Nino B. Cocchiarella

Two Views of the Logic of Plurals and a Reduction of One to the Other

Abstract. There are different views of the logic of plurals that are now in circulation, two of which we will compare in this paper. One of these is based on a two-place relation of *being among*, as in 'Peter is among the juveniles arrested'. This approach seems to be the one that is discussed the most in philosophical journals today. The other is based on Bertrand Russell's early notion of a class as many, by which is meant not a class as one, i.e., as a single entity, but merely a plurality of things. It was this notion that Russell used to explain plurals in his 1903 *Principles of Mathematics*; and it was this notion that I was able to develop as a consistent system that contains not only a logic of plurals but also a logic of mass nouns as well. We compare these two logics here and then show that the logic of the *Among* relation is reducible to the logic of classes as many.

Keywords: Plurals, Pluralities, Classes as many, The among relation,
The "is one of" relation.

There are different views of the logic of plurals that are now in circulation.[1] One of these is based on a two-place relation of *being among* , as in 'Peter is among the juveniles arrested'.[2] This approach seems to be the one that is discussed the most in philosophical journals today. The other is based on Bertrand Russell's early notion of a class as many, by which is meant not a class as one, i.e., as a single entity, but a mere plurality of things. It was this notion that I developed in 2002 as a provably consistent system that contains not only a logic of plurals but also a logic of mass nouns as well.[3] It also contains, as we show in this paper, the plural logic based on the *Among* relation. We will first briefly describe and compare these two logics here and then show that the logic of the *Among* relation as described in Linnebo [13] is reducible to the logic of classes as many.

---

[1] See, e.g., Boolos [2], Schein [18], Cocchiarella [4], McKay [14], Linnebo [13], Yi [20] and [21], and Oliver and Smiley [15,16].

[2] The alternative reading for the Among relation is: *is one of*. We will use only 'Among' in this paper.

[3] See Cocchiarella [4,6, Chap. 11, 7].

Presented by **Richmond Thomason**; *Received* July 2, 2014

Studia Logica (2015) 103: 757–780
DOI: 10.1007/s11225-014-9591-y

We will first briefly discuss the plural logic based on the *Among* relation as described by Linnebo.[4] Then we will briefly explain the basics of the logic of classes as many, and finally we will show how the logic of the *Among* relation is reducible to the logic of classes as many.

1. One Among Many

The logic of plurals as based on the *Among* relation has been developed as an extension of standard first-order predicate logic with identity. The extension involves adding a new type of quantifiable variable for what nowadays are called pluralities (or what Bertrand Russell much earlier called classes as many). When these plural variables are attached to a quantifier they are assumed to represent our use in ordinary language of plural quantifier phrases, such as 'all republicans who voted against the bill', 'some democrats who voted for the bill', etc. Pluralities are also what plural definite descriptions denote, as in 'the juveniles arrested last night'.

Note that these examples illustrate two different roles for pluralities, one as the *referents* of plural quantifier phrases such as 'all republicans who voted against the bill' and 'some democrats who voted for the bill', which are plural quantifier (noun) phrases of English, and the other as the *denotata* of plural noun phrases, and in particular of plural definite descriptions such as 'the juveniles arrested last night'. The important difference here is that the latter kind of phrase occur as "terms" or "arguments" of predicates, and in particular as arguments (on the right-hand side) of the predicate 'is among'. Linnebo does not include a formal account of the logic of plural definite descriptions in his account of the logic of the *Among* relation, but some of the other works cited on this logic do.[5] It is convenient not having to deal with such definite descriptions here because that will simplify the details in our proof of the reduction theorem given later. The logic of classes as many, however, like some of the other works on the Among relation, does include such an account as part of its logic of names.

Plural (noun) quantifier phrases do not occur as "terms", of course, but only as quantifier phrases. Also, strictly speaking, Linnebo's logic of the *Among* relation does not affix quantifiers to complex plural noun phrases such as 'republicans who voted against the bill', which (as we will see) is how

[4] We have chosen Linnebo's paper because it is a quick and easy read, and also because it is readily accessible by the internet.

[5] See, e.g., Yi [21] and Oliver and Smiley [15,16].

they are represented in the logic of classes as many. Linnebo and others who work with the logic of the *Among* relation analyze these phrases in analogy to the way that most elementary logic texts deal with singular quantifier phrases, namely as a conjunction of predicates, e.g., as in

$$Republican(xx) \wedge Voted\text{-}against\text{-}the\text{-}bill(xx),$$

where xx is a plural variable, which of course suffices. In the plural logic of classes as many, on the other hand, quantifiers are affixed to complex plural noun phrases as well as to simple plurals, and so does the definite description operator (which we take as a special type of quantifier).

The plural variables of the *Among* plural logic are contrasted with the so-called "individual" or singular variables, for which we will use x, y, z, with or without numerical subscripts. The plural variables, as indicated above, are written as xx, yy, and zz with or without numerical subscripts. Despite appearances, each plural variable, is of course a single variable, not two variables. Apparently, writing a single plural variable as xx or yy is supposed to be suggestive of the fact that these variables have pluralities as their values.

Now it is noteworthy that a plurality in both the *Among* logic and the logic of classes as many can be made up of a single object, despite our usual view of thinking of them as consisting of two or more individual objects.[6] In other words, each individual object constitutes a plurality and can be a value of the plural variables. Thus, quantification over pluralities, as in $(\forall xx)\varphi$ or $(\exists yy)\psi$ includes quantification over single objects, so that $(\forall xx)\varphi \rightarrow (\forall x)\varphi$ is valid in this logic, though of course the converse is not.[7]

Why have plural variables as well as the usual first-order variables? If plural variables can have single objects as values as well as pluralities, then why can't first-order variables have pluralities as well as single objects as values? Certainly there is nothing in ordinary English that suggests that our use of plurals means a shift to a different ontological category from that involved in our use of singular expressions—especially given that single objects are among the values of the plural variables. Plural terms, as well as singular terms, can occur as arguments of predicates, and in English both can occur as subjects as well as direct and indirect objects. It is quite different

[6] That in fact is how I thought of the situation in my 2002 paper; but this was later changed in my 2009 paper. The change ensures that the plural 'some' is dual to the plural 'all'.

[7] Linnebo does not in fact stipulate this formula as an axiom, and it does not seem to be derivable from the axioms he does list.

if one were to add predicate variables and quantifiers binding them, because predicates clearly have a different functional role grammatically from that of noun phrases, plural or singular, such as proper names and plural common nouns, or plural and singular definite descriptions—a difference, moreover, upon which a good deal of the history of philosophy and metaphysics has been based.

The implicit assumption of this approach to plural logic, apparently, is that standard first-order predicate logic (with identity) is exclusively a logic of singular reference and therefore not an appropriate medium for representing plurals or what are now also called pluralities. Certainly, beginning with Gottlob Frege, that is the way first-order logic has been interpreted throughout the 20th Century. But must it be interpreted that way, especially given that a logic of plurals had not been developed until recently? We do not think so, and we reject this assumption in our account of pluralities as classes as many. It does not follow, moreover, that having pluralities as values of the first-order variables means that we are ontologically committed to a new type or category of entity, namely pluralities, over and above single objects. Admittedly, that is exactly what is suggested by the logic of the *Among* relation with its separate category or type of variables for pluralities along with quantifiers binding such, which means that the variables cannot be taken only as schema letters. In contrast, on our view where pluralities are taken as values of the first-order variables, and in particular where pluralities are not taken as constituting a separate ontological category, we maintain that pluralities are ontologically nothing over and above the single objects of which they are constituted. This does not mean that all our commonsense talk about pluralities, whether in terms of plural predication or plural reference, can be reduced to talk of singulars. The situation is similar to the irreducibility of mental states to brain states. That is, even though we cannot reduce all our common-sense talk about mental states to talk about brain states, it does not follow that mental states are an ontological type over and above that of brain states, and that we must then be ontologically committed to some form of dualism. Perhaps what is needed in these matters is a revised characterization of ontological commitment.

In any case, as already indicated, the quantifiers $\forall$ and $\exists$ are affixed to (or indexed by) plural variables, as in $(\forall xx)$ and $(\exists yy)$, as well as to so-called singular variables in the plural logic of the *Among* relation. Predicate constants and constants corresponding to each type of variable can be added when needed in particular applications. So-called “non-distributive” predicates, i.e., predicates such as ‘surround’ as in McKay’s example ‘The students surrounded the Pentagon’, are not (and should not be) excluded. In this example, the Pentagon was (supposedly) surrounded by many students (i.e., a plurality of students), none of which individually (i.e., alone) can be said to have surrounded the Pentagon. With a distributive predicate, such as 'mortal' as in 'All men are mortal', it follows that each individual man is mortal if all men are.

The one primitive constant of this logic is a two place-predicate $\prec$ for the relation of *being among*. Unlike ordinary two-place predicates between individuals, however, this predicate is said to result in a well-formed (meaningful) formula only when a singular term occurs on the left (or first-argument position) and a plural term occurs on the right (in the second-argument position), as in $x \prec yy$. It is not clear why only a plurality term (variable or constant) can occur in the right-hand position of $\prec$. After all, single objects, i.e., individuals, are also values of the plurality variable yy, in which case one would think that $x \prec x$ would be meaningful, and valid as well. In any case the first-order language with $\prec$ as a primitive and with both distributive and non-distributive predicates allowed is called *PFO+*, where *PFO* stands for Plural First-Order logic, which is what Linnebo calls it. Of course, one might well ask whether *PFO* really is just a *first-order* logic given that plural variables and quantifiers binding such are part of its language and logic.

The most important axiom schema for the *Among* relation is the following conditional comprehension principle:

$$(\exists x)\varphi(x) \to (\exists yy)(\forall x)[x \prec yy \leftrightarrow \varphi(x)], \qquad \text{(Comp)}$$

where φ is a formula of *PFO+* that contains x (and possibly other variables) free but contains no occurrences of the plural variable yy. The antecedent is essential here because there is no such thing as an empty plurality. So, before one can posit the "existence" of a plurality of x such that $\varphi(x)$, we need to know that some object x is such that $\varphi(x)$. The second axiom in fact stipulates that no plurality is empty:

$$(\forall yy)(\exists x)(x \prec yy).$$

The final axiom schema is an axiom of extensionality:

$$(\forall yy)(\forall zz)[(\forall x)(x \prec yy \leftrightarrow x \prec zz) \to (\varphi(yy) \leftrightarrow \varphi(zz))].$$

What this says in effect is that co-extensive pluralities are indiscernible, which is an indirect way of saying that they are identical. This is not a problem so long as we are not extending the language to include tense and modal operators. But then even if we were to extend the language to include such operators, Linnebo and other defenders of this approach to plural logic do not find this problematic. That is because they maintain that a plurality "must include precisely the objects that it in fact includes," where 'must' is interpreted with necessary force.[8] In other words Linnebo and the other defenders of this position are willing to accept the following two principles:

$$x \prec yy \rightarrow \Box(x \prec yy),$$
$$\neg(x \prec yy) \rightarrow \Box\neg(x \prec yy).$$

To use Linnebo's example, the individual people who are among the people now wearing shoes are necessarily among the people now wearing shoes (ibid.). Plural common nouns, in other words, such as 'people who are now wearing shoes', are "rigid" in the same way that proper names are said to be rigid in modal logic.

In our view, this position confuses pluralities with sets, which have their being in their members and not in the plural concepts expressed by such phrases as 'people now wearing shoes'. Clearly, a plural phrase and the concept it expresses can refer to (or denote) different pluralities at different times, and certainly in different possible worlds as well. One of the main reasons for constructing a logic of plurals is to characterize the implicit logic of natural language where pluralities are what plural noun phrases denote or refer to. That is certainly why philosophers beginning with Russell were initially interested in a logic of plurals. And if so, then don't we use those phrases to refer to or denote different pluralities over time and in different possible worlds? Shouldn't a logic of plurals represent the way it is used in natural language?

The above two principles, incidentally, can easily be avoided by simply restricting the above extensionality axiom to extensional formulas, i.e., formulas in which no tense or modal operators occur. Leibniz's law for singular terms can be left without change, because that law applies only to singular terms. In any case, regardless whether or not the followers of the *Among* plural logic accept this suggested modification, the above theses are emphatically rejected in our alternative approach of the logic of classes as many where pluralities are seen as what plural concepts refer to or denote.

2. The Logic of Names (Noun Phrases)

The logic of classes as many is an extension of a more basic logic called the logic of names, where by a name we mean a noun phrase consisting of either

[8]Linnebo [13].

a proper name, definite description, a common noun, complex or simple, or a verbal noun, which is used when reference is to events. The theory of reference implicit in this logic differs from the standard account of analytic philosophy in that it recognizes general as well as singular reference, whereas the standard view recognizes only singular reference, i.e., reference involving the use of a proper name or (singular) definite description. On our account, because a basic form of judgment is expressed by an assertion that consists of a noun phrase and a verb phrase, the noun phrase is taken as having a referential role regardless whether or not it is a proper name, a definite description, or a quantifier noun phrase. A definite description, incidentally, is counted in our theory as a quantifier phrase on a par with a universal or existential quantifier phrase, and, because our first-order logic is free of existential presuppositions, so is the use of a proper name.

A proper name, such as ‘George’, for example, can occur as part of a quantifier phrase, as in $(\exists x George)F(x)$, which indicates that the name ‘George’ is being used with existential presupposition. It is noteworthy, moreover, that in our logic of classes as many, which is an extension of our logic of names, all names, proper or common, can be transformed into “terms”, i.e., arguments of predicates, and one result of such a transformation is that the formula $(\exists x George)F(x)$ turns out to be equivalent to the more standard free-logic expression: $(\exists x)[x = George \wedge F(x)]$.

Definite descriptions, as far as their logical syntax is concerned, are also quantifier phrases, and they are like indefinite descriptions in that regard.[9] Both definite and indefinite descriptions are quantifier phrases and as such both can be used as referential expressions, though of course they differ semantically in their particular referential roles.

A quantifier phrase is made up of two parts, the first being a determiner such as ‘every’, ‘some’, the indefinite article ‘a’, and the definite article ‘the’—and others as well, such as ‘most’ ‘few’, etc., which we will not deal with here. The second part of a quantifier phrase is a common noun, or what we call a *common name*, which could be a mass noun, a count noun, or a gerund in its role as a verbal noun (enabling us to refer to a kind of event). A count noun can be simple, such as, e.g., ‘politician’, or complex, such as ‘politician who is conservative’, where the complexity is the result of affixing a qualifying relative clause, such as ‘who (or that, or which) is conservative’, to the head noun. Similarly, a mass noun can be simple, such as, e.g., ‘water’, or complex as with ‘water that is polluted’. Because our

---

[9] We agree in this respect with Gareth Evans who held a similar view of definite descriptions in his [8, p. 57].

concern here is with plurals, we will not deal with either gerunds or mass nouns in this paper. The interested reader can find our development of the logic of mass nouns in Cocchiarella [7].

Definite descriptions, incidentally, can also be used with or without existential presupposition. We will use $\exists_1$ for the definite description operator when it is used with such a presupposition. Our analysis of such a use agrees in essentials with Bertrand Russell's analysis. Thus, e.g., although we symbolize 'The A is (are) F' as $(\exists_1 xA)F(x)$, the truth conditions of this formula are given in the following biconditional (which we assume as an axiom schema)[10]:

$$(\exists_1 xA)F(x) \leftrightarrow (\exists xA)[(\forall yA)(y = x) \wedge F(x)].$$

The logic of names contains absolute as well as relative quantifier phrases, i.e., relative quantifier phrases such as $(\forall xA)$ and $(\exists xA)$, where A is a name, common or proper, and complex or simple. We will use the standard quantifier forms $(\forall x)$ and $(\exists y)$ for the absolute quantifier phrases. As indicated, we will use x, y, z, etc., with or without numerical subscripts, as first-order variables and A, B, C, with or without numerical subscripts, as name variables. Complex names are formed by adjoining so-called "defining" or restricting relative clauses to names. We will use '/', as in $A/\varphi x$ to represent the adjunction of a formula φx to the name A (which may itself be complex). We read $A/\varphi x$ as 'A that is φx'. Thus, e.g., the quantifier phrase representing reference to a politician who is conservative would be symbolized as $(\exists x Politician/Conservative(x))$.

Names and *formulas* are inductively defined simultaneously as follows: (1) every name variable (or constant) is a *name*; (2) for all first-order variables $x, y, (x = y)$ is a *formula*; and if φ, ψ are formulas, B is a name (complex or simple), and x and C are a first-order and a name variable respectively, then (3) $\neg\varphi$, (4) $(\varphi \to \psi)$, (5) $(\forall x)\varphi$, (6) $(\forall xB)\varphi$, (7) $(\exists_1 xB)\varphi$, and (8) $(\forall C)\varphi$ are *formulas*, and (9) B/φ and (10) $/\varphi$ are *names*. The existential quantifier and other sentential connectives are understood as defined in the usual way. We assume the usual definitions of bondage and freedom for first-order variables and of the proper substitution of one such variable

[10] We use the dual quantifier expression $\forall_1$ for the use of a definite description that is without existential presupposition, as in 'The student who writes the best essay will receive a grade of A' in a context in which two or more students might write the best essays equally well. The axiom scema for $\forall_1$ is:

$$(\forall_1 xA)F(x) \leftrightarrow (\forall xA)[(\forall yA)(y = x) \to F(x)].$$

for another in a formula. We assume as well the definitions of bondage and freedom of occurrences of name variables in formulas, and the proper substitution in a formula φ of a name variable (or constant) B for free occurrences of a name variable C.[11]

As noted, the logic of names consists of free first-order predicate logic (with identity), which we will assume hereafter. Quantifiers, as indicated in our definition of formulahood, apply to name variables as well as to first-order variables. The axioms for these are parallel to those for monadic second-order predicate logic, namely:

$$(\forall C)\varphi \to \varphi(B[x]/C), \quad \text{where } B \text{ is free for } C \text{ in } \varphi \text{ with respect to } x;$$

$$\chi \to (\forall C)\chi, \quad \text{where } C \text{ is not free in } \chi \text{; and}$$

$$\chi \to (\forall x)\chi, \quad \text{where } x \text{ is not free in } \chi.$$

Two additional axioms show how the relative quantifier phrases are connected with the absolute quantifiers:

$$(\forall xA)\varphi \leftrightarrow (\forall x)[(\exists yA)(x = y) \to \varphi], \quad \text{where } x, y \text{ are different variables;}$$

$$(\forall xA/\psi)\varphi \leftrightarrow (\forall xA)[\psi \to \varphi].$$

Finally, as primitive inference rules we assume *modus ponens* and universal generalization for absolute quantifiers indexed by either a first-order or a name variable. The rule of universal generalization for relative quantifiers is derivable. The logic of names as briefly described here is equivalent, incidentally, to monadic second-order predicate logic. For proof of the details see [3]. We might also note that Stanislaw Leśniewski's logic of names, which he also called *ontology*, is reducible to this logic of names.[12]

The logic of names, like the plural logic of the *Among* relation, extends (free) first-order logic by including a new logico-grammatical category with its own constants, both simple and complex, and corresponding variables that can be affixed to quantifiers. But, unlike the logic of the *Among* relation, the category of names is an essential component of a key grammatical category of natural language, namely the category of noun phrases. It is this category that complements verb phrases, and it is by means of this category that reference, both general and singular, is achieved. This kind of extension, in other words, allows for a more natural representation of our speech and mental acts than does a separate category of plurals as in the logic of *Among*.

[11] For details see Cocchiarella [3].

[12] For proof of this claim see Cocchiarella [3] or [6, Chap. 10].

3. Classes as Many (Pluralities)

Now, to obtain the logic of classes as many, we extend the logic of names by allowing for a transformation (specifically a "nominalization") of the names that occur as parts of quantifier phrases into "terms" that can be substituends of the first-order variables that occur as arguments of first-order predicates. As already noted, this does affect how we interpret the ontological commitments of first-order logic. In our view, just as the nominalization of predicate expressions has come about through the evolution of language and culture—and hence our introduction and use of abstract intensional objects—so too our use of "nominalized" common noun phrases, complex or simple—and hence our introduction and use of pluralities or classes as many—is a similar development of culture and language. The logic of both forms of nominalization is part of a more general ontological framework that I have called conceptual realism.[13]

With this transformation, proper and common names (count nouns) are taken as denoting pluralities (including pluralities of one). We need this feature in a plural logic because predicates can be true of pluralities as arguments no less so than of single objects, i.e., it is a feature that allows us to denote pluralities as well as to quantify over and refer to them. In the case of proper names this means that a proper name can now occur as a "term", i.e., as an argument of predicates, just the way it does in standard free logic where it will either denote nothing or at most a single object. In the case of a common count noun, such as 'man', we similarly obtain a "term", e.g., 'mankind', which might denote nothing (at a given time, e.g., after a nuclear holocaust), or which might denote a plurality such as the totality of all humans alive today.

The transformation ("nominalization") of simple common names and name variables into terms is no different than that for proper names. But in order to transform a complex common name into a complex term we need a variable-binding operator that functions in the way that the λ-operator functions in the construction of complex predicates. We will use the cap-notation with brackets, $[\hat{x}A/\ldots x\ldots]$, for this purpose. Accordingly, where A is a name, proper or common, complex or simple, we take $[\hat{x}A]$ to be a complex name in which the variable x is bound and that can occur either as part of a quantifier phrase or as a complex term. Thus, where A is a name and φ is a formula, $[\hat{x}A], [\hat{x}A/\varphi]$, and $[\hat{x}/\varphi]$ are names in which all of the

[13] See, e.g., Cocchiarella [6] for an account of this ontology.

free occurrences of x in A and φ are bound. When they occur as terms, we read these expressions as follows:

$[\hat{x}A]$ is read as '(the) things (i.e., single objects) that are A' (or just '(the) A's');

$[\hat{x}A/\varphi]$ is read as '(the) A's that are φ'; and

$[\hat{x}/\varphi]$ is read as '(the) things (single objects) that are φ'.

It should be noted here that only single objects, i.e., things or individuals, can be members of classes as many.

The simultaneous inductive definition of names and formulas given in the last section is now understood to be extended to include names of this complex form along with n-place predicate constants (for $n \in \omega$) as well.[14] Note that we now have formulas of the form $(\forall y[\hat{x}A])\varphi(y/x)$, as well as those of the form $(\forall xA)\varphi x$ and $(\forall yA(y/x))\varphi(y/x)$.[15] The first of these forms is reducible to the last because of the addition of the following axiom schema to the axioms for the simple logic of names:

$$(\forall y[\hat{x}A])\varphi \leftrightarrow (\forall yA(y/x))\varphi, \qquad \text{where } y \text{ does not occur in } A.$$

The existential counterpart to this axiom, namely,

$$(\exists y[\hat{x}A])\varphi \leftrightarrow (\exists yA(y/x))\varphi$$

is theorem 8 of the logic of classes as many. We will later use an instance of (a rewrite of) this theorem schema in the proof of our counterpart of the comprehension principle of the logic of *Among*. There are other axioms and theorems as well, needless to say, the description of which we will not go into here. But the interested reader can find them in Cocchiarella [4] or the appendix of Cocchiarella [7].[16]

The motivation for these axioms is a result of three important features of the notion of a class as many as described by Russell in his account of plurals in his 1903 *Principles of Mathematics*. The first feature is that a vacuous common name, i.e., a common name that names nothing, has no plurality as its denotatum, which is not the same as denoting the empty class as many—because in fact there is no such thing as an empty plurality. Thus, according to Russell, "there is no such thing as the null class, though

[14] See Cocchiarella [4] for the full definition or see the appendix of Cocchiarella [7].

[15] We take $A(y/x)$ and $\varphi(y/x)$ to be the result of properly substituting y for x in A and φ, respectively. The slash '/ ' in these expressions is not to be confused of course with the slash in the formation of a complex name $A/\varphi x$.

[16] For a number of interesting theorems of the logic of classes as many, see Cocchiarella [4].

there are null class-concepts."[17] The second feature is that the denotatum of a plural common name that names just one thing is just that one thing, or, in other words, single objects are themselves pluralities, which means that single objects are members of themselves as classes as many. This corresponds to each individual being "among" itself in the plural logic of *Among*. Indeed, it provides the basic rationale for single objects being pluralities, a rationale that cannot be found independently in the logic of *Among*.

Russell's third feature is that, unlike sets, classes as many are literally made up of their members, i.e., they are merely pluralities (*Vielheiten*), and as such they cannot themselves be members of classes as many. Thus, according to Russell, "though terms may be said to belong to ... [a] class [as many], the class [as a plurality] must not be treated as itself a single logical subject."[18] It is this feature of not being a member of any class as many—unless it is itself a single object and therefore a member of itself—that partly characterizes the non-individuality of a class as many as a mere plurality having no being beyond the objects that make it up, but which nevertheless can be referred to by means of plural quantifier phrases or denoted by "nominalized" plural noun phrases occurring as terms of predicates.

Now it is just as natural, we claim, to speak of membership in a class as many in the sense of being one among the many that make up that class as it is of membership in a set, or class as one. We can define this notion of being a member of a class as many, or being among, as follows.

Def: $x \in y \leftrightarrow (\exists A)[(y = A) \wedge (\exists z A)(x = z)]$.

Note that in the definition of $\in$ the occurrence of A in $(y = A)$ is as a term denoting the plurality of things that fall under the name concept that A stands for, whereas the occurrence of A in the quantifier phrase $(\exists z A)$ stands for the name concept itself. With membership understood in this way we can define inclusion, proper or otherwise, in the usual way.

Def: $x \subseteq y \leftrightarrow (\forall z)[z \in x \rightarrow z \in y]$.

Def: $x \subset y \leftrightarrow x \subseteq y \wedge y \nsubseteq x$.

Russell's paradox is not derivable in this logic, incidentally. Instead of leading to a contradiction, the Russell class as many, $[\hat{x}/(\exists A)(x = A \wedge x \notin A)]$, is easily shown not to exist (as a value of the bound first-order

[17]Russell [17, §70].

[18]Russell [17, §70].

variables).[19] Similarly, the empty class as many, namely, the class of *things* that are not self-identical, $[\hat{x}/x \neq x]$, also does not exist (as a value of the bound first-order variables); and neither does the universal class, $[\hat{x}/(x = x)]$, if there are at least two single objects, i.e., *individuals*. Following Nelson Goodman [9], we will also call individuals *atoms*.[20] The (complex) name for atoms is defined as follows:

Def: $Atom = [\hat{x}/\neg(\exists y)(y \subset x)]$.

This definition of an atom goes back to Goodman and the so-called Leonard–Goodman calculus of individuals, which when formulated within a free logic turns out to be reducible to our present logic of classes as many.[21] We retain this terminology here because Goodman's nominalistic dictum that things are identical if they have the same atoms is provable in the logic of classes as many. That is,

$$(\forall x)(\forall y)[(\forall z Atom)(z \in x \leftrightarrow z \in y) \rightarrow x = y]$$

is a theorem of the logic of classes as many. Indeed, not only is this dictum provable but it is a consequence of the unqualified extensionality principle,

$$(\forall z)(z \in x \leftrightarrow z \in y) \rightarrow x = y,$$

which is taken as an axiom of the logic of classes as many.

The extensionality axiom is a natural assumption for the concept of a class as many. After all, if plurality A is made up of the same single objects as plurality B, then they must be the same plurality (regardless of the difference, if any, between the concept A and the concept B). But perhaps one might argue that if we were to extend the system to include a tense or modal logic as well, then we would seem to be committed to the two theses that Linnebo finds unproblematic for pluralities, namely, that if x is among a plurality A, then necessarily x is among that plurality, and similarly that if x is not among the plurality A, then necessarily x is not among that plurality[22]:

[19] It should be remembered that in free logic being a substituend of free first-order variables is not the same as denoting a value of the bound first-order variables. In free logic, in other words, some terms may denote nothing.

[20] For details on these matters see Cocchiarella [4] or [6, Chap. 11].

[21] See Leonard and Goodman [12]. See Eberle [10, Chap. 2], for a reconstruction of the calculus of individuals in a free first-order logic.

[22] As is well-known, various notions of necessity can be defined within tense logic. So this result would apply in tense logic as well.

$$x \in A \to \Box(x \in A),$$
$$x \notin A \to \Box(x \notin A).$$

This result, we have said, should be rejected in a logic of plurals, and certainly it should be rejected in the logic of classes as many. After all, common name concepts generally change the pluralities they refer to or denote over time, and certainly over different possible worlds.[23] Names of animals and plants that have become extinct, for example, will no longer denote what they once may have denoted, and they will no doubt refer to or denote different pluralities in different possible worlds. The claim that common name concepts cannot refer to or denote different pluralities over time or in different possible worlds is certainly a consequence we do not want, and, we maintain, it should be rejected.

Of course, as noted, this result depends on extending our logic of classes as many to include a tense or modal logic as well. There is no problem, in other words, so long as we restrict ourself to a strictly extensional logic. But then, actually there is no problem even with a tense or modal logic added to the system—so long as we restrict the applications of Leibniz's law for pluralities to strictly extensional contexts. Doing so, moreover, does not mean that the full, unqualified version of Leibniz's law does not apply to atoms, i.e., single objects. Indeed, the natural solution to this problem is to have two versions of Leibniz's law, one restricted to extensional contexts, and the other applicable to all contexts but only for atoms. The extensional identity of pluralities would then not lead to their being necessarily identical. Thus, the end result is that we have two versions of Leibniz's law, one for all extensional contexts, which is derivable by induction on formulas from the atomic case:

$x = y \to (\varphi \to \psi)$, where φ, ψ are atomic formulas and ψ is obtained from φ by replacing an occurrence of y by x[24]

and the other for all contexts:

$$(\exists z Atom)(x = z) \wedge (\exists z Atom)(y = z) \to [x = y \to (\varphi \leftrightarrow \psi)],$$

where ψ is obtained from φ by replacing one or more free occurrences of x by free occurrences of y.

[23] Common names will *refer to* pluralities when they occur as parts of quantifier phrases, and they will *denote* the same pluralities when they occur as terms.

[24] The full version of Leibniz's law is derivable from this and the other axioms by a simple induction on extensional formulas, i.e., formulas in which no intensional operators occur.

This last axiom is redundant, we want to emphasize, if we do not add any nonextensional contexts to the logic of classes as many. Finally, we note that the difference between how Leibniz's law applies to atoms and how it applies to pluralities is significant in how it distinguishes ontologically the individuality of atoms from the mere plurality of classes as many.

Finally, we should note that although most common names are not "rigid" in the pluralities they refer to or denote, nevertheless the common name 'Atom' is "rigid", i.e., atoms are necessarily atoms:

$$Atom =_{df} [\hat{x}/\Box\neg(\exists y)(y \subset x)].$$

4. Plural Reference and Plural Predication

What is generally called plural quantification in the literature corresponds to what in our conceptualist framework of general as well as singular reference we call plural reference, a terminology that we will continue here. Plural reference and predication are important, if not central, motivating features of the logic of classes as many.

Now there are two parts to our analysis of plural reference and plural predication, which we will briefly review here.[25] The first deals with a logical analysis of plural reference and predication in our speech and mental acts. The second deals with the logical forms that represent the truth conditions of those acts in terms of our logic of classes as many. The logical forms representing our speech and mental acts are a part of the deductive machinery of our overall logic only insofar as they are connected by meaning postulates to the logical forms that represent their truth conditions in the logic of classes as many.

We extend the simultaneous inductive definition of the meaningful (well-formed) expressions of the logic of classes as many to include the following clauses, which are designed to represent plural reference and predication in our speech and mental acts. Because we are concerned only with plurals in the present paper, we restrict our definition to count nouns.

1. if A is a common count noun, then A^P is a *plural name*;
2. if A is a common count noun, x is a first-order variable, and φx is a formula, then $[\hat{x}A/\varphi x]^P$ and $[\hat{x}/\varphi x]^P$ are *plural names*;
3. if $A/\varphi(x)$ is a (complex) common count noun, then $(A/\varphi x)^P$ is

[25] For a more detailed account see Cocchiarella [6, Chap. 11].

$A^P/[\lambda x\varphi x]^P(x)$ and $[\hat{x}A/\varphi x]^P$ is $[\hat{x}A^P/[\lambda x\varphi x]^P(x)]$;

4. if F is a one-place predicate constant, or of the form $[\lambda x\varphi(x)]$, then F^P is a one-place *plural predicate constant*; and
5. if A^P is a plural name, x is a first-order variable, and φ is a formula, then $(\forall xA^P)\varphi$ and $(\exists xA^P)\varphi$ are formulas.

In regard to clause (5), we read, e.g., '$(\forall xRepublican^P)$' as the plural phrase 'all republicans' and '$(\exists xR\ epublican^P)$' as the plural phrase 'some republicans', and similarly '$(\forall xRepublican^P/Conservative)(x)$' as 'all republicans who are conservative', or more simply 'all conservative republicans', and '$(\exists xRepublican^P/Conservative(x))$' as 'some conservative republicans', etc. We note that a plural name is not a name *simpliciter* (in the logic) and that unlike the latter there is no rule for the "nominalization" (or transformation) of a plural name into a term. We also note that only *monadic* predicates are pluralized. With the addition of λ-abstracts, a two-place relation R can be pluralized in either its first- or second-argument position, or even in both, by using a λ-abstract, as, e.g.,

$$[\lambda xR(x,y)]^P,$$

$$[\lambda yR(x,y)]^P,$$

$$[\lambda x[\lambda y[R(x,y)]^P(y)]^P,$$

respectively; and a similar observation applies to n-place predicates for $n > 2$.

We can now represent the plural references and predications we express in our speech acts in a natural and intuitive way. Also, given the following meaning postulates, we can then represent the truth conditions of these speech acts in terms of our logic of classes as many. The first meaning postulate is for the plural 'Some':

$$(\exists xA^P)\varphi x \leftrightarrow (\exists x/x \subseteq A)\varphi x, \qquad (SmCount)$$

and the second is for the plural 'All':

$$(\forall xA^P)\varphi x \leftrightarrow (\forall x/x \subseteq A)\varphi x. \qquad (AllCount)$$

Thus, for example, the truth conditions of the sentence 'All republicans are conservative', which can be symbolized as:

$$(\forall xRepublican^P)Conservative^P(x),$$

can now be represented as:

$$(\forall x/x \subseteq Republican))Conservative^P(x),$$

where $x \subseteq Republican$ means that x is a subplurality of the totality (or class as many) of republicans. Then, given a meaning postulate to the effect that the predicate adjective 'conservative' is distributive, i.e., the postulate

$$Conservative^P(x) \leftrightarrow (\forall y/y \in x)Conservative(y),$$

it follows that the original sentence 'All republicans are conservative' is equivalent to 'Every republican is conservative', i.e.,

$$(\forall y Republican)Conservative(y).$$

A similar analysis, which we will not go into here, can then be given for 'Some republicans are conservative'. Non-distributive predicates, such as in 'The students surrounded the Pentagon' will of course be irreducible in their application to a plurality. Where A is a name symbol for 'the Pentagon', this sentence can be symbolized as follows:

$$(\exists_1 x Student^P)[\lambda x Surrounded(x, A)]^P(x),$$

which, by *(SmCount),* initially can be reduced to

$$(\exists_1 x/x \subseteq [\hat{y}Student])[\lambda x Surrounded(x, A)]^P(x).$$

But because 'surrounded' is non-distributive in its first-argument position, no further reduction can be given. More examples can be found in Cocchiarella [6, Chap. 11].[26]

Before concluding this brief description of the logic of classes as many, we note that one possible application of the logic of plurals suggested by Linnebo is that pluralities might be used in set theory as a way of quantifying over collections of sets, or what are usually called proper (or ultimate) classes. Indeed, such a use of classes as many as pluralities had already been formalized by Bell [1]. It is noteworthy that Bell's system was shown to be reducible to the above logic of classes as many in [4].

5. The Reduction of the Plural Logic of Among

We turn now to the reduction of the plural logic of *Among* to the logic of classes as many. We first describe a translation function from Linnebo's

---

[26] In Cocchiarella [4–6], I took there to be at least two single objects in each plurality; that is, pluralities consisting of just a single object were not considered pluralities. I corrected that position in [7].

plural logic to the logic of classes as many. Then, in the next section we show that the translation of every theorem of Linnebo's plural logic is a theorem of the logic of classes as many, and hence that Linnebo's plural logic is reducible to the logic of classes as many.

We first describe the formal language $\mathrm{L}_{\mathrm{PFO}}$ of Linnebo's Plural First-order Logic with or without nondistributive predicates or predicates that take plural arguments. As already indicated, the logic of classes as many can accommodate both distributive and nondistributive predicates.

(1) **singular and plural terms**:
singular variables x_i
plural variables xx_i
singular constants a_i
plural constants aa_i

(2) **two dyadic predicates**:
$=$ (the identity sign).
$\prec$ (the relation of "is one of").
nonlogical predicate constants R_i^n (for n-place relations).

(3) **Formulas**:
Atomic formulas:
$R_i^n(t_1, \ldots, t_n)$ for each n-place predicate and singular terms $t_{1,..,}t_n$.
$t \prec T$ when t is a singular term and T is a plural term.

Complex formulas:
$\neg\varphi, (\varphi \wedge \psi)$ are formulas when φ, ψ are formulas.

$\exists v\varphi$ and $\exists vv\varphi$ are formulas when φ is a formula, v is a singular variable and vv is a plural variable.

Note: Other connectives and operators are understood as abbreviations in the usual way. As noted in our first section, unlike the logic of classes as many, there are no complex terms in Linnebo's system, and in particular no complex terms based on a variable-binding operator, as well as no analysis of plural definite descriptions. We also note here that Linnebo assumes standard first-order logic with identity as the background first-order logic part of his system. This contrasts with the free first-order logic of the logic of classes as many. As a result, the translation function formulated here must take into consideration that the free singular variables of Linnebo's plural logic have values that "exist" in his logic, i.e., their values are the values of bound singular variables, and that each value of the free plural variables is a plurality that consists of at least one single object. To accommodate this difference makes the translation function seem more complicated than it really is.

5.1. The Translation Function

The translation function f is defined on all terms, predicates and formulas of Linnebo's plural logic.

(Note: We assume a 1-1 correlation between plural variables xx_i and name variables A_i.)

1) For all singular variables $x_i, f(x_i) =_{df} x_i$.
2) For all plural variables $xx_i, f(xx_i) =_{df} A_i$.
3) For each n-place predicate $R_i^n, f(R_i^n) =_{df} R_i^n$.
4) For each atomic formula $R_i^n(t_1, \dots, t_n)$, where R_i^n is other than $\prec$,

$f(R_i^n(t_1, \dots, t_n)) =_{df} (\exists y Atom)(y = x_{j_1}) \wedge \cdots \wedge (\exists y Atom)(y = x_{j_k}) \wedge (\exists y)(y \in f(xx_{i_1})) \wedge \cdots \wedge (\exists y)(y \in f(xx_{i_k})) \rightarrow R_i^n(f(t_1), \dots, f(t_n))$,
where $x_{j_1}, \dots, x_{j_k}$ are all the (free) singular variables among the terms $t_1, \dots, t_n$, and $xx_{i_1}, \dots, xx_{i_k}$ are all of the (free) plural variables among the terms $t_1, \dots, t_n$, and y is a singular variable not occurring in $R_i^n(t_1, \dots, t_n)$.
5) For each singular variable x and each plural variable xx,

$$f(x_i \prec xx_i) =_{df} (\exists y Atom)(y = x_i) \wedge (\exists y Atom)(y \in f(xx_i)) \rightarrow x_i \in f(xx_i),$$

where $\in$ is as defined in the logic of classes as many, and y is other than x_i.

6) For all formulas φ,

$f(\neg\varphi) =_{df} (\exists y Atom)(y = x_{j_1}) \wedge \cdots \wedge (\exists y Atom)(y = x_{j_k}) \wedge (\exists y)(y \in f(xx_{i_1})) \wedge \cdots \wedge (\exists y)(y \in f(xx_{i_m})) \rightarrow \neg f(\varphi)$,

where $x_{j_1}, \dots, x_{j_k}$ are all the singular variables and $xx_{i_1}, \dots, xx_{i_m}$ are all of the plural variables occurring free in φ, and y is a singular variable not occurring in φ.
7) For all formulas φ, ψ,
$f(\varphi \wedge \psi) =_{df} (\exists y Atom)(y = x_{j_1}) \wedge \cdots \wedge (\exists y Atom)(y = x_{j_k}) \wedge (\exists y)(y \in f(xx_{i_1})) \wedge \cdots \wedge (\exists y)(y \in f(xx_{i_m})) \rightarrow f(\varphi) \wedge f(\psi)$,

where $x_{j_1}, \dots, x_{j_k}$ are all the singular variables and $xx_{i_1}, \dots, xx_{i_m}$ are all of the plural variables occurring free in φ or ψ, and y is a singular variable not occurring in φ or ψ.

8) For each formula φ and singular variable x_i,

$f(\exists x_i \varphi) =_{df} (\exists y Atom)(y = x_{j_1}) \wedge \cdots \wedge (\exists y Atom)(y = x_{j_k}) \wedge (\exists y)(y \in f(xx_{i_1})) \wedge \cdots \wedge (\exists y)(y \in f(xx_{i_m})) \rightarrow (\exists x_i Atom) f(\varphi)$, where $x_{j_1}, \ldots, x_{j_k}$ are all the singular variables and $xx_{i_1}, \ldots, xx_{i_j}$ are all of the plural variables occurring free in $\exists x_i\varphi$, and y is a singular variable not occurring in $\exists x_i\varphi$.

9) For each formula φ and plural variable xx_i,

$f(\exists xx_i\varphi) =_{df} (\exists yAtom)(y = x_{j_1}) \wedge \cdots \wedge\ (\exists yAtom)(y = x_{j_k}) \wedge (\exists y)(y \in f(xx_{i_1})) \wedge \cdots \wedge\ (\exists y)(y \in f(xx_{i_m})) \rightarrow (\exists A_i)[(\exists yAtom)(y \in A_i) \wedge\ f(\varphi)]$,

where $x_{j_1}, \ldots, x_{j_k}$ are all the singular variables and $xx_{i_1}, \ldots, xx_{i_m}$ are all of the plural variables occurring free in $\exists xx_i\varphi, y$ is a singular variable not occurring in $\exists xx_i\varphi$, and A_i is the name variable corresponding to the plural variable xx_i.

5.2. Logical Axioms and Inference Rules for the Logic FPO

Linnebo assumes a natural deduction system as a background logic, but the system is not described, and we remain unsure just what the rules for his plural logic are. In any case, for our purposes of showing that the translation of every theorem of Linnebo's plural logic is a theorem of our logic of classes as many, it is preferable to use an axiomatic formulation of the background logic. The axiomatic version of standard first-order logic with identity we will use here is Tarski's substitution-free axiom set.[27]

Axioms for standard first-order logic with identity:

1) All tautologous formulas.

2) $\varphi \rightarrow \forall x_i\varphi$, where x_i is a singular variable not occurring free in φ.

3) $\forall x_i(\varphi \rightarrow \psi) \rightarrow (\forall x_i\varphi \rightarrow \forall x_i\psi)$.

4) $\exists x_i(x_j = x_i)$, where $j \neq i$.

5) $x_i = x_j \rightarrow (\varphi \rightarrow \psi)$, where φ is an atomic formula and ψ is obtained from φ by replacing an occurrence of x_j by an occurrence of x_i.[28]

Note: As inference rules we assume modus ponens and universal generalization (of what is provable), i.e.,

If $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$, then $\vdash \psi$, and
If $\vdash \varphi$, then $\vdash \forall x_i\varphi$ and $\vdash \forall xx_i\varphi$.

---

[27] See Tarski [19] and Kalish and Montague [11].

[28] Leibniz's law, (LL), for all formulas, atomic or otherwise, follows from axioms (5) and the other axioms by a simple induction on formulas. Then by universal generalization of (LL), axioms (3) and (1), and then (2), the schema: $\forall x_i\varphi(x) \rightarrow \varphi(y)$ follows. The identity thesis $x_i = x_i$ is provable from axioms (5) and (1).

The f-transform of axioms (1)–(3) remain unchanged in structure except for having antecedent clauses regarding the existence and non-emptiness of the free singular and plural variables occurring free in the formulas in question, and therefore, because (1)–(3) are also axioms of the logic of classes as many, those f-transforms are theorems of the free first-order logic of the logic of classes as many. The f-transform of axiom 4 becomes:

$$(\exists x_k Atom)(x_j = x_k) \to \exists x_i(x_j = x_i),$$

where x_i, x_j, x_k are distinct (singular) object variables. This result is trivially provable in the logic of classes as many.

Axiom (5) becomes:

$$\chi_1 \ldots \wedge\ \chi_n \wedge (\exists y Atom)(x_i = y) \wedge (\exists y Atom)(x_j = y) \to \\ (x_i = x_j \to [f(\varphi) \to f(\psi)]),$$

where $\chi_1, \ldots, \chi_n$ are the existence statements as described above for the remaining free variables occurring in φ and ψ. This formula is equivalent to:

$$\chi_1 \ldots \wedge\ \chi_n \to (\forall x_i Atom)(\forall x_j Atom)(x_i = x_j \to [f(\varphi) \to f(\psi)]),$$

the consequent of which is trivially provable in the logic of classes as many, and therefore so is the entire formula.

Except for replacing singular variables by plural variables, the axioms for the plural quantifiers are entirely similar to the above, and the proof of the translations of these axioms into the logic of classes as many is entirely similar. The inference rules of Linnebo's plural logic lead only from provable translations to provable translations.

5.3. The Plural Axioms of the Plural Logic FPO

Every instance of the extensionality axiom schema in Linnebo's logic, namely:

$$(\forall yy)(\forall zz)[(\forall x)(x \prec yy \leftrightarrow x \prec zz) \to (\varphi(yy) \leftrightarrow \varphi(zz))],$$

is easily seen to be a consequence of the following instance of the extensionality axiom of the logic of classes as many:

$$(\forall A)(\forall B)[(\forall x)(x \in A \leftrightarrow x \in B) \to A = B].$$

The f-translation of Linnebo's extensionality axiom would have an initial conditional clause regarding the variables occurring free in φ, which we can ignore here because the consequent of the whole conditional is provable in the logic of classes as many. The f-translation of the indiscernibility clause in

Linnebo's axiom schema, i.e., the biconditional $\varphi(yy) \leftrightarrow \varphi(zz)$, is of course a consequence of Leibniz's law applied to $A = B$.

The remaining two axioms of Linnebo's plural logic are the comprehension principle,

$$\exists u\varphi(u) \to \exists xx\forall u[u \prec xx \leftrightarrow \varphi(u)] \tag{comp}$$

and an axiom stipulating that every plurality is nonempty:

$$\forall xx\exists x(x \prec xx).$$

The f-translation of this axiom is:

$$(\forall A)(\exists xAtom)[(\exists yAtom)(y = x) \wedge (\exists yAtom)(y \in A) \to x \in A],$$

where $f(xx) = A$. This f-translation says in effect that every nonempty class as many is nonempty, which is trivially provable.

The f-translation of **(comp)** is as follows:

$$\chi_1 \cdots \wedge \ \chi_n \to$$
$$[(\exists uAtom)f(\varphi(u)) \to (\exists A)((\exists yAtom)(y \in A) \wedge (\forall uAtom)[u \in A \leftrightarrow f(\varphi)])],$$

where $\chi_1, \ldots, \chi_n$ are the existence statements as described above for all the free variables in $\exists u\varphi(u)$. Suppose the antecedent $\chi_1 \wedge \cdots \wedge \ \chi_n$ and also that $(\exists uAtom)f(\varphi(u))$ are given, and let $A = [\hat{u}/((\exists yAtom)(u = y) \wedge f(\varphi(u))]$. Finally, let $\psi(u)$ be $f(\varphi(u))$, which, by hypothesis, means that $(\exists uAtom)\psi(u)$ is given. It then suffices to show that

$$(\exists yAtom)(y \in A) \wedge (\forall uAtom)[u \in A \leftrightarrow \psi(u)])$$

follows. Assume now that u is an atom. Accordingly, if $u \in A$, i.e., if $u \in [\hat{u}/(\exists yAtom)(u = y) \wedge \psi(u)]$, then, by definition of $\in$ in the logic of classes as many, $(\exists z[\hat{u}/(\exists yAtom)(u = y) \wedge \psi(u)])(z = u)$, and by a rewrite of Theorem 8 of the same logic, namely the schema,

$$(\exists z[\hat{u}B])\chi \leftrightarrow (\exists zB(z/u))\chi,$$

(substituting $[\hat{u}/(\exists yAtom)(u = y) \wedge \psi(u)]$ for $[\hat{u}B]$), $(\exists z/(\exists yAtom)(z = y) \wedge \psi(z))(z = u)$ follows, from which $(\exists yAtom)(u = y) \wedge \psi(u)$ then follows, which is the left-to-right direction of the biconditional to be shown. For the right-to-left direction, assume $\psi(u)$ and show that $u \in A$ follows, i.e., that $u \in [\hat{u}/(\exists yAtom)(u = y) \wedge \psi(u)]$ follows. Note that by assumption $(\exists yAtom)(u = y)$, and therefore we have the conjunction $(\exists yAtom)(u = y) \wedge \psi(u)$. Now by assumption and theorem 8 again we have:

$$(\exists z[\hat{u}/(\exists yAtom)(u = y) \wedge \psi(u)])(u = z),$$

and hence, by definition of $\in$, $u \in [\hat{u}/(\exists yAtom)(u = y) \wedge \psi(u)]$, i.e., $u \in A$. We conclude then that

$$(\forall uAtom)[u \in A \leftrightarrow \psi(u)],$$

which what was to be shown. Finally, we note that by assumption

$$(\exists uAtom)f(\varphi(u)),$$

that is, $(\exists uAtom)\psi(u)$, and therefore by the above biconditional

$$(\exists uAtom)(u \in A).$$

It follows accordingly by existential generalization on A that the f-translation of **(comp)** is provable in the logic of classes as many.

6. Conclusion

The f-translation of every theorem of Linnebo's plural logic is a theorem of my logic of classes as many, and hence Linnebo's plural logic is reducible to my logic classes as many.

References

[1] Bell, J., Sets and classes as many, *Journal of Philosphical Logic* 29:585–601, 2000.
[2] Boolos, G., To be is to be a value of a variable (or to be some values of some variables), *Journal of Philosophy* 81:430–450, 1984.
[3] Cocchiarella, N. B., A conceptualist interpretation of Leśniewski's ontology, *History and Philosophy of Logic* 22:29–43, 2001.
[4] Cocchiarella, N. B., On the logic of classes as many, *Studia Logica* 70:303–338, 2002.
[5] Cocchiarella, N. B., Denoting concepts, reference, and the logic of names, classes as many, groups, and plurals, *Linguistics and Philosophy* 28:135–179, 2005.
[6] Cocchiarella, N. B., *Formal Ontology and Conceptual Realism*, Synthese Library vol. 339, Springer, Dordrecht, 2007.
[7] Cocchiarella, N. B., Mass nouns in a logic of classes as many, *Journal of Philosophical Logic* 38(3):343–361, 2009.
[8] Evans, G., *The Varieties of Reference*, Clarendon Press, Oxford, 1982.
[9] Goodman, N., A world of individuals, in *The Problem of Universals*, University of Notre Dame Press, Notre Dame, 1956, reprinted in Goodman's *Problems and Projects*, The Bobbs-Merrill Co., Inc., Indianapolis and New York, 1972.
[10] Eberle, R. A., *Nominalistic Systems*, Synthese Library, D. Reidel Publishing Company, Dordrecht, 1970.
[11] Kalish, D., and R. M. Montague, On Tarski's formalization of predicate logic with identity, *Archiv für mathematische Logik und Grundlagenforschung* 7:81–101, 1965.

[12] Leonard, H. S., and N. Goodman, The calculus of individuals, *The Journal of Symbolic Logic* 5:45–55, 1940.

[13] Linnebo, Ø., Plural quantification, 2004, *Stanford Encyclopedia of Philosophy*, revised 2012.

[14] McKay, T., *Plural Predication*, Oxford University Press, Oxford, 2006.

[15] Oliver, A., and T. Smiley, *Journal of Philosophical Logic* 35:317–348, 2006.

[16] Oliver, A., and T. Smiley, *Plural Logic*, Oxford University Press, Oxford, 2013.

[17] Russell, B., 1903, *The Principles of Mathematics*, second edition, Norton & Co., New York, 1938.

[18] Schein, B., *Plurals and Events*, MIT Press, Cambridge, MA, 1993.

[19] Tarski, A., A simplified formulation of predicate logic with identity, *Archiv für mathematische Logik und Grundlagenforschung* 7:61–79, 1965.

[20] Yi, B.-U., The logic and meaning of plurals, Part I, *Journal of Philosophical Logic* 34:459–506, 2005.

[21] Yi, B.-U., The logic and meaning of plurals, Part II, *Journal of Philosophical Logic* 35:239–288, 2005.

N. B. Cocchiarella
Indiana University
Bloomington, IN
USA
cocchiar@indiana.edu