Petr Cintula
Carles Noguera

A Note on Natural Extensions in Abstract Algebraic Logic

Abstract. Transfer theorems are central results in abstract algebraic logic that allow to generalize properties of the lattice of theories of a logic to any algebraic model and its lattice of filters. Their proofs sometimes require the existence of a natural extension of the logic to a bigger set of variables. Constructions of such extensions have been proposed in particular settings in the literature. In this paper we show that these constructions need not always work and propose a wider setting (including all finitary logics and those with countable language) in which they can still be used.

Keywords: Abstract algebraic logic, Consequence relations, Natural extensions, Transfer theorems.

1. Introduction

Abstract algebraic logic (AAL) has evolved into an independent field of mathematical logic that provides systematic theories to deal with the multiplicity of logical systems according to their relation with corresponding algebraic semantics (for comprehensive monographs and surveys see [5–8]).

Some of the main results of AAL are those labeled as *transfer theorems*, which can be described as theorems that show that a property of the lattice of theories of the logic also holds in any matricial model of the logic (i.e. the property is *transferred* from the syntax to the semantics of the logic). Czelakowski proved a general *transfer principle* encompassing a great deal of such results (though not all; examples can be found e.g. in [2,3,5,6]). Namely, he proved in [5, Theorem 1.7.1] that in a finitary protoalgebraic logic any property of the lattice of its theories expressed by a universal sentence of elementary lattice theory can be transferred.

The proofs of transfer results, including the general transfer principle of Czelakowski, can be rather involved and often require to add new variables to the language of the logic which, roughly speaking, give a syntactical means

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to refer in the logic to the elements of an arbitrary algebra. In those proofs it is usually important to make sure that the logic obtained in the extended language does not differ too much from the original one.

This is made precise in [5] by the notion of *natural extension*. The proof of the transfer principle in [5, Theorem 1.7.1] and other results (see e.g. [4]) require the existence of natural extensions of a given logic for arbitrary new sets of variables. A syntactical definition of a candidate of such extension was given by Shoesmith and Smiley [10] and was claimed to work in general by Czelakowski in an exercise of his book [5, Exercise 0.3.3].

In this paper we prove that the construction does not work in full generality and identify a technical restriction that needs to be added to ensure the existence of natural extensions. We also present a counterexample of a logic not satisfying the restriction and for which the construction does not yield a natural extension.

2. Natural Extensions

A *propositional language*[1] $\mathcal{L}$ is given by an arbitrary infinite set of variables *Var* and a collection (with no restriction on the cardinality) of propositional connectives of arbitrary finite arities. By $\boldsymbol{Fm}_{\mathcal{L}}$ we denote the free term algebra in the language $\mathcal{L}$ and by $Fm_{\mathcal{L}}$ its universe, i.e. the set of $\mathcal{L}$-formulae. Given $\Gamma \subseteq Fm_{\mathcal{L}}$, by $V(\Gamma)$ denote the set of variables occurring in formulae from Γ. An endomorphism of $\boldsymbol{Fm}_{\mathcal{L}}$ is called an $\mathcal{L}$*-substitution*. Each mapping $\sigma\colon Var \to Fm_{\mathcal{L}}$ determines univocally an $\mathcal{L}$-substitution, which unambiguously we will also denote as σ.

A *logic* L in a language $\mathcal{L}$ is a structural consequence relation on $Fm_{\mathcal{L}}$, i.e. a relation, denoted as $\vdash_{\mathrm{L}}$, between subsets of $Fm_{\mathcal{L}}$ and elements of $Fm_{\mathcal{L}}$ such that for each $\Gamma \cup \Delta \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$:

- $\varphi \vdash_{\mathrm{L}} \varphi$. (Reflexivity)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\Gamma, \Delta \vdash_{\mathrm{L}} \varphi$. (Monotonicity)
- If $\Gamma \vdash_{\mathrm{L}} \psi$ for each $\psi \in \Delta$ and $\Delta \vdash_{\mathrm{L}} \varphi$, then $\Gamma \vdash_{\mathrm{L}} \varphi$. (Cut)
- If $\Gamma \vdash_{\mathrm{L}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathrm{L}} \sigma(\varphi)$ for each $\mathcal{L}$-substitution σ. (Structurality)

Given a propositional language $\mathcal{L}$, an $\mathcal{L}$*-matrix* is a pair $\mathbf{A} = \langle \boldsymbol{A}, F\rangle$ where $\boldsymbol{A}$ is an algebra of type $\mathcal{L}$ and $F \subseteq A$ is a subset called the *filter* of

[1] All notions considered in the paper mostly agree with the definitions used in [5]. We work with consequence relations rather than consequence operators and in the definition of logic we do not restrict the cardinalities of the sets of variables and connectives.

the matrix. An **A**-*evaluation* is a homomorphism $e\colon \boldsymbol{Fm}_{\mathcal{L}} \to \boldsymbol{A}$. Each $\mathcal{L}$-matrix **A** defines a logic $\models_{\mathbf{A}}$ in the following way: for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $\Gamma \models_{\boldsymbol{A}} \varphi$ iff for each **A**-evaluation e, if $e[\Gamma] \subseteq F$ then $e(\varphi) \in F$.

Definition 2.1. The cardinality of a logic L, denoted as card(L), is the smallest cardinal κ such that for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ we have: if $\Gamma \vdash_{\mathrm{L}} \varphi$, then there is $\Gamma_0 \subseteq \Gamma$ with $|\Gamma_0| < \kappa$ such that $\Gamma_0 \vdash_{\mathrm{L}} \varphi$. A logic L is finitary if $\mathrm{card}(\mathrm{L}) = \omega$.

Definition 2.2. Let L be a logic in a language $\mathcal{L}$ with variables *Var*. Consider a logic L' in the language $\mathcal{L}'$ which has the same connectives as $\mathcal{L}$ and variables $Var' \supseteq Var$. We say that*:*

- L' *is an* extension *of* L *if for every* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ *we have:* $\Gamma \vdash_{\mathrm{L}} \varphi$ *implies* $\Gamma \vdash_{\mathrm{L}'} \varphi$*,*
- L' *is a* conservative extension *of* L *if for every* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ *we have:* $\Gamma \vdash_{\mathrm{L}} \varphi$ *iff* $\Gamma \vdash_{\mathrm{L}'} \varphi$*,*
- L' *is a* natural extension *of* L *if it is a conservative extension and* $\mathrm{card}(\mathrm{L}') = \mathrm{card}(\mathrm{L})$*.*

The next definition introduces a candidate for a natural extension of a given logic in a given set of variables, in the same way as in [5, Exercise 0.3.3] (which was inspired by a previous definition in [10]).

Definition 2.3. Let $\mathcal{L}$ be a propositional language with a set of variables *Var* and L a logic in $\mathcal{L}$. Consider the language $\mathcal{L}'$ which has the same connectives as $\mathcal{L}$ and variables $Var' \supseteq Var$. We define a relation $\mathrm{L}^{Var'}$ by setting for any $\Gamma' \cup \{\varphi'\} \subseteq Fm_{\mathcal{L}'}$*:*

$$\Gamma' \vdash_{\mathrm{L}^{Var'}} \varphi' \quad \textit{iff} \quad \textit{there are } \Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}} \textit{ and a mapping } h\colon Var \to Fm_{\mathcal{L}'} \textit{ such that } h(\varphi) = \varphi',\ h[\Gamma] \subseteq \Gamma',\ \textit{and } \Gamma \vdash_{\mathrm{L}} \varphi.$$

Lemma 2.4. *Let* $\mathcal{L}$*,* $\mathcal{L}'$*,* L*, and* $\mathrm{L}^{Var'}$ *be as in the previous definition. Then for any* $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ *we have:*

1. $\Gamma \vdash_{\mathrm{L}^{Var'}} \varphi$ *iff* $\Gamma \vdash_{\mathrm{L}} \varphi$*.*

Furthermore for any $\Gamma' \cup \Delta' \cup \{\varphi'\} \subseteq Fm_{\mathcal{L}'}$ *we have:*

2. $\varphi' \vdash_{\mathrm{L}^{Var'}} \varphi'$

3. If $\Gamma' \vdash_{\mathrm{L}^{Var'}} \varphi'$*, then* $\Gamma', \Delta' \vdash_{\mathrm{L}^{Var'}} \varphi'$

4. If $\Gamma' \vdash_{\mathrm{L}^{Var'}} \varphi'$*, then* $\sigma[\Gamma'] \vdash_{\mathrm{L}^{Var'}} \sigma(\varphi')$ *for each* $\mathcal{L}'$*-substitution* σ*.*

5. If $\Gamma' \vdash_{\mathrm{L}^{Var'}} \varphi'$*, then there is* $\Gamma'_0 \subseteq \Gamma'$ *such that* $|\Gamma'_0| < \mathrm{card}(\mathrm{L})$ *and* $\Gamma'_0 \vdash_{\mathrm{L}^{Var'}} \varphi'$*.*

Proof. The right-to-left direction in the first claim is trivial. To show the converse assume that $\Gamma \vdash_{\mathrm{L}^{Var'}} \varphi$ and let $\bar{\Gamma} \cup \{\bar{\varphi}\} \subseteq Fm_{\mathcal{L}}$ and h be witnesses of that fact according to the definition. Then h must be such that $h[V(\bar{\Gamma} \cup \{\bar{\varphi}\})] \subseteq Var$ (because $h[\bar{\Gamma} \cup \{\bar{\varphi}\}] \subseteq \Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$). Define $h' \colon Var \to Var$ as $h'(v) = h(v)$ if $v \in V(\bar{\Gamma} \cup \{\bar{\varphi}\})$ and $h'(v) = v$ otherwise. We have $h'[\bar{\Gamma}] \vdash_{\mathrm{L}} h'(\bar{\varphi})$. Observing that $h'[\bar{\Gamma}] = h[\bar{\Gamma}] \subseteq \Gamma$ and $h'(\bar{\varphi}) = h(\bar{\varphi}) = \varphi$ we obtain: $\Gamma \vdash_{\mathrm{L}} \varphi$.

Claims 2 and 3 are trivial. To prove claim 4 assume that $\Gamma' \vdash_{\mathrm{L}^{Var'}} \varphi'$ and let Γ, φ, and h be some witnesses provided by the definition. Next consider an $\mathcal{L}'$-substitution σ and note that the triple $\sigma \circ h$, Γ, and φ witnesses the fact that $\sigma[\Gamma'] \vdash_{\mathrm{L}^{Var'}} \sigma(\varphi')$.

To prove the final claim assume that $\Gamma' \vdash_{\mathrm{L}^{Var'}} \varphi'$ and that h, Γ, and φ are witnessing this fact. Due to the cardinality of L there has to be $\Gamma_0 \subseteq \Gamma$ such that $|\Gamma_0| < \mathrm{card}(\mathrm{L})$ and $\Gamma_0 \vdash_{\mathrm{L}} \varphi$. To complete the proof just observe that h, Γ_0, and φ witnesses the fact that $h[\Gamma_0] \vdash_{\mathrm{L}^{Var'}} \varphi'$ and also $h[\Gamma_0] \subseteq \Gamma'$ and $|h[\Gamma_0]| < \mathrm{card}(\mathrm{L})$. ■

Thus we have seen that $\mathrm{L}^{Var'}$ is a structural, reflexive, monotonic relation which conservatively extends L and has cardinality card(L) (we have proved that its cardinality is at worst card(L) and, due to the conservativity, it cannot be better). Nonetheless, we cannot yet conclude, as claimed in [5, Exercise 0.3.3], that $\mathrm{L}^{Var'}$ is a natural extension of L, because we have not shown that $\mathrm{L}^{Var'}$ is indeed a logic (it still may fail to satisfy Cut). Before we show under which conditions $\mathrm{L}^{Var'}$ is a logic we need the following easy set-theoretic lemma.

Lemma 2.5. *Let V, X, Y be sets such that $X, Y \subseteq V$, $|X| \leq |Y|$, and $\omega \leq |Y|$. Then there is a one-to-one function $g \colon V \to V$ and a function $\bar{g} \colon V \to V$ such that $g[X] \subseteq Y$ and $(\bar{g} \circ g)(x) = x$ for each $x \in X$. Furthermore, if $|X| < |V|$, then g can be taken as a bijection and $\bar{g} = g^{-1}$.*

Proof. Let us first assume that $|X| < |V|$ and take an arbitrary set $Y' \subseteq Y$ such that $|Y'| = |X|$. Observe that $|V \setminus X| = |V| = |V \setminus Y'|$ and g can be taken as the union of the bijections $X \to Y'$ and $V \setminus X \to V \setminus Y'$.

If $|X| = |V| = |Y|$, let us consider a set $Y' \subseteq Y$ such that $|Y'| = |Y \setminus Y'| = |Y|$. Thus $|V \setminus X| \leq |V| = |Y| = |Y \setminus Y'| \leq |V \setminus Y'|$. Let us now construct g as the union of a bijection $X \to Y'$ and an injection $V \setminus X \to V \setminus Y'$. The function $\bar{g}$ is constructed as $\bar{g}(v) = p$ if there is p such that $g(p) = v$ and $\bar{g}(v) = v$ otherwise. ■

THEOREM 2.6. *Let $\mathcal{L}$ be a propositional language with a set of variables Var, L a logic in $\mathcal{L}$, and $\mathit{Var}' \supseteq \mathit{Var}$. Further assume that either $|\mathit{Var}'| = |\mathit{Var}|$ or $\mathrm{card}(\mathrm{L}) \leq |\mathit{Var}|^+$. Then:*

1. *$\mathrm{L}^{\mathit{Var}'}$ is a natural extension of L.*
2. *If L' is a natural extension of L in the language $\mathcal{L}'$, then $\mathrm{L}' = \mathrm{L}^{\mathit{Var}'}$.*

PROOF. Let us denote $|\mathit{Var}|$ as κ and $\mathrm{card}(\mathrm{L})$ as κ_0. All we need to prove is that $\mathrm{L}^{\mathit{Var}'}$ enjoys Cut. Assume that $\bar{\Gamma} \vdash_{\mathrm{L}^{\mathit{Var}'}} \bar{\Delta}$ (i.e., $\bar{\Gamma} \vdash_{\mathrm{L}^{\mathit{Var}'}} \delta$ for any $\delta \in \bar{\Delta}$) and $\bar{\Delta} \vdash_{\mathrm{L}^{\mathit{Var}'}} \varphi'$; we need to prove that $\bar{\Gamma} \vdash_{\mathrm{L}^{\mathit{Var}'}} \varphi'$.

First we show that there are sets of formulae $\Delta' \subseteq \bar{\Delta}$ and $\Gamma' \subseteq \bar{\Gamma}$ such that $\Gamma' \vdash_{\mathrm{L}^{\mathit{Var}'}} \Delta'$, $\Delta' \vdash_{\mathrm{L}^{\mathit{Var}'}} \varphi'$, and $|V(\Gamma' \cup \Delta' \cup \{\varphi'\})| \leq \kappa$. In the case that $|\mathit{Var}'| = |\mathit{Var}|$ the claim is trivial (just take $\Delta' = \bar{\Delta}$ and $\Gamma' = \bar{\Gamma}$). In the other case we assume that $\kappa_0 \leq \kappa^+$ and so there has to be $\Delta' \subseteq \bar{\Delta}$ such that $|\Delta'| \leq \kappa$ and $\Delta' \vdash_{\mathrm{L}^{\mathit{Var}'}} \varphi'$. Analogously for each $\delta \in \Delta'$ there is $\Gamma'_\delta \subseteq \bar{\Gamma}$ such that $|\Gamma'_\delta| \leq \kappa$ and $\Gamma'_\delta \vdash_{\mathrm{L}^{\mathit{Var}'}} \delta$. Thus we have a set $\Gamma' = \bigcup_{\delta \in \Delta'} \Gamma_\delta \subseteq \bar{\Gamma}$ such that $|\Gamma'| \leq \kappa$ and $\Gamma' \vdash_{\mathrm{L}^{\mathit{Var}'}} \Delta'$ and so $|V(\Gamma' \cup \Delta' \cup \{\varphi'\})| \leq \kappa$.

Thus we can apply Lemma 2.5 for $V = \mathit{Var}'$, $Y = \mathit{Var}$, and $X = V(\Gamma' \cup \Delta' \cup \{\varphi'\})$ and obtain functions $g, \bar{g}\colon \mathit{Var}' \to \mathit{Var}'$ such that $g(\chi) \in \mathit{Fm}_{\mathcal{L}}$ and $(\bar{g} \circ g)(\chi) = \chi$ for each $\chi \in \Gamma' \cup \Delta' \cup \{\varphi'\}$.

Next, by structurality (of $\mathrm{L}^{\mathit{Var}'}$) and conservativity (of $\mathrm{L}^{\mathit{Var}'}$ over L), we have $g[\Gamma'] \vdash_{\mathrm{L}} g[\Delta']$ and $g[\Delta'] \vdash_{\mathrm{L}} g(\varphi')$ and so $g[\Gamma'] \vdash_{\mathrm{L}} g(\varphi')$. Thus, we also have $g[\Gamma'] \vdash_{\mathrm{L}^{\mathit{Var}'}} g(\varphi')$ and, applying the substitution $\bar{g}$, we obtain $\Gamma' \vdash_{\mathrm{L}^{\mathit{Var}'}} \varphi'$. The monotonicity of $\mathrm{L}^{\mathit{Var}'}$ completes the proof.

Let us prove now the second claim. Take $\Gamma' \cup \{\varphi'\} \subseteq \mathit{Fm}_{\mathcal{L}'}$. First we show that if $\Gamma' \vdash_{\mathrm{L}'} \varphi'$, then there are h, Γ, and φ witnessing that $\Gamma' \vdash_{\mathrm{L}^{\mathit{Var}'}} \varphi'$. We know that there is $\Gamma'_0 \subseteq \Gamma'$ such that $|\Gamma'_0| < \kappa_0$ and $\Gamma'_0 \vdash_{\mathrm{L}'} \varphi'$. Clearly, we have $|V(\Gamma'_0 \cup \{\varphi'\})| \leq \kappa$. Thus we can apply Lemma 2.5 for $V = \mathit{Var}'$, $Y = \mathit{Var}$, and $X = V(\Gamma'_0 \cup \{\varphi'\})$ and obtain functions $g, \bar{g}\colon \mathit{Var}' \to \mathit{Var}'$ such that $g(\chi) \in \mathit{Fm}_{\mathcal{L}}$ and $(\bar{g} \circ g)(\chi) = \chi$ for each $\chi \in \Gamma'_0 \cup \{\varphi'\}$. Then, taking h as the restriction of $\bar{g}$ to Var, $\Gamma = g[\Gamma'_0]$, and $\varphi = g(\varphi')$, we have the desired witnesses. Indeed, $\Gamma \cup \{\varphi\} \subseteq \mathit{Fm}_{\mathcal{L}}$, $h[\Gamma] = (\bar{g} \circ g)[\Gamma'_0] = \Gamma'_0 \subseteq \Gamma'$, $h(\varphi) = (\bar{g} \circ g)(\varphi') = \varphi'$; finally, we obtain $g[\Gamma'_0] \vdash_{\mathrm{L}} g(\varphi')$ due to structurality (of L') and conservativity (of L' over L).

For the converse implication take any $\Gamma' \cup \{\varphi'\} \subseteq \mathit{Fm}_{\mathcal{L}'}$ and assume that $\Gamma' \vdash_{\mathrm{L}^{\mathit{Var}'}} \varphi'$. Then there have to be a set $\Gamma \cup \{\varphi\} \subseteq \mathit{Fm}_{\mathcal{L}}$ and a mapping $h\colon \mathit{Var} \to \mathit{Fm}_{\mathcal{L}'}$ such that $h(\varphi) = \varphi'$, $h[\Gamma] \subseteq \Gamma'$, and $\Gamma \vdash_{\mathrm{L}} \varphi$. Thus $\Gamma \vdash_{\mathrm{L}'} \varphi$ and so, by extending h to an $\mathcal{L}'$-substitution (in an arbitrary way) and applying structurality, we obtain $h[\Gamma] \vdash_{\mathrm{L}'} h(\varphi)$. Therefore, by monotonicity, $\Gamma' \vdash_{\mathrm{L}'} \varphi'$. ∎

Therefore we have seen that the construction does indeed give a natural extension (actually the only possible one) if either $|Var| = |Var'|$ or $\mathrm{card}(\mathrm{L}) \leq |Var|^+$. In the first case, this is not surprising because having sets of variables with the same cardinal, we obtain isomorphic algebras of formulae and the construction is essentially just a renaming of variables. If $|Var| < |Var'|$ the situation is more interesting and we need to assume that $\mathrm{card}(\mathrm{L}) \leq |Var|^+$. In Proposition 2.8 we show the non-triviality of this condition by constructing a logic that does not satisfy it; however the vast majority of logics considered in the literature do satisfy this restriction, in particular:

- when L is a finitary logic, because then $\mathrm{card}(\mathrm{L}) = \omega$ and we are assuming that *Var* is always infinite,
- when $|Var| = |Fm_{\mathcal{L}}|$ (e.g. if the set of logical connectives is countable), because always $\mathrm{card}(\mathrm{L}) \leq |Fm_{\mathcal{L}}|^+$.

Therefore, since it is formulated for finitary logics only, Czelakowski's proof of the transfer principle in [5, Theorem 1.7.1], although based on a flawed exercise, still works.

Before we show that the assumptions of Theorem 2.6 cannot be dropped, we need to prove one more lemma which shows an equivalent formulation of $\mathrm{L}^{Var'}$ à la Łoś–Suszko [9] which can be seen as a strengthening of another one of Czelakowski's exercises on natural extensions [5, Exercise 0.3.5]. Note that, unlike in that exercise, we prove it not only in the case that $\mathrm{card}(\mathrm{L}) \leq |Var|$ (which also entails that $\mathrm{L}^{Var'}$ is a natural extension of L, due to Theorem 2.6) but also in the case that $|Var| < |Var'|$ (which will be needed for our counterexample in which $\mathrm{L}^{Var'}$ is not a natural extension of L).

Lemma 2.7. *Let $\mathcal{L}$, $\mathcal{L}'$,* L, *and* $\mathrm{L}^{Var'}$ *be as in Definition 2.3. Further assume that either $|Var| < |Var'|$ or* $\mathrm{card}(\mathrm{L}) \leq |Var|$. *Then for any $\Gamma' \cup \{\varphi'\} \subseteq Fm_{\mathcal{L}'}$ we have:*

$$\Gamma' \vdash_{\mathrm{L}^{Var'}} \varphi' \quad \textit{iff} \quad \begin{array}{l}\textit{there are } \Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}} \textit{ and a bijection } h\colon Var' \to Var' \\ \textit{such that } h(\varphi) = \varphi',\ h[\Gamma] \subseteq \Gamma',\ \textit{and } \Gamma \vdash_{\mathrm{L}} \varphi.\end{array}$$

Proof. The right-to-left direction is trivial (the bijection $h\colon Var' \to Var'$ determines, in particular, a mapping from *Var* to $Fm_{\mathcal{L}'}$). We show the reverse one: assume that $\Gamma' \vdash_{\mathrm{L}^{Var'}} \varphi'$ and let Γ, φ, and h be witnesses as given by the definition. W.l.o.g. we can assume that $|\Gamma| < \mathrm{card}(\mathrm{L})$.

If we assume that $|Var| < |Var'|$, then $|V(h[\Gamma \cup \{\varphi\}])| \leq |Var| < |Var'|$. Assume now that $\mathrm{card}(\mathrm{L}) \leq |Var|$. If *Var* is uncountable, then clearly $|V(h[\Gamma \cup \{\varphi\}])| \leq \omega \times |\Gamma| = \max\{\omega, |\Gamma|\} < |Var| \leq |Var'|$. If $|Var| = \omega$, then Γ is finite and we also obtain that $|V(h[\Gamma \cup \{\varphi\}])| < |Var| \leq |Var'|$.

Thus, in any case, we can apply Lemma 2.5 for $V = Var'$, $Y = Var$, and $X = V(h[\Gamma \cup \{\varphi\}])$ and obtain a bijection $g\colon Var' \to Var'$ for which we have $g[V(h[\Gamma \cup \{\varphi\}])] \subseteq Var$.

We show that g^{-1}, $g[h[\Gamma]]$ and $g(h(\varphi))$ are the needed witnesses for our characterization: indeed g^{-1} is a bijection; $g[h[\Gamma]] \cup \{g(h(\varphi))\} \subseteq Fm_{\mathcal{L}}$, $g^{-1}[g[h[\Gamma]]] = h[\Gamma] \subseteq \Gamma'$, $g^{-1}(g(h(\varphi))) = h(\varphi) = \varphi'$. The fact that $g \circ h$ gives a substitution of $Fm_{\mathcal{L}}$ entails than $g[h[\Gamma]] \vdash_{\mathrm{L}} g(h(\varphi))$. ■

We can now build an example that shows that the conditions of Theorem 2.6 cannot be dropped, i.e. the statement of [5, Exercise 0.3.3] is not valid in general. Let us first take two sets of variables $Var \subseteq Var'$ such that $|Var| < |Var'|$: $Var = \{v_n \mid n \in \omega\}$ and $Var' = \{v_\alpha \mid \alpha \in \aleph_1\}$. Let $\mathcal{L}$ be a propositional language with a nullary connective $\bar{\alpha}$ for each $\alpha \in \aleph_1$ and two binary connectives $\approx$ and $\not\approx$. Let $\mathbf{A}$ be the $\mathcal{L}$-matrix with domain $\aleph_1$, filter $F = \{0\}$, and operations defined as: $\bar{\alpha}^{\mathbf{A}} = \alpha$ for each $\alpha \in \aleph_1$, $\approx^{\mathbf{A}} (a, b) = 0$ if $a = b$ and 1 otherwise, and $\not\approx^{\mathbf{A}} (a, b) = 0$ if $a \neq b$ and 1 otherwise. Let S be the logic $\models_{\mathbf{A}}$ and let $\mathrm{S}^{Var'}$ be the relation defined in Lemma 2.4.[2]

Proposition 2.8. *$\mathrm{card}(\mathrm{S}) > |Var|^+$ and $\mathrm{S}^{Var'}$ does not satisfy Cut.*

Proof. Consider the set of formulae $\Delta = \{v_0 \not\approx \bar{\alpha} \mid \alpha \in \aleph_1 \setminus \{0\}\} \subseteq Fm_{\mathcal{L}}$ and note that $\Delta \vdash_{\mathrm{S}} v_0 \approx \bar{0}$. Indeed, if e is an arbitrary $\mathbf{A}$-evaluation and $\beta = e(v_0)$, then either $\beta = 0$ and the conclusion is valid or $\beta \neq 0$ in which case one of the premises is not valid. Now consider an arbitrary subset $\Delta_0 \subseteq \Delta$ of cardinality $\aleph_0$. Take $B = \{\alpha \in \aleph_1 \mid v_0 \not\approx \bar{\alpha} \in \Delta_0\}$. Clearly, any $\mathbf{A}$-evaluation e such that $e(v_0) \notin B \cup \{0\}$ shows that $\Delta_0 \nvdash_{\mathrm{S}} v_0 \approx \bar{0}$. Thus $\mathrm{card}(\mathrm{S}) = \aleph_2 > \aleph_1 = |Var|^+$.

Now consider the set of $\mathcal{L}'$-formulae $\Gamma' = \{v_\alpha \approx \bar{\alpha}, v_\alpha \not\approx v_0 \mid \alpha \in \aleph_1 \setminus \{0\}\}$. We prove the following three claims:

- $\Delta \vdash_{\mathrm{S}^{Var'}} v_0 \approx \bar{0}$
- $\Gamma' \vdash_{\mathrm{S}^{Var'}} \Delta$, i.e., $\Gamma' \vdash_{\mathrm{S}^{Var'}} v_0 \not\approx \bar{\alpha}$ for each $\alpha \in \aleph_1 \setminus \{0\}$
- $\Gamma' \nvdash_{\mathrm{S}^{Var'}} v_0 \approx \bar{0}$.

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[2]The logic S is rather artificial, but it is tailored to provide a simple counterexample without distraction to any possible additional properties. One could also find more natural examples: consider, for instance, any prototypical fuzzy logic (see e.g. [1]) given by an algebra defined over the real unit interval (e.g. Łukasiewicz or Gödel–Dummett logics) and enrich it with the so-called Monteiro–Baaz projection (interpreted as $\triangle x = 1$ if $x = 1$ or 0 otherwise, which allows to define the connectives $\approx$ and $\not\approx$) and a truth constant $\bar{r}$ for each $r \in [0, 1]$ (interpreted as $\bar{r} = r$, which would play the rôle of $\bar{\alpha}$).

The first claim is trivial. To show the second one, fix any $\alpha \in \aleph_1 \setminus \{0\}$ and take the set $\Gamma = \{v_1 \approx \bar{\alpha}, v_1 \not\approx v_0\}$ and a mapping h such that $h(v_0) = v_0$ and $h(v_1) = v_\alpha$ and observe that $h[\Gamma] \subseteq \Gamma'$, $h(v_0 \not\approx \bar{\alpha}) = v_0 \not\approx \bar{\alpha}$ and $\Gamma \vdash_{\mathrm{S}} v_0 \not\approx \bar{\alpha}$ (trivial, as the premises are satisfied only if $e(v_0) \neq e(v_1) = \alpha$).

For the final claim, assume for a contradiction that $\Gamma' \vdash_{\mathrm{S}^{Var'}} v_0 \approx \overline{0}$. Then (using the characterization of $\mathrm{S}^{Var'}$ proved in Lemma 2.7) there must be formulae $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ (hence Γ has only countably many variables) and a bijection h in Var' such that $h[\Gamma] \subseteq \Gamma'$, $h(\varphi) = v_0 \approx \overline{0}$, and $\Gamma \vdash_{\mathrm{S}} \varphi$. Let us denote $h^{-1}(v)$ as $\hat{v}$. Then $\varphi = \hat{v}_0 \approx \overline{0}$ and there has to be a countable set $B \subseteq \aleph_1 \setminus \{0\}$ such that $\Gamma \subseteq \hat{\Gamma} = \{\hat{v}_\alpha \approx \bar{\alpha}, \hat{v}_\alpha \not\approx \hat{v}_0 \mid \alpha \in B\}$. Thus we would also have $\hat{\Gamma} \vdash_{\mathrm{S}} \hat{v}_0 \approx \overline{0}$, which can be easily refuted by setting $e(\hat{v}_\alpha) = \alpha$ for each $\alpha \in B$ and $e(\hat{v}_0) \notin B \cup \{0\}$; a contradiction. ■

Thus, in our example $\mathrm{S}^{Var'}$ is not even a logic and so it cannot be a natural extension of S in the language $\mathcal{L}'$. However, there would still be an easy way to define a natural extension of S in that language. Take now $\mathbf{A}$ as an $\mathcal{L}'$-matrix and define the logic S′ in $\mathcal{L}'$ as $\models_{\mathbf{A}}$. Then S′ is a natural extension of S. Indeed, the conservativity is trivial, thus $\mathrm{card}(\mathrm{S}') \geq \mathrm{card}(\mathrm{S})$ and, since trivially $\mathrm{card}(\mathrm{S}') \leq |Fm_{\mathcal{L}'}|^+$, we obtain $\mathrm{card}(\mathrm{S}') = \aleph_2 = \mathrm{card}(\mathrm{S})$. Thus the question whether each logic has a natural extension to languages of arbitrary cardinality remains open; we have only shown that Shoesmith–Smiley construction is not universally applicable.

Finally, it is worth noting that our example does not refute Czelakowski's claim in [5, Exercise 0.3.4] that all natural extensions (on a fixed set of variables) of a given logic coincide. We can obtain this fact, under our additional assumption, as a trivial corollary of Theorem 2.6.

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P. Cintula
Institute of Computer Science
Academy of Sciences of the Czech Republic
Pod vodárenskou věží 2
182 07 Prague, Czech Republic
http://www.cs.cas.cz/cintula
cintula@cs.cas.cz

C. Noguera
Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
Pod vodárenskou věží 4
182 08 Prague, Czech Republic
http://www.carlesnoguera.cat
noguera@utia.cas.cz