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Duality for the Logic of Quantum Actions

Abstract. In this paper we show a duality between two approaches to represent quantum structures abstractly and to model the logic and dynamics therein. One approach puts forward a "quantum dynamic frame" (Baltag et al. in Int J Theor Phys, 44(12):2267–2282, 2005), a labelled transition system whose transition relations are intended to represent projections and unitaries on a (generalized) Hilbert space. The other approach considers a "Piron lattice" (Piron in Foundations of Quantum Physics, 1976), which characterizes the algebra of closed linear subspaces of a (generalized) Hilbert space. We define categories of these two sorts of structures and show a duality between them. This result establishes, on one direction of the duality, that quantum dynamic frames represent quantum structures correctly; on the other direction, it gives rise to a representation of dynamics on a Piron lattice.

Keywords: Quantum logic, Piron lattice, Modal logic, Labelled transition system, Duality, Orthomodular lattice.

1. Introduction

There is a long tradition of investigating dualities between algebraic structures and "spatial" structures, showing that categories of certain algebras and of certain spaces are equivalent to each other, except that they have opposite directions of morphisms. A classic example is the Stone duality between Boolean algebras and fields of sets [19]; see [12] for the Stone duality and vast extensions thereof. For dualities seen in modal logic, such as one between complete atomic Boolean algebras with operators and Kripke frames, see [7, Chapter 5] and [21, Sect. 5].[1] In this paper we build further on this tradition and study the duality of two different quantum structures, *Piron lattices* [18] and *quantum dynamic frames* [3], which are abstractions

[1] While duality plays a crucial role both in Kripke semantics and in locale theory [12], the two fields use the term "frame" to mean quite different structures. In this article we always use "frame" along the terminology of Kripke semantics.

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of Hilbert spaces. Hilbert spaces are among the standard tools for representing quantum systems, and these abstractions highlight essential properties of quantum systems.

Piron lattices provide an algebraic perspective on Hilbert spaces and focus on *testable properties* of the system. Testable properties of a physical system can be represented as closed linear subspaces of a Hilbert space, with the one-dimensional subspaces being the states of the system. These states form the atoms of an atomic lattices of closed linear subspaces. A Piron lattice is such a lattice with the appropriate constraints for it to capture the abstract structure of a generalized Hilbert space [18]; a Piron lattice that satisfies "Mayet's condition" [15] captures the structure of an infinite dimensional Hilbert space over the complex numbers, reals, or quaternions.[2] Such lattices highlight the algebraic properties of a physical system, where joins and meets correspond to the disjunction and conjunction of the properties being tested, and orthocomplementation corresponds to the negation of the property. This sets the foundation for an algebraic semantics of quantum logic. However, on the surface, this logical structure is static and timeless.

Quantum dynamic frames provide a *dynamic* perspective on quantum systems. The underlying objects are states, and they are related to each other via actions that transform the system from one state to another. In this way a quantum dynamic frame is a type of labelled transition system [3]. Relations are constructed from atomic actions that are either projections (corresponding to tests) or unitary evolutions (reversible actions). These quantum dynamic frames are used for reasoning about quantum programs via the "logic of quantum programs" [4], a natural extension of Hoare logic and propositional dynamic logic, which are used for reasoning about classical programs [11].

Given that the Piron lattices focus on testable properties and appear static and the quantum dynamic frames focus on states and actions and are clearly dynamic, it might seem that these two approaches are scarcely related. We show in this paper that these two approaches are categorically dual to each other, that is, the category of one is equivalent to the dual (opposite) category of the other. A first step was given in [3], where it was

[2] See [1] for a review of the relationship between Piron lattices and Hilbert spaces. Also, see [20] for how "propositional systems", which are slightly more general than Piron lattices (see Definition 2.1), are categorically equivalent to "Hilbert geometries" and "Hilbert lattices"; the former generalize projective geometries given by one-dimensional subspaces of Hilbert spaces, and the latter generalize lattices of all subspaces, and not just closed ones, of Hilbert spaces. See [13] for the underlying orthomodular lattice structure of traditional quantum logic.

observed that a quantum dynamic frame gives rise to a Piron lattice and vice versa. This relationship concerns just the objects of the categories. We provide a detailed and complete proof of this observation, provide full categorical structure for both Piron lattices and quantum dynamic frames, and show that these categories are dual to each other. For each of the frames and the lattices, we consider two types of morphisms. One type is that defined by Moore [16] for two simpler categories: state spaces (symmetric anti-reflexive frames that separate points) and property lattices (complete atomistic orthocomplemented lattices). These categories are weaker than the ones we consider in this paper as they do not capture superpositions which are important to quantum theory. However, the definition of the morphisms used by Moore can be used for our categories as well. We also define stronger types of morphisms for both the Piron lattices and quantum dynamic frames. As these morphisms are strictly stronger than Moore's, we refer to them as *strong* and the Moore morphisms as *weak*. Both Piron lattice morphisms act directly on properties, while both quantum dynamic frame morphisms act directly on states. These two types of morphisms are dual to each other (have reverse arrows), as is noted in the morphism of state property spaces discussed in [11].

Our duality result shows that quantum dynamic frames and Piron lattices form categories that are essentially the same (except for the direction of morphisms). We also show this relation can be restricted to the objects satisfying Mayet's condition. As Piron lattices satisfying Mayet's condition have already been shown to be equivalent to Hilbert spaces, this result clarifies the close relationship quantum dynamic frames have with Hilbert spaces. The structures of both quantum dynamic frames and Piron lattices are each a focal point of quantum logic, and hence our duality adds a new perspective on the formal relation between these different quantum structures.

Our paper is structured as follows. We define two categories of Piron lattices in Sect. 2.1, one with homomorphisms defined by Moore, and the other with homomorphisms preserving more structure. We define two categories of quantum dynamic frames in Sect. 2.2, similarly with two sorts of structure-preserving maps. We define functors from the Piron lattices to the quantum dynamic frames in Sect. 3.1, and opposite ones in Sect. 3.2. These then form dualities, as is proven in Sect. 3.3, and, in Sect. 3.4, we restrict these dualities to the categories of objects satisfying Mayet's condition. Finally, Sect. 4 concludes the paper and points to future work.

2. The Categories

In this section, we define categories of Piron lattices and of quantum dynamic frames.[3] In fact we provide two categories, $\mathbb{L}_w$ and $\mathbb{L}_s$, of Piron lattices, and also two categories, $\mathbb{F}_w$ and $\mathbb{F}_s$, of quantum dynamic frames. In each case the two categories share the same objects, but one (viz., $\mathbb{L}_w$ or $\mathbb{F}_w$) has more morphisms than the other (viz., $\mathbb{L}_s$ or $\mathbb{F}_s$); or, in other words, morphisms in the former preserve less structure than ones in the latter.

2.1. Categories of Piron Lattices

Any Hilbert space $\mathcal{H}$ gives rise to a lattice $(L, \leq)$, where L is the family of closed linear subspaces and $\leq$ is set-inclusion $\subseteq$; moreover, the orthocomplement in $\mathcal{H}$ gives a map $-^{\perp} : L \to L$. Piron [18] axiomatized lattices $(L, \leq, -^{\perp})$ that arise from Hilbert spaces in this way— lattices satisfying his axioms (in Definition 2.1 below) are now called Piron lattices. As he proved, Piron lattices of height at least 4 correspond to (generalized) Hilbert spaces of dimension at least 4. In this section we define two categories, $\mathbb{L}_w$ and $\mathbb{L}_s$, of Piron lattices. They share the same objects, but $\mathbb{L}_w$ has more morphisms than $\mathbb{L}_s$.

2.1.1. Piron Lattices. Here is a set of axioms of a Piron lattice. Lattices satisfying certain subsets of the axioms have useful names as well.

Definition 2.1. A *bounded lattice* is a lattice with a greatest element I ("top") and a least element O ("bottom"). An *ortholattice* $\mathfrak{L}$ is a bounded lattice $(L, \leq)$ that satisfies (1) below. An *orthomodular lattice* $\mathfrak{L}$ is an ortholattice $(L, \leq, -^{\perp})$ that satisfies (2). A *propositional system* $\mathfrak{L}$ is an orthomodular lattice $(L, \leq, -^{\perp})$ that satisfies (3)–(5). Lastly, a *Piron lattice* $\mathfrak{L}$ is a propositional system $(L, \leq, -^{\perp})$ that satisfies (6).

(1) **Orthocomplement:** $\mathfrak{L}$ is equipped with a map $-^{\perp} : L \to L$ such that
   (a) $p^{\perp\perp} = p$;
   (b) $p \leq q$ implies $q^{\perp} \leq p^{\perp}$;
   (c) $p \wedge p^{\perp} = O$ and $p \vee p^{\perp} = I$.

(2) **Weak Modularity:** $q \leq p$ implies $p[q] = q$, where $p[q] := p \wedge (p^{\perp} \vee q)$.

(3) **Completeness:** For any $A \subseteq L$, its meet $\bigwedge A$ and join $\bigvee A$ are in L.

Call $a \in L$ an *atom* if $a \neq O$ and either $p = O$ or $p = a$ holds for every $p \in L$ such that $p \leq a$. Write $\mathrm{At}(\mathfrak{L})$ for the set of atoms of $\mathfrak{L}$.

[3] See [2] for an exposition of category theory.

(4) **Atomicity:** For any $p \neq O$, there is an $a \in \mathrm{At}(\mathfrak{L})$ such that $a \leq p$.

(5) **Covering Law:** If $a \in \mathrm{At}(\mathfrak{L})$ and $a \not\leq p^{\perp}$ then $p[a] \in \mathrm{At}(\mathfrak{L})$.[4]

(6) **Superposition Principle:** For any two distinct $a, b \in \mathrm{At}(\mathfrak{L})$, there is a $c \in \mathrm{At}(\mathfrak{L})$, distinct from both a and b, such that $a \vee c = b \vee c = a \vee b$.[5]

Atoms are meant to correspond to one-dimensional subspaces, or rays, of a Hilbert space; so they satisfy, for instance:

(7) $a \not\leq p$ iff $a \wedge p < a$ iff $a \wedge p = O$, for any atom a.

The fact that closed linear subspaces in general are certain sets of rays is expressed in Piron lattices by

PROPOSITION 2.2. *Let $\mathfrak{L}$ be an orthomodular lattice satisfying Completeness and Atomicity. Then $\mathfrak{L}$ is* atomistic, *meaning that every $p \in L$ has $p = \bigvee [\![p]\!]$, where $[\![p]\!] := \{a \in \mathrm{At}(\mathfrak{L}) \mid a \leq p\}$.*

PROOF. First observe that $p = q$ if both $q \leq p$ and $[\![p]\!] \subseteq [\![q]\!]$, as follows. Suppose the antecedents. $[\![p]\!] \subseteq [\![q]\!]$ means that, for any $a \in \mathrm{At}(\mathfrak{L})$, if $a \leq p$ then $a \leq q$, which implies $a \not\leq q^{\perp}$, for otherwise $a \leq q \wedge q^{\perp} = O$. Therefore no $a \in \mathrm{At}(\mathfrak{L})$ satisfies $a \leq p \wedge q^{\perp}$, that is, $p \wedge q^{\perp} = O$, i.e., $p^{\perp} \vee q = I$. Hence $p = p \wedge (p^{\perp} \vee q) = q$ by $q \leq p$ and Weak Modularity.

Let $q = \bigvee [\![p]\!]$. Then it holds that both $q \leq p$ and $[\![p]\!] \subseteq [\![q]\!]$. Here $\bigvee [\![p]\!] \leq p$ because $a \leq p$ for all $a \in [\![p]\!]$; and if $a \in [\![p]\!]$ then $a \leq \bigvee [\![p]\!]$. ■

The connective $p[q] := p \wedge (p^{\perp} \vee q)$ defined in (2) is sometimes [8] called the "Sasaki projection". The monotone map $p[-] : L \to L$ expresses, in Hilbert-space terms, the direct-image[6] operation under the projector onto the subspace p; this should make the conceptual meaning of (2) and (5) transparent. There are many properties an expression of projectors must satisfy, and the following (the rest of this subsubsection) are some of them.

(8) $p[q] = p \wedge (p^{\perp} \vee q) \leq p$.

(9) $p[a] \wedge a^{\perp} = p \wedge (p^{\perp} \vee a) \wedge a^{\perp} = (p \wedge a^{\perp}) \wedge (p \wedge a^{\perp})^{\perp} = O$.

(10) If $q \leq p^{\perp}$ then $p[q] = p \wedge (p^{\perp} \vee q) = p \wedge p^{\perp} = O$.

LEMMA 2.3. *For any $a, b \in \mathrm{At}(\mathfrak{L})$, $a \not\leq b^{\perp}$ is equivalent to $b[a] = b$ and to*

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[4] In an orthomodular lattice, this statement of the Covering Law is equivalent to that in [18]. See [18] or [5] for proofs.

[5] Usually a Piron lattice is defined with the property called irreducibility instead of (6); see, for example, [22]. Yet a propositional system satisfies (6) iff it is irreducible.

[6] Because of the view of Sasaki projection as a direct-image, we use the notation standardly used for such. On page 12, we define direct-image for arbitrary functions.

(a) $p[a] = b$ *for some* $p \in L$.

PROOF. If $a \not\leq b^\perp$, then $b[a] \neq O$ by the Covering Law, whereas $b[a] \leq b$ by (8) for atom b, and hence $b[a] = b$. Also, $b[a] = b$ obviously implies (a). Finally, if (a) $p[a] = b$, then $a \not\leq b^\perp$, for otherwise $b \leq a^\perp$ and (9) would imply $b = b \wedge a^\perp = p[a] \wedge a^\perp = O$ for atom b. ■

In an ortholattice $\mathfrak{L}$, define $[p]q := p^\perp \vee (p \wedge q) = (p[q^\perp])^\perp$, the so-called "Sasaki hook", obtaining a monotone map $[p]- : L \to L$. This expresses the inverse-image operation under the projector onto p.[7] In fact, Weak Modularity amounts to the adjunction[8] formed by direct image $p[-]$ and inverse image $[p]-$ (just as in $f[-] \dashv f^{-1}[-]$ for any function f):

THEOREM 2.4. Coecke and Smets [8] *An ortholattice* $\mathfrak{L}$ *satisfies Weak Modularity iff every* $p[-]$ *is left adjoint to* $[p]-$ *(written* $p[-] \dashv [p]-$*).*

$-[-]$ and $[-]-$ are also meant to generalize conjunction and implication (classical logic has $p \wedge - \dashv p \Rightarrow -$ for classical implication $\Rightarrow$). They are supposed to mean, respectively, the following:

- $p[q]$: We may have moved to the current state by testing whether p or not (and receiving the answer "Yes") when q was the case.
- $[p]q$: If we test p and receive the answer "Yes", then q will be the case.

These may help make sense of the unit and counit laws of the adjunction, $q \leq [p](p[q])$ and $p[[p]q] \leq q$. In fact, it is useful to observe that Weak Modularity amounts to the equalities among the following six terms.

$$\begin{array}{ccc} p[[p]q] & \stackrel{\text{def}}{=} & p \wedge (p^\perp \vee (p^\perp \vee (p \wedge q))) \\ & & \| \\ p[p \wedge q] & \stackrel{\text{def}}{=} & p \wedge (p^\perp \vee (p \wedge q)) \\ {\scriptstyle \text{Weak Modularity}}\ \rotatebox{90}{$\leq$} & \lneq & \| {\scriptstyle \text{def}} \\ p \wedge q & & p \wedge [p]q \end{array} \tag{11}$$

[7]Here inverse images are meant to include the kernel.

[8]An *adjunction* between two partially ordered sets $(S_1, \leq_1)$ and $(S_2, \leq_2)$ (often called a *Galois connection*) is a pair of monotone maps $L : S_1 \to S_2$ and $R : S_2 \to S_1$ such that $L(x) \leq_2 y$ iff $x \leq_1 R(y)$ for all $x \in S_1$ and $y \in S_2$, or, equivalently, for which the "unit" law $x \leq_1 RL(x)$ and the "counit" law $LR(y) \leq_2 y$ hold. Such L and R are called *left* and *right adjoints* to each other. In fact, adjunction is defined for categories $\mathbb{C}$, $\mathbb{D}$ and functors $L : \mathbb{C} \to \mathbb{D}$ and $R : \mathbb{D} \to \mathbb{C}$ in general, by requiring certain conditions (that hold trivially in the case of posets) on the correspondence between morphisms $f : L(C) \to D$ and $g : C \to R(D)$, or on natural transformations $\eta : 1_\mathbb{C} \to RL$ (unit) and $\epsilon : LR \to 1_\mathbb{D}$ (counit). See [2, Definition 9.6 and Proposition 10.1]. We mention a general adjunction in Theorem 3.24.

2.1.2. Morphisms of Piron Lattices. There are options as to how to define morphisms of Piron lattices. The first one is due to Moore [16,17].

Definition 2.5. A function $h : L_1 \to L_2$ is a *weak homomorphism between two Piron lattices* $(L_1, \leq_1, -^{\perp_1})$ and $(L_2, \leq_2, -^{\perp_2})$ if the following hold:

(12) $h(\bigwedge_1 A) = \bigwedge_2 h[A]$ for any $A \subseteq L_1$.

(13) *(Moore's condition)* if b is an atom of L_2, then there exists an atom a of L_1, such that $b \leq_2 h(a)$.

These homomorphisms form a category [16,17]: the composition of two weak homomophisms is again a weak homomorphism, and the identity map is the identity. Let us write $\mathbb{L}_w$ for this category of Piron lattices and weak homomorphisms. As a second option we consider the smaller class of morphisms that also preserve orthocomplement, and as a consequence preserve arbitrary joins including the bottom.

Definition 2.6. A weak homomorphism $k : L_1 \to L_2$ is a *strong homomorphism between Piron lattices* $(L_i, \leq_i, -^{\perp_i})$ $(i = 1, 2)$ if k moreover satisfies

(14) $k(p^{\perp_1}) = k(p)^{\perp_2}$ for all $p \in L_1$.

Clearly, the composition of two strong homomorphisms also preserves the orthocomplement, and the identity map is a strong homomorphism; so Piron lattices and strong homomorphisms form a subcategory, $\mathbb{L}_s$, of $\mathbb{L}_w$ that is "wide" in the sense of sharing all the objects. Note that strong homomorphisms preserve I, O, $\vee$, $-[-]$ and $[-]-$, since they preserve $\wedge$ and $-^{\perp}$.

2.2. Categories of Quantum Dynamic Frames

We now proceed to define two categories $\mathbb{F}_w$ and $\mathbb{F}_s$ for quantum dynamic frames. Similarly to the Piron lattice categories we defined earlier, they have the same objects and only differ in their morphisms.

2.2.1. Quantum Dynamic Frames. Any Hilbert space $\mathcal{H}$ gives rise to a Kripke frame: a tuple $(\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P\in\mathcal{L}})$ where Σ is the set of rays in $\mathcal{H}$; $\mathcal{L}$ is the family of closed linear subspaces of $\mathcal{H}$, with each subspace expressed as a set of rays; and, for each closed linear subspace $P \in \mathcal{L}$, $\xrightarrow{P?}$ is the relation on Σ such that $s \xrightarrow{P?} t$ iff the projection of s onto P in $\mathcal{H}$ is t. With these projections we can also define the non-orthogonality relation: s is not orthogonal to t, written $s \to t$, iff $s \xrightarrow{P?} t$ for some $P \in \mathcal{L}$. So s is orthogonal to t iff $s \nrightarrow t$. Then we can furthermore define the orthocomplement $\sim A$ of any subset $A \subseteq \Sigma$: $s \in \sim A$ iff $s \nrightarrow t$ for all $t \in A$.

The axioms of a quantum dynamic frame given by Baltag and Smets [3] aim at characterizing Kripke frames $(\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P \in \mathcal{L}})$ that can be abstracted away from Hilbert spaces in this manner.

Definition 2.7. A *quantum dynamic frame* $\mathfrak{F}$ is a tuple $(\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P \in \mathcal{L}})$ such that Σ is a set, $\mathcal{L} \subseteq \mathcal{P}(\Sigma)$, and $\xrightarrow{P?} \subseteq \Sigma \times \Sigma$ for each $P \in \mathcal{L}$, and that satisfies the following, where $\rightarrow = \bigcup_{P \in \mathcal{L}} \xrightarrow{P?}$:

(15) $\mathcal{L}$ is closed under arbitrary intersection.

(16) $\mathcal{L}$ is closed under orthocomplement, where the orthocomplement of $A \subseteq \Sigma$ is $\sim A := \{s \in \Sigma \mid s \nrightarrow t \text{ for all } t \in A\}$.

(17) **Atomicity:** For any $s \in \Sigma$, $\{s\} \in \mathcal{L}$.

(18) **Adequacy:** For any $s \in \Sigma$ and $P \in \mathcal{L}$, if $s \in P$, then $s \xrightarrow{P?} s$.

(19) **Repeatability:** For any $s, t \in \Sigma$ and $P \in \mathcal{L}$, if $s \xrightarrow{P?} t$, then $t \in P$.

(20) **Self-Adjointness:** For any $s, t, u \in \Sigma$ and $P \in \mathcal{L}$, if $s \xrightarrow{P?} t \rightarrow u$, then there is a $v \in \Sigma$ such that $u \xrightarrow{P?} v \rightarrow s$.

(21) **Covering Property:** Suppose $s \xrightarrow{P?} t$ for $s, t \in \Sigma$ and $P \in \mathcal{L}$. Then, for any $u \in P$, if $u \neq t$ then $u \rightarrow v \nrightarrow s$ for some $v \in P$; or, contrapositively, $u = t$ if $u \rightarrow v$ implies $v \rightarrow s$ for all $v \in P$.

(22) **Proper Superposition:** For any $s, t \in \Sigma$ there is a $u \in \Sigma$ such that $s \rightarrow u \rightarrow t$.

The above definition differs from the one given in [3] in three ways. First, we have added (16), since (15) and (17)–(22) do not ensure (16).[9] Secondly, the axiom called Mayet's condition is part of the definition in [3], but we treat it as an additional axiom; we will discuss it in Sect. 3.4. Lastly, and perhaps most importantly, even though frames have unitary operators as part of their structure in [3], they do not in our definition. We will show how we deal with unitaries in Sect. 2.2.2.

The following series of lemmas show some basic properties of quantum dynamic frames. They will be used to show the duality result later on, but

[9] For a counterexample, take an arbitrary Hilbert space $\mathcal{H}$ of dimension greater than 2; let Σ be the set of one-dimensional subspaces; let $\mathcal{L}$ consist exactly of $\varnothing$, Σ, and all singletons $\{s\} \subseteq \Sigma$; and let relations $\xrightarrow{\{s\}?}$ be the obvious ones. Since $s \rightarrow t$ iff $s \xrightarrow{\{t\}?} t$, it is easy to verify that $(\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P \in \mathcal{L}})$ satisfies (15), (17)–(22) but not (16).

will also help with conceptual understanding of Definition 2.7. We start with one of fundamental properties of the relation $\to$, which expresses non-orthogonality.

LEMMA 2.8. $\to$ *is reflexive and symmetric.*

PROOF. By Atomicity and Adequacy, $s \xrightarrow{\{s\}?} s$. So, assuming $s \to t$, we have $s \xrightarrow{\{s\}?} s \to t$. By Self-Adjointness, there is a $u \in \Sigma$ such that $t \xrightarrow{\{s\}?} u \to s$. By Repeatability, $u \in \{s\}$, so $u = s$. This means that $t \xrightarrow{\{s\}?} s$, so $t \to s$. ■

This justifies writing $s \perp t$ and $s \not\perp t$ for $s \not\to t$ and $s \to t$, our expression of the symmetric relations of orthogonality and non-orthogonality. Note that $s \perp t$ iff $s \in {\sim}\{t\}$, since ${\sim}A = \{s \in \Sigma \mid s \perp t \text{ for all } t \in A\}$.

An important consequence of Lemma 2.8 is the following. Let us define the modal operators $\Box, \Diamond : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ using $\to$ as accessibility; i.e.,

$$\Box A = \{s \in \Sigma \mid t \in A \text{ whenever } s \to t\},$$
$$\Diamond A = \{s \in \Sigma \mid s \to t \text{ for some } t \in A\} = \neg\Box\neg A,$$

where $\neg$ is the set complement $\Sigma \setminus -$. Clearly, $\Box$ and $\Diamond$ are monotone, and

$${\sim}A = \{s \in \Sigma \mid t \notin A \text{ whenever } s \not\perp t\} = \Box\neg A = \neg\Diamond A.$$

Then Lemma 2.8 implies the following proposition. There, (23) and (24) are the modal-logical expressions of reflexivity and symmetry, respectively; (25) is another way of putting (24), and entails (26) immediately. (27) is by a classic result in [6]; also see [10] for "orthologic", the logic of ortholattices, and its modal-logical representation.

PROPOSITION 2.9. *The monotone maps* $\Box, \Diamond : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ *satisfy:*

(23) $\Box A \subseteq A \subseteq \Diamond A$ *for every* $A \subseteq \Sigma$.

(24) $A \subseteq \Box\Diamond A = {\sim}{\sim}A$ *for every* $A \subseteq \Sigma$.

(25) $\Diamond \dashv \Box$.

(26) $\Box\Diamond\Box = \Box$, *i.e.*, ${\sim}{\sim}{\sim} = {\sim}$.

(27) *Moreover,* $\mathcal{L}_{\sim\sim} = \{A \subseteq \Sigma \mid {\sim}{\sim}A = A\} = \{{\sim}A \mid A \subseteq \Sigma\}$ *forms an ortholattice with the top* Σ, *the bottom* $\varnothing$, *orthocomplement* $\sim$, $\wedge = \cap$ *and* $\vee = \sqcup$, *which is defined by* $A \sqcup B = {\sim}{\sim}(A \cup B) = \Box(\Diamond A \cup \Diamond B) = {\sim}({\sim}A \cap {\sim}B)$ *for any* $A, B \subseteq \Sigma$.

Indeed, as we will show (in Proposition 2.18), $\mathcal{L}_{\sim\sim} = \mathcal{L}$. Its "$\subseteq$" part is easily obtained by (15)–(17):

LEMMA 2.10. *For any* $A \subseteq \Sigma$, ${\sim}A = \bigcap_{t \in A} {\sim}\{t\} \in \mathcal{L}$.

For the "$\supseteq$" part, we need to reflect upon $\xrightarrow{P?}$, which purports to express a projector on Hilbert spaces. We show that it is a partial function (Corollary 2.13) and that $s \xrightarrow{P?} t$ means that t is the closest state to s inside P, in the sense that s and t are orthogonal to the same states in P (Proposition 2.15).

LEMMA 2.11. *$s \xrightarrow{P?} t$ implies*

(28) *$t \in P$ and, for all $u \in P$, if $t \not\perp u$ then $s \not\perp u$.*

PROOF. Suppose $s \xrightarrow{P?} t$. By Repeatability, $t \in P$. Assume $t \not\perp u$ for $u \in P$. Then, by $s \xrightarrow{P?} t \to u$, Self-Adjointness yields $v \in \Sigma$ such that $u \xrightarrow{P?} v \to s$. Since $u \in P$ and, by Lemma 2.8, $u \to w$ implies $w \to u$ for all $w \in P$, the Cover Property (with $u \xrightarrow{P?} v$) implies $u = v$. So $v \to s$ yields $s \not\perp u$. ■

Combining this with the Covering Property, we have

LEMMA 2.12. *$s \xrightarrow{P?} v$ implies that v is the unique $t \in \Sigma$ satisfying* (28).

COROLLARY 2.13. *$\xrightarrow{P?}$ is a partial function for each $P \in \mathcal{L}$.*

LEMMA 2.14. *If $s \to u$ for some $u \in P$, then $s \xrightarrow{P?} t \to u$ for some unique $t \in P$.*

PROOF. Suppose $s \to u$ for some $u \in P$. Then $u \xrightarrow{P?} u \to s$ by Adequacy and the symmetry of $\to$. Hence by Self-Adjointness $s \xrightarrow{P?} t \to u$ for some $t \in P$. The uniqueness is by Corollary 2.13. ■

PROPOSITION 2.15. *$s \xrightarrow{P?} t$ is equivalent to each of* (28) *and*

(a) *$t \in P$ and, for all $u \in P$, $t \not\perp u$ iff $s \not\perp u$.*

PROOF. (a) obviously implies (28). We then show (28) implies $s \xrightarrow{P?} t$. Suppose (28). Then $s \not\perp t$ since $t \not\perp t$ by Lemma 2.8. So Lemma 2.14 yields $v \in P$ such that $s \xrightarrow{P?} v$, where $v = t$ by Lemma 2.12 since t satisfies (28).

Lastly, to show $s \xrightarrow{P?} t$ implies (a), suppose $s \xrightarrow{P?} t$. Then the assertion $t \in P$ and the "only if" part of (a) are Lemma 2.11; so, for the "if", assume $s \not\perp u$ for $u \in P$. By Lemma 2.14, there is a $v \in \Sigma$ such that $s \xrightarrow{P?} v \not\perp u$.

But $s \xrightarrow{P?} t$ and Corollary 2.13 imply $v = t$ and so $t \not\perp u$. ■

This characterization of $\xrightarrow{P?}$ leads to the characterization of $\mathcal{L}$ as the family $\mathcal{L}_{\sim\sim}$ of fixed points of $\sim\sim$. First, writing $s \sqcup t$ for $\{s\} \sqcup \{t\} = \sim\sim\{s, t\} = \Box\Diamond\{s, t\}$, observe

LEMMA 2.16. *Suppose* $s \xrightarrow{P?} t \in P$ *and that there is no* $u \in s \sqcup t$ *such that* $s \to u \in {\sim}P$. *Then* $s = t$.

PROOF. By (24), $s, t \in \{s, t\} \subseteq {\sim}{\sim}\{s, t\} = s \sqcup t$; so $t \xrightarrow{s \sqcup t?} t$ by Adequacy. It is therefore enough by the Covering Property to show that $u \to t$ for all $u \in s \sqcup t$ such that $s \to u$. Fix such u; by the supposition, $u \notin {\sim}P$, i.e., $u \in \neg{\sim}P = \Diamond P$. Hence Lemma 2.14 yields some $v \in P$ such that $u \xrightarrow{P?} v$ and so $u \to v$. Then $u \in s \sqcup t = \Box\Diamond\{s, t\}$ implies $v \in \Diamond\{s, t\}$, i.e., either $v \to s$ or $v \to t$; this entails $u \to t$, since Proposition 2.15 with $s \xrightarrow{P?} t \in P$ and $u \xrightarrow{P?} v \in P$ implies that $v \not\to s$ iff $v \not\to t$ iff $u \not\to t$. ■

LEMMA 2.17. ${\sim}{\sim}P = P$ *for every* $P \in \mathcal{L}$.

PROOF. (24) implies $P \subseteq {\sim}{\sim}P$. Also, ${\sim}{\sim}P = \Box\Diamond P \subseteq \Diamond P$ by (23). Fix any $s \in {\sim}{\sim}P \subseteq \Diamond P$. Then Lemma 2.14 yields $t \in P$ with $s \xrightarrow{P?} t \in P$, whereas $s \in {\sim}{\sim}P$ means that $s \to u \in {\sim}P$ for no u. Hence Lemma 2.16 implies $s = t \in P$. ■

This and Lemma 2.10, combined with Proposition 2.9, establish

PROPOSITION 2.18. $\mathcal{L} = \{A \subseteq \Sigma \mid {\sim}{\sim}A = A\} = \{{\sim}A \mid A \subseteq \Sigma\}$, *and it forms an ortholattice* $(\mathcal{L}, \subseteq, \cap, \sqcup, \Sigma, \varnothing, {\sim})$.

The following import of Propositions 2.18 and 2.15 is worth observing. That is, when orthogonality $\perp$ is abstracted from a quantum dynamic frame, $\perp$ gives back $\mathcal{L}$ and $\xrightarrow{P?}$ using $\sim$. Here is another characterization of $\xrightarrow{P?}$, using the frame version of the Sasaki projection $P[Q] := P \cap ({\sim}P \sqcup Q)$.

PROPOSITION 2.19. $s \xrightarrow{P?} t$ *iff* $P[\{s\}] = \{t\}$.

PROOF. Recall from Proposition 2.15 that $s \xrightarrow{P?} t$ iff (28). Observe

$$\begin{aligned} (28) &\iff t \in P \text{ and, for all } u \in P,\ u \perp s \text{ implies } t \perp u \\ &\iff t \in P \text{ and } t \perp u \text{ for all } u \in P \cap {\sim}\{s\} \\ &\iff t \in P \cap {\sim}(P \cap {\sim}\{s\}) \\ &\iff t \in P \cap ({\sim}P \sqcup \{s\}) = P[\{s\}] \qquad \text{(by Lemma 2.17).} \end{aligned}$$

So $P[\{s\}] = \{t\}$ implies (28) and so $s \xrightarrow{P?} t$. On the other hand, if $s \xrightarrow{P?} t$ and (28), then Lemma 2.12 implies $P[\{s\}] = \{t\}$. ■

2.2.2. Morphisms of Quantum Dynamic Frames. We discuss two options for morphisms on quantum dynamic frames. The first option is due to Moore [16]. First, given a partial function $f : \Sigma_1 \rightharpoonup \Sigma_2$ and any $A \subseteq \Sigma_2$, define the "weakest preimage" of A under f as

$$f^{-1}[A] := \{s \in \Sigma_1 \mid \text{either } f(s) \text{ is undefined or defined and } f(s) \in A\},$$ [10]

and observe that $f^{-1}[-] : \mathcal{P}(\Sigma_2) \to \mathcal{P}(\Sigma_1)$ is right adjoint to the direct-image operation $f[-] : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2) :: B \mapsto \{f(s) \mid s \in B \text{ and } f(s) \text{ is defined}\}$.

Definition 2.20. A partial function $f : \Sigma_1 \rightharpoonup \Sigma_2$ is a *weak map between quantum dynamic frames* $(\Sigma_i, \mathcal{L}_i, \{\xrightarrow{P?}_i\}_{P\in\mathcal{L}_i})$ $(i = 1, 2)$ if $f^{-1}[-]$ "preserves testability", meaning that $f^{-1}[P] \in \mathcal{L}_1$ for all $P \in \mathcal{L}_2$,

Quantum dynamic frames and weak maps form a category, $\mathbb{F}_w$, where identity maps are identity morphisms. Another option of morphisms is bounded morphisms, a familiar concept in modal logic (see [7]). These maps preserve the structure of quantum dynamic frames in the sense of preserving all modal formulas.

Definition 2.21. A function $g : \Sigma_1 \to \Sigma_2$ is a *strong map between two quantum dynamic frames* $(\Sigma_i, \mathcal{L}_i, \{\xrightarrow{P?}_i\}_{P\in\mathcal{L}_i})$ $(i = 1, 2)$ if g is a bounded morphism with respect to $\to_i$, that is,

(29) if $s \to_1 t$, then $g(s) \to_2 g(t)$; and

(30) if $g(s) \to_2 t$, then there exists $u \in \Sigma_1$ such that $g(u) = t$ and $s \to_1 u$.

It is easy to see that identity maps are strong maps and that strong maps are composable. So quantum dynamic frames and strong maps form a category, $\mathbb{F}_s$. (It may be interesting to observe that every strong map with a nonempty domain is surjective by Proper Superposition.)

Bounded morphisms can be characterized by the $\Box$ operator as follows, a characterization commonly found in modal logic. A proof is found, e.g., in [7] (see the proofs of Proposition 5.51 (iv) and Proposition 5.52 (iv) in [7]).

Proposition 2.22. *A function $g : \Sigma_1 \to \Sigma_2$ is a bounded morphism (with respect to $\to_i$) if and only if g^{-1} commutes with $\Box$, in the sense that, for all $B \subseteq \Sigma_2$, $g^{-1}\Box_2 B = \Box_1 g^{-1} B$.*

An immediate consequence is

[10] The notation $f^{-1}[-]$ disagrees with the definition that may be more standard, in which $f^{-1}[A]$ does not contain the "kernel" of f. Our $f^{-1}[A]$ contains the kernel.

PROPOSITION 2.23. *g^{-1} of any strong map $g : \Sigma_1 \to \Sigma_2$ preserves $\sim$ (and therefore $\sqcup$ and $-[-]$ as well).*

PROOF. g^{-1} preserves $\neg$ since g is a total function, and preserves $\Box$ by Proposition 2.22. So g^{-1} preserves $\sim = \Box\neg$. ∎

This in turn immediately shows that $\mathbb{F}_s$ is a wide subcategory of $\mathbb{F}_w$.

PROPOSITION 2.24. *Every strong map $g : \Sigma_1 \to \Sigma_2$ is a weak map.*

PROOF. If $P \in \mathcal{L}_2$, then $\sim\sim P = P$ by Lemma 2.17, and so Proposition 2.23 implies $\sim\sim g^{-1}[P] = g^{-1}[\sim\sim P] = g^{-1}[P]$, which means that $g^{-1}[P] \in \mathcal{L}_1$ by Proposition 2.18. ∎

One may wonder how much structure of quantum dynamic frames is preserved by morphisms of $\mathbb{F}_w$ or of $\mathbb{F}_s$, since the definition of $\mathbb{F}_w$-morphism does not involve $\xrightarrow{P?}$, and that of $\mathbb{F}_s$-morphism involves neither $\mathcal{L}$ nor $\xrightarrow{P?}$. The following should give some reassurance:

PROPOSITION 2.25. *Given quantum dynamic frames $(\Sigma_i, \mathcal{L}_i, \{\xrightarrow{P?}_i\}_{P\in\mathcal{L}_i})$ for $i = 1, 2$, any function $g : \Sigma_1 \to \Sigma_2$ is an isomorphism in $\mathbb{F}_s$, iff (a)–(c) below hold, and iff (a) and (d) hold.*

(a) *g is a bijection.*

(b) *For any $A \subseteq \Sigma_1$, $A \in \mathcal{L}_1$ iff $g[A] \in \mathcal{L}_2$.*

(c) *For any $s, t \in \Sigma_1$ and $P \in \mathcal{L}_1$, $s \xrightarrow{P?} t$ iff $g(s) \xrightarrow{g[P]?} g(t)$.*

(d) *For any $s, t \in \Sigma_1$, $s \to t$ iff $g(s) \to g(t)$.*

PROOF. For "only if" of the first "iff", take an isomorphism g of $\mathbb{F}_s$. (a) is obvious, and (b) is by Proposition 2.24. (c) holds because Propositions 2.19 and 2.23 (along with (a)) imply that $s \xrightarrow{P?} t$ iff $P[\{s\}] = \{t\}$ iff $g[P][\{g(s)\}] = \{g(t)\}$ iff $g(s) \xrightarrow{g[P]?} g(t)$.

"If" of the first "iff" and the second "iff" are straightforward. ∎

The characterization in terms of (a)–(c) makes it clear that $\mathbb{F}_s$ provides the right notion of isomorphism for $(\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P\in\mathcal{L}})$. In contrast, isomorphisms in $\mathbb{F}_w$ are rather too weak.[11]

In addition, functions $g : \Sigma \to \Sigma$ satisfying (a) and (d) correspond to unitary and antiunitary operators on the Hilbert space corresponding to Σ,

[11]Consider the quantum dynamic frame $(\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P\in\mathcal{L}})$ of a two-dimensional Hilbert space. $\mathcal{L}$ consists of $\varnothing$, Σ and all the singletons. This means that any arbitrary permutation on Σ, regardless of $\xrightarrow{P?}$, is an isomorphism in $\mathbb{F}_w$.

as implied by Wigner's theorem.[12] (In fact, (a) and (d) for $g : \Sigma \to \Sigma$ define "unitaries" in [3].) This justifies our omission of unitaries from the structure of objects, since unitaries can be recovered as automorphisms.

3. Dualities

In this section we show dualities between the categories of Piron lattices and those of quantum dynamic frames. In general, a *duality* between two categories $\mathbb{C}$ and $\mathbb{D}$ is a pair of contravariant functors $F : \mathbb{C}^{\mathrm{op}} \to \mathbb{D}$ and $G : \mathbb{D}^{\mathrm{op}} \to \mathbb{C}$, such that $F \circ G$ is naturally isomorphic to the identity functor $1_{\mathbb{D}}$ on $\mathbb{D}$ and $G \circ F$ is naturally isomorphic to the identity functor $1_{\mathbb{C}}$ on $\mathbb{C}$.[13] Here we first define contravariant functors between $\mathbb{F}_w$ and $\mathbb{L}_w$ and between $\mathbb{F}_s$ and $\mathbb{L}_s$, and then show that they form dualities between the corresponding pairs of categories.

3.1. From Piron Lattices to Quantum Dynamic Frames

In this subsection, we define a contravariant functor $F : {\mathbb{L}_w}^{\mathrm{op}} \to \mathbb{F}_w$. Its restriction to $\mathbb{L}_s$ gives another functor $F_s : {\mathbb{L}_s}^{\mathrm{op}} \to \mathbb{F}_s$; we may write F_w for F when the distinction needs emphasizing.

3.1.1. Mapping of Objects. Recall that, given any Piron lattice $\mathfrak{L} = (L, \leq, -^{\perp})$, we write $\mathrm{At}(\mathfrak{L})$ for its set of atoms and $[\![p]\!] = \{a \in \mathrm{At}(\mathfrak{L}) \mid a \leq p\}$ for every $p \in L$. Now define $F(\mathfrak{L})$ to be the structure $(\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P\in\mathcal{L}})$ given by

- $\Sigma = \mathrm{At}(\mathfrak{L})$;
- $\mathcal{L} = \{[\![p]\!] \subseteq \Sigma \mid p \in L\}$;
- for each $[\![p]\!] \in \mathcal{L}$, the relation $\xrightarrow{[\![p]\!]?} \subseteq \Sigma \times \Sigma$ such that, for any $a, b \in \mathrm{At}(\mathfrak{L})$, $a \xrightarrow{[\![p]\!]?} b$ iff $p[a] = b$.

Fixing an arbitrary Piron lattice $\mathfrak{L} = (L, \leq, -^{\perp})$ (for the duration of this subsubsection), we are going to show that $F(\mathfrak{L})$ actually forms a quantum dynamic frame, that is, we will verify that it satisfies the axioms (15)–(22)

[12] For a short proof of this theorem, see Section 4 of [9]. For more about the significance of this theorem to theoretic physics, see Section 3-2 of [18].

[13] Given two functors $F_1, F_2 : \mathbb{C} \to \mathbb{D}$, a natural transformation η from F_1 to F_2 is a family of morphisms $\eta_X : F_1(X) \to F_2(X)$ for all objects X of $\mathbb{C}$ such that for any morphism $f : X \to Y$ of $\mathbb{C}$, $\eta_Y \circ F_1(f) = F_2(f) \circ \eta_X$. Then η is moreover called a *natural isomorphism* if each component η_X is an isomorphism of $\mathbb{D}$.

of a quantum dynamic frame one by one. It is useful to rewrite Lemma 2.3 as (31), as well as to observe (32):

(31) $a \to b$ if and only if $a \not\leq b^\perp$, for any $a, b \in \mathrm{At}(\mathfrak{L})$.

(32) Since $a \wedge a^\perp = O$, (7) implies $a \not\leq a^\perp$, and so $a \to a$ by (31).

Observe that $[\![-]\!]$ is in fact a monotone map preserving a lot of structure.

LEMMA 3.1. $[\![-]\!] : L \to \mathcal{P}(\Sigma)$ *is an order embedding.*

PROOF. $[\![p]\!] \subseteq [\![q]\!]$ implies $p = \bigvee[\![p]\!] \leq \bigvee[\![q]\!] = q$ by Proposition 2.2, whereas $p \leq q$ obviously entails $[\![p]\!] \subseteq [\![q]\!]$. ■

LEMMA 3.2. $[\![-]\!] : L \to \mathcal{P}(\Sigma)$ *preserves all meets and orthocomplement.*

PROOF. $[\![\bigwedge_{i\in I} p_i]\!] = \bigcap_{i\in I}[\![p_i]\!]$ because, for any $a \in \mathrm{At}(\mathfrak{L})$,

$$a \leq \bigwedge_{i\in I} p_i \iff a \leq p_i \text{ for all } i \in I \iff a \in [\![p_i]\!] \text{ for all } i \in I \iff a \in \bigcap_{i\in I}[\![p_i]\!].$$

$[\![p^\perp]\!] = \sim[\![p]\!]$ because, for any $a \in \mathrm{At}(\mathfrak{L})$,

$$\begin{aligned} a \leq p^\perp &\iff \bigvee[\![p]\!] = p \leq a^\perp && \text{(by Proposition 2.2)} \\ &\iff b \leq a^\perp, \text{ i.e., } a \leq b^\perp, \text{ for all } b \in [\![p]\!] \\ &\iff a \not\to b \text{ for all } b \in [\![p]\!] && \text{(by (31))} \\ &\iff a \in \sim[\![p]\!] \end{aligned}$$

■

LEMMA 3.3. $F(\mathfrak{L})$ *satisfies* (15), (16), (17) *Atomicity,* (18) *Adequacy, and* (19) *Repeatability.*

PROOF. (15) and (16) are by Lemma 3.2. Let $a, b \in \mathrm{At}(\mathfrak{L})$ and $[\![p]\!] \in \mathcal{L}$. (17): $\{a\} = [\![a]\!] \in \mathcal{L}$. (18): If $a \in [\![p]\!]$, i.e. $a \leq p$, then Weak Modularity implies $p[a] = a$, i.e., $a \xrightarrow{[\![p]\!]?} a$. (19): If $a \xrightarrow{[\![p]\!]?} b$, then (8) means that $b = p[a] \leq p$, i.e., $b \in [\![p]\!]$. ■

LEMMA 3.4. $F(\mathfrak{L})$ *satisfies* (20) *Self-Adjointness: Given any* $a, b, c \in \mathrm{At}(\mathfrak{L})$ *and* $[\![p]\!] \in \mathcal{L}$, *suppose* $a \xrightarrow{[\![p]\!]?} b \to c$. *Then* $c \xrightarrow{[\![p]\!]?} d \to a$ *for some* $d \in \mathrm{At}(\mathfrak{L})$.

PROOF. $b \leq p$ and $b \not\leq c^\perp$ by Lemma 3.3 (19) and by (31); hence $p \not\leq c^\perp$ and so $c \not\leq p^\perp$. Hence $p[c]$ is an atom by the Covering Law. While $c \xrightarrow{[\![p]\!]?} p[c]$ by definition, we claim $p[c] \to a$.

Suppose that $p[c] \not\to a$; Then (31) implies $p[c] \leq a^{\perp}$ and so $a \leq (p[c])^{\perp} = [p]c^{\perp}$. This implies, since $a \xrightarrow{[\![p]\!]?} b$, that $b = p[a] \leq p[[p]c^{\perp}] \leq c^{\perp}$ by (11). So (31) implies $b \not\to c$, contradicting $b \to c$. ■

LEMMA 3.5. *$F(\mathfrak{L})$ satisfies (21) Covering Property: Given any $a, b, c \in \mathrm{At}(\mathfrak{L})$ and $[\![p]\!] \in \mathcal{L}$, suppose $a \xrightarrow{[\![p]\!]?} b$, $c \neq b$ and $c \in [\![p]\!]$. Then $c \to d \twoheadrightarrow a$ for some $d \in [\![p]\!]$.*

PROOF. Since $c \neq b$ are both atoms, $c \not\leq b$, and so $b^{\perp}[c]$ is an atom by the Covering Law. By definition, $c \xrightarrow{[\![b^{\perp}]\!]?} b^{\perp}[c]$ and so $c \to b^{\perp}[c]$. We claim $b^{\perp}[c] \in [\![p]\!]$ and $b^{\perp}[c] \leq a^{\perp}$, which implies $b^{\perp}[c] \twoheadrightarrow a$ by (31).

Since $b = p[a] \leq p$ by (8) and $c \leq p$ by supposition, $b^{\perp}[c] = b^{\perp} \wedge (b \vee c) \leq b \vee c \leq p$, and so $b^{\perp}[c] \in [\![p]\!]$. Also, $b^{\perp}[c] \leq b^{\perp} = (p[a])^{\perp} = [p]a^{\perp}$ by (8). Therefore $b^{\perp}[c] \leq p \wedge [p]a^{\perp} \leq a^{\perp}$ by (11). ■

LEMMA 3.6. *Given any $a, b \in \mathrm{At}(\mathfrak{L})$, there is a $c \in \mathrm{At}(\mathfrak{L})$ such that $a \not\leq c^{\perp}$ and $c \not\leq b^{\perp}$. So, by (31), $F(\mathfrak{L})$ satisfies (22) Proper Superposition.*

PROOF. If $a \not\leq b^{\perp}$ then $c = a$ works by (32); so assume $a \leq b^{\perp}$. It follows that $a \neq b$ by (32). So, by the Superposition Principle, there is a $c \in \mathrm{At}(\mathfrak{L})$ such that $c \neq a$, $c \neq b$ and $a \vee b = a \vee c = b \vee c$. Then observe $a \not\leq c^{\perp}$, for otherwise $a \leq b^{\perp} \wedge c^{\perp} = (b \vee c)^{\perp} = (a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$, contradicting (32). Similarly, from $b \leq a^{\perp}$ we have $b \not\leq c^{\perp}$, i.e., $c \not\leq b^{\perp}$. ■

Lemmas 3.3 through 3.6 establish

THEOREM 3.7. *$F(\mathfrak{L})$ is a quantum dynamic frame.*

3.1.2. Mapping of Morphisms. To define how F acts on morphisms, we start with the following observation. Given an $\mathbb{L}_w$-morphism $h : L_1 \to L_2$, since h preserves all meets, by the adjoint functor theorem there is a monotone map,

$$\ell_h : L_2 \to L_1 :: y \mapsto \bigwedge_{y \leq_2 h(x)} x,$$

that is left adjoint to h as a monotone map, $\ell_h \dashv h$, that is, for any $x \in L_1$ and $y \in L_2$, $\ell_h(y) \leq_1 x$ iff $y \leq_2 h(x)$. Moreover observe

LEMMA 3.8. *Let $h : L_1 \to L_2$ be an $\mathbb{L}_w$-morphism. Then ℓ_h maps each atom to either an atom or O_1.*

PROOF. For each $b \in \mathrm{At}(L_2)$, (13) yileds $a \in \mathrm{At}(L_1)$ such that $b \leq_2 h(a)$, which by $\ell_h \dashv h$ implies $\ell_h(b) \leq_1 a$, so $\ell_h(b)$ is either an atom or O_1. ■

We define $F(h) : F(L_2) \rightharpoonup F(L_1)$ for any $\mathbb{L}_w$-morphism $h : L_1 \to L_2$ to be the restriction of ℓ_h to the atoms of L_2 that ℓ_h maps to atoms of L_1.

LEMMA 3.9. *Given any $\mathbb{L}_w$-morphism $h : L_1 \to L_2$, $F(h) : F(L_2) \rightharpoonup F(L_1)$ has $F(h)^{-1}[[\![p]\!]] = [\![h(p)]\!]$ for any $[\![p]\!] \in \mathcal{L}_1$ of $F(L_1)$.*

PROOF. Lemma 3.8 and $\ell_h \dashv h$ imply

$$\begin{aligned} F(h)^{-1}[[\![p]\!]] &= \{b \in \mathrm{At}(L_2) \mid \text{either } \ell_h(b) = O_1 \text{ or } \ell_h(b) \in [\![p]\!]\} \\ &= \{b \in \mathrm{At}(L_2) \mid \ell_h(b) \leq_1 p\} \\ &= \{b \in \mathrm{At}(L_2) \mid b \leq_2 h(p)\} = [\![h(p)]\!]. \end{aligned}$$

■

PROPOSITION 3.10. (a) *For any $\mathbb{L}_w$-morphism h, $F(h)$ is an $\mathbb{F}_w$-morphism.*
(b) *F preserves identity morphisms and composition.*

PROOF. (a) $F(h)^{-1}[-]$ preserves testability since Lemma 3.9 means that, for any $[\![p]\!] \in \mathcal{L}_1$, $F(h)^{-1}[[\![p]\!]] = [\![h(p)]\!] \in \mathcal{L}_2$.

(b) For any Piron lattice L, we have $\ell_{1_L}(y) = \bigwedge_{y \leq x} x = y$, that is, $F(1_L) = 1_{F(L)}$. Given any two weak homomorphisms $h_1 : L_1 \to L_2$ and $h_2 : L_2 \to L_3$, we have $F(h_2 \circ h_1) = F(h_1) \circ F(h_2)$ because $\ell_{h_i} \dashv h_i$ implies

$$(\ell_{h_1} \circ \ell_{h_2})(y) = \bigwedge_{\ell_{h_2}(y) \leq_2 h_1(x)} x = \bigwedge_{y \leq_3 h_2 \circ h_1(x)} x = \ell_{h_2 \circ h_1}(y).$$

■

This and Theorem 3.7 mean that F is a contravariant functor from $\mathbb{L}_w$ to $\mathbb{F}_w$. We define another functor F_s by restricting F to $\mathbb{L}_s$. Then we have $F_s : \mathbb{L}_s{}^{\mathrm{op}} \to \mathbb{F}_s$, since F_s lands in $\mathbb{F}_s$, as in Proposition 3.12.

LEMMA 3.11. *Let $k : L_1 \to L_2$ be an $\mathbb{L}_s$-morphism and suppose* $\mathrm{At}(L_2) \neq \varnothing$. *Then ℓ_k maps atoms to atoms.*

PROOF. Since $p \leq k \circ \ell_k(p)$ by $\ell_k \dashv k$, $\ell_k(p) = O_1$ implies $p \leq k \circ \ell_k(p) = k(O_1) = O_2$. Thus, for any $b \in \mathrm{At}(L_2)$, $\ell_k(b) \neq O_1$, and so $\ell_k(b) \in \mathrm{At}(L_1)$ by Lemma 3.8. ■

PROPOSITION 3.12. *For any $\mathbb{L}_s$-morphism k, $F(k)$ is an $\mathbb{F}_s$-morphism.*

PROOF. Given any strong homomorphism $k : L_1 \to L_2$, we prove (29) and (30) of Definition 2.21 for $F(k) : F(L_2) \to F(L_1)$. (29): Observe that, since k preserves $-^{\perp}$ and $\ell_k \dashv k$, any $b \in \mathrm{At}(L_2)$ has $b \leq_2 k \circ \ell_k(b) = k \circ \ell_k(b)^{\perp\perp} = k(\ell_k(b)^{\perp})^{\perp}$ and so $k(\ell_k(b)^{\perp}) \leq_2 b^{\perp}$. Hence $\ell_k(b_0) \leq_1 \ell_k(b_1)^{\perp}$

for $b_0, b_1 \in \mathrm{At}(L_2)$ implies, by $\ell_k \dashv k$, that $b_0 \leq_2 k(\ell_k(b_1)^\perp) \leq_2 {b_1}^\perp$. Thus, by (31), $b_0 \rightarrow_2 b_1$ implies $F(k)(b_0) = \ell_k(b_0) \rightarrow_1 \ell_k(b_1) = F(k)(b_1)$.

(30): Suppose $F(k)(b) \rightarrow_1 a$ for $b \in \mathrm{At}(L_2)$ and $a \in \mathrm{At}(L_1)$. Then $\ell_k(b) = F(k)(b) \not\leq_1 a^\perp$ by (31), and so $b \not\leq_2 k(a^\perp) = k(a)^\perp$ because $\ell_k \dashv k$ and k preserves $-^\perp$. Therefore, by the Covering Law, $k(a)[b] \in \mathrm{At}(L_2)$, so $b \xrightarrow{[\![k(a)[b]]\!]?}_2 k(a)[b]$ and hence $b \rightarrow_2 k(a)[b]$. Moreover, (8) implies $k(a)[b] \leq_2 k(a)$ and so $\ell_k(k(a)[b]) \leq_1 a$ by $\ell_k \dashv k$. But, because $\ell_k(k(a)[b])$ is an atom by Lemma 3.11, $F(k)(k(a)[b]) = \ell_k(k(a)[b]) = a$. ■

3.2. From Quantum Dynamic Frames to Piron Lattices

In this subsection, we define a contravariant functor $G_w = G : {\mathbb{F}_w}^{\mathrm{op}} \rightarrow \mathbb{L}_w$, and obtain another $G_s : {\mathbb{F}_s}^{\mathrm{op}} \rightarrow \mathbb{L}_s$ as the restriction to $\mathbb{F}_s$.

3.2.1. Mapping of Objects. Given any quantum dynamic frame $\mathfrak{F} = (\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P\in\mathcal{L}})$, we define $G(\mathfrak{F})$ as $(\mathcal{L}, \subseteq, \sim)$. We will show that this $G(\mathfrak{F})$ forms a Piron lattice, that is, we will verify that it satisfies (1)–(6). In Proposition 2.18, we established that any $G(\mathfrak{F})$ is an ortholattice; so we carry on to show that $G(\mathfrak{F})$ satisfies the other axioms of a Piron lattice, (2)–(6).

We will use the laws of ortholattice as well as the laws in Proposition 2.9 without particular reference.

Lemma 3.13. *$G(\mathfrak{F})$ satisfies (2) Weak Modularity: if $Q \subseteq P$ for any $Q, P \in \mathcal{L}$, then $P[Q] = Q$.*

Proof. $Q \subseteq P$ implies $Q \subseteq P[Q]$ in any ortholattice. For $P[Q] \subseteq Q$, first observe $P[Q] = P \cap \Box\neg(P \cap \Box\neg Q) \subseteq P \cap (\neg P \cup \Diamond Q) = P \cap \Diamond Q$. Fix $s \in P[Q] \subseteq P \cap \Diamond Q$; so $s \in P$, and Lemma 2.14 yields some $t \in Q$ with $s \xrightarrow{Q?} t$. Then $s = t \in Q$ by Lemma 2.16, since there is no $u \in s \sqcup t$ such that $s \rightarrow u \in {\sim}Q$, as follows. $s \in P$ and $t \in Q \subseteq P$ imply $s \sqcup t \subseteq P$, where $\sqcup$ is the join of $\mathcal{L}$. Therefore $s \in P[Q] \subseteq {\sim}P \sqcup Q = {\sim}(P \cap {\sim}Q)$ implies $(s \sqcup t) \cap {\sim}Q \subseteq P \cap {\sim}Q \subseteq {\sim}\{s\}$, that is, $(s \sqcup t) \cap \Diamond\{s\} \cap {\sim}Q = \varnothing$. ■

Thus $G(\mathfrak{F})$ is an orthomodular lattice; it will be useful shortly to note that $G(\mathfrak{F})$ therefore satisfies the following consequence of (11):

$$p^\perp \vee (p[q]) = (p \wedge [p]q^\perp)^\perp = (p \wedge q^\perp)^\perp = p^\perp \vee q. \tag{33}$$

Lemma 3.14. *$G(\mathfrak{F})$ satisfies (3) Completeness and (4) Atomicity.*

Proof. (3): $\mathcal{L}$ has all meets by (15). Given any $\{P_i \in \mathcal{L}\}_{i\in I}$, $\sim\sim\bigcup_{i\in I} P_i$ is its join in $\mathcal{L}$, because, for every $Q \in \mathcal{L}$,

$$P_i \subseteq Q \text{ for all } i \in I \iff \bigcup_{i\in I} P_i \subseteq Q \iff \sim\sim\bigcup_{i\in I} P_i \subseteq Q.$$

Here "$\Leftarrow$" of the second equivalence is by $\bigcup_{i\in I} P_i \subseteq \sim\sim\bigcup_{i\in I} P_i$; "$\Rightarrow$" is because $\sim\sim$ is monotone and $\sim\sim Q = Q$. Thus $\mathcal{L}$ has all joins.

(4): By (17), singletons $\{s\} \in \mathcal{L}$ serve as atoms. ■

Lemma 3.15. *$G(\mathfrak{F})$ satisfies (5) the Covering Law: if $\{s\} \not\subseteq P \in \mathcal{L}$, then $(\sim P)[\{s\}]$ is a singleton.*

Proof. Since $s \in \neg P = \neg\sim\sim P = \Diamond\sim P$, Lemma 2.14 yields $t \in \sim P$ with $s \xrightarrow{\sim P?} t$, which implies $(\sim P)[\{s\}] = \{t\}$ by Proposition 2.19. ■

Lemma 3.16. *Suppose $s \neq t$ and $u \neq s$ for $s, t \in \Sigma$ and $u \in s \sqcup t$. Then $s \sqcup u = s \sqcup t$.*

Proof. Since $\sqcup$ is the join in $\mathcal{L}$, $s, u \in s \sqcup t$ implies $s \sqcup u \subseteq s \sqcup t$, and so

$$(\sim\{s\})[\{u\}] = \sim\{s\} \cap (s \sqcup u) \subseteq \sim\{s\} \cap (s \sqcup t) = (\sim\{s\})[\{t\}].$$

Yet, by $s \neq u$ and $s \neq t$, Lemma 3.15 implies that both $(\sim\{s\})[\{u\}]$ and $(\sim\{s\})[\{t\}]$ are singletons. Therefore $(\sim\{s\})[\{u\}] = (\sim\{s\})[\{t\}]$. Hence (33) implies

$$s \sqcup u = \{s\} \sqcup ((\sim\{s\})[\{u\}]) = \{s\} \sqcup ((\sim\{s\})[\{t\}]) = s \sqcup t.$$

■

Lemma 3.17. *$G(\mathfrak{F})$ satisfies (6) the Superposition Principle: If $s, t \in \Sigma$ are distinct, then there is a $u \in \Sigma$ distinct from s and t with $s \sqcup u = t \sqcup u = s \sqcup t$.*

Proof. By Lemma 3.16, it is enough to find some $u \in s \sqcup t$ distinct from s and t. We consider two cases: Case 1: $s \not\perp t$. Since $s \xrightarrow{s\sqcup t?} s \neq t \in s \sqcup t$ (by Adequacy), the Covering Property yields some $u \in s \sqcup t$ such that $u \not\to s$, which implies $u \neq s$ by $s \to s$ (Lemma 2.8) and $u \neq t$ by $t \to s$.

Case 2: $s \perp t$. Proper Superposition yields $v \in \Sigma$ such that $s \to v \to t$. Then $v \in \Diamond\{t\} \subseteq \Diamond(s \sqcup t)$, so Lemma 2.14 yields $u \in s \sqcup t$ with $v \xrightarrow{s\sqcup t?} u$. Since $s, t \in s \sqcup t$, therefore by Proposition 2.15 $s \to v \to t$ implies $s \to u \to t$, which means that $s \neq u \neq t$ because $s \not\to t$. ■

By Lemmas 3.13, 3.14, 3.15 and 3.17 as well as Proposition 2.18, we have

Theorem 3.18. *$G(\mathfrak{F})$ is a Piron lattice.*

3.2.2. Mapping of Morphisms. Here we define how G acts on morphisms. Given an $\mathbb{F}_w$-morphism $f : \Sigma_1 \rightharpoonup \Sigma_2$, we can define $G(f) : \mathcal{L}_2 \to \mathcal{L}_1$ as $f^{-1}[-]$ (restricted to $\mathcal{L}_2$), because f being an $\mathbb{F}_w$-morphism means that $G(f)(P) \in \mathcal{L}_1$ for all $P \in \mathcal{L}_2$. Then $G : {\mathbb{F}_w}^{\mathrm{op}} \to \mathbb{L}_w$ is a functor, by Theorem 3.18 and

Proposition 3.19. (a) *For an $\mathbb{F}_w$-morphism f, $G(f)$ is an $\mathbb{L}_w$-morphism.*
(b) *G preserves identity morphisms and composition.*

Proof. (a): $f^{-1}[-] : \mathcal{P}(\Sigma_2) \to \mathcal{P}(\Sigma_1)$, as a right adjoint, preserves all intersections. So $G(f)$ preserves all meets. Moore's condition holds since it amounts to the triviality that, for every $s \in \Sigma_1$, there is a $t \in \Sigma_2$ such that either $f(s)$ is undefined or else $f(s) = t$. (b) follows simply because $f \mapsto f^{-1}[-]$ is a powerset functor (from the category of partial functions). ■

We define a functor $G_s : {\mathbb{F}_s}^{\mathrm{op}} \to \mathbb{L}_s$ as the restriction of G to $\mathbb{F}_s$, since it lands in $\mathbb{L}_s$ by Proposition 2.23.

3.3. Natural Isomorphisms of the Functors

Now that we have described the two pairs of functors, $F : {\mathbb{L}_w}^{\mathrm{op}} \to \mathbb{F}_w$ and $G : {\mathbb{F}_w}^{\mathrm{op}} \to \mathbb{L}_w$ on the one hand and $F_s : {\mathbb{F}_s}^{\mathrm{op}} \to \mathbb{L}_s$ and $G_s : {\mathbb{F}_s}^{\mathrm{op}} \to \mathbb{L}_s$ on the other, we are ready to prove that each pair forms a duality.

For a set Σ, write $\eta_\Sigma : \Sigma \to \mathcal{P}(\Sigma) :: s \mapsto \{s\}$. Then, for any quantum dynamic frame $\mathfrak{F} = (\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P \in \mathcal{L}})$, it is straightforward to check that $FG(\mathfrak{F}) = (\Sigma', \mathcal{L}', \{\xrightarrow{Q?}\}_{Q \in \mathcal{L}'})$ consists of $\Sigma' = \eta_\Sigma[\Sigma]$, $\mathcal{L}' = \{\eta_\Sigma[P] \mid P \in \mathcal{L}\}$, and $\eta_\Sigma(s) \xrightarrow{\eta_\Sigma[P]?} \eta_\Sigma(t)$ iff $s \xrightarrow{P?} t$ (by Proposition 2.19). So, defining $\eta_{\mathfrak{F}} := \eta_\Sigma$, Proposition 2.25 implies

Lemma 3.20. *Each $\eta_{\mathfrak{F}} : \mathfrak{F} \to FG(\mathfrak{F})$ is an isomorphism in $\mathbb{F}_s$.*

Furthermore,

Lemma 3.21. *η is a natural transformation from $1_{\mathbb{F}_w}$ to $F_w \circ G_w$.*

Proof. Given an $\mathbb{F}_w$-morphism $f : \mathfrak{F}_1 \rightharpoonup \mathfrak{F}_2$, we have $G(f) = f^{-1}[-]$, and its left adjoint $\ell_{G(f)}$ has $\ell_{G(f)}(\{s\}) = \bigcap_{s \in f^{-1}[P], P \in \mathcal{L}_2} P$. If $f(s)$ is undefined, then $s \in f^{-1}[\varnothing]$ for $\varnothing \in \mathcal{L}_2$, and so $\ell_{G(f)}(\{s\}) = \varnothing$. If $f(s)$ is defined, then $s \in f^{-1}[P]$ iff $f(s) \in P$, whereas $\{f(s)\} \in \mathcal{L}_2$ by Lemma 3.3 (17); thus $\ell_{G(f)}(\{s\}) = \{f(s)\}$. Therefore $FG(f)(\{s\})$ is $\{f(s)\}$ if $f(s)$ is defined and otherwise undefined. This clearly makes $FG(f) \circ \eta_{\mathfrak{F}_1} = \eta_{\mathfrak{F}_2} \circ f$. ■

Thus, η is a natural isomorphism both from $1_{\mathbb{F}_w}$ to $F_w \circ G_w$ and from $1_{\mathbb{F}_s}$ to $F_s \circ G_s$. On the other hand, given any Piron lattice $\mathfrak{L} = (L, \leq, -^{\perp})$, write $GF(\mathfrak{L}) = (\mathcal{L}, \subseteq, \sim)$ and define $\tau_{\mathfrak{L}} : \mathfrak{L} \to GF(\mathfrak{L})$ by $[\![-]\!] : L \to \mathcal{L}$.

Lemma 3.22. *Each* $\tau_{\mathfrak{L}} : \mathfrak{L} \to GF(\mathfrak{L})$ *is an isomorphism in* $\mathbb{L}_s$.

Proof. Lemma 3.1 means, because $[\![-]\!]$ is onto $\mathcal{L} = \{[\![p]\!] \mid p \in L\}$, that $\tau_{\mathfrak{L}}$ is an order isomorphism; so $\tau_{\mathfrak{L}}$ satisfies (12) and (13). Also, it satisfies (14) by Proposition 3.2. Hence $\tau_{\mathfrak{L}}$ is an isomorphism in $\mathbb{L}_s$ and so in $\mathbb{L}_w$. ■

Lemma 3.23. τ *is a natural transformation from* $1_{\mathbb{L}_w}$ *to* $G_w \circ F_w$.

Proof. Given any $h : \mathfrak{L}_1 \to \mathfrak{L}_2$, Lemma 3.9 implies $GF(h) \circ \tau_{\mathfrak{L}_1}(p) = GF(h)([\![p]\!]) = F(h)^{-1}[[\![p]\!]] = [\![h(p)]\!] = \tau_{\mathfrak{L}_2} \circ h(p)$. ■

Thus τ is a natural isomorphism both from $1_{\mathbb{L}_w}$ to $G_w \circ F_w$ and from $1_{\mathbb{L}_s}$ to $G_s \circ F_s$. Moreover, it is easy to check that $F\tau_{\mathfrak{L}} \circ \eta_{F\mathfrak{L}} :: a\ (\in \mathrm{At}(\mathfrak{L})) \mapsto \{a\} = [\![a]\!] \mapsto \bigwedge_{[\![a]\!] \subseteq [\![p]\!]} p = a$ and that $G\eta_{\mathfrak{F}} \circ \tau_{G\mathfrak{F}} :: P\ (\in \mathcal{L}) \mapsto [\![P]\!] \mapsto {\eta_{\mathfrak{F}}}^{-1}[[\![P]\!]] = \{s \in \Sigma \mid \{s\} \in [\![P]\!]\} = P$; thus $F\tau \circ \eta_F = 1_F$ and $G\eta \circ \tau_G = 1_G$. Therefore we have established

Theorem 3.24. (F, G, η, τ) *and* (F_s, G_s, η, τ) *form dualities between* $\mathbb{F}_w$ *and* $\mathbb{L}_w$ *and between* $\mathbb{F}_s$ *and* $\mathbb{L}_s$, *respectively. Moreover,* $G \dashv F$ *with* η *unit and* τ *counit, where we write* $F : \mathbb{L}^{\mathrm{op}} \to \mathbb{F}$ *and* $G : \mathbb{F} \to \mathbb{L}^{\mathrm{op}}$.

3.4. Mayet's Condition

The duality result we have just proven extends to certain (full) subcategories of $\mathbb{L}_w$, $\mathbb{L}_s$, $\mathbb{F}_w$ and $\mathbb{F}_s$; namely, the categories of Piron lattices and quantum dynamic frames that satisfy the property called *Mayet's condition* [15]. As mentioned in the introduction, this condition added to a Piron lattice captures the structure of an infinite dimensional Hilbert space over the complex numbers, reals, or quaternions.

Definition 3.25. By a *strong automorphism*, let us mean an isomorphism, either of $\mathbb{L}_s$ or of $\mathbb{F}_s$, on the same object. A Piron lattice $\mathfrak{L} = (L, \leq, -^{\perp})$ is said to satisfy Mayet's condition if there is a strong automorphism $k : L \to L$ such that

(34) there is a $p \in L$ such that $k(p) < p$, and

(35) there is a $q \in L$ such that there are at least two distinct atoms below q and $k(r) = r$ for all $r \leq q$.

A quantum dynamic frame $\mathfrak{F} = (\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P \in \mathcal{L}})$ is said to satisfy Mayet's condition if there is a strong automorphism $g : \Sigma \to \Sigma$ such that

(36) there is a $P \in \mathcal{L}$ such that $g^{-1}[P] \subset P$, and

(37) there is a $Q \in \mathcal{L}$ that has at least two distinct elements and such that $g(s) = s$ for all $s \in Q$.

Using this condition let us define a full subcategory $\mathbb{L}_w^M$ (respectively, $\mathbb{L}_s^M$, $\mathbb{F}_w^M$ or $\mathbb{F}_s^M$) of $\mathbb{L}_w$ (respectively, $\mathbb{L}_s$, $\mathbb{F}_w$ or $\mathbb{F}_s$); that is, the objects of $\mathbb{L}_w^M$ are the objects of $\mathbb{L}_w$ satisfying Mayet's condition, whereas any pair of objects $\mathfrak{L}_1$, $\mathfrak{L}_2$ of $\mathbb{L}_w^M$ has the same set of morphisms as it has in $\mathbb{L}_w$. Then $\mathbb{L}_w^M$ and $\mathbb{L}_s^M$ are dual to $\mathbb{F}_w^M$ and to $\mathbb{F}_s^M$, respectively, which follows from

Proposition 3.26. *A Piron lattice $\mathfrak{L}$ satisfies Mayet's condition iff $F(\mathfrak{L})$ satisfies Mayet's condition. A quantum dynamic frame $\mathfrak{F}$ satisfies Mayet's condition iff $G(\mathfrak{F})$ satisfies Mayet's condition.*

Proof. We first show the two "only if" parts. Suppose $\mathfrak{L} = (L, \leq, -^{\perp})$ satisfies Mayet's condition and let $k : L \to L$ be a strong automorphism that satisfies (34) and (35). While $F(k)$ is a strong automorphism, it satisfies (36) and (37) as follows. We have $p, q \in L$ as in (34) and (35). Then $F(k)^{-1}[[\![p]\!]] = [\![k(p)]\!] \subset [\![p]\!]$ by Lemmas 3.9 and 3.1. By (35), $[\![q]\!]$ has at least two elements; also each $s \in [\![q]\!]$ has $s = k(s)$, which implies by $\ell_k \dashv k$ that $\ell_k(s) \leq s$ and so $F(k)(s) = \ell_k(s) = s$ by Lemma 3.11.

Suppose $\mathfrak{F} = (\Sigma, \mathcal{L}, \{\xrightarrow{P?}\}_{P \in \mathcal{L}})$ satisfies Mayet's condition and let $g : \Sigma \to \Sigma$ be a strong automorphism that satisfies (36) and (37). While $G(g)$ is a strong automorphism, it satisfies (34) and (35) as follows. (36) means that there is a $P \in \mathcal{L}$ such that $G(g)(P) = g^{-1}[P] \subset P$. We have Q as in (37); then it contains two dinstinct singletons, and $R \subseteq Q$ implies $G(g)(R) = g^{-1}(R) = R$ since g restricted to Q is the identity.

Now the "if" parts follow from the "only if" parts because Mayet's condition is stable under isomorphisms in $\mathbb{L}_s$ and in $\mathbb{F}_s$. For the first "if", for instance, if $F(\mathfrak{L})$ satisfies Mayet's condition, then by the second "only if" $GF(\mathfrak{L})$ satisfies it as well, and so does $\mathfrak{L}$. ∎

It immediately follows from this fact that the functors F_w, G_w, F_s, and G_s restrict to the subcategories $\mathbb{L}_w^M$, $\mathbb{L}_s^M$, $\mathbb{F}_w^M$ and $\mathbb{F}_s^M$, and moreover, by Theorem 3.24 (and the fullness of these subcategories),

Theorem 3.27. *The pair (F_w^M, G_w^M) forms a duality between $\mathbb{F}_w^M$ and $\mathbb{L}_w^M$ and the pair (F_s^M, G_s^M) forms a duality between $\mathbb{F}_s^M$ and $\mathbb{L}_s^M$.*

4. Conclusions and Future Work

This paper provides duality results connecting Piron lattices and quantum dynamic frames. We have defined a functor F_w from $\mathbb{L}_w$ to $\mathbb{F}_w$ and a functor G_w from $\mathbb{F}_w$ to $\mathbb{L}_w$. We similarly have functors F_s and G_s between $\mathbb{L}_s$ and $\mathbb{F}_s$. We have shown that (F_s, G_s) forms a duality between categories $\mathbb{L}_s$ and $\mathbb{F}_s$ and that (F_w, G_w) forms a duality between categories $\mathbb{L}_w$ and $\mathbb{F}_w$. We have also shown that these dualities are preserved when restricting these categories to those objects that satisfy Mayet's condition. Future work may involve forming dualities between algebraic and set theoretic quantum structures that are even richer. Adding probability to the setting may be a useful step to take, and there exist dualities involving probability already, such as [14]. Another line of future investigation is to develop categories and duality or correspondence results relating to a variation of quantum dynamic frames that do not have parametrized relations, but rather just the non-orthogonality relation.

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