoonup x)$ (third axiom) $\Rightarrow \Box_1 x \leq \Box_2 x \vee x$ (first axiom) $\Rightarrow \Box_1 x \leq \Box_2 x$ (first axiom). In the same manner we can show that $\Box_2 x \leq \Box_1 x$ and hence $\Box_1 x = \Box_2 x$.

Let us suppose that there exist two diamond operations $\Diamond_1$ and $\Diamond_2$. Then $\Diamond_1 x \rightharpoonup \Diamond_2 x = \Diamond_1(x \rightharpoondown \Diamond_2 x) \rightharpoonup \Diamond_2(x \rightharpoondown \Diamond_2 x)$ (sixth axiom). But $\Diamond_1(x \rightharpoondown \Diamond_2 x) \rightharpoonup \Diamond_2(x \rightharpoondown \Diamond_2 x) \geq (x \rightharpoondown \Diamond_2 x) \rightarrow \Diamond_2(x \rightharpoondown \Diamond_2 x) = x \rightharpoondown \Diamond_2 x = x$ (fifth and sixth axiom). Hence, $\Diamond_1 x \wedge (\Diamond_1 x \rightharpoonup \Diamond_2 x) \geq x \wedge \Diamond_1 x = \Diamond_1 x$ (fifth axiom), i.e. $\Diamond_1 x \wedge \Diamond_2 x = \Diamond_1 x$ (since $\Diamond_1 x \wedge (\Diamond_1 x \rightharpoonup \Diamond_2 x = \Diamond_1 x \wedge \Diamond_2 x)$. Analogically we have $\Diamond_1 x \wedge \Diamond_2 x = \Diamond_2 x$. So, $\Diamond_1 x = \Diamond_2 x$.

Now we show the behavior of $\Box$ and $\Diamond$ in a chain MG^2-algebra. Denote $x \prec y$ if y covers x (i.e. there is no element between x and y different from x and y with $x \leq y$). Let us consider any chain G^2-algebra C and define the operations $\Box$ and $\Diamond$ as follows: for every $a \in C$ if $a = 1$, then $\Box a = a$, if $a \neq 1$, then $\Box a = b$, where b covers a (i.e. there is no element between a and b different from a and b with $a \leq b$); for every $a \in C$ if $a = 0$, then $\Diamond a = a$, if $a \neq 0$, then $\Diamond a = b$, where a covers b. Notice, that there is the only way to define in C the operations $\Box$ and $\Diamond$ satisfying the axioms of MG^2-algebra. Indeed, if $x \neq 1$, then $\Box x > x$. If $\Box x \leq x$, then $\Box x \rightharpoonup x = 1 \neq x$ (that contradicts to the third axiom). Analogically, if $x \neq 0$, then $\Diamond x < x$. Indeed, if $\Diamond x \geq x$, then $x \rightharpoondown \Diamond x = 0 \neq \Diamond x$ (that contradicts to the fifth axiom). Let us suppose $a, b, c \in C$ and $a < b < c$, and $\Box a = c$. Then, according to the second axiom, $c = \Box a \leq b \vee (b \rightharpoonup a) = b$, that contradicts to the condition $b < c$. Now let $\Diamond c = a$. Then, according to the eighth axiom, $c \leq \Box\Diamond c = \Box a = b$, that contradicts to the condition $b < c$.

At last we show that the last axiom holds in C. If $x \leq y$, the axiom is trivially holds. If $x > y$, then $x \rightharpoonup y = y$ and $\Diamond y \leq \Diamond x \rightharpoonup \Diamond y$ according to the property of implication $\rightharpoonup$.

Let us denote the variety (and the category, as well) of all MG^2-algebras by $\mathbf{MG}^2$.

An algebra $(A; \vee, \wedge, \Diamond, -, 0, 1)$ is said to be GL-algebra (or *diagonalizable algebra*) if $(A; \vee, \wedge, -, 0, 1)$ is a Boolean algebra and the unary operation $\Diamond$ satisfies the following conditions: (1) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$, (2) $\Diamond 0 = 0$, (3) $\Diamond a \leq \Diamond(a \wedge -\Diamond a)$.

Let us denote by **GL** the variety and the category of GL-algebras with GL-algebra homomorphisms.

Now we shall describe a duality for MG^2-algebras, but first we give a description of a duality for GL- and GL^2-algebras. Let (X, R) be a Kripke frame, where R is at least transitive. We shall say that a subset $Y \subset X$ is an *upper cone* (or cone) if $x \in Y$ and xRy imply $y \in Y$. The concept of a *lower cone* is defined dually. A subset $Y \subset X$ is called *a bicone* if it is an upper cone and a lower cone at the same time.

We say that (X, τ, R) is a descriptive GL-frame [1,16] if R is a transitive binary relation on X and

(1) X is a Stone space (i.e. 0-dimensional, Hausdorff and compact topological space);
(2) $R(x)$ and $R^{-1}(x)$ are closed sets for every $x \in X$;
(3) for every clopen A of X, $R^{-1}(A)$ is a clopen;
(4) for every clopen A of X and every $x \in A$ there is an element $y \in A \setminus R^{-1}(A)$ such that either xRy or $x \in A \setminus R^{-1}(A)$.

Let $X_i = (X_i, \tau_i, R_i)$, $i = 1, 2$. A map $f : X_1 \to X_2$ from a GL-frame X_1 to a GL-frame X_2 is said to be *strongly isotone* (or p-morphism) if

$$f(x)R_2 y \Leftrightarrow (\exists z \in X_1)(xR_1 z \& f(z) = y).$$

Hereinafter instead of (X, τ, R) we will write (X, R) or simply X.

Let us denote by **KGL** (the class and) the category of descriptive GL-frames and continuous strongly isotone maps.

An algebra $(A; \vee, \wedge, -, \Diamond_1, \Diamond_2, 0, 1)$ is said to be a GL^2-algebra if (1) $(A; \vee, \wedge, -, \Diamond_1, 0, 1)$ is a GL-algebra (2) $(A; \vee, \wedge, -, \Diamond_2, 0, 1)$ is a GL-algebra and (3) $\Box_1^\rho \Diamond_2^\rho x = \Diamond_2^\rho x$, $\Diamond_2^\rho \Box_1^\rho x = \Box_1^\rho x$, where $\Box_1^\rho x = \Box_1 x \wedge x$, $\Diamond_2^\rho x = \Diamond_2 x \vee x$.

Let us denote by $\mathbf{GL^2}$ the variety and the category of GL^2-algebras with GL^2-algebra homomorphisms.

Now we shall describe a duality for GL-algebras defining two contravariant functors: $\mathcal{GL}$ and $\mathcal{KGL}$ [1]. Let $X \in \mathbf{KGL}$ and $A \in \mathbf{GL}$. The set $\mathcal{GL}(X)$ of all clopen subsets of X is closed under the set union, intersection, complementation and the operator R^{-1} [1]. The set $\mathcal{KGL}(A)$ of all ultrafilters of $A \in \mathbf{GL}$ with a relation $xRy \Leftrightarrow (\forall a \in A)(a \in y \Rightarrow \Diamond a \in x)$, topologized by taking the family of sets $h(a) = \{F \in \mathcal{KGL}(A) : a \in F\}$ as a base, is an object of **GL** [1].

Furthermore, setting $\mathcal{KGL}(h) = h^{-1} : \mathcal{KGL}(B) \to \mathcal{KGL}(A)$ for every morphism $h : A \to B$ of **GL**, and $\mathcal{GL}(f) = f^{-1} : \mathcal{GL}(Y) \to \mathcal{GL}(X)$ for every morphism $f : X \to Y$ of **KGL**, we obtain contravariant functors $\mathcal{GL} : \mathbf{GL} \to \mathbf{KGL}$ and $\mathcal{KGL} : \mathbf{KGL} \to \mathbf{GL}$ [1].

THEOREM 2.1. [1] *The functors $\mathcal{GL}$ and $\mathcal{KGL}$ establish a dual equivalence between the categories* **GL** *and* **KGL**.

As the duality between category of closure algebras and the category of descriptive frames is extended to the category of closure algebras with conjugate operators and the category of symmetric Kripke frames [11], we can extend the duality between the categories **GL** and **KGL** to the categories $\mathbf{GL^2}$ and $\mathbf{KGL^2}$ (which will be defined below).

We say that (X, R) is a GL^2-frame if (X, R) and $(X, \widetilde{R})$ (where $x\widetilde{R}y \Longleftrightarrow yRx$) are both GL-frames. Let us denote by $\mathbf{KGL^2}$ the category of GL^2-frames and continuous strongly isotone maps $f : X_1 \to X_2$ such that $xR_2f(z)R_2y \Leftrightarrow (\exists x', y')(x'R_1zR_1y')$ and $f(z') = x$, $f(y') = y$ for $X_1, X_2 \in \mathbf{KGL^2}$ [11]. A GL^2-frame (X, R) is *descriptive* if GL-frames (X, R) and $(X, \widetilde{R})$ are descriptive.

Taking into account that GL-algebra is a Boolean algebra, and GL^2-algebra is also a Boolean algebra as well, and, moreover, (X, R) and $(X, \widetilde{R})$ are both descriptive GL-frames, we can extend the duality between **GL** and **KGL** to the duality between $\mathbf{GL^2}$ and $\mathbf{KGL^2}$. Let $(A, \vee, \wedge, \Diamond, -, 0, 1)$ be a GL-algebra. Then the algebra $(A, \vee, \wedge, \Diamond^\rho, -, 0, 1)$, where $\Diamond^\rho x = x \vee \Diamond x$, is a closure algebra (or $S4$-algebra). The set $\{x \in A : \Box^\rho x = x\}$ of open elements of A forms a Heyting algebra. The Boolean algebra $B(T)$ generated by T (inside A) is called *trapharet algebra* [12, 20]. It is well-known that any

Heyting algebra is embedded into trapharet algebra of some GL-algebra [12, 20]. The letter is called GL-algebra envelope of this Heyting algebra. Notice that T is closed under the operation $\Box(= -\Diamond-)$. The same procedure we can realize for GL^2-algebra and MG^2-algebra as well.

Let X be an object of $\mathbf{KGL^2}$ and $A \in \mathbf{GL^2}$. $\mathcal{GL}^2(X)$ of all clopen subsets of X is closed under the set union, intersection, complementation and the operators R^{-1} and R since (X, R) and $(X, \widetilde{R})$ are both descriptive GL-frames. So, $\mathcal{GL}^2((X; R)) = (\mathcal{GL}^2(X); \cup, \cap, -, R^{-1}, R, \emptyset, X)$ is an object of $\mathbf{GL^2}$ (i.e. GL^2-algebra) that is $(\mathcal{GL}^2(X); \cup, \cap, -, R^{-1}, \emptyset, X)$ and $(\mathcal{GL}^2(X); \cup, \cap, -, R, \emptyset, X)$ are GL-algebras. It is obvious that all axioms of GL^2-algebra are satisfied. Indeed, the conditions (1) and (2) are immediate consequence of [22] (Corollary 1). As to identity (3), it is obvious that for every $Y \subset X$ $-R_\rho^{-1} - (R_\rho(Y)) = R_\rho(Y)$, $R_\rho(-R_\rho^{-1} - (Y)) = -R_\rho^{-1} - (Y)$, because $R_\rho(Y)$ and $-R_\rho^{-1} - (Y)$ are upper cones, where R_ρ is reflexive closure of the relation R. The set $\mathcal{KGL}^2(A)$ of all ultrafilters of $A \in \mathbf{GL^2}$ with a relation $xRy \Leftrightarrow (\forall a \in A)(a \in y \Rightarrow \Diamond_1(a) \in x)$, topologized by taking the family of sets $h(a) = \{F \in \mathcal{KGL}^2(A) : a \in F\}$ as a base of clopen sets, is an object of $\mathbf{KGL^2}$. Furthermore, setting $\mathcal{KGL}^2(h) = h^{-1} : \mathcal{KGL}^2(B) \to \mathcal{KGL}^2(A)$ for every morphism $h : A \to B$ of $\mathbf{GL^2}$ (which is at the same time Boolean homomorphism), and $\mathcal{GL}^2(f) = f^{-1} : \mathcal{GL}^2(Y) \to \mathcal{GL}^2(X)$ for every morphism $f : X \to Y$ of $\mathbf{KGL^2}$ (which is at the same time morphism of $\mathcal{KGL}$), we obtain contravariant functors $\mathcal{GL}^2 : \mathbf{GL} \to \mathbf{KGL^2}$ and $\mathcal{KGL}^2 : \mathbf{KGL^2} \to \mathbf{GL^2}$.

Now we describe a duality for MG^2-algebras. A Kripke frame (X, R) is called an *MG^2-frame* if: (1) (X, R) is a GL^2-frame, (2) (X, R_ρ) is a poset, (3) $R_\rho^{-1}(x)$, $R_\rho(x)$ are chains, where R_ρ is the reflexive closure of R. MG^2-frame is *descriptive* if corresponding GL^2-frame is descriptive. Let $\mathbf{KMG^2}$ be the category of descriptive MG^2-frames and continuous strongly isotone mappings. Notice, that if a descriptive MG^2-frame (X, R) is finite, then the binary relation R is irreflexive. Notice, also, that if (X, R) is an MG^2-frame, then (X, R_ρ) is a strongly symmetric Kripke frame [11]. Such frames are dual to Skolem algebras, that is, to the algebraic counterparts of symmetric intuitionistic logic Int^2. Taking into consideration that for any MG^2-frame (X, R) the frame (X, R_ρ) is a disjoint union of chains we see that (X, R_ρ) is a dual object of a G^2-algebra. In other words MG^2-frames are disjoint unions of chains.

Taking into account mentioned above facts we arrived to the following

PROPOSITION 2.2. *The category* $\mathbf{MG^2}$ *of* MG^2*-algebras and algebraic homomorphisms is dually equivalent to the category* $\mathbf{KMG^2}$ *of* MG^2*-frames and continuous strongly isotone maps.*

Now we illustrate this duality (for some detail we refer to [11,16]) and describe two contravariant functors: $\mathfrak{M} : \mathbf{MG^2} \to \mathbf{KMG^2}$ and $\mathfrak{S} : \mathbf{KMG^2} \to \mathbf{MG^2}$. Let $X \in \mathbf{KMG^2}$ and $A \in \mathbf{MG^2}$. The set $\mathfrak{S}(X)$ of all clopen cones of X is closed under the following operations: the set union, intersection, $U \rightharpoonup V = -(R^{-1}(U - V) \cup (U - V))$, $U \rightharpoondown V = R(U - V) \cup (U - V)$, $\Box U = -R^{-1}(-U)$, $\Diamond U = R(U)$. Moreover, the algebra $(\mathfrak{S}(X), \cup, \cap, \rightharpoonup, \rightharpoondown, \Box, \Diamond, \emptyset, X)$ satisfies to the axioms of MG^2-algebras and, consequently, it is an MG^2-algebra. Furthermore, for any morphism $f : (X_1, R_1) \to (X_2, R_2)$ in $\mathbf{KMG^2}$, $\mathfrak{S}(f) = f^{-1} : \mathfrak{S}(X_2) \to \mathfrak{S}(X_1)$ is a homomorphism. The restriction follows from the fact that $\mathfrak{S}(X_i)$ $(i = 1, 2)$ is G-algebra because $\mathfrak{S}(X_i)$ $(i = 1, 2)$ is G^2-algebra. Moreover, $\mathfrak{S}(X_i)$ $(i = 1, 2)$ is MG^2-algebra. Since $\mathfrak{S}(X_i)$ $(i = 1, 2)$ is G-algebra $\mathfrak{S}(f)$ is a homomorphism preserving Gödel operations. Notice that $f^{-1}(Y)$ is a clopen for every clopen Y of X_1, and the set $\{f^{-1}(Y) : Y \text{ is upper cone and clopen of } X_1\}$, which is Gödel algebra, is closed under the operations $\rightharpoondown$, $\Box$, $\Diamond$. On the other hand, for each MG^2-algebra A, the set $\mathfrak{M}(A)$ of all prime filters of A with binary relation R on it, defined in the following way: $xRy \Leftrightarrow (\forall a \in A)(\Box a \in x \Rightarrow a \in y)$, and topologized by taking the family $\beta(a) = \{F \in \mathfrak{M}(A) : a \in F\}$, for $a \in A$, and their complements as a subbase of clopen sets is an object of $\mathbf{KMG^2}$; and for each MG^2-algebra homomorphism $h : A \to B$, $\mathfrak{M}(h) = h^{-1} : \mathfrak{M}(B) \to \mathfrak{M}(A)$ is a morphism of $\mathbf{KMG^2}$. More precisely, if $A \in \mathbf{MG^2}$, then A is G^2-algebra (and G-algebra as well). Therefore, $\mathfrak{M}\mathfrak{S}(A) \cong A$. Analogically, since an object X of $\mathbf{KMG^2}$ is a descriptive G^2-frame, we have that $\mathfrak{S}(X)$ is a G^2-algebra such that $\Diamond$ and $\Box$ satisfy the axioms of MG^2-algebras. So, we have two contravariant functors: $\mathfrak{M} : \mathbf{MG^2} \to \mathbf{KMG^2}$ and $\mathfrak{S} : \mathbf{KMG^2} \to \mathbf{MG^2}$. These functors establish a dual equivalence between the categories $\mathbf{MG^2}$ and $\mathbf{KMG^2}$.

Now we consider congruences in an MG^2-algebra. Let us introduce some abbreviations: $\neg x = x \rightharpoonup 0$, $\ulcorner x = 1 \rightharpoondown x$.

Let T be an MG^2-algebra. A subset $F \subset T$ is said to be a Skolem filter [11,30], if F is a filter (i.e. $1 \in F$, if $x \in F$ and $x \leq y$, then $y \in F$, if $x, y \in F$, then $x \wedge y \in F$) and if $x \in F$, then $\neg\ulcorner x \in F$. The equivalence relation $x \equiv y \Leftrightarrow (x \rightharpoonup y) \wedge (y \rightharpoonup x) \in F$ is a congruence relation for the Skolem reduct of T [30]. In [30] it has been shown also that there is a lattice isomorphism between the lattice of all congruences of a Skolem algebra and the lattice of all Skolem filters of the Skolem algebra. A Skolem filter F is said to be $\Diamond$-*filter* if in addition it satisfies the following condition: $x \rightharpoonup y \in F \Rightarrow \Diamond x \rightharpoonup \Diamond y \in F$.

Since $\Diamond$-filter is a Skolem filter (and, hence, a lattice filter as well), we have that the equivalence relation $\equiv$ preserves lattice operations and operations $\rightharpoonup$ and $\rightharpoondown$. We say that for any elements $x, y \in A$ $x \equiv y$ iff $(x \rightharpoonup y) \wedge (y \rightharpoonup x) \in F$. Observe that the equivalence relation $x \equiv y$ preserves the operations $\Box$ and $\Diamond$. Indeed, $(x \rightharpoonup y) \leq \Box(x \rightharpoonup y) \leq (\Box x \rightharpoonup \Box y)$ (according to the first and fourth axiom). So, if $x \rightharpoonup y \in F$, then $\Box x \rightharpoonup \Box y \in F$. Analogically, if $y \rightharpoonup x \in F$, then $\Box y \rightharpoonup \Box x \in F$. Hence $\Box x \equiv \Box y$. Let $x \rightharpoonup y \in F$. Then $\Diamond x \rightharpoonup \Diamond y \in F$ since it is a $\Diamond$-filter. Analogically we show that if $y \rightharpoonup x \in F$, then $\Diamond y \rightharpoonup \Diamond x \in F$. So, $\Diamond x \equiv \Diamond y$.

Now let $\equiv$ is a congruence relation on an MG^2-algebra A. Then $F = \{x \in A : x \equiv 1\}$ is a $\Diamond$-filter. Indeed, it is obvious that F is Skolem filter. Now suppose that $x \rightharpoonup y \equiv 1$. Then $x \rightharpoonup y \equiv 1$ implies $\Diamond(x \rightharpoonup y) \equiv \Diamond 1$. Then $\Diamond 1 \rightharpoonup \Diamond(x \rightharpoonup y) \equiv \Diamond 1 \rightharpoonup \Diamond 1 = 1$. Since $\Diamond(x \rightharpoonup y) \leq \Diamond x \rightharpoonup \Diamond y$, we have $\Diamond 1 \rightharpoonup \Diamond(x \rightharpoonup y) \leq \Diamond 1 \rightharpoonup (\Diamond x \rightharpoonup \Diamond y) \leq 1$. So, $\Diamond 1 \rightharpoonup (\Diamond x \rightharpoonup \Diamond y) \equiv 1$. But, $\Diamond 1 \rightharpoonup (\Diamond x \rightharpoonup \Diamond y) = (\Diamond 1 \wedge \Diamond x) \rightharpoonup \Diamond y) = \Diamond x \rightharpoonup \Diamond y$. From here we deduce that $\Diamond x \rightharpoonup \Diamond y \equiv 1$.

So we have

Theorem 2.3. *Let T be an MG^2-algebra. The lattice of all congruences of the algebra T is isomorphic to the lattice of all $\Diamond$-filters of the algebra T.*

According to the duality between the category of G^2-algebras and the category of (descriptive) G^2-frames (= strongly symmetric Kripke frames) the lattice of all congruences of an G^2-algebra T is anti-isomorphic to the lattice (by the inclusion relation $\subseteq$) of all closed bicones of the G^2-frame (X, R) corresponding to T [11]. Analogically we have

Theorem 2.4. *Let T be an MG^2-algebra and (X, R) the MG^2-frame corresponding to T. Then the lattice of all congruences of T is anti-isomorphic to the lattice of all closed bicones of (X, R) (ordered by inclusion $\subseteq$).*

Proof. Let T be an MG^2-algebra and (X, R) the MG^2-frame corresponding to T. Identify the elements of T with corresponding clopens of X. Let F be a $\Diamond$-filter of T. Then $V = \bigcap\{Y : Y \in F\}$ is a closed cone of X. Taking into account that F is a Skolem filter we have $Y \in F \Rightarrow \neg_\Gamma Y \in F$ for every $Y \in T$. Notice that $\neg_\Gamma x \leq x$ [30, Theorem 1.3]. But, since X is a disjoint union of chains, we have that $\neg_\Gamma Y = -R_\rho^{-1}(R_\rho(-Y))$ is a bicone of X. So, $V = \{\neg_\Gamma Y : Y \in F\}$ is closed bicone.

Conversely, if V is a closed bicone, then the set F_V of all clopen cones of X that contain V forms $\Diamond$-filter. Indeed, it is obvious that F is a filter. It is also

Skolem filter because if a clopen Y contains V, then $\neg\ulcorner Y(= -R_\rho^{-1}(R_\rho(-Y)))$ also contains V. Also, by the set-theoretical operations we deduce that if V contains clopens Y_1 and Y_2 and $Y_1 \rightharpoonup Y_2 \in F$, then $V \subset \Diamond Y_1 \rightharpoonup \Diamond Y_2$.

It is obvious that if V_1, V_2 are closed bicones and $V_1 \subset V_2$, then $F_{V_1} \supset F_{V_2}$. ∎

Theorem 2.5. *The variety* $\mathbf{MG^2}$ *is generated by its finite members.*

Proof. Let $P = Q$ be an equation that does not hold in the variety $\mathbf{MG^2}$. We will show that there exists a finite MG^2-algebra in which $P = Q$ does not hold. Observe, that if an MG^2-algebra A is subdirectly irreducible, then the MG^2-frame (X, R) corresponding to A is linearly ordered [11] and A is a chain as well.

Notice that any linearly ordered MG^2-algebra A is subdirectly irreducible. Indeed, the only non-trivial $\Diamond$-filter of A is $\{1\}$ since $\neg\ulcorner a = 0$ for any $a \neq 1$. But $\{1\}$ corresponds to the trivial congruence relation on A.

So, if some equation $P = Q$ does not hold in the variety $\mathbf{MG}^2$, then it does not hold in some subdirectly irreducible MG^2-algebra A which is a chain. Let $a_1, \ldots, a_m \in A$ be the elements of A where the equation $P = Q$ is refuted, i.e. $(P \rightharpoonup Q) \wedge (Q \rightharpoonup P) \neq 1$ on the elements $a_1, \ldots, a_m$. It is clear that the set of elements of A generated by the elements $a_1, \ldots, a_m$ by means of the operations $\wedge, \vee, \rightharpoonup, \rightharpoondown$ is finite, say $D_1(\subset A)$. Add to this set D_1 all values of subterms of the term $(P \rightharpoonup Q) \wedge (Q \rightharpoonup P)$ (on the elements $a_1, \ldots, a_m$). Notice, that the obtained set is also finite, say having n elements. Observe that any finite chain algebra can be converted in a unique way into MG^2-algebra, where the operations $\Box$ and $\Diamond$ are defined uniquely (see example in this section). It is easy to check, that, if we take in such a way obtained finite n-element chain MG^2-algebra, then $(P \rightharpoonup Q) \wedge (Q \rightharpoonup P) \neq 1$ on the corresponding elements of the MG^2-algebra. From here we deduce the statement of the theorem. ∎

In that way we have proved that the variety $\mathbf{MG^2}$ is generated by its finite members. Now we define the logic MG^2 in algebraic terms: MG^2 is the set of all formulas valid in all finite totally ordered MG^2-algebras. This definition of the logic MG^2 is equivalent to the Kripke semantic definition given in the introduction.

As follows from the duality (Proposition 2.2) there is a one-to-one correspondence between the homomorphic images of an MG^2-algebra T and the closed bicones of the MG^2-frame (X, R) corresponding to it, and between subalgebras of MG^2-algebra T and correct partitions of (X, R) (for details see [16]), where *a correct partition* of a MG^2-frame (Y, R) is an equivalence relation E on Y, such that

- E is a closed equivalence relation, i.e. the E-saturation[1] of any closed subset is closed;
- E-saturation of any upper cone (down cone) is an upper cone (a down cone);
- $(\forall x, y \in Y)(E(x) \cap R^{-1}(E(y)) \neq \emptyset \Rightarrow E(x) \subseteq R^{-1}(E(y))$;
  $(\forall x, y \in Y)(E(x) \cap R(E(y)) \neq \emptyset \Rightarrow E(x) \subseteq R(E(y))$;
- for every $x, y \in Y$, if $\neg(xEy)$, then there exists a saturated clopen upper cone (down cone) U such that $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.
- there is a MG^2-frame (Z, Q) and a strongly isotone onto map $f : Y \to Z$ such that $Kerf = E$.

Remark. Notice that if $f : Y \to Z$ is a strongly isotone onto map, then $E(= Kerf)$ is a correct partition. In this case all E-saturated clopen upper cones of (Y, R) form a subalgebra of the algebra of all clopen upper cones (Y, R), i.e. the E-saturated clopen upper cones of (Y, R) are closed under all operations of MG^2-algebra. It is provided by the conditions of the definition, since for any saturated clopen subset $U \subset Y$ we have that the sets $R(U)$ and $R^{-1}(U)$ are saturated clopen sets; and operations $\rightharpoonup$ and $\rightharpoondown$ are defined by means of R and R^{-1}. About correct partition (bisimulation equivalence in other terminology) and correspondence between subalgebras and correct partition we refer also to [2,3,12] for Heyting and monadic Heyting algebras.

Let (X, R) be a MG^2-frame and $x \in X$. A *chain out* of x is a linearly ordered subset of $R(x) \cup \{x\}$ with the least element x; the *depth of* x is the maximum cardinality of a chain out of x denoted by $depth(x)$.

Notice, that if (X, R) is a disjoint union of n-element chain MG^2-frames (X_i, R_i) $(i = 1, \ldots, m)$, then there exists a unique order isomorphism $\varphi_{ij} : X_i \to X_j$ for every $i, j \leq m$. Then the maximal correct proper partition of (X, R) will be a partition, for which any class containing x contains also the element $\varphi_{ij}(x)$ for every $i, j \leq m$. If (X, R) is a disjoint union of an n-element chain MG^2-frame and an m-element chain MG^2-frame with $n \neq m$, then there does not exist any non-trivial correct partition (see the example below Fig. 1).

Indeed, let us consider any finite n-element strongly symmetric Kripke frame (X, R) and E is non-trivial correct partition. Let us suppose that $E(a)$

[1] If $V \subseteq Y$ then E-saturation of V is $E(V) = \bigcup_{x \in V} E(x)$, where $E(x)$ is a class of all elements E-equivalent to x.

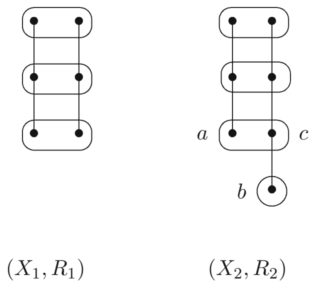


Figure 1. Correct/incorrect partition

is a nontrivial E-equivalence class such that a is the smallest element of this class and there exists $b \in E(a)$ such that aRb. Then $E(a) \cap R(E(b)) \neq \emptyset$ (since it contains b), but $E(a) \nsubseteq R(E(b))$ (since $a \notin R(E(b))$) that contradicts to the correctness of E.

Let us consider strongly symmetric Kripke frame (X, R) which is disjoint union of three-element linearly ordered frame, say Y, and four-element linearly ordered frame, say Z, and E is non-trivial equivalence relation, where E-equivalent elements are inside ovals (or circle), see Fig. 1. Let a be the smallest element of Y (three-element chain) and b be the smallest element of Z (four-element chain). Then, $R^{-1}(a) \cup \{a\}$ is a down cone, but $E(R^{-1}(a) \cup \{a\})$ is not down cone because $b \notin E(R^{-1}(a) \cup \{a\})$ which contradicts to the correctness of E.

Therefore, if we have disjoint union of (say) two finite strongly symmetric Kripke frames, then non-trivial equivalence relation E will be correct partition only in the case when the frames have the same number of elements and any elements from $E(x)$ have the same depth. Let us illustrate correct partition by discussing two examples. Two MG^2-frames, (X_1, R_1) and (X_2, R_2), are depicted in Fig. 1, where (X_1, R_1) is a disjoint union of two three-element MG^2-frames and (X_2, R_2) is a disjoint union of three- and four-element MG^2-frames. The elements in the inside of ovals and circles are E-equivalent. Then the partition for (X_1, R_1) is correct, but the partition for (X_2, R_2) is not correct, since $E(a) \cap R(E(b)) \neq \emptyset$ and $E(a) \nsubseteq R(E(b))$, where a is the bottom element of three-element chain and b is the bottom element of four-element chain. Notice, that we can not take as an equivalence class three bottom elements (the elements a, b and the element c which covers b)), since in this case $R(\{a, b, c\})$ is not E-saturated, because it contains c and does not contain a, b.

Recall some basic facts from universal algebra. Let $\mathbf{K}$ be a variety. Recall that an algebra $A \in \mathbf{K}$ is said to be a *free algebra* over $\mathbf{K}$, if there exists a set $A_0 \subset A$ such that A_0 generates A and every mapping f from A_0 to any algebra $B \in \mathbf{K}$ has a unique extension to a homomorphism h from A to B. In this case A_0 is said to be *a set of free generators* of A. If the set of free generators is finite, then A is said to be a *free algebra on finitely many generators*. We denote a free algebra A with $m \in (\omega + 1)$ free generators by $F_{\mathbf{K}}(m)$. We shall omit the subscript $\mathbf{K}$ if the variety $\mathbf{K}$ is known.

We can characterize the m-generate free algebra A on the generators $g_1, \ldots, g_m$ over the variety $\mathbf{K}$ in the following way: the algebra A is a free algebra on the generators $g_1, \ldots, g_m$ iff for any m variable equation $P(x_1, \ldots, x_m) = Q(x_1, \ldots, x_m)$, the equation holds in the variety $\mathbf{K}$ iff the equation $P(g_1, \ldots, g_m) = Q(g_1, \ldots, g_m)$ is true in the algebra A on the free generators [4,24]. We also formulate the characterization of m-generate free algebra A on the generators $g_1, \ldots, g_m$ over the variety $\mathbf{K}$ given in [19], §26, Theorem 1: *A is a free algebra in $\mathbf{K}$ on the generators $g_1, \ldots, g_m$ if and only if for any equality $P(x_1, \ldots, x_m) = Q(x_1, \ldots, x_m)$, the later belongs the equational theory of $\mathbf{K}$ if and only if $P(g_1, \ldots, g_m) = Q(g_1, \ldots, g_m)$ is true in A.*

Let $\mathbf{K}$ be any variety of algebras. An algebra A is said to be *a retract* of the algebra B, if there are homomorphisms $\varepsilon : A \to B$ and $h : B \to A$ such that $h\varepsilon = Id_A$, where Id_A denotes the identity map over A. An algebra $A \in \mathbf{K}$ is called *projective* $\mathbf{K}$, if for any $B, C \in \mathbf{K}$, any epimorphism (that is an onto homomorphism) $\gamma : B \to C$ and any homomorphism $\beta : A \to C$, there exists a homomorphism $\alpha : A \to B$ such that $\gamma\alpha = \beta$. In varieties, projective algebras are characterized as retracts of free algebras. A subalgebra A of $F_{\mathbf{K}}(m)$ is said to be a *projective subalgebra* if there exists an endomorphism $h : F_{\mathbf{K}}(m) \to F_{\mathbf{K}}(m)$ such that $h(F_{\mathbf{K}}(m)) = A$ and $h(x) = x$ for every $x \in A$.

Now we will give some topological facts for descriptive frames which will be very useful for a description of free and projective MG^2-algebras.

Claim. Let (X, R) is a descriptive MG^2-frame and $Y \subset X$ a dense subset of X (i.e. $clY = X$, where cl is the closure operator of the space X), then the family $\{Y \cap V : V$ is a closed and open upper subset of X$\}$ forms an MG^2-algebra, which is isomorphic to the MG^2-algebra of all closed and open upper subsets of X.

The correctness of this assertion follows from the general topological property: if V is a clopen of X and $Y \subset X$ is a dense subset of X, then $cl(Y \cap V) = V$. Indeed, $cl(Y \cap V) = V$ for any closed and open (clopen) upper subset V of X, where $Y \subset X$ is a dense subset of X, and (Y, R_Y)

is a Kripke frame (not a descriptive MG^2-frame, where R_Y is the restriction of R to the subset Y). Observe, that the set of all clopen subsets of X forms Boolean algebra which is a base for clopen sets of the topology τ of the Stone space X. Let Y be a dense subset of X. Then the family $\{Y \cap V : V$ is closed and open subset of X$\}$ forms a Boolean algebra which is isomorphic to the Boolean algebra of the clopen subsets of the Stone space (= zero-dimensional, Hausdorf and compact space) X. The topological space Y, with the base $\{Y \cap V : V$ is a closed and open subset of X$\}$ of open subsets, is Hausdorff and zero-dimensional, but not compact (in general). The compactification of the space Y by the base $\{Y \cap V : V$ is a closed and open subset of X$\}$ gives a Hausdorf, zero-dimensional and compact space κY, which is homeomorphic to the Stone space X. Recall, that the compactification of the space Y is a pair $(\kappa Y, \kappa)$, where κY is a compact space and $\kappa : Y \to \kappa Y$ a homeomorphic embedding such that $cl\kappa(Y) = \kappa Y$. In the sequel, we identify Y with the homeomorphic image $\kappa(Y) \subset \kappa Y$. Extending this consideration to the relation R_Y (using the duality) we get MG^2-frame $(\kappa Y, \kappa R_Y)$ which is isomorphic to (X, R). For detail information on the topological spaces and compactifications we refer to [9] (where Stone space is zero-dimensional, Hausdorf and compact space). So, to describe a free algebra it is enough to have a proper dense subset of the space X.

3. Free MG^2-algebras

This section is devoted to a description of the finitely generated free MG^2-algebras and a study some of their properties.

At first we describe the one-generated free MG^2-algebra. We will describe a frame (X, R) such that the free one-generated MG^2-algebra may be obtained as a subalgebra of the MG^2-algebra of all upper cones (X, R). (X, R) is depicted in Fig. 2 and this subalgebra is generated by a single upper cone depicted by the encircled points. The above description of one-generated free MG^2-algebra we can generalize to the m-generated case $(m > 1)$.

Let (C_n^m, R_n^m) $(0 \leq m \leq n > 0)$ be an MG^2-frame,[2] where C_n^m is an n-element set $\{c_1^m, \dots, c_n^m\}$ and R_n^m is an irreflexive and transitive relation such that $c_1^m R_n^m c_2^m \dots c_{n-1}^m R_n^m c_n^m$. Let $X_n = \coprod_{m=0}^n C_n^m$ be a disjoint union of C_n^m, $R_n = \bigcup_{m=0}^n R_n^m$ and $(X, R) = \bigcup_{n=1}^\infty (X_n, R_n)$. Let g_n^m $(0 \leq m \leq n > 0)$ be the (unique) m-element upper cone of C_n^m and $g_n = \{g_n^0, \dots, g_n^n\}$. Then

[2]Notice, that if we have a finite frame, then we have discrete topology on it.

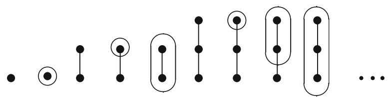


Figure 2. A part of free cyclic MG^2-frame

$G = \bigcup_{n=1}^{\infty} g_n \subset X$. A part of X is depicted in the Fig. 2, where the generator is represented by circles or ovals. So, we have (X, R) and $G \subset X$.

Let $(T, \cup, \cap, \rightharpoonup, \neg\!\!\rightharpoondown, \Box, \Diamond, \emptyset, X)$ be the algebra generated by G, within the algebra of all upper cones of (X, R), by means of the following operations: the union $\cup$, the intersection $\cap$, $A \rightharpoonup B = -R_\rho^{-1} - (-A \cup B)$, $A \rightharpoondown B = R_\rho(A \cap -B)$, $\Box(A) = -R^{-1} - (A)$, $\Diamond(A) = R(A)$ for any upper cones of A and B of X_n, where R_ρ is a reflexive closure of the relation R.

Observe, that if A is an upper cone of a MG^2-frame, then $\Box A \supseteq A$ and $\Diamond A \subseteq A$ (because of irreflexivity of R).

Lemma 3.1. *The MG^2-algebra $T_n^m = \mathfrak{S}(C_n^m) = (Con(C_n^m), \cup, \cap, \rightharpoonup, \rightharpoondown, \Box, \Diamond, \emptyset, C_n^m)$ is generated by any element of T_n^m, where $Con(C_n^m)$ is the set of all upper cones of (C_n^m, R_n^m), $\cup$ is the union, $\cap$ is the intersection, $A \rightharpoonup B = -(R_n^m)_\rho^{-1} - (-A \cup B)$, $A \rightharpoondown B = (R_n^m)_\rho(A \cap -B)$, $\Box A = -(R_n^m)^{-1} - (A)$, $\Diamond A = R_n^m(A)$.*

Proof. Observe that MG^2-algebra T_n^m is generated by the empty set $\emptyset$. Indeed, $\emptyset < \Box(\emptyset) < \cdots < \Box^n(\emptyset)$, hence $\emptyset$ generates T_n^m. Also, using $\neg$ and $\wedge$, from any element of T_n^m we can obtain $\emptyset$ and this is way any element of T_n^m generates this algebra. ■

Theorem 3.2. *The algebra*

$$(T, \cup, \cap, \rightharpoonup, \rightharpoondown, \Box, \Diamond, \emptyset, X)$$

is freely generated in $\mathbf{MG}^2$ *by a single element.*

Proof. It is obvious that $(T, \cup, \cap, \rightharpoonup, \rightharpoondown, \Box, \Diamond, \emptyset, X)$ is an MG^2-algebra, which is a subdirect product of subdirectly irreducible MG^2-algebras T_n^m corresponding to the MG^2-frames (C_n^m, R_n^m) $(0 \leq m \leq n > 0$. Therefore any true MG^2-algebra equation holds in T. Now, let us suppose that some equation with one variable $P = Q$ does not hold in the variety $\mathbf{MG^2}$. Then $P = Q$ does not hold in some finite subdirectly irreducible MG^2-algebra A. A MG^2-frame corresponding to A is isomorphic to some finite MG^2-frame, say (Y, R), which is isomorphic to (C_n^m, R_n^m) for some non-negative integer n. Obviously, (C_n^m, R_n^m) is a bicone of (X, R). Therefore, there exists a homomorphism $h : T \to A$, sending the generator of T to the generator of A. From here we conclude that the equation $P = Q$ does not hold in T. ■

Suppose $(X; R)$ is an MG^2-frame, $A = \mathfrak{S}(X)$ and $g_1, \ldots, g_n \in A$. Now we will present a criterion to decide whether the algebra A is generated by $g_1, \ldots, g_n$. Our criterion extends the analogous one for descriptive intuitionistic frames from [15] to MG^2-frames.

Denote by $\mathbf{n}$ the set $\{1, \ldots, n\}$. Given $g_1, \ldots, g_n$ and given $p \subseteq \mathbf{n}$, we define G_p to be the set of all $x \in X$ such that for $i = 1, \ldots, n$, $x \in g_i$ iff $i \in p$, and given $x \in X$ we set $Col(x) = \{i \in \mathbf{n} : x \in g_i\}$.

It is obvious that $\{G_p\}_{p \subseteq \mathbf{n}}$ is a partition of X which we call the *colouring* of X.[3] A point $x \in G_p$ is said to have the *colour* p, written as $Col(x) = p$. Let us note that $g_i = \bigcup\{G_p : i \in p\}$, $i = 1, \ldots, n$.

LEMMA 3.3. *Suppose E is a correct partition of X and $\{g_1, \ldots, g_n\}$ is an arbitrary set of subsets of X. The following two conditions are mutually equivalent:*

(1) Every g_i is E-saturated, that is $E(g_i) = g_i (1 \leq i \leq n)$;

(2) Every class G_p is E-saturated, that is $E(G_p) = G_p (p \subseteq \mathbf{n})$.

PROOF. Easy. ■

THEOREM 3.4. (Colouring Theorem) *A is generated by $g_1, \ldots, g_n$ iff for every non-trivial correct partition E of $X (= \mathfrak{M}(A))$, there exists an equivalence class of E containing points of different colours.*

PROOF. Let us identify A with the set of clopen upper cones of $X (= \mathfrak{M}(A))$ and $g_1, \ldots, g_n$ with corresponding clopen upper cones of X. Suppose A is generated by $g_1, \ldots, g_n$ and E is a non-trivial correct partition of X. Consider the set A_E of E-saturated elements of A. Notice, that A_E is the subalgebra of E-saturated upper cones of X which is a subalgebra of A corresponding to the partition E and, since E is non-trivial, $A_E \neq A$. Since $g_1, \ldots, g_n$ generate A and E is a proper correct partition of X, there exists $i \leq n$ such that $g_i \notin A_E$. Therefore there exists $p \subseteq \mathbf{n}$ such that G_p is not E-saturated. But then there exists $x \in G_p$ such that $E(x) \cap G_p \neq \emptyset$ and $E(x) \cap -G_p \neq \emptyset$. Hence $E(x)$ contains points of different colour.

Conversely, suppose A is not generated by $g_1, \ldots, g_n$. Denote by A_0 the least subalgebra of A containing $g_1, \ldots, g_n$. Obviously A_0 is a proper subalgebra of A and the correct partition E of X corresponding to A_0 is non-trivial.[4] Moreover, since $g_1, \ldots, g_n \in A_0$, $E(g_i) = g_i (1 \leq i \leq n)$ and hence

[3] It is also clear that G_p are the atoms of the Boolean algebra $\mathcal{B}(g_1, \ldots, g_n)$ generated (in the Boolean algebra of all subsets of X) by $g_1, \ldots, g_n \subset X$.

[4] E is defined on X by putting xEy iff $x \in U \Leftrightarrow y \in U$ for every $U \in A_0$.

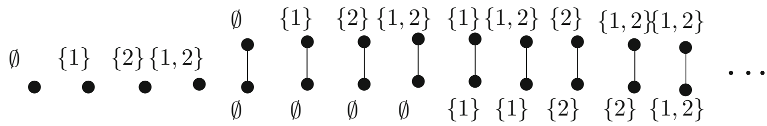


Figure 3. Colouring of MG^2-frame

$E(G_p) = G_p(p \subseteq \mathbf{n})$. But then every equivalence class of E contains only points of the same colour. ■

We now turn to the general case and describe the m-generated free MG^2-algebra. We do so by constructing a Kripke frame and by specifying a subalgebra of the MG^2-algebra of its upper cones.

Let $X(m)$ be a disjoint union of finite linearly ordered MG^2-frames such that for any positive integer n the number of n-element chains is defined in the following way. The number of n-element chains in $X(m)$ is defined to be the number of colorings of the n-element chain (X, R) that satisfy, for all $x, y \in X$, if xRy, then $Col(x) \subseteq Col(y)$.

Let $G_i = \{x \in X(m) : i \in Col(x)\}$, $i = 1, \ldots, m$ and $(T(m), \cup, \cap, \rightharpoonup, \rightharpoondown, \Box, \Diamond, \emptyset, X)$ be an algebra of all upper cones of $X(m)$ generated by $G_i = \{x \in X(m) : i \in Col(x)\}$, $i = 1, \ldots, m$ by means of the operations: $\cup$, $\cap$, $A \rightharpoonup B = -R_\rho^{-1} - (-A \cup B)$, $A \rightharpoondown B = R_\rho(A \cap -B)$, $\Box A = -(R^{-1} - (A)$, $\Diamond A = R(A)$. Then the MG^2-algebra $(T(m), \cup, \cap, \rightharpoonup, \rightharpoondown, \Box, \Diamond, \emptyset, X)$ will be the m-generated free algebra in the variety $\mathbf{MG^2}$. In the one-generated case, depicted in Fig. 2, the elements inside circles or ovals are colored in color $\{1\}$ and other elements in color $\emptyset$.

In Fig. 3 the part of $X(2)$ containing the one and two-element chains is depicted. The elements of these chains is colored by $p \subset \{1, 2\}$ colors.

Now we give some facts concerning to the representation of m-generated MG^2-algebra as a subalgebra of inverse limit of the family of m-generated free MG^2-algebras in the subvarieties of $\mathbf{MG^2}$ generated by finite number of finite chain MG^2-algebra. Moreover, we will give a characterization of finitely presented MG^2-algebras.

We denote by $\mathbf{MG^2_n}$ the variety of MG^2-algebras generated by $\{S_1, \ldots, S_n\}$, where S_i $(1 \leq i \leq n)$ is the $(i+1)$-element linearly ordered MG^2-algebra, i.e. $\mathfrak{M}(S_i)$ is linearly ordered i-element MG^2-frame. This subvariety of the variety $\mathbf{MG^2}$ can be picked out by the identity: $\Box^n 0 = 1$. Notice that $\mathbf{MG^2_n}$ is a locally finite variety and $\mathbf{MG^2}$ is generated by $\bigcup_{n \in \omega} \mathbf{MG^2_n}$.

Observe that the m-generated free MG^2_n-algebra $F_{\mathbf{MG^2_n}}(m)$ is a homomorphic image of $F_{\mathbf{MG^2}}(m)$ such that dually the MG^2-frame $X_n(m)$ of $F_{\mathbf{MG^2_n}}(m)$ is the bicone of the MG^2-frame $X(m)$ of $F_{\mathbf{MG^2}}(m)$ containing the i-element linearly ordered subframes for $i \leq n$. It is clear that $F_{\mathbf{MG^2_n}}(m)$ is finite.

We recall that if $\mathbf{K}$ is a variety of algebras and Ω is a set of m-ary $\mathbf{K}$-equations, then $F_{\mathbf{K}}(m, \Omega)$ is the object free over $\mathbf{K}$ with respect to the conditions Ω on the generators (see [19]).

An algebra A is called *finitely presented* if A is finitely generated, by the generators $a_1, \ldots, a_m \in A$, and there exist a finite number of equations $P_1(x_1, \ldots, x_m) = Q_1(x_1, \ldots, x_m), \ldots, P_n(x_1, \ldots, x_m) = Q_n(x_1, \ldots, x_m)$ holding in A on the generators $a_1, \ldots, a_m \in A$ such that if there exists an m-generated algebra B, with generators $b_1, \ldots, b_m \in B$, such that the equations $P_1(x_1, \ldots, x_m) = Q_1(x_1, \ldots, x_m), \ldots, P_n(x_1, \ldots, x_m) = Q_n(x_1, \ldots, x_m)$ hold in B on the generators $b_1, \ldots, b_m \in B$, then there exists a unique homomorphism $h : A \to B$ sending a_i to b_i.

Observe that we can rewrite any equation $P(x_1, \ldots, x_m) = Q(x_1, \ldots, x_m)$ in the variety $\mathbf{MG^2}$ into an equivalent one $P(x_1, \ldots, x_m) \leftrightarrow Q(x_1, \ldots, x_m) = 1$, where $P(x_1, \ldots, x_m) \leftrightarrow Q(x_1, \ldots, x_m) = (P(x_1, \ldots, x_m) \rightharpoonup Q(x_1, \ldots, x_m)) \wedge (Q(x_1, \ldots, x_m) \rightharpoonup P(x_1, \ldots, x_m))$. So, for $\mathbf{MG^2}$ we can replace n equations by one

$$\bigwedge_{i=1}^{n} P_i(x_1, \ldots, x_m) \leftrightarrow Q_i(x_1, \ldots, x_m) = 1.$$

Lemma 3.5. *Let P be an m-ary polynomial. Then there is a principal $\Diamond$-filter J of $F_{\mathbf{MG^2}}(m)$ such that $F_{\mathbf{MG^2}}(m, \{P = 1\}) \cong F_{\mathbf{MG^2}}(m)/J$.*

Proof. Let $J = \{x : x \in F_{\mathbf{MG^2}}(m), x \geq \neg\ulcorner P(g_1, \ldots, g_m)\}$, where $g_1, \ldots, g_m$ are free generators of $F_{\mathbf{MG^2}}(m)$, that is $\Diamond$-filter since $\neg \ulcorner P(g_1, \ldots, g_m)$ corresponds to clopen bicone. Then $g_1/J, \ldots, g_m/J$ are generators of $F_{\mathbf{MG^2}}(m)/J$, which has required universal property. Let A be an MG^2-algebra generated by $a_1, \ldots, a_m$, $P(a_1, \ldots, a_m) = 1$ and $f : F_{\mathbf{MG^2}}(m) \to A$ be the homomorphism such that $f(g_i) = a_i$, $i = 1, \ldots, m$. Then $P(g_1, \ldots, g_m) \in f^{-1}(\{1\})$ and $J \subset f^{-1}(\{1\})$. By the homomorphism theorem there is a homomorphism $f' : F_{\mathbf{MG^2}}(m)/J \to A$ such that $f'\pi_J = f$, where π_J is a natural homomorphism from $F_{\mathbf{MG^2}}(m)$ to $F_{\mathbf{MG^2}}(m)/J$. It should be clear that f' is needed homomorphism extending the map $g_i/J \to a_i$. Observe that the same statement holds for $F_{\mathbf{MG^2}}(m, E)$ where $E = \{P(x_1, \ldots, x_m) = 1\}$. ∎

Lemma 3.6. *Let* $0 \neq u \in F_{\mathbf{MG^2}}(m)$. *Then* $J = \{x : x \geq \neg \ulcorner u\}$ *is a proper* $\diamondsuit$*-filter in* $F_{\mathbf{MG^2}}(m)$ *such that* $F_{\mathbf{MG^2}}(m)/J \cong F_{\mathbf{MG^2}}(m, \{P = 1\})$ *for some* m*-ary polynomial* P.

Proof. Let J be a $\diamondsuit$-filter satisfying the condition of the Lemma. Then $u = P(g_1, \dots, g_m)$ for some polynomial P, where $g_1, \dots, g_m$ are free generators. We have that $F_{\mathbf{MG^2}}(m)/J$ is generated by $g_1/J, \dots, g_m/J$, and that

$$P(g_1/J, \dots, g_m/J) = P(g_1, \dots, g_m)/J = 1_{F(m)/J}.$$

The rest can be verified as in the proof of Lemma 3.5. ■

Theorem 3.7. *If* $A \in \mathbf{MG^2}$ *is finite and generated by* m *elements, then there is a principal filter* J *such that* $A \cong F_{\mathbf{MG^2}}(m)/J$.

Proof. Let $A \in \mathbf{MG^2}$ be finite and suppose that A is generated by $a_1, \dots, a_m \in A$. Let P_{a_i} be the m-ary polynomial x_i, $i = 1, \dots, n$, and in general let P_x be a polynomial such that $P_x(a_1, \dots, a_m) = x$, for each $x \in A$. Let Ω be the collection of equations of the type $P_x \vee P_y = P_{x \vee y}$, $P_x \wedge P_y = P_{x \wedge y}$, $P_x \rightharpoonup P_y = P_{x \rightharpoonup y}$, $P_x \rightharpoondown P_y = P_{x \rightharpoondown y}$, $\Box P_x = P_{\Box x}$, $\diamondsuit P_x = P_{\diamondsuit x}$ and $P_0 = 0$, $P_1 = 1$. Then $A \cong F_{\mathbf{MG^2}}(m, \Omega)$. For if B is generated by $b_1, \dots, b_m$ and $b_1, \dots, b_m$ satisfy Ω then $\{P_x(b_1, \dots, b_m) : x \in A\} = B$ and the map $f : A \to B$ defined by $f(x) = P_x(b_1, \dots, b_m)$ is a homomorphism extending the map $a_i \mapsto b_i$, $i = 1, \dots, m$. Since Ω is finite, the theorem follows. ■

Theorem 3.8. *If* $A \in F_{\mathbf{MG^2}}(m)$ *is finitely presented with the collection of symbols* $z_1, \dots, z_m$ *and an equation* $P(z_1, \dots, z_m) = 1$, *then there is a principal* $\diamondsuit$*-filter* J *of* $F_{\mathbf{MG^2}}(m)$ *such that* $A \cong F_{\mathbf{MG^2}}(m)/J$.

Proof. Let $B \in \mathbf{MG^2}$ and B is generated $b_1, \dots, b_m$ with $P(b_1, \dots, b_m) = 1$. Then the map $g_i \mapsto b_i$, $i = 1, \dots, m$ and $g_1, \dots, g_m$ are free generators of $F_{\mathbf{MG^2}}(m)$, can be extended to a homomorphism of $F_{\mathbf{MG^2}}(m)$ onto B. That is, there exists a $\diamondsuit$-filter F such that $B \cong F_{\mathbf{MG^2}}(m)/F$. Since $P(b_1, \dots, b_m) = 1$, we have $P(g_1, \dots, g_m) \in F$. Let J be the principal $\diamondsuit$-filter of $F_{\mathbf{MG^2}}(m)$ generated by $P(g_1, \dots, g_m)$. It is obvious that $J \subseteq F$. By the homomorphism theorem there is a homomorphism of $F_{\mathbf{MG^2}}(m)/J$ onto $F_{\mathbf{MG^2}}(m)/F$ and hence a homomorphism of $F_{\mathbf{MG^2}}(m)/J$ onto B. Hence, $F_{\mathbf{MG^2}}(m)/J$ is a finitely presented algebra with collection of generating symbols $z_1, \dots, z_m$ and the equation $P(z_1, \dots, z_m) = 1$. According to [23] (Chapter 5, Corollary 2) $A \cong F_{\mathbf{MG^2}}(m)/J$. ■

From the Lemmas 3.5, 3.6 and Theorem 3.8 we have

Theorem 3.9. *An* m*-generated* MG^2*-algebra* A *is finitely presented iff*

$$A \cong F_{\mathbf{MG^2}}(m)/[u),$$

where $[u)$ is a principal $\Diamond$-filter generated by some element $u \in F_{\mathbf{MG^2}}(m)$.

Now we represent m-generated free MG^2-algebra by means of inverse limit, and so we construct inverse system, which will be useful for characterization of finitely generated projective MG^2-algebras.

THEOREM 3.10. *$F_{\mathbf{MG^2}}(m)$ is isomorphic to a subalgebra of an inverse limit $F_\infty(m)$ of a chain of order type ω^* of finite algebras, for $m \in \omega$, and the finite algebras are isomorphic to $F_{\mathbf{MG^2_n}}(m)$.*

PROOF. Let $g_1^{(n)}, \ldots, g_m^{(n)}$ be the free generators of $F_{\mathbf{MG^2_n}}(m)$, $m \in \omega$. By Theorem 3.7 there is a principal $\Diamond$-filter J, that is generated by $\Box^n 0$, such that $F_{\mathbf{MG^2}}(m)/J \cong F_{\mathbf{MG^2_n}}(m)$, where g_i/J corresponds to $g_i^{(n)}$ for $i = 1, \ldots, m$. In fact, there is a chain of filters $J_1 \leq J_2 \leq J_3 \leq \cdots$, where $J_i \leq J_j$ iff $J_i \supset J_j$, such that for each $n \in \omega$ there is an isomorphism $\sigma_n : F_{\mathbf{MG^2}}(m)/J_n \to F_{\mathbf{MG^2_n}}(m)$, with the property that $\sigma_n \circ h_n = f_n$, f_n being the homomorphism $F_{\mathbf{MG^2}}(m) \to F_{\mathbf{MG^2_n}}(m)$ satisfying $f_n(g_i) = g_i^{(n)}$, $i = 1, \ldots, m$, and h_n being the projection $F_{\mathbf{MG^2}}(m) \to F_{\mathbf{MG^2}}(m)/J_n$, defined by $g_i \to g_i/J_n$. Observe, that the inclusion $J_j \subset J_i$ with $j > i$ is based on the inequality $\Box^i 0 < \Box^j 0$ which valid in $F_{\mathbf{MG^2}}(m)$.

Let $\pi_{kl} : F_{\mathbf{MG^2}}(m)/J_k \to F_{\mathbf{MG^2}}(m)/J_l$ for $k \geq l \geq 1$ be the homomorphism defined by $x/J_k \mapsto x/J_l$ $(J_k \subset J_l)$ (See Fig. 4). Then $\pi_{lt} \circ \pi_{kl} = \pi_{kt}$ for $k \geq l \geq t \geq 1$. Thus, we have an inverse system $U_m^{\mathbf{MG^2}} = \{F_{\mathbf{MG^2}}(m)/J_k, \pi_{kl} : k \geq l \geq 1\}$ and its inverse limit $F_\infty(m) = \lim U_m^{\mathbf{MG^2}}$ exists, since the quotients $F_{\mathbf{MG^2}}(m)/J_k$, $k \geq 1$, are finite. $F_\infty(m) = \{(x_k)_k \in \prod_{k=1}^{\infty} F_{\mathbf{MG^2}}(m)/J_k : \pi_{lt}(x_l) = x_t, l \geq t \geq 1\}$.

Let $\pi_n : F_\infty(m) \to F_{\mathbf{MG^2}}(m)/J_n$ be the canonical projections and $z_i = (g_i/J_1, g_i/J_2, \ldots)$, $i = 1, \ldots, m$. Then $z_i \in F_\infty(m)$. We consider the subalgebra of $F_\infty(m)$ generated by $\{z_1, \ldots, z_m\}$ and show that it is isomorphic to $F_{\mathbf{MG^2}}(m)$ by proving that it has the required universal property. Since $\mathbf{MG^2} = \bigcup_{n \in \omega} \mathbf{MG^2_n}$, we only need to show that every map $z_i \mapsto a_i$ to an algebra A generated by $\{a_1, \ldots, a_m\}$ which belongs to some $\mathbf{MG^2_n}$ can be extended to a homomorphism from the algebra A_1 generated by $\{z_1, \ldots, z_m\}$ to A. But if $A \in \mathbf{MG^2_n}$, then there is a homomorphism $h : F_{\mathbf{MG^2_n}}(m) \to A$ such that $h(g_i^{(n)}) = a_i$ $i = 1, \ldots, m$ and $h \circ \sigma_n \circ \pi_n|_{A_1} : A_1 \to A$ is the needed homomorphism extending the map $z_i \mapsto a_i$, $i = 1, \ldots, m$. ■

$$F_{\mathbf{MG^2}}(m)/J_1 \xleftarrow{\pi_{12}} F_{\mathbf{MG^2}}(m)/J_2 \xleftarrow{\pi_{23}} F_{\mathbf{MG^2}}(m)/J_3 \xleftarrow{\pi_{34}} \cdots F_\infty(m) \longleftarrow F_{\mathbf{MG^2}}(m)$$

Figure 4. Inverse system of factors of $F_{\mathbf{MG^2}}(m)$

4. Projective MG^2-algebras

This section is devoted to a study of finitely generated projective MG^2-algebras.

THEOREM 4.1. *For every m and n any subalgebra A of the algebra $F_{\mathbf{MG^2_n}}(m)$ is a retract of $F_{\mathbf{MG^2_n}}(m)$, i.e. for any injective homomorphism $\varepsilon : A \to F_{\mathbf{MG^2_n}}(m)$ there is a surjective homomorphism $h : F_{\mathbf{MG^2_n}}(m) \to A$ such that $h\varepsilon = Id_A$.*

PROOF. We will prove this Theorem using duality. Let $(X_n(m), R_n)$ be the MG^2-frame corresponding to the m-generated free MG^2_n-algebra $F_{\mathbf{MG^2_n}}(m)$, i.e. $(X_n(m), R_n) = \mathfrak{M}(F_{\mathbf{MG^2_n}}(m))$. Let E be the correct partition of $(X_n(m), R_n)$ corresponding to the subalgebra $\varepsilon(A)$ of the algebra $F_{\mathbf{MG^2_n}}(m)$. Then $(X_n(m)/E, R_n^E)$ is an MG^2-frame corresponding to the algebra A, where $(\forall V_1, V_2 \in X_n(m)/E)\ V_1 R_n^E V_2 \Leftrightarrow (\exists v_1, v_2)(v_1 \in V_1, v_2 \in V_2)(v_1 R_n v_2)$. Notice, that $X_n(m)$ is a finite set (as well as $X_n(m)/E$). The algebra A is isomorphic to the saturated upper cones of $X_n(m)$. Identify the elements of A with the corresponding saturated cones of $X_n(m)$. Let U be a join irreducible bicone of $X_n(m)/E$ (i.e. if $U = U_1 \cup U_2$, where U_1, U_2 are bicones of $X_n(m)/E$, then either $U = U_1$ or $U = U_2$). The bicone U contains a join irreducible bicone X_U of $X_n(m)$. The disjoint union $X = \coprod_{E(U)=U} X_U$ of all such kind of bicones is a bicone of $(X_n(m), R_n)$. It is clear that $(X_n(m)/E, R_n^E) \cong (X, R'_n)$, where R'_n is the restriction of R_n on $X(\subset X_n(m))$. To see this phenomenon more sharply let us consider the MG^2-frame (X_1, R_1) depicted in the Fig. 1. (X_1, R_1) is a cardinal sum of two three-element chains. The elements in the inside of ovals are equivalent. The only saturated bicone U of the MG^2-frame is (X_1, R_1). Obviously that $U(= X_1)$ contains a three-element bicone, say X_U, and $(X_1/E, R_1^E) \cong (X, R'_1)$, where $X = X_U$ and R'_1 is the restriction of R_1 on X. So, we have a strongly isotone embedding $f : X \to X_n(m)$ and strongly isotone onto map $g : X_n(m) \to X$, corresponding to the existing correct partition E, such that $gf = Id_X$, where $g(x) = E(x) \cap X$. So, according to the duality, for a given injective homomorphism $\varepsilon : A \to F_n(m)$ there exists a surjective homomorphism $h : F_n(m) \to A$ such that $h\varepsilon = Id_A$, where $\varepsilon = \mathfrak{S}(g)$ and $h = \mathfrak{S}(f)$. ■

From this theorem we it follows

COROLLARY 4.2. *Any m-generated subalgebra A of the m-generated free MG^2_n-algebra $F_{\mathbf{MG^2_n}}(m)$ is projective.*

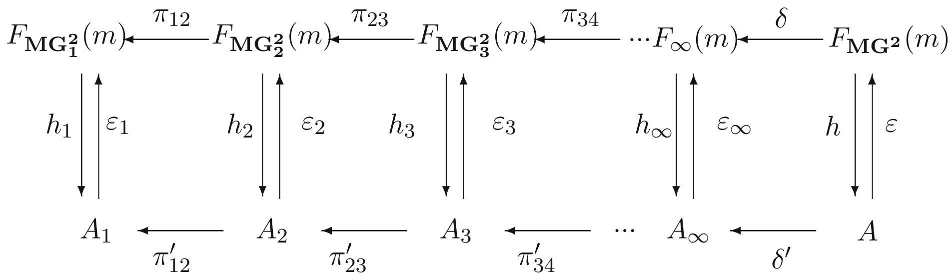


Figure 5. Inverse system MG^2-algebras

Proof. The proof immediately follows from Theorem 4.1. ■

From Theorems 3.10, 4.1 and Corollary 4.2 follows

Theorem 4.3. *Any m-generated subalgebra of the m-generated free MG^2-algebra $F_{\mathbf{MG^2}}(m)$ is projective.*

Proof. Let A be an m-generated MG^2-subalgebra of $F_{\mathbf{MG^2}}(m)$. We represent $F_{\mathbf{MG^2}}(m)$ as a subalgebra of the inverse limit $F_\infty(m)$ of the inverse system $\{F_{\mathbf{MG^2_k}}(m), \pi_{kl} : k \geq l \geq 1\}$ (Theorem 3.10). Let $A_n = \pi_n(A) \subset F_{\mathbf{MG^2_n}}(m)$ be the subalgebra of $F_{\mathbf{MG^2_n}}(m)$ that is the image of the canonical projection $\pi_n : F_{\mathbf{MG^2}}(m) \to F_{\mathbf{MG^2_n}}(m)$. Then the subalgebra A of $F_{\mathbf{MG^2}}(m)$ can be represented as a subalgebra of an inverse limit A_∞ of inverse system $\{A_k, \pi'_{kl} : k \geq l \geq 1\}$, where π'_{kl} is a restriction of π_{kl} on the subalgebra $A_l \subset F_{\mathbf{MG^2_l}}(m)$ (see Fig. 5). Let $\varepsilon : A \to F_{\mathbf{MG^2}}(m)$ ($\varepsilon_n : A_n \to F_{\mathbf{MG^2_n}}(m)$) be an embedding (identity map). According to the Theorem 4.1 for the embedding ε_n there exists surjective homomorphism $h_n : F_{\mathbf{MG^2_n}}(m) \to A_n$ such that $h_n \varepsilon_n = Id_{A_n}$. Dually, we have the diagram depicted in the Fig. 5, where we denote by f^* its dual $\mathfrak{S}(f)$. Using the duality we have $\varepsilon^*_{i+1} = \pi'^*_{i(i+1)} \varepsilon^*_i$, $\pi^*_{i(i+1)} h^*_i = h^*_{i+1} \pi'^*_{i(i+1)}$ and hence $\pi_{i(i+1)} \varepsilon_{i+1} = \varepsilon_i \pi'_{i(i+1)}$, $h_i \pi_{i(i+1)} = \pi'_{i(i+1)} h_{i+1}$ (Fig. 6).

Let us consider a fragment of this diagram represented in the Fig. 7. Observe that $\pi^*_{n(n+1)} : X_n(m) \to X_{n+1}(m)$ is an embedding, i.e. $\pi^*_{n(n+1)}(X_n(m))$ is a subbicone of $X_{n+1}(m)$. Then $E_{n+1} \cap X^2_n(m) = E_n$, where $Ker \varepsilon^*_i = E_i$ for $i = n, n+1$.

Observe, that $\pi^*_{n(n+1)} h^*_n = h^*_{n+1} \pi'^*_{n(n+1)}$. Let $h = (h_1, h_2, h_3, \ldots)$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots)$. Let $a_i = (a^{(i)}_1, a^{(i)}_2, a^{(i)}_3, \ldots)$, $i = 1, \ldots, m$, be generators of A. Then $h\varepsilon(a_i) = (h_1\varepsilon_1(a^{(i)}_1), h_2\varepsilon_2(a^{(i)}_2), h_3\varepsilon_3(a^{(i)}_3), \ldots) = (a^{(i)}_1, a^{(i)}_2, a^{(i)}_3, \ldots) = a_i$. So, A is projective. ■

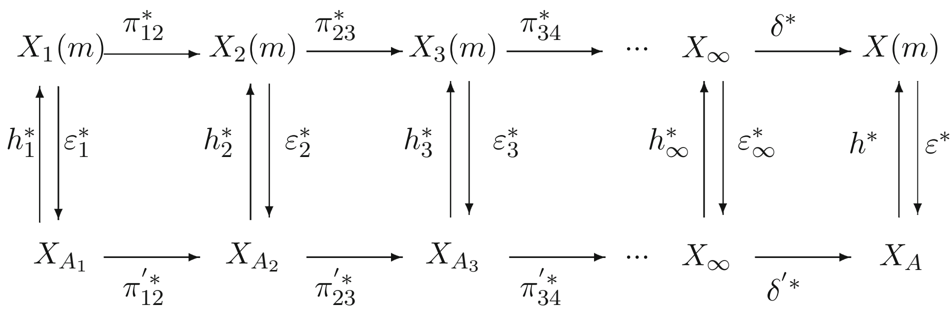


Figure 6. Direct system of MG^2-frames

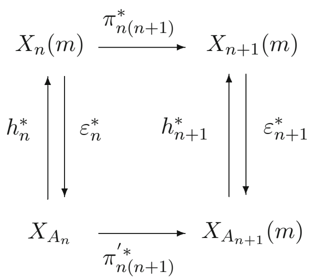


Figure 7. A part of diagram of direct system of MG^2-frames

MG^2-frame $X_3(1)$ with a correct partition is depicted in the Fig. 8, where non-trivial classes are represented by elements inside rectangle.

Notice, that any finite MG^2-algebra is not projective, since it is not a retract of any free MG^2-algebra. Indeed, $F_{\mathbf{MG^2}}(m)$ have no finite subalgebras. If A is a subalgebra of $F_{\mathbf{MG^2}}(m)$ then there exists non-trivial correct partition with non-trivial equivalence class containing infinitely many elements having different depths.

Proposition 4.4. [7,25] *Let* $\mathbf{V}$ *be a variety and* $F_{\mathbf{V}}(m)$ *be an* m*-generated free algebra of the variety* $\mathbf{V}$*, and let* $g_1, \ldots, g_m$ *be its free generators. Then an* m*-generated subalgebra* A *of* $F_{\mathbf{V}}(m)$ *with generators* $a_1, \ldots, a_m \in A$ *is projective if and only if there exist polynomials* $P_1(x_1, \ldots, x_m), \ldots, P_m(x_1, \ldots, x_m)$ *such that*

$$P_i(g_1, \ldots, g_m) = a_i$$

and

$$P_i(P_1(x_1, \ldots, x_m), \ldots, P_m(x_1, \ldots, x_m)) = P_i(x_1, \ldots, x_m),\ i = 1, \ldots, m.$$

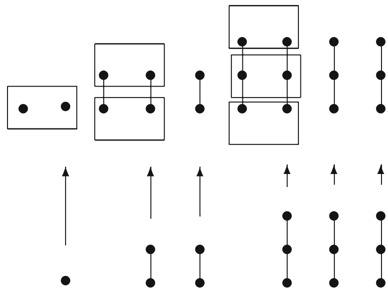


Figure 8. MG^2-frame $X_3(1)$ with a correct partition

From the Proposition we obtain that in $F_{\mathbf{V}}(m)$ holds

$$P_i(P_1(g_1,\dots,g_m),\dots,P_m(g_1,\dots,g_m)) = P_i(g_1,\dots,g_m) = a_i,$$

$i = 1,\dots,m$, i.e. $P_i(a_1,\dots,a_m) = a_i$ in A. This suggests to consider the free object $F_{\mathbf{V}}(m,\Omega)$ over the variety $\mathbf{V}$ with respect to the set of equations $\Omega = \{P_1(x_1,\dots,x_m) = x_1,\dots,P_1(x_1,\dots,x_m) = x_m\}$.

Adapting the result obtained in [25] for the variety $\mathbf{MG^2}$ it holds the following

Theorem 4.5. *If A is an n-generated projective MG^2-algebra, then A is finitely presented.*

Proof. Since A is n-generated projective MG^2-algebra, A is a retract of $F_{\mathbf{MG^2}}(n)$, i.e. there exist homomorphisms $h : F_{\mathbf{MG^2}}(n) \to A$ and $\varepsilon : A \to F_{\mathbf{MG^2}}(n)$ such that $h\varepsilon = Id_A$, $h(g_i) = a_i$, and moreover, according to Proposition 4.4, there exist n polynomials $P_1(x_1,\dots,x_n),\dots,P_n(x_1,\dots,x_n)$ such that

$$P_i(g_1,\dots,g_n) = \varepsilon(a_i) = \varepsilon h(g_i)$$

and

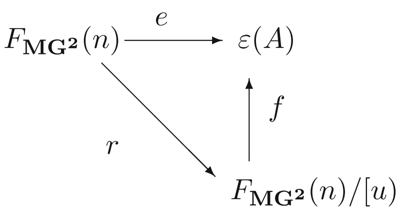
$$P_i(P_1(x_1,\dots,x_n),\dots,P_n(x_1,\dots,x_n)) = P_i(x_1,\dots,x_n),\ i = 1,\dots,n,$$

where $g_1,\dots,g_n$ are the free generators of $F_{\mathbf{MG^2}}(n)$. Observe that $h(g_1),\dots,h(g_n)$ are generators of A, where $h(g_i) = a_i \in A$, $i = 1,\dots,n$. Let e be the endomorphism $\varepsilon h : F_{\mathbf{MG^2}}(n) \to F_{\mathbf{MG^2}}(n)$. This endomorphism has properties: $ee = e$ and $e(x) = x$ for every $x \in \varepsilon(A)$.

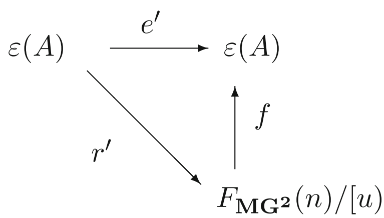
Let us consider the set of equations $\Omega = \{P_i(x_1,\dots,x_n) \leftrightarrow x_i = 1 : i = 1,\dots,n\}$ and let $u = \bigwedge_{i=1}^{n}(P_i(g_1,\dots,g_n) \leftrightarrow g_i) \in F_{\mathbf{MG^2}}(n)$, where $x \leftrightarrow y$ is an abbreviation of $(x \rightharpoonup y) \wedge (y \rightharpoonup x)$. Then, according to Theorem 3.9, $F_{\mathbf{MG^2}}(m)/[u] \cong F_{\mathbf{MG^2}}(n,\Omega)$. We will show that $A \cong F_{\mathbf{MG^2}}(n,\Omega)$. Observe that the equations from Ω are true in A on the elements a_i and $\varepsilon(a_i) = e(g_i)$, $i = 1, \ldots, n$. Indeed, since e is an endomorphism

$$e(u) = \bigwedge_{i=1}^{n} e(g_i) \leftrightarrow P_i(e(g_1), \ldots, e(g_n)).$$

But taking into account that $e(g_i) = P_j(g_1, \ldots, g_n)$, we have $P_i(e(g_1), \ldots, e(g_n)) = P_i(P_1(g_1, \ldots, g_n), \ldots, P_n(g_1, \ldots, g_n)) = P_i(g_1, \ldots, g_n) = \varepsilon h(g_i) = e(g_i)$, for $i = 1, \ldots, n$. Hence $e(u) = 1$ and $u \in e^{-1}(1)$, i.e. $[u] \subseteq e^{-1}(1)$. Therefore there exists a homomorphism $f : F_{\mathbf{MG^2}}(n)/[u] \to \varepsilon(A)$ such that the diagram



commutes, i.e. $fr = e$, where r is a natural homomorphism sending x to $x/[u]$. Now consider the restrictions e' and r' on $\varepsilon(A) \subseteq F_{\mathbf{MG^2}}(n)$ of e and r respectively.



Then $fr' = e'$. But $e' = Id_{\varepsilon(A)}$. Therefore $fr' = Id_{\varepsilon(A)}$. We conclude that r' is an injection. Finally we show that r' is surjective by proving that $r(\varepsilon(a_i)) = r(g_i)$ for all i. That is, we show that $\varepsilon(a_i) \leftrightarrow g_i \in [u)$. Indeed $e(g_i) = P_i(g_1, \ldots, g_n)$ and $g_i \leftrightarrow P_i(g_1, \ldots, g_n) = g_i \leftrightarrow e(g_i)$, where $e(g_i) = \varepsilon h(g_i)$. So $g_i \leftrightarrow P_i(g_1, \ldots, g_n) \geq \bigwedge_{i=1}^{n} g_i \leftrightarrow P_i(g_1, \ldots, g_n)$, i.e. $g_i \leftrightarrow P_i(g_1, \ldots, g_n) \in [u)$. Hence r' is an isomorphism between $\varepsilon(A)$ and $F(n)/[u)$. Consequently $A(\cong \varepsilon(A))$ is finitely presented. ■

Hereby, we arrived to the following

Theorem 4.6. *An m-generated MG^2-subalgebra of the m-generated free MG^2-algebra $F_{\mathbf{MG^2}}(m)$ is finitely presented if and only if it is projective.*

Analogous results were obtained by Ghilardi for MV_n-algebras (alias n-valued Lukasiewicz algebras), Gödel algebras [18].

The rest of the paper is devoted to analysis of the m-generated free MG^2-algebra $F_{\mathbf{MG^2}}(m)$ and its Kripke frames.

Recall that (X,R) is an MG^2-frame, then X is a Stone space, i.e. 0-dimensional, Hausdorff and compact space. Let $(\kappa X(m), \kappa R)$ be the MG^2-frame corresponding to the algebra $F_{\mathbf{MG^2}}(m)$. Then $\kappa X(m)$ is the set of all prime filters of $F_{\mathbf{MG^2}}(m)$. And the Boolean algebra generated by all clopen upper cones of $\kappa X(m)$ forms the basis for the topology of the space $\kappa X(m)$. $\kappa X(m)$ contains the set of all principal prime filters of $F_{\mathbf{MG^2}}(m)$ which we identify with $X(m)$. These principal prime filters are generated by upper cones of C_n^m. We will show that $X(m)$ is a dense subset of $\kappa X(m)$.

THEOREM 4.7. *Any finite cone of* $(X(m), R)$ *is an element of* $F_{\mathbf{MG^2}}(m)$.

PROOF. Recall that the binary relation R is irreflexive and $\diamondsuit(A) = R(A)$ for $A \subset X(m)$. So, notice, that if A is an upper cone, then $\diamondsuit(A) = A - minA$, where $minA$ is the set of all minimal elements of A. From this observation we have that $\neg(\diamondsuit(X))$ consists of the set of all one-element bicones (which is finite and isomorphic to $(X_1(m))$), and $\neg(\diamondsuit^2(X))$ consists of the set all one- and two-element bicones (which is finite and isomorphic to $(X_2(m))$); and $\neg(\diamondsuit^n(X))$ consists of the set all i-element bicones (which is finite and isomorphic to $(X_i(m))$) where $i \leq n$ and $\neg Y = -R_\rho^{-1}(Y)$, $\diamondsuit^{k+1}(Y) = \diamondsuit(\diamondsuit^k(Y))$. So, any finite cone of $(X(m), R)$ is an element of $F_{\mathbf{MG^2}}(m)$. ■

COROLLARY 4.8. *Any singleton* $\{x\}$, *for* $x \in X(m)$, *is closed and open (clopen) in* $\kappa X(m)$.

PROOF. Any element x of any irreducible bicone of $X(m)$ represents a principal prime filter generated by $R(x) \cup \{x\}$. From here we conclude that a singleton $\{x\}$, for $x \in X$, is closed and open (clopen). ■

COROLLARY 4.9. $X(m)$ *is a dense subset of* $\kappa X(m)$.

PROOF. Since X is the unit (top) element of $F_{\mathbf{MG^2}}(m)$, we have that $clX = \kappa X$, where cl is the closure operator of the space κX. ■

From this Corollary we conclude that if Y is an element of $F_{\mathbf{MG^2}}(m)$, then clY will be a clopen subset of κX which we denote by κY.

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