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Hilbert Algebras with a Modal Operator $\Diamond$

Abstract. A Hilbert algebra with supremum is a Hilbert algebra where the associated order is a join-semilattice. This class of algebras is a variety and was studied in Celani and Montangie (2012). In this paper we shall introduce and study the variety of $H^{\vee}_{\Diamond}$-algebras, which are Hilbert algebras with supremum endowed with a modal operator $\Diamond$. We give a topological representation for these algebras using the topological spectral-like representation for Hilbert algebras with supremum given in Celani and Montangie (2012). We will consider some particular varieties of $H^{\vee}_{\Diamond}$-algebras. These varieties are the algebraic counterpart of extensions of the implicative fragment of the intuitionistic modal logic $\mathbf{IntK}_{\Diamond}$. We also determine the congruences of $H^{\vee}_{\Diamond}$-algebras in terms of certain closed subsets of the associated space, and in terms of a particular class of deductive systems. These results enable us to characterize the simple and subdirectly irreducible $H^{\vee}_{\Diamond}$-algebras.

Keywords: Hilbert algebras, Modal operators, Topological representation, Simple and subdirectly irreducible algebras.

1. Introduction

The Classical Modal logics are extensions of classical logic with new operators, called modal operators. Similarly, the Intuitionistic Modal logics are extensions of the Intuitionistic logic **Int** with new modal operators. In contrast with the classical case, the intuitionistic box modality $\Box$ and the intuitionistic diamond modality $\Diamond$ are not interdefinable. So, we have more possibilities of defining many different Intuitionistic Modal logics. We can consider intuitionistic modal logics with a modal operator $\Box$, for instance the logic $\mathbf{IntK}_{\Box}$ axiomatized by adding the following axioms to **Int**: $\Box(\phi \wedge \psi) \leftrightarrow \Box\phi \wedge \Box\psi$ and $\Box\top = \top$ (see [4, 19, 31], and [32]). It is possible to consider Intuitionistic Modal logics with a modal operator $\Diamond$, for instance the logic $\mathbf{IntK}_{\Diamond}$, axiomatized with the axioms $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$ and $\top \rightarrow \neg\Diamond\bot$. Extensions of $\mathbf{IntK}_{\Box}$ and $\mathbf{IntK}_{\Diamond}$ were studied in [4, 19], and [31]. Finally, we can study intuitionistic modal logics with the two modal operators $\Box$ and $\Diamond$. For example, the logic $\mathbf{IntK}_{\Box\Diamond}$ is the smallest logic containing both $\mathbf{IntK}_{\Box}$ and

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$\mathbf{IntK}_\Diamond$. Extensions of $\mathbf{IntK}_{\Box\Diamond}$ were studied in [1,4,19–21,26,28,29], and [32]. Note that, unlike the classical modal logics, the formula $\Diamond\phi$ cannot be considered as the abbreviation of $\neg\Box\neg\phi$. Similarly, the formula $\Box\phi$ cannot be considered as the abbreviation of $\neg\Diamond\neg\phi$. Thus, the normal intuitionistic modal logics $\mathbf{IntK}_\Box$ and $\mathbf{IntK}_\Diamond$ have different behaviors.

Some of these considerations can also be made if we want to study some fragments of the logic **Int** with modal operators. For example, we can study the $\{\rightarrow\}$-fragment of **Int** with a modal operator $\Box$, or we can study the $\{\rightarrow, \vee, \bot\}$-fragment of **Int** with a modal operator $\Diamond$. In [11], we started with the study of $\{\rightarrow, \Box\}$-fragment of the normal intuitionistic modal logic $\mathbf{IntK}_\Box$. This fragment is denoted by $\mathbf{IntK}_\Box^{\rightarrow}$, and their algebraic semantics is the variety $\mathrm{Hil}_\Box$ of Hilbert algebras with a necessity modal operator $\Box$. Other interesting fragment is the $\{\rightarrow, \vee, \bot, \Diamond\}$-fragment of $\mathbf{IntK}_\Diamond$. As in Intuitionistic Modal logics, the behavior of these fragments is very different, and thus it is interesting to investigate each of them.

Heyting algebras with modal operators are the algebraic counterpart of the intuitionistic modal logics $\mathbf{IntK}_\Box$, $\mathbf{IntK}_\Diamond$ and $\mathbf{IntK}_{\Box\Diamond}$ (see [1,17,26, 28] or [29]. Similarly, Hilbert algebras with modal operators are the algebraic counterpart of some implicative fragments of some intuitionistic modal logics.

The main purpose of this paper is to develop the topological representation of class of algebras associated with the $\{\rightarrow, \vee, \bot, \Diamond\}$-fragment of $\mathbf{IntK}_\Diamond$, called $H_\Diamond^\vee$-algebras. This representation is based on the ideas and techniques on topological representation for distributive semilattices, implicative semilattices and Hilbert algebras using spectral-like topological spaces with a fixed basis. What makes the topological dualities a powerful mathematical tool is that it allows us to use topology in the study of algebra (and vice versa). Many algebraic notions have their dual translation in terms of nice topological notions. The basic idea underlying the completeness results of many (propositional) logics is based on duality theory since the canonical model of a propositional logic is the dual of the Lindenbaum–Tarski algebra of the logic.

We recall that the first to develop a topological representation by means of spectral spaces for bounded distributive lattices was M. Stone in [30]. Later, H. Priestley in [27] proved that there is a duality between certains ordered topological spaces, called *Priestley spaces*, and bounded distributive lattices. Through both versions we can have a duality for Boolean algebras. In [22] George Grätzer gives a spectral-like topological representation (not a full duality) for distributive semilattices and extends the representation for bounded distributive lattices given by Stone. The representation of Grätzer is used in [7] to develop a full duality between distributive semilattices and certain spectral-like topological spaces, called DS-spaces (see also [6] and [15]). A DS-space is a *sober* topological space $\langle X, \mathcal{T}\rangle$ such that the collection $\mathcal{KO}(X)$ of compact and open subsets forms a basis for the topology $\mathcal{T}$ (see [15] for others equivalent definitions). The family $D(X) = \{X - U : U \in \mathcal{KO}(X)\}$ is closed under finite intersection, and in [22] it is proven that $\langle D(X), \cap, X\rangle$ is a distributive semilattice, called the dual distributive semilattice of $\langle X, \mathcal{T}\rangle$. If $\langle X, \mathcal{T}\rangle$ is compact, then $\emptyset \in D(X)$, i.e., $\langle D(X), \cap, \emptyset, X\rangle$ is a *bounded* distributive semilattice. In addition, if for each $U, V \in D(X)$ we have $U \Rightarrow V = (U \cap V^c]^c \in D(X)$, then $\langle D(X), \Rightarrow, \cap, X\rangle$ is an implicative semilattice (see [6]).

On other hand, Priestley-like dualities for bounded distributive semilattices has been recently developed by G. Bezhanishvili and R. Jansana in [2], and by G. Hansoul and C. Poussart in [24]. Following [2], a *generalized Priestley space* is a quadruple $X = \langle X, \leq, \mathcal{T}, X_0\rangle$ such that $\langle X, \leq, \mathcal{T}\rangle$ is a Priestley space, and X_0 is a dense subset of X satisfying additional conditions. Moreover, if $\mathrm{CloUp}(X)$ is the bounded distributive lattice of all clopen and increasing subsets of $\langle X, \leq, \mathcal{T}\rangle$, then the family de subsets $X^* = \{U \in \mathrm{CloUp}(X) : \max(X - U) \subseteq X_0\}$ is closed under $\cap$, and $\langle X^*, \cap, \emptyset, X\rangle$ is a bounded distributive lattice. In [2] G. Bezhanishvili and R. Jansana proves that the category of generalized Priestley spaces is dually equivalent to the category of bounded distributive semilattices. Moreover, if X is a generalized Priestley space, then the space $\langle X_0, \mathcal{T}_0\rangle$ is a DS-space, where $\mathcal{T}_0$ is the topology generated by the basis $\{X_0 - U : U \in X^*\}$. Since the category of bounded distributive semilattices is dually equivalent to the category of compact DS-spaces and to the category of generalized Priestley spaces, it is possible to prove that these two topological categories are equivalent. This fact is similar to the known result that assert that the category of Priestley spaces is equivalent to the category of the spectral spaces.

Topological dualities for Hilbert algebras have been also recently developed. A topological duality for the subvariety of Tarski algebras extending the known duality for Boolean algebras was investigated in [8]. Moreover, in [9] and [10] is develop a topological duality for Hilbert algebras, and Hilbert algebras with supremum. These dualities are based on sober spaces with a fixed basis of compact subsets. If $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is the dual space of a Hilbert algebra and the fixed basis $\mathcal{K}$ is the set of *all* open and compact sets, i.e., $\mathcal{K} = \mathcal{KO}(X)$, then $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is the dual space of an implicative semilattice. Thus, the duality given in [9] and [10] extends the duality given in [6]. Moreover, following the ideas of [3] and [12] a Priestley-style duality for a categories having Hilbert as objects is studied in [13].

In this paper we shall define the variety $\mathrm{Hil}_\Diamond^\vee$ of bounded Hilbert algebras with supremum endowed with a modal operator $\Diamond$, and we will give a spectral-like duality for this class of algebras. In a future work we shall study a Priestley-like duality for $\mathrm{Hil}_\Diamond^\vee$.

The paper is organized as follows. In Section 2 we will recall the definitions and some basic properties of Hilbert algebras and Hilbert algebras with supremum. We will recall the topological representation and duality of Hilbert algebras with supremum developed in [10]. In Section 3 we will introduce the bounded Hilbert algebras with supremum endowed with a unary operator $\Diamond$, or $H_\Diamond^\vee$-algebras for short. We will develop the topological representation and duality for $H_\Diamond^\vee$-algebras using the representation given in [10]. In Section 4 we shall characterize the $H_\Diamond^\vee$-algebras that satisfy certain equations by means of first-order conditions defined in the dual space. Finally, in Section 5 we will study the congruences of $H_\Diamond^\vee$-algebras, called $\Diamond$-congruences, and introduce the notion of closed deductive systems. Thus, we prove that the lattice of $\Diamond$-congruences and of closed deductive systems are isomorphic and they are dually isomorphic with certain closed subsets. We shall apply these results to determine the simple and subdirectly irreducible algebras of the variety $\mathrm{Hil}_\Diamond^\vee$.

2. Preliminaries

In this section we will fix the terminology adopted in this paper and introduce the main definitions of Hilbert algebra together with the standard concepts and known results on these algebras that are useful for this paper.

Let $\langle X, \leq \rangle$ be a poset. The set of all increasing subsets of X is denoted by $\mathcal{P}_{\leq}(X)$. For each $Y \subseteq X$, the increasing (decreasing) set generated by Y is $[Y) = \{x \in X : \exists y \in Y : y \leq x\}$ ($(Y] = \{x \in X : \exists y \in Y : x \leq y\}$). If $Y = \{y\}$, then we will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively. A subset $K \subseteq X$ is called *dually directed* if for any $x, y \in K$ there exists $z \in K$ such that $z \leq x$ and $z \leq y$.

Consider a pair $\langle X, \mathcal{K} \rangle$ where X is a set and $\emptyset \neq \mathcal{K} \subseteq \mathcal{P}(X)$. We define a relation $\leq_{\mathcal{K}} \subseteq X \times X$ by

$$x \leq_{\mathcal{K}} y \text{ iff } \forall W \in \mathcal{K}(x \notin W \text{ then } y \notin W). \tag{1}$$

It is easy to see that $\leq_{\mathcal{K}}$ is a reflexive and transitive relation. Define the operators sat and cl on $\mathcal{P}(X)$ as follows. For each $Y \subseteq X$, let

$$\mathrm{sat}(Y) = \bigcap \{W : Y \subseteq W \ \& \ W \in \mathcal{K}\},$$

and

$$\mathrm{cl}(Y) = \bigcap \{X - W : Y \cap W = \emptyset \ \&\ W \in \mathcal{K}\}.$$

The set $\mathrm{sat}(Y)$ is the *saturation* of Y, and $\mathrm{cl}(Y)$ is the *closure* of Y. A topological space $\langle X, \mathcal{T}\rangle$ with a base $\mathcal{K}$ we will denote by $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$. If $\mathcal{K}$ is a basis of a topology $\mathcal{T}$ defined on X, then $\leq_{\mathcal{K}}$ is the *specialization dual order* of X. In this case the relation $\leq_{\mathcal{K}}$ can be defined as $x \leq y$ iff $x \in \mathrm{cl}(\{y\}) = \mathrm{cl}(y)$. Recall that $\leq_{\mathcal{K}}$ is an order when the space is T_0. In this case, $\mathrm{cl}(Y) = [Y)$, $\mathrm{sat}(Y) = (Y]$, and every open (resp. closed) subset is a decreasing (resp. increasing) subset respect to $\leq_{\mathcal{K}}$.

Let $\langle X, \mathcal{T}\rangle$ be a topological space. The set of all closed subsets of $\langle X, \mathcal{T}\rangle$ is denoted by $\mathcal{C}(X)$. An arbitrary non-empty subset Y of X is *irreducible* if for every $Z, W \in \mathcal{C}(X)$ such that $Y \subseteq Z \cup W$, implies $Y \subseteq Z$ or $Y \subseteq W$. A topological space $\langle X, \mathcal{T}\rangle$ is *sober* if, for every irreducible closed set Y, there exists a *unique* $x \in X$ such that $\mathrm{cl}(\{x\}) = Y$. Notice that a sober space is automatically T_0.

Hilbert Algebras with Supremum

It is known that the variety of Hilbert algebras is the algebraic semantics of the positive implicative fragment $\mathbf{Int}^{\rightarrow}$ of the intuitionistic propositional calculus **Int** (see [16, 18] or [25]). Similarly, it is possible to conclude that the variety of Hilbert algebras with supremum is the algebraic semantics of the $\{\rightarrow, \vee\}$-fragment of **Int**.

Definition 1. A Hilbert algebra is an algebra $A = \langle A, \rightarrow, 1\rangle$ of type $(2, 0)$ such that the following axioms hold in A:

1. $a \rightarrow (b \rightarrow a) = 1$,
2. $(a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = 1$,
3. $a \rightarrow b = 1 = b \rightarrow a$ implies $a = b$.

In [18] Diego proves that the class of Hilbert algebras form a variety which is denoted by Hil. Note that the relation $\leq$ defined in a Hilbert algebra A by $a \leq b$ if and only if $a \rightarrow b = 1$ is a partial order on A with top element 1. We shall say that a Hilbert algebra A is *bounded* if there exists $0 \in A$ such that $0 \rightarrow a = 1$, for every $a \in A$. The variety of bounded Hilbert algebras is denoted by Hil^0.

Let A be a Hilbert algebra. A subset D of A is a *deductive system, or implicative filter,* if $1 \in D$, and if $a, a \rightarrow b \in D$ then $b \in D$. The set of all deductive systems of a Hilbert algebra A is denoted by $\mathrm{Ds}(A)$. The deductive system generated by a set C is $\langle C\rangle = \bigcap\{D \in \mathrm{Ds}(A) : C \subseteq D\}$. If $C = \{a\}$, then we write $\langle a\rangle = \{b \in A : a \leq b\}$. A deductive system D is *irreducible* if and only if for any $D_1, D_2 \in \mathrm{Ds}(A)$ such that $D = D_1 \cap D_2$, it follows that $D = D_1$ or $D = D_2$. The set of all irreducible deductive systems of a Hilbert algebra A is denoted by $X(A)$. Recall that a deductive system D is irreducible iff for every $a, b \in A$ such that $a, b \notin D$ there exists $c \notin D$ such that $a, b \leq c$ (see [18] or [5]). A decreasing subset I of A is an *order-ideal* of A if for each $a, b \in I$ there exists $c \in I$ such that $a \leq c$ and $b \leq c$. The set of all order-ideals of A is denoted by $\mathrm{Id}(A)$.

THEOREM 2. *[5] Let A be a Hilbert algebra. Let $D \in \mathrm{Ds}(A)$ and let $I \in \mathrm{Id}(A)$ such that $D \cap I = \emptyset$. Then there exists $x \in X(A)$ such that $D \subseteq x$ and $x \cap I = \emptyset$.*

Let $\langle X, \leq\rangle$ be a poset. It is known that $\langle \mathcal{P}_{\leq}(X), \Rightarrow_{\leq}, X\rangle$ is a Hilbert algebra where the implication $\Rightarrow_{\leq}$ is defined by

$$U \Rightarrow_{\leq} V = (U \cap V^c]^c = \{x : [x) \cap U \subseteq V\} \tag{2}$$

for $U, V \in \mathcal{P}_{\leq}(X)$.

We will now introduce the definition of Hilbert algebras where the associated order is a join-semilattice. It was studied in [10].

DEFINITION 3. An algebra $A = \langle A, \to, \vee, 1\rangle$ of type $(2, 2, 0)$ is a Hilbert algebra with supremum, or $H^{\vee}$-algebra for short, if

1. $\langle A, \to, 1\rangle$ is a Hilbert algebra.
2. $\langle A, \vee, 1\rangle$ is a join-semilattice with top element 1.
3. For all $a, b \in A$, $a \to b = 1$ if and only if $a \vee b = b$.

If there exists an element $0 \in A$ such that $0 \to a = 1$, for all $a \in A$, then we say that A is an $H^{\vee}$-algebra with bottom element or an $H_0^{\vee}$-algebra.

The variety of Hilbert algebras with supremum will be denoted by $\mathrm{Hil}^{\vee}$. The variety of Hilbert algebras with supremum with bottom element will be denoted by $\mathrm{Hil}_0^{\vee}$.

Let $A, B \in \mathrm{Hil}$. A mapping $h : A \to B$ is a *semi-homomorphism* if $h(1) = 1$, and $h(a \to b) \leq h(a) \to h(b)$, for all $a, b \in A$. A mapping $h : A \to B$ is a homomorphism if it is semi-homomorphism such that $h(a) \to h(b) \leq h(a \to b)$, for all $a, b \in A$.

Let $A, B \in \mathrm{Hil}^{\vee}$. A $\vee$*-semi-homomorphism between* A and B is a semi-homomorfism $h : A \to B$ such that it preserves the operation $\vee$, i.e., $h(a \vee b) = h(a) \vee h(b)$ for all $a, b \in A$. Similarly, if h is a homomorphism that preserves the join, it will be called a $\vee$-*homomorphism*. Denoted by $\mathrm{Hil}^{\vee}\mathcal{S}$ the category of $H^{\vee}$-algebras and $\vee$ -semi-homomorphisms, and by $\mathrm{Hil}^{\vee}\mathcal{H}$ the category of $H^{\vee}$-algebras and $\vee$-homomorphisms.

We note that if $\langle A, \rightarrow, \vee, 1\rangle$ is an $H^{\vee}$-algebra, then a subset I is an order-ideal iff I is an ideal of the join-semilattice $\langle A, \vee\rangle$.

2.1. $H^{\vee}$-Spaces

In [9] and [10] a full duality for Hilbert algebras and for Hilbert algebras with supremum was developed, where the dual spaces of Hilbert algebras are compactly based sober spaces. In this subsection we recall the duality for Hilbert algebras with supremum.

Definition 4. An $H^{\vee}$-*space* is a topological space $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ such that:

H1. $\mathcal{K}$ is a base of open and compact subsets for a topology $\mathcal{T}_{\mathcal{K}}$ on X,

H2. For every $U, V \in \mathcal{K}$, $\mathrm{sat}(U \cap V^c) \in \mathcal{K}$,

H3. $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is sober,

H4. $U \cap V \in \mathcal{K}$, for all $U, V \in \mathcal{K}$.

We note that by (1), every open subset of $\mathcal{K}$ is decreasing with respect to induced order $\leq_{\mathcal{K}}$. We note also that $U \Rightarrow_{\leq_{\mathcal{K}}} V = \mathrm{sat}(U \cap V^c)^c$, for all $U, V \in \mathcal{P}_{\leq_{\mathcal{K}}}(X)$, because $\mathrm{sat}(Y) = \{x \in X : \exists y \in Y : x \leq_{\mathcal{K}} y\} = (Y]_{\leq_{\mathcal{K}}}$, for any $Y \subseteq X$. In [10] it was proved that if $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ be an $H^{\vee}$-space then

$$D(X) = \langle D(X), \cup, \Rightarrow_{\leq_{\mathcal{K}}}, X\rangle$$

is a subalgebra of the Hilbert algebra with supremum $\langle \mathcal{P}_{\leq_{\mathcal{K}}}(X), \cup, \Rightarrow_{\leq_{\mathcal{K}}}, X\rangle$, where $D(X) = \{U : U^c \in \mathcal{K}\}$.

Let A be an $H^{\vee}$-algebra. Let us consider the poset $\langle X(A), \subseteq\rangle$ and the mapping $\varphi : A \rightarrow \mathcal{P}_{\subseteq}(X(A))$ defined by $\varphi(a) = \{x \in X(A) : a \in x\}$. Then A is isomorphic to the subalgebra $D(X(A)) = \{\varphi(a) : a \in A\}$ of the $H^{\vee}$-algebra $\langle \mathcal{P}_{\subseteq}(X(A)), \cup, \Rightarrow_{\subseteq}, X(A)\rangle$. From the results on representation on $H^{\vee}$-algebra given in [10] it follows that the family $\mathcal{K}_A = \{\varphi(a)^c : a \in A\}$ is a basis for a topology $\mathcal{T}_{\mathcal{K}_A}$ and the pair $\langle X(A), \mathcal{T}_{\mathcal{K}_A}\rangle$ is an $H^{\vee}$-space, called the *dual space of* A. If A is an $H_0^{\vee}$-algebra, then $\varphi(0) = \emptyset$. So, $X(A) = \varphi(0)^c \in \mathcal{K}_A$ and consequently the $H^{\vee}$-space $\langle X(A), \mathcal{T}_{\mathcal{K}_A}\rangle$ is compact. Moreover, if $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is an $H^{\vee}$-space, then $\langle D(X), \cup, \Rightarrow, X\rangle \in \mathrm{Hil}^{\vee}$, and $\varepsilon(x) = \{U \in D(X) : x \in U\}$ belongs to $X(D(X))$, for each $x \in X$. Thus, the mapping $\varepsilon : X \rightarrow X(D(X))$ is well-defined and it is an homeomorphism between the topological spaces $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ and $\left\langle X(D(X)), \mathcal{T}_{\mathcal{K}_{D(X)}}\right\rangle$.

Definition 5. Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1}\rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2}\rangle$ be H-spaces. The relation $R \subseteq X_1 \times X_2$ is an *H-relation* if $R^{-1}(U) = \{x \in X_1 : R(x) \cap U \neq \emptyset\} \in \mathcal{K}_1$, for every $U \in \mathcal{K}_2$, and $R(x)$ is a closed subset of X_2, for all $x \in X_1$.

An H-relation $R \subseteq X_1 \times X_2$ is an *H-functional relation* if $(x, y) \in R$, then there exists $z \in X_1$ such that $x \leq z$ and $R(z) = [y)$.

An H-relation $R \subseteq X_1 \times X_2$ is *irreducible* if, for every $x \in X_1$ such that $R(x) \neq \emptyset$ we have that $R(x)$ is an irreducible closed subset of X_2.

In [9] it was proved that if $\langle X_1, \mathcal{T}_{\mathcal{K}_1}\rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2}\rangle$ are H-spaces and $R \subseteq X_1 \times X_2$ is an H-relation, then the mapping $h_R : D(X_2) \to D(X_1)$ defined by

$$h_R(U) = \{x \in X_1 \mid R(x) \subseteq U\}$$

for each $U \in D(X_2)$, is a semi-homomorphism.

Let A, B be Hilbert algebras and $h : A \to B$ be a semi-homomorphism. In [9] it was proved that the relation $R_h \subseteq X(B) \times X(A)$ defined by

$$(x, y) \in R_h \textit{ iff } h^{-1}(x) \subseteq y \tag{3}$$

is an H-relation. The following result given in [10], characterizes the $\vee$-semi-homomorphisms via R_h and it allowed us to prove that the category $\mathrm{Hil}^{\vee}\mathcal{S}$ is dually isomorphic to the category of $H^{\vee}$-spaces with irreducible H-relations.

Theorem 6. *Let $h : A \to B$ be a semi-homomorphism defined between the $H^{\vee}$-algebras A and B. Then, the following conditions are equivalents:*

1. *h preserves the operation $\vee$,*
2. *the relation R_h is irreducible,*
3. *$h^{-1}(x) \in X(A)$ or $h^{-1}(x) = A$, for all $x \in X(B)$.*

In [10] it was shown that the irreducible H-functional relations between $H^{\vee}$-spaces can be characterized by means of special partial functions defined between $H^{\vee}$-spaces called H-partial functions. An *H-partial function* is a partial map $f : X_1 \to X_2$ with domain $\mathrm{dom}(f)$, defined between the $H^{\vee}$-spaces $\langle X_1, \mathcal{T}_{\mathcal{K}_1}\rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2}\rangle$ such that:

1. $[f(x)) = f([x))$ for each $x \in \mathrm{dom}(f)$.
2. $x \in \mathrm{dom}(f)$ iff there exists $y \in X_2$ such that $f([x)) = [y)$.
3. If $U \in \mathcal{K}_2$, then $\left(f^{-1}(U)\right] \in \mathcal{K}_1$.

Consequently, the category $\mathrm{Hil}^{\vee}\mathcal{H}$ is dually isomorphic to the category of $H^{\vee}$-spaces with irreducible H-functional relations and to the category of $H^{\vee}$-spaces with H-partial functions.

The ideals of an $H^{\vee}$-algebra A also have a topological characterization.

Definition 7. Let $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ be an $H^{\vee}$-space. A subset $Y \subseteq X$ is *open directed* iff there exists a subset $\mathcal{B} \subseteq D(X)$ such that $Y = \bigcup\{U : U \in \mathcal{B}\}$.

The set of all open directed subsets of $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is noted by $\mathrm{Od}(X)$. In [10] it was proved that given a subset $Y \subseteq X$, $Y \in \mathrm{Od}(X)$ *iff* $Y = \bigcup I(Y)$, where $I(Y) = \{U \in D(X) : U \subseteq Y\}$. The next result was shown in [10].

Proposition 8. *Let* $A \in \mathrm{Hil}^{\vee}$ *and let* $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ *its* $H^{\vee}$*-space dual. The mapping* $\beta : \mathrm{Id}(A) \to \mathrm{Od}(X)$ *defined by*

$$\beta(I) = \{x \in X : x \cap I \neq \emptyset\}$$

for each $I \in \mathrm{Id}(A)$*, is a lattice-isomorphism.*

3. $H^{\vee}_{\Diamond}$-Algebras: Representation and Duality

In this section we will introduce the Hilbert algebras with an operator $\Diamond$, or $H^{\vee}_{\Diamond}$-algebras, and we shall extend the results on representation of Hilbert algebras to the case of $H^{\vee}_{\Diamond}$-algebras.

Definition 9. An algebra $\langle A, \Diamond\rangle$ is an $H^{\vee}_{\Diamond}$-algebra if $\langle A, \vee, \to, 0, 1\rangle \in \mathrm{Hil}^{\vee}_{0}$ and $\Diamond$ is a unary operator of A that satisfies the following conditions:

($\Diamond$1) $\Diamond 0 = 0$,

($\Diamond$2) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$.

Note that under these conditions $\Diamond$ is monotone. The variety of $H^{\vee}_{\Diamond}$-algebras is denoted by $\mathrm{Hil}^{\vee}_{\Diamond}$. The variety $\mathrm{Hil}^{\vee}_{\Diamond}$ corresponds to the $\{\vee, \to, 0, \Diamond\}$-reduct of the variety of Heyting algebras with a modal operator $\Diamond$ (see [26,31], or [28]).

Let $\langle A, \Diamond\rangle, \langle B, \Diamond\rangle \in \mathrm{Hil}^{\vee}_{\Diamond}$. We say that a map $h : A \to B$ commutes with $\Diamond$ if $h(\Diamond a) = \Diamond h(a)$ for all $a \in A$. Denote by $\mathrm{Hil}^{\vee}_{\Diamond}\mathcal{S}$ the category whose objects are $H^{\vee}_{\Diamond}$-algebras and whose morphisms are $\vee$-semi-homomorphisms that commute with $\Diamond$ and $h(0) = 0$. Also denoted by $\mathrm{Hil}^{\vee}_{\Diamond}\mathcal{H}$ the category of $H^{\vee}_{\Diamond}$-algebras and $\vee$-homomorphisms that commute with $\Diamond$ and $h(0) = 0$.

Let $\langle A, \Diamond\rangle \in \mathrm{Hil}^{\vee}_{\Diamond}$ and $C \subseteq A$. We define the set $\Diamond^{-1}(C) = \{a \in A : \Diamond a \in C\}$. Note that by the monotony of $\Diamond$, for each decreasing (increasing) subset C of A the set $\Diamond^{-1}(C)$ is a decreasing (increasing) subset of A. Moreover,

$\Diamond^{-1}(I)$, $(\Diamond(I)] \in \mathrm{Id}(A)$ for each $I \in \mathrm{Id}(A)$. Thus, for all $x \in X(A)$, we have that $\Diamond^{-1}(x^c) = \Diamond^{-1}(x)^c \in \mathrm{Id}(A)$.

3.1. Duality for Objects

We proceed with the representation for $H^{\vee}_{\Diamond}$-algebras. Let X be a set and $S \subseteq X \times X$. Let us consider the mapping $\Diamond_S : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by:

$$\Diamond_S(U) = \{x \in X : S(x) \cap U \neq \emptyset\} = S^{-1}(U).$$

Now, we define the topological spaces associated with the $H^{\vee}_{\Diamond}$-algebras.

Definition 10. The triple $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ is an $H^{\vee}_{\Diamond}$-space if $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is an $H^{\vee}$-space and S is a binary relation on X such that:

1. $X \in \mathcal{K}$.
2. For each $x \in X$, $S(x)^c \in \mathrm{Od}(X)$.
3. If $U \in D(X)$, then $\Diamond_S(U) \in D(X)$.

Lemma 11. *Let $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ be an $H^{\vee}_{\Diamond}$-space. Then*

1. *$S(x)$ is a compact subset of $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ for each $x \in X$.*
2. *$\left(\leq^{-1} \circ S\right) = (S\circ \ \leq^{-1}) = S$, where $\leq$ is $\leq_{\mathcal{K}}$.*

Proof. (1) Let $x \in X$ and let $L \subseteq \mathcal{K}$. Consider $S(x) \subseteq \bigcup\{U : U \in L\}$. This is, $\bigcap\{U^c : U \in L\} \subseteq S(x)^c$. As $S(x)^c \in \mathrm{Od}(X)$ it follows that

$$S(x)^c = \bigcup\{V \in D(X) : V \subseteq S(x)^c\}$$

and so, $\bigcap\{U^c : U \in L\} \cap \bigcap\{V^c : V \in D(X) \text{ and } V \subseteq S(x)^c\} = \emptyset$. Note that $\{V^c : V \in D(X) \text{ and } V \subseteq S(x)^c\}$ is a dually directed subset of $\mathcal{K}$. Indeed, let $U, V \in D(X)$ such that $V \subseteq S(x)^c$ and $U \subseteq S(x)^c$. Since $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ is an $H^{\vee}$-space, $U \cup V \in D(X)$. Moreover, $U \cup V \subseteq S(x)^c$ and $(U \cup V)^c \subseteq U^c, V^c$. As $\bigcap\{U^c : U \in L\}$ is a closed subset of $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$, by Theorem 3.3 of [10], there exists $V_0 \in D(X)$ such that $V_0 \subseteq S(x)^c$ and $\bigcap\{U^c : U \in L\} \cap V_0^c = \emptyset$, this is,

$$V_0^c \subseteq \bigcup\{U : U \in L\}.$$

As V_0^c is a compact subset of $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$, there exist $U_1, ..., U_n \in L$ such that $S(x) \subseteq V_0^c \subseteq U_1 \cup ... \cup U_n$.

(2) By the reflexivity of $\leq^{-1}$, we have immediately that $S \subseteq \left(\leq^{-1} \circ S\right)$. Let $x, y \in X$. If $(x, y) \in (\leq^{-1} \circ S)$, there exists $z \in X$ such that $z \leq x$ and $(z, y) \in S$. Suppose that $y \notin S(x)$. As $S(x)^c \in \mathrm{Od}(X)$, there exists

$U \in D(X)$ such that $U \subseteq S(x)^c$ and $y \in U$. By assumption, $y \in S(z)$ and consequently, $z \in \Diamond_S(U)$. As $\Diamond_S(U) \in D(X)$ and $z \leq x$, we have that $x \in \Diamond_S(U)$. This is, $S(x) \cap U \neq \emptyset$ which is a contradiction. Thus, $\left(\leq^{-1} \circ S\right) = S$. By the reflexivity of $\leq^{-1}$, we get that $S \subseteq S\circ \leq^{-1}$. Let $(x, y) \in S\circ \leq^{-1}$, i.e., there exists $z \in X$ such that xSz and $y \leq z$. Now, suppose that $y \notin S(x)$. So, there exists $V \in D(X)$ such that $V \subseteq S(x)^c$ and $y \in V$. Thus, $z \in V$ and consequently, $z \notin S(x)$, which is impossible. So, $(S\circ \leq^{-1}) = S$. ■

In [26] it was proved that if $\langle X, \leq\rangle$ is a poset and S a binary relation defined on X, then the condition $\left(\leq^{-1} \circ S\right) \subseteq \left(S\circ \leq^{-1}\right)$ is equivalent to obtaining that $\mathcal{P}_{\leq}(X)$ is closed under $\Diamond_S$. Consequently, we have the following result.

Lemma 12. *If $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ is an $H^{\vee}_{\Diamond}$-space, then*

$$\langle \mathcal{P}_{\leq}(X), \Diamond_S\rangle = \langle \mathcal{P}_{\leq}(X), \cup, \Rightarrow, \Diamond_S, \emptyset, X\rangle$$

is an $H^{\vee}_{\Diamond}$-algebra, and $\langle D(X), \Diamond_S\rangle = \langle D(X), \cup, \Rightarrow, \Diamond_S, \emptyset, X\rangle$ is a subalgebra of $\langle \mathcal{P}_{\leq}(X), \Diamond_S\rangle$.

Let $\langle A, \Diamond\rangle$ be an $H^{\vee}_{\Diamond}$-algebra. Let us consider the binary relation S_A defined on $X(A)$ by:

$$(x, y) \in S_A \textit{ iff } y \subseteq \Diamond^{-1}(x).$$

Lemma 13. *Let $\langle A, \Diamond\rangle \in \mathrm{Hil}^{\vee}_{\Diamond}$ and $x \in X(A)$. Then, $\Diamond a \in x$ iff there exists $y \in X(A)$ such that $(x, y) \in S_A$ and $a \in y$.*

Proof. Suppose that $\Diamond a \in x$. This is, $a \notin \Diamond^{-1}(x^c)$. As $\Diamond^{-1}(x^c) \in \mathrm{Id}(A)$, we get $[a) \cap \Diamond^{-1}(x)^c = \emptyset$. By Theorem 2, there exists $y \in X(A)$ such that $y \subseteq \Diamond^{-1}(x)$ and $a \in y$. The reciprocal is immediate. ■

Proposition 14. *If $\langle A, \Diamond\rangle \in \mathrm{Hil}^{\vee}_{\Diamond}$, then $\langle X(A), \mathcal{T}_{\mathcal{K}_A}, S_A\rangle$ is an $H^{\vee}_{\Diamond}$-space.*

Proof. In [10] it was proved that $\langle X(A), \mathcal{T}_{\mathcal{K}_A}\rangle$ is an $H^{\vee}$-space. We will prove that S_A satisfies the conditions of Definition 10:

1. Since $0 \in A$, $\varphi(0) = \emptyset \in D(X(A))$. So, $X \in \mathcal{K}_A$.
2. Let $x, y \in X(A)$.

$$\begin{aligned} y \in S_A(x)^c \quad &\textit{iff} \quad y \nsubseteq \Diamond^{-1}(x) \quad \textit{iff} \quad y \cap \Diamond^{-1}(x)^c \neq \emptyset \\ &\textit{iff} \quad y \in \beta\left(\Diamond^{-1}(x)^c\right). \end{aligned}$$

By Proposition 8, $S_A(x)^c \in \mathrm{Od}(X(A))$.

3. Note that if $U \in D(X(A))$ then there exists $a \in A$ such that $U = \varphi(a)$. By Lemma 13, $\Diamond_{S_A}(U) = \Diamond_{S_A}(\varphi(a)) = \varphi(\Diamond a) \in D(X(A))$. ■

Theorem 15. *For each $H^{\vee}_{\Diamond}$-algebra $\langle A, \Diamond\rangle$ there exists an $H^{\vee}_{\Diamond}$-space $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ such that $\langle A, \Diamond\rangle$ is isomorphic to $\langle D(X), \Diamond_S\rangle$.*

Proof. Consider the triple $\langle X(A), \mathcal{T}_{\mathcal{K}_A}, S_A\rangle$. In [10] it was proved that $\varphi : A \to D(X(A))$ is an isomorphism of $H^{\vee}$-algebras and $\langle X(A), \mathcal{T}_{\mathcal{K}_A}\rangle$ is an $H^{\vee}$-space. As $\langle A, \Diamond\rangle$ is an $H^{\vee}_{\Diamond}$-algebra, by Proposition 14 we have that $\varphi(\Diamond a) = \Diamond_{S_A}(\varphi(a))$ for all $a \in A$ and $\langle X(A), \mathcal{T}_{\mathcal{K}_A}, S_A\rangle$ is an $H^{\vee}_{\Diamond}$-space. By Lemma 12, $\langle D(X(A)), \Diamond_{S_A}\rangle$ is an $H^{\vee}_{\Diamond}$-algebra and consequently, φ is an isomorphism between the $H^{\vee}_{\Diamond}$-algebras $\langle A, \Diamond\rangle$ and $\langle D(X(A)), \Diamond_{S_A}\rangle$. ∎

We note that if $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ is an $H^{\vee}_{\Diamond}$-space, $\left\langle X(D(X)), \mathcal{T}_{\mathcal{K}_{D(X)}}, S_{D(X)}\right\rangle$ is the $H^{\vee}_{\Diamond}$-space of $\langle D(X), \Diamond_S\rangle$.

Theorem 16. *Let $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ be an $H^{\vee}_{\Diamond}$-space. Then, the mapping $\varepsilon_X : X \to X(D(X))$ is an homeomorphism between $H^{\vee}_{\Diamond}$-spaces such that*

$$(x, y) \in S \text{ iff } (\varepsilon_X(x), \varepsilon_X(y)) \in S_{D(X)},$$

for every $x, y \in X$.

Proof. In [9] it was proved that ε_X is an homeomorphism between the $H^{\vee}$-spaces $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ and $\left\langle X(D(X)), \mathcal{T}_{\mathcal{K}_{D(X)}}\right\rangle$. Moreover, since $\langle D(X), \Diamond_S\rangle \in \mathrm{Hil}^{\vee}_{\Diamond}$, Proposition 14 shows that $\left\langle X(D(X)), \mathcal{T}_{\mathcal{K}_{D(X)}}, S_{D(X)}\right\rangle$ is an $H^{\vee}_{\Diamond}$-space, where for every $F, P \in X(D(X))$, $(F, P) \in S_{D(X)}$ iff $P \subseteq \Diamond_S^{-1}(F)$. Let $x, y \in X$. We only need to show that $(x, y) \in S$ iff $(\varepsilon_X(x), \varepsilon_X(y)) \in S_{D(X)}$. Assume that $(x, y) \in S$. To prove that $\varepsilon_X(y) \subseteq \Diamond_S^{-1}(\varepsilon_X(x))$, let $U \in D(X)$ such that $U \in \varepsilon_X(y)$. So, $y \in U$ and as $y \in S(x)$, we obtain that $x \in \Diamond_S(U)$. As $\Diamond_S(U) \in D(X)$, $\Diamond_S(U) \in \varepsilon_X(x)$ and so, $U \in \Diamond_S^{-1}(\varepsilon_X(x))$. Conversely, suppose that $y \notin S(x)$. As $S(x)^c \in \mathrm{Od}(X)$, and $y \in S(x)^c = \bigcup\{U \in D(X) : U \subseteq S(x)^c\}$, there exists $U \in D(X)$ such that $U \in \varepsilon_X(y)$ and $U \cap S(x) = \emptyset$. This is, $x \notin \Diamond_S(U)$. So, there exists $U \in \varepsilon_X(y)$ such that $\Diamond_S(U) \notin \varepsilon_X(x)$. Thus, $\varepsilon_X(y) \nsubseteq \Diamond_S^{-1}(\varepsilon_X(x))$, i.e., $(\varepsilon_X(x), \varepsilon_X(y)) \notin S_{D(X)}$. ∎

3.2. Duality for Morphisms

In order to complete the duality, we need to assign an appropriate irreducible H-relation for each $\vee$-semi-homomorphism that commutes with $\Diamond$ and preserves the bottom element between $H^{\vee}_{\Diamond}$-algebras.

Definition 17. Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1}, S_1\rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2}, S_2\rangle$ be $H^{\vee}_{\Diamond}$-spaces. Let $R \subseteq X_1 \times X_2$ a relation. We shall say that R is a *strong irreducible H-relation* if it is an irreducible H-relation such that $R(x) \neq \emptyset$, for each $x \in X_1$.

It is easy to see that

$$\forall x \in X_1 \, (R(x) \neq \emptyset) \quad \textit{iff} \;\; h_R(\emptyset) = \emptyset.$$

Thus, a binary relation R defined between $H^{\vee}_{\Diamond}$-spaces is a strong irreducible H-relation iff h_R is a $\vee$-semi-homomorphism which preserves the bottom element.

Notice that each strong irreducible H-relation $R \subseteq X_1 \times X_2$ provides us with a function $f_R : X_1 \to X_2$. Indeed, as $\langle X_1, \mathcal{T}_{\mathcal{K}_1} \rangle$ is sober, and for all $x \in X_1$, $R(x)$ is an irreducible closed nonempty subset of X_2, then for each $x \in X_1$ there exists a unique $y \in X_2$ such that

$$R(x) = \mathrm{cl}\,(\{y\}) = \mathrm{cl}(y) = [y).$$

Therefore, we can define a function $f_R : X_1 \to X_2$ such that

$$f_R(x) = y \quad \textit{iff} \;\; R(x) = \mathrm{cl}(y) = [y).$$

Note that for each $x \in X_1$, we have that $R(x) = [f_R(x))$. Moreover, if we take every $U \in \mathcal{K}_2$, we obtain that

$$\begin{aligned} R^{-1}(U) &= \{x \in X_1 : (x, y) \in R \text{ for some } y \in U\} \\ &= \{x \in X_1 : [f_R\,(x)) \cap U \neq \emptyset\} \\ &= \{x \in X_1 : f_R\,(x) \in U\} = f_R^{-1}\,(U)\,, \end{aligned}$$

and as R is an H-relation, $f_R^{-1}\,(U) \in \mathcal{K}_1$.

Remark 18. We note that if R is a strong irreducible H-relation between the $H^{\vee}_{\Diamond}$-spaces $\langle X_1, \mathcal{T}_{\mathcal{K}_1}, S_1 \rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2}, S_2 \rangle$, then

$$h_R(U) = f_R^{-1}(U),$$

for any $U \in D(X_2)$. Indeed, let $x \in h_R(U)$, for some $U \in D(X_2)$. Then $x \in h_R(U)$ iff $R(x) = [f_R(x)) \subseteq U$ iff $f_R(x) \in U$ iff $x \in f_R^{-1}(U)$.

Definition 19. Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1}, S_1 \rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2}, S_2 \rangle$ be $H^{\vee}_{\Diamond}$-spaces. Let $R \subseteq X_1 \times X_2$ be a strong irreducible H-relation. We shall say that R is a $\Diamond$-*relation* if it satisfies the following conditions:

$(MF1)$ If $(x, y) \in S_1$, then $(f_R(x), f_R(y)) \in S_2$.

$(MF2)$ $h_R(\Diamond_{S_2}(U)) \subseteq \Diamond_{S_1}(h_R(U))$, for every $U \in D(X_2)$.

Denote by $\mathcal{M}_{\Diamond}\mathcal{SR}^{\vee}$ the category whose objects are $H^{\vee}_{\Diamond}$-spaces and whose morphisms are $\Diamond$-relations.

Proposition 20. *Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1}, S_1\rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2}, S_2\rangle$ be $H^{\vee}_{\Diamond}$-spaces. Let $R \subseteq X_1 \times X_2$ be a strong irreducible H-relation. Then, R satisfies the condition $(MF1)$ iff $\Diamond_{S_1}(h_R(U)) \subseteq h_R(\Diamond_{S_2}(U))$, for every $U \in D(X_2)$.*

Proof. $\Rightarrow$) Let $U \in D(X_2)$ and $x \in \Diamond_{S_1}(h_R(U))$. So, $S_1(x) \cap h_R(U) \neq \emptyset$, i.e., there exists $w \in X_1$ such that $w \in S_1(x)$ and $w \in h_R(U) = f_R^{-1}(U)$. This is, $f_R(w) \in U$. As $(x, w) \in S_1$, by $(MF1)$, $f_R(w) \in S_2(f_R(x))$. Thus, $S_2(f_R(x)) \cap U \neq \emptyset$ and hence, $f_R(x) \in \Diamond_{S_2}(U)$. So, $x \in f_R^{-1}(\Diamond_{S_2}(U)) = h_R(\Diamond_{S_2}(U))$.

$\Leftarrow$) Let $x \in X_1$. Suppose that there exists $y \in S_1(x)$ such that $f_R(y) \notin S_2(f_R(x))$. Since $S_2(f_R(x))^c \in \mathrm{Od}(X_2)$, we obtain that

$$f_R(y) \in S_2(f_R(x))^c = \bigcup \{U \in D(X_2) : U \subseteq S_2(f_R(x))^c\}\,.$$

There exists $U \in D(X_2)$ such that $U \subseteq S_2(f_R(x))^c$ and $f_R(y) \in U$. Thus, $y \in h_R(U)$. As $S_1(x) \cap h_R(U) \neq \emptyset$, we get $x \in \Diamond_{S_1}(h_R(U))$. By assumption, $x \in h_R(\Diamond_{S_2}(U))$, i.e., $f_R(x) \in \Diamond_{S_2}(U)$. Thus, $U \cap S_2(f_R(x)) \neq \emptyset$, which contradicts that $U \subseteq S_2(f_R(x))^c$. ∎

By Proposition 20 and Theorem 5.12 in [10], we obtain the following result.

Corollary 21. *Let $\langle X_1, \mathcal{T}_{\mathcal{K}_1}, S_1\rangle$ and $\langle X_2, \mathcal{T}_{\mathcal{K}_2}, S_2\rangle$ be $H^{\vee}_{\Diamond}$-spaces and $R \subseteq X_1 \times X_2$ be a $\Diamond$-relation. Then, h_R is a morphism of $\mathrm{Hil}^{\vee}_{\Diamond}\mathcal{S}$.*

Now, we will study the relation R_h defined in (3) when h is a morphism of $\mathrm{Hil}^{\vee}_{\Diamond}\mathcal{S}$.

Proposition 22. *Let $\langle A, \Diamond\rangle$, $\langle B, \Diamond\rangle \in \mathrm{Hil}^{\vee}_{\Diamond}$. Let $h : A \to B$ be a $\vee$-semi-homomorphism such that it commutes with $\Diamond$ and preserves the bottom element. Then, R_h is a $\Diamond$-relation.*

Proof. By Theorem 6, R_h is an irreducible H-relation. We prove that R_h is a strong irreducible H-relation showing that $R_h(x) \neq \emptyset$, for each $x \in X(B)$. As h is a $\vee$-semi-homomorphism, by item 3 of Theorem 6 we get that $h^{-1}(x) \in X(A)$ or $h^{-1}(x) = A$ for every $x \in X(B)$. Suppose that there exists $x \in X(B)$ such that $h^{-1}(x) = A$. So, we have $0 \in h^{-1}(x)$. This is, $h(0) = 0 \in x$ and so, $x = A$, which is impossible. Thus, $h^{-1}(x) \in X(A)$ and consequently, $h^{-1}(x) \in R_h(x)$ for every $x \in X(B)$. Since R_h is a strong irreducible H-relation, we get that $R_h(x) = \left[h^{-1}(x)\right)$, for each $x \in X(B)$. Then we have a function

$$f_{R_h} : X(B) \to X(A) \text{ such that } f_{R_h}(x) = h^{-1}(x).$$

By Lemma 3.5 of [9], we have that $h_{R_h} \circ \varphi_A = \varphi_B \circ h$, i.e., $h_{R_h}(\varphi_A(a)) = \varphi_B(h(a))$, for all $a \in A$. We will prove that R_h satisfies the conditions of Definition 19.

We note that for each $a \in A$:

$$\begin{array}{rclcl} h_{R_h}(\Diamond_{S_A}(\varphi_A(a))) & = & h_{R_h}(\varphi_A(\Diamond a)) & = & \varphi_B(h(\Diamond a)) \\ & = & \varphi_B(\Diamond h(a)) & = & \Diamond_{S_B}\varphi_B(h(a)) \\ & = & \Diamond_{S_B} h_{R_h}(\varphi_A(a)). & & \end{array}$$

Then by Definition 19 and Proposition 20 we have that R_h is a $\Diamond$-relation. ■

From the previous Proposition it is immediately evident that the categories $\mathrm{Hil}^{\vee}_{\Diamond}\mathcal{S}$ and $\mathcal{M}_{\Diamond}\mathcal{SR}^{\vee}$ are dually equivalent.

Let $\mathrm{Hil}^{\vee}_{\Diamond}\mathcal{H}$ be the category with $H^{\vee}_{\Diamond}$-algebras and $\vee$-homomorphisms that commute with $\Diamond$ and preserve the bottom element, and let $\mathcal{M}_{\Diamond}\mathcal{SF}^{\vee}$ be the category whose objects are $H^{\vee}_{\Diamond}$-spaces and whose morphisms are irreducible H-functional relations that are $\Diamond$-relations. As the category $\mathrm{Hil}^{\vee}\mathcal{H}$ is dually isomorphic to the category of $H^{\vee}$-spaces with irreducible H-functional relations, by previous results we can affirm that $\mathrm{Hil}^{\vee}_{\Diamond}\mathcal{H}$ and $\mathcal{M}_{\Diamond}\mathcal{SF}^{\vee}$ are dually equivalent.

4. Some Subvarieties of $H^{\vee}_{\Diamond}$-Algebras

Let $\langle A, \Diamond\rangle \in \mathrm{Hil}^{\vee}_{\Diamond}$. For each $n \in \mathbb{N}_0$, we define inductively the formula $\Diamond^n$ as $\Diamond^0 a = a$ and $\Diamond^{n+1}a = \Diamond(\Diamond^n a)$. The variety of $H^{\vee}_{\Diamond}$-algebras generated by a finite set of identities Γ will be denoted by $\mathrm{Hil}^{\vee}_{\Diamond} + \{\Gamma\}$.

We shall consider some particular varieties of $H^{\vee}_{\Diamond}$-algebras. These varieties are the algebraic counterpart of extensions of the implicative fragments of the intuitionistic modal logic $\mathbf{IntK}_{\Diamond}$ (see [1,4,26,28,29], and [31]).

Consider the following identities:

$$\begin{array}{ll} \Diamond\mathbf{T} & a \to \Diamond a \approx 1, \\ \Diamond\mathbf{4} & \Diamond^2 a \to \Diamond a \approx 1, \\ \Diamond\mathbf{5} & \Diamond(\Diamond a \to \Diamond b) \to (\Diamond a \to \Diamond b) \approx 1. \end{array}$$

We note that $\mathrm{Hil}^{\vee}_{\Diamond} + \{\Diamond\mathbf{T}, \Diamond\mathbf{5}\}$ is a subvariety of $\mathrm{Hil}^{\vee}_{\Diamond} + \{\Diamond\mathbf{T}, \Diamond\mathbf{4}\}$. In fact. Let $\langle A, \Diamond\rangle \in \mathrm{Hil}^{\vee}_{\Diamond} + \{\Diamond\mathbf{T}, \Diamond\mathbf{5}\}$. Since $a \leq \Diamond a$ for all $a \in A$, in particular for $a = 1$. So, $\Diamond 1 = 1$. Thus, for all $b \in A$ we have that:

$$1 = \Diamond(\Diamond 1 \to \Diamond b) \to (\Diamond 1 \to \Diamond b) = \Diamond(1 \to \Diamond b) \to (1 \to \Diamond b) = \Diamond^2 b \to \Diamond b.$$

We shall establish that certain additional conditions defined in an $H_{\Diamond}^{\vee}$-algebra correspond to additional properties defined in the dual space. For this, we will use the following result.

LEMMA 23. *Let $\langle A, \Diamond\rangle \in \mathrm{Hil}_{\Diamond}^{\vee}$ and let $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ be its dual $H_{\Diamond}^{\vee}$-space. Let $n \in \mathbb{N}$ and $x \in X$. Then, $\Diamond^n a \in x$ iff there exists $y \in X$ such that $(x, y) \in S^n$ and $a \in y$.*

PROOF. The proof is by induction of n. For $n = 1$ is valid by Lemma 13. Suppose the claim holds for n. Consider $\Diamond^{n+1} a = \Diamond(\Diamond^n a) \in x$. By Lemma 13, there exists $y \in X$ such that $y \in S(x)$ and $\Diamond^n a \in y$. By assumption, there exists $z \in X$ such that $(y, z) \in S^n$ and $a \in z$. Thus, there exists $z \in X$ such that $(x, z) \in S^{n+1}$ and $a \in z$. Now, suppose that there exists $y \in X$ such that $(x, y) \in S^{n+1}$ and $a \in y$. So, there exists $z \in X$ such that $(x, z) \in S^n$ and $(z, y) \in S$. Since $a \in y$, we obtain that $\Diamond a \in z$ and as $(x, z) \in S^n$, we have $\Diamond^n(\Diamond a) = \Diamond^{n+1} a \in x$, by assumption. ■

Let $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ be an $H_{\Diamond}^{\vee}$-space. Following the notation used in [19], we denote by ϱ and ϱ' the next first-order conditions:

$$\varrho \Leftrightarrow \forall x \forall y \forall z \left[xSy \wedge xSz \Rightarrow \exists t(y \leq t \wedge tSz \wedge \forall u(tSu \Rightarrow xSu))\right],$$
$$\varrho' \Leftrightarrow \forall x \forall y \left[xSy \Rightarrow \exists t(y \leq t \wedge tSx \wedge xSt)\right],$$

REMARK 24. Note that if S is reflexive and transitive then ϱ and ϱ' are equivalent. Suppose that ϱ is satisfied in $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ and let $x, y \in X$ such that xSy. As S is reflexive, xSy and xSx. By ϱ, there exists $t \in X$ such that $y \leq t$, tSx and for all $u \in X$, tSu implies xSu. In particular, as tSt, we have that xSt. Thus, there exists $t \in X$ such that $y \leq t$, tSx and xSt.

Reciprocally, assume that ϱ' is satisfied in $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ and let $x, y, z \in X$ such that xSy and xSz. If xSy then there exists $t \in X$ such that $y \leq t$, tSx and xSt. As tSx and xSz, by transitivity of S, result tSz. Now, let $u \in X$ such that tSu. Since xSt, by transitivity of S we obtain that xSu. Thus, there exists $t \in T$ such that $y \leq t$, tSz and for all $u \in X$, tSu implies xSu.

THEOREM 25. *Let $\langle A, \Diamond\rangle \in \mathrm{Hil}_{\Diamond}^{\vee}$ and $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ its dual $H_{\Diamond}^{\vee}$-space. The following statements are satisfied:*

1. $\Diamond^n a \to a \approx 1$ *for all* $a \in A$ *iff* $\forall x, y[(x, y) \in S^n \Rightarrow y \subseteq x]$, *for all* $n \in \mathbb{N}$.
2. $A \in \mathrm{Hil}_{\Diamond}^{\vee} + \{\Diamond\mathbf{T}\}$ *iff* S *is reflexive.*
3. $A \in \mathrm{Hil}_{\Diamond}^{\vee} + \{\Diamond\mathbf{4}\}$ *iff* S *is transitive.*
4. $A \in \mathrm{Hil}_{\Diamond}^{\vee} + \{\Diamond\mathbf{5}\}$ *iff* ϱ *is satisfied in* $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$.

5. $A \in \mathrm{Hil}^{\vee}_{\Diamond} + \{\Diamond\mathbf{T},\Diamond\mathbf{5}\}$ *iff* ϱ' *is satisfied in* $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ *and* S *is reflexive and transitive.*

Proof. We will prove only the assertions (1), (4) and (5). The other proofs are analogous.

(1) Assume that $\Diamond^n a \leq a$ for all $a \in A$. Consider $(x, y) \in S^n$ and let $b \in y$. By Lemma 23, $\Diamond^n b \in x$. As $\Diamond^n b \leq b$, we get that $b \in x$. So, $y \subseteq x$. Now, assume that $(x, y) \in S^n$ implies $y \subseteq x$ and suppose that there exists $a \in A$ such that $\Diamond^n a \nleq a$. So, there exists $x \in X$ such that $\Diamond^n a \in x$ and $a \notin x$. By Lemma 23, there exists $y \in X$ such that $(x, y) \in S^n$ and $a \in y$. Under the above assumption, $y \subseteq x$ and so, $a \in x$, which is a contradiction.

(4) Assume that $\Diamond(\Diamond a \to \Diamond b) \leq \Diamond a \to \Diamond b$, for all $a, b \in A$. Suppose that $(x, y) \in S$ and $(x, z) \in S$. To show that there exists an element $t \in X$ such that $y \leq t$ and $z \in S(t)$, we will consider the deductive system $\langle y \cup \Diamond z\rangle$ and the order-ideal $\left(\Diamond\left(\Diamond^{-1}(x^c)\right)\right]$, and we will prove that $\langle y \cup \Diamond z\rangle \cap \left(\Diamond\left(\Diamond^{-1}(x^c)\right)\right] = \emptyset$. Suppose that the contrary holds, this is, there exists $a \in A$ such that $a \in \langle y \cup \Diamond z\rangle$ and $a \in \left(\Diamond\left(\Diamond^{-1}(x^c)\right)\right]$. So, there are $b \in y$, $c \in z$ and $d \in \Diamond^{-1}(x^c)$ such that $b \to (\Diamond c \to \Diamond d) = 1 \in y$. Thus, $\Diamond c \to \Diamond d \in y$ and as $y \subseteq \Diamond^{-1}(x)$, $\Diamond(\Diamond c \to \Diamond d) \in x$. By assumption, $\Diamond c \to \Diamond d \in x$. Since $z \subseteq \Diamond^{-1}(x)$, we get $\Diamond c \in x$, and so $\Diamond d \in x$, which is a contradiction. Thus, there exists $t \in X$ such that $y \subseteq t$, $\Diamond z \subseteq t$ and $\Diamond\left(\Diamond^{-1}(x^c)\right) \cap t = \emptyset$. This is, $z \subseteq \Diamond^{-1}(t)$ and $\Diamond^{-1}(x^c) \subseteq \Diamond^{-1}(t^c)$. Let $u \in X$ such that tSu, i.e., $u \subseteq \Diamond^{-1}(t)$. As $\Diamond^{-1}(t) \subseteq \Diamond^{-1}(x)$, we get xSu. Thus, there exists $t \in X$ such that $y \subseteq t$, tSz and for all $u \in X$, if tSu then xSu.

Conversely, suppose that there exist $a, b \in A$ such that $\Diamond(\Diamond a \to \Diamond b) \nleq \Diamond a \to \Diamond b$. So, there exists $x \in X$ such that $\Diamond(\Diamond a \to \Diamond b) \in x$, $\Diamond a \in x$ and $\Diamond b \notin x$. By Lemma 13, there are elements $y, z \in S(x)$ such that $\Diamond a \to \Diamond b \in y$ and $a \in z$. By assumption, there exists $t \in X$ such that $y \subseteq t$, $z \subseteq \Diamond^{-1}(t)$ and tSu implies xSu for all $u \in X$. So, $\Diamond a \to \Diamond b \in t$, $\Diamond a \in t$ and consequently, $\Diamond b \in t$. By Lemma 13, there exists $u \in X$ such that $(t, u) \in S$ and $b \in u$. So, $(x, u) \in S$ and so $\Diamond b \in x$, which is impossible.

(5) It is immediately by items (2), (3), (4) and Remark 24. ∎

5. Congruences of $H^{\vee}_{\Diamond}$-Algebras

A well-known result given by A. Diego and A. Monteiro (see [16,18,25] or [23]) ensures that the lattice of congruences of a Hilbert algebra is isomorphic to the lattice of the deductive systems. This result can be extended to Hilbert algebras with supremum as was proved in [14].

Let $A = \langle A, \rightarrow, \vee, 1\rangle$ be an $H^{\vee}$-algebra. A *congruence* of A is an equivalence relation $\theta \subseteq A \times A$ compatible with the implication $\rightarrow$ and with $\vee$. If A is an $H_0^{\vee}$-algebra, then every equivalence relation $\theta \subseteq A \times A$ compatible with the implication $\rightarrow$ is also compatible with the negation $\neg$, because $\neg a = a \rightarrow 0$, for every $a \in A$. In [14] it was proved that θ is a congruence of A if and only if θ is an equivalence relation compatible with the implication $\rightarrow$. Thus the lattice of the congruences of $\langle A, \rightarrow, \vee, 1\rangle$ is the same as the lattice of the congruences of $\langle A, \rightarrow, 1\rangle$. So, we denote the lattice of the congruences of A by $\mathrm{Con}(A)$.

Let A be a Hilbert algebra. Let $\theta \in \mathrm{Con}(A)$. The equivalence class

$$[1]_\theta = \{a \in A \mid (a, 1) \in \theta\},$$

is a deductive system. Reciprocally, if $D \in \mathrm{Ds}\,(A)$, then

$$\theta(D) = \left\{(a, b) \in A^2 \mid a \rightarrow b, b \rightarrow a \in D\right\},$$

is a congruence of A. The lattice $\mathrm{Ds}\,(A)$ and $\mathrm{Con}(A)$ are isomorphic under the mutually inverse mappings $\theta \rightarrow [1]_\theta$, and $D \rightarrow \theta(D)$ (see [18,23], or [16]).

If L is a lattice, we denote by L^d the lattice with the dual order and if two lattices L_1 and L_2 are isomorphic, we write $L_1 \cong L_2$.

Let A be a Hilbert algebra and $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$ its dual H-space. If $D \in \mathrm{Ds}\,(A)$, then $\mu\,(D) = \{x \in X(A) : D \subseteq x\}$ is a closed subset of $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$. If Y is a closed subset of $\langle X, \mathcal{T}_{\mathcal{K}}\rangle$, then $\pi(Y) = \{a \in A : Y \subseteq \varphi(a)\}$ is a deductive system of A. Moreover, if $Y \in \mathcal{C}(X)$, then $\mu\,(\pi(Y)) = Y$, and if $D \in \mathrm{Ds}\,(A)$, then $\pi\,(\mu\,(D)) = D$. Thus, μ is a dual isomorphism between $\mathrm{Ds}\,(A)$ and $\mathcal{C}(X)$. Therefore

$$\mathrm{Con}(A) \cong \mathrm{Ds}\,(A) \cong \mathcal{C}(X)^d.$$

Let $\langle A, \Diamond\rangle \in \mathrm{Hil}_{\Diamond}^{\vee}$. We say that θ is a $\Diamond$-congruence iff θ is an equivalence relation defined on A which is compatible with $\rightarrow$, $\vee$ and $\Diamond$. Denote by $\mathrm{Con}_{\Diamond}\,(A)$ the lattice of congruences of $\langle A, \Diamond\rangle$.

Definition 26. Let $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ be an $H_{\Diamond}^{\vee}$-space. A subset $Y \subseteq X$ is said to be S-*maximum* if for all $x \in Y$, $\max S(x) \subseteq Y$.

It is clear that X and $\emptyset$ are trivially S-maximum sets and it is easy to check from the above definition that the intersection and the union of any family of S-maximum sets is again an S-maximum set. So, we can conclude that the set of all S-maximum subsets of X is a complete sublattice of $\mathcal{P}(X)$. In particular, the S-maximum and closed subsets of X, ordered by inclusion, form a sublattice of $\mathcal{C}(X)$ closed under arbitrary intersections which shall be denoted by $\mathcal{C}_{\max}(X)$.

Let $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ be an $H^{\vee}_{\Diamond}$-space. The family of all S-maximum and closed subsets of X defines a topology

$$\mathcal{T}_S = \{X - Y : Y \in \mathcal{C}_{\max}(X)\}$$

on X, such that their closed subsets are exactly the members of $\mathcal{C}_{\max}(X)$. We shall denote by $\mathrm{cl}_{\max}(Y)$ the closure of a subset $Y \subseteq X$, when X is endowed with the topology $\mathcal{T}_S$. Note that $\mathrm{cl}_{\max}(Y) \in \mathcal{C}_{\max}(X)$. In particular, if $Y \in \mathcal{C}_{\max}(X)$, then $\mathrm{cl}_{\max}(Y) = \mathrm{cl}(Y)$. A subset Y of X is called *S-closed* (*S-dense*), if it is a *closed* (*dense*) *subset* of X with respect to the topology $\mathcal{T}_S$.

Remark 27. Let $\langle A, \Diamond\rangle \in \mathrm{Hil}^{\vee}_{\Diamond}$ and $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ its dual $H^{\vee}_{\Diamond}$-space. Note that

$$\textit{if } S(x) \neq \emptyset \textit{ then } \max S(x) \neq \emptyset, \textit{ for } x \in X.$$

To prove this, we will see that every chain in $S(x)$ has an upper bound in $S(x)$. Let $x \in X$ and let $\{y_i\}_{i\in\alpha}$ be a totally ordered family of $S(x)$. Let $z = \bigcup\{y_i : i \in \alpha\}$. We will prove that $z \in X$. It is clear that z is an increasing subset and that if $a \vee b \in z$ then $a \in z$ or $b \in z$. Let $a, a \to b \in z$. So, there are $i, j \in \alpha$ such that $a \in y_i$ and $a \to b \in y_j$. Without loss of generality, we may assume that $i \leq j$ and so, that $y_i \subseteq y_j$. Thus, $a, a \to b \in y_j$ and hence, $b \in y_j$. Thus, $b \in z$ and consequently, $z \in X$. We shall prove that $z \in S(x)$. Let $a \in z$. So, there exists $i \in \alpha$ such that $a \in y_i$. As $y_i \in S(x)$, it follows that $a \in \Diamond^{-1}(x)$. So, every chain in $S(x)$ has an upper bound in $S(x)$ and by Zorn's Lemma, there is $m \in \max S(x)$.

Now we shall give a characterization of the $\Diamond$-congruences applying the duality.

Theorem 28. *Let* $\langle A, \Diamond\rangle \in \mathrm{Hil}^{\vee}_{\Diamond}$ *and* $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ *its dual* $H^{\vee}_{\Diamond}$*-space. Then,*

$$\mathcal{C}_{\max}(X)^d \cong \mathrm{Con}_{\Diamond}(A).$$

Proof. We shall prove that the map $\Phi : \mathcal{C}_{\max}(X) \to \mathrm{Con}_{\Diamond}(A)$ defined by

$$\Phi(Y) = \{(a, b) \in A \times A : a \to b, b \to a \in \pi(Y)\}$$

is an anti-isomorphism, where $\pi(Y) = \{a \in A : Y \subseteq \varphi(a)\}$.

Let $Y \in \mathcal{C}_{\max}(X)$. As $\pi(Y) \in \mathrm{Ds}(A)$, result that $\Phi(Y) = \theta(\pi(Y)) \in \mathrm{Con}(A)$. To prove that Φ is well-defined we need to show that $\Phi(Y)$ preserves the operator $\Diamond$.

Let $(a, b) \in \Phi(Y)$. We will prove that $(\Diamond a, \Diamond b) \in \theta(Y)$. For this, we show that $\Diamond a \to \Diamond b \in \pi(Y)$, i.e., $Y \subseteq \varphi(\Diamond a) \Longrightarrow \varphi(\Diamond b) = \{x \in X : [x) \cap \varphi(\Diamond a) \subseteq$

$\varphi(\Diamond b)\}$. Let $x \in Y$ and let $y \in X$ such that $y \in [x) \cap \varphi(\Diamond a)$. So, $x \subseteq y$ and $\Diamond a \in y$. By Lemma 13, there exists $z \in X$ such that $z \in S(y)$ and $a \in z$. By Remark 27, there exists $w \in \max S(y)$ such that $z \subseteq w \subseteq \Diamond^{-1}(y)$ and since $a \in z$, we get $a \in w$. On the other hand, as $x \in Y$ and Y is an increasing set, we get $y \in Y$ and since $\max S(y) \subseteq Y$, we have $w \in Y$. By assumption, $a \to b \in \pi(Y)$ and so, $[w) \cap \varphi(a) \subseteq \varphi(b)$. As $w \in \varphi(a)$, we have $w \in \varphi(b)$. Thus, $b \in w \subseteq \Diamond^{-1}(y)$ and consequently $\Diamond b \in y$, i.e., $y \in \varphi(\Diamond b)$. So, $\Diamond a \to \Diamond b \in \pi(Y)$. By a similar argument we can prove that $\Diamond b \to \Diamond a \in \pi(Y)$. So, we have proved that $\Phi(Y) \in \mathrm{Con}_{\Diamond}(A)$ for each $Y \in \mathcal{C}_{\max}(X)$.

Let $Y, W \in \mathcal{C}_{\max}(X)$. It is clear that if $Y \subseteq W$ then $\Phi(W) \subseteq \Phi(Y)$. To prove that the map Φ is one-to-one, assume that $\Phi(Y) = \Phi(W)$ and suppose that $Y \neq W$. Without loss of generality, we assume that there exists $x \in Y$ such that $x \notin W$. As W is a closed set, there exists $a \in A$ such that $W \subseteq \varphi(a)$ and $x \notin \varphi(a)$. Thus, $a = 1 \to a \in \pi(W)$ and as $\pi(W) \in \mathrm{Ds}(A)$, $1 = a \to 1 \in \pi(W)$. So, $(1, a) \in \Phi(W) = \Phi(Y)$ and consequently, $a \in \pi(Y)$. This is, $Y \subseteq \varphi(a)$ and so, $x \in \varphi(a)$, which is a contradiction.

Thus, to complete the proof, we need to prove that Φ is onto. Let $\theta \in \mathrm{Con}_{\Diamond}(A)$ and

$$Z = \bigcap \{\varphi(a) : a \in [1]_\theta\} = \{x \in X : [1]_\theta \subseteq x\}.$$

It is clear that $Z \in \mathcal{C}(X)$. We will show that Z is an S-maximum set. On the contrary, let $x \in Z$, $y \in \max S(x)$ and suppose that $y \notin Z$. Hence, there exists $a \in [1]_\theta$ such that $a \notin y$. We prove that

$$\langle y \cup \{a\} \rangle \cap \Diamond^{-1}(x^c) = \emptyset.$$

Suppose that the contrary holds, this is, there exists $b \in A$ such that $b \in \langle y \cup \{a\} \rangle \cap \Diamond^{-1}(x^c)$. So, there exists $c \in y$ such that $c \to (a \to b) = 1 \in y$ and $\Diamond b \notin x$. As $c \in y$, we get $a \to b \in y$, and since $y \in S(x)$, $\Diamond(a \to b) \in x$. On the other hand, since $(a, 1) \in \theta$ we get $(a \to b, b) \in \theta$. Thus, $(\Diamond(a \to b), \Diamond b) \in \theta$ and so, $(\Diamond(a \to b) \to \Diamond b, 1) \in \theta$. Hence, $\Diamond(a \to b) \to \Diamond b \in [1]_\theta$ and as $[1]_\theta \subseteq x$, $\Diamond(a \to b) \to \Diamond b \in x$. As $\Diamond(a \to b) \in x$, we obtain that $\Diamond b \in x$, which is a contradiction. Thus, there exists $w \in X$ such that $\langle y \cup \{a\} \rangle \subseteq w$ and $w \cap \Diamond^{-1}(x^c) = \emptyset$. This is, $y \subseteq w$ and $w \in S(x)$. As $y \in \max S(x)$, $y = w$ and so, $a \in y$ which is impossible. We have proved that $\max S(x) \subseteq Z$, for all $x \in Z$. Thus, $Z \in \mathcal{C}_{\max}(X)$.

Now, we will prove that $\Phi(Z) = \theta$. Note that $Z \subseteq \varphi(a)$ iff $a \in [1]_\theta$. Indeed, it is clear that if $a \in [1]_\theta$ then $Z \subseteq \varphi(a)$. Now, suppose that $a \notin [1]_\theta$. So, $[1]_\theta \cap (a] = \emptyset$. So, there exists $x \in X$ such that $[1]_\theta \subseteq x$ and $a \notin x$. This is, $Z \nsubseteq \varphi(a)$. Thus,

$$\begin{array}{ccccc} (a,b)\in\Phi(Z) & \Longleftrightarrow & a\to b, b\to a\in\pi(Z) & \Longleftrightarrow \\ Z\subseteq\varphi(a\to b),\varphi(b\to a) & \Longleftrightarrow & a\to b, b\to a\in[1]_\theta & \Longleftrightarrow \\ (a\to b,1),(b\to a,1)\in\theta & \Longleftrightarrow & (a,b)\in\theta. & \end{array}$$

■

Now we will study the deductive systems in $H_\Diamond^\vee$-algebras that are in correspondence with $\Diamond$-congruences.

DEFINITION 29. Let $\langle A,\Diamond\rangle\in \mathrm{Hil}_\Diamond^\vee$. A deductive system F of A is a *closed deductive system* if $a\to b\in F$ implies $\Diamond a\to\Diamond b\in F$, for every $a,b\in A$.

The lattice of all closed deductive systems of $\langle A,\Diamond\rangle\in \mathrm{Hil}_\Diamond^\vee$ is denoted by $\mathrm{Ds}_\Diamond(A)$.

PROPOSITION 30. *Let* $\langle A,\Diamond\rangle\in \mathrm{Hil}_\Diamond^\vee$ *and* $\langle X,\mathcal{T}_\mathcal{K},S\rangle$ *its dual* $H_\Diamond^\vee$*-space. Then,* $\mathrm{Ds}_\Diamond(A)\cong\mathcal{C}_{\max}(X)^d$.

PROOF. Let $\mu:\mathrm{Ds}_\Diamond(A)\to\mathcal{C}_{\max}(X)$ such that $\mu(F)=\{x\in X: F\subseteq x\}$ for all $F\in\mathrm{Ds}_\Diamond(A)$. We know that $\mu(F)$ is a closed subset of $\langle X,\mathcal{T}_\mathcal{K}\rangle$ for all $F\in\mathrm{Ds}(A)$. It is clear that if $F,D\in\mathrm{Ds}_\Diamond(A)$ such that $F\subseteq D$ then $\mu(D)\subseteq\mu(F)$. Now, we prove that $\mu(F)$ is an S-maximum set for each $F\in\mathrm{Ds}_\Diamond(A)$. Let $x\in\mu(F)$. Suppose that there exists $y\in\max S(x)$ such that $y\notin\mu(F)$, i.e., $F\nsubseteq y$. Then, there exists $f\in F$ such that $f\notin y$. Let us consider the deductive system $\langle y\cup\{f\}\rangle$. Since $y\in\max S(x)$,

$$\langle y\cup\{f\}\rangle\nsubseteq\Diamond^{-1}(x).$$

Thus, there is $d\in A$ such that $d\in\langle y\cup\{f\}\rangle$ and $d\notin\Diamond^{-1}(x)$. This is, there exists $q\in y$ such that $f\to(q\to d)=1$ and $\Diamond d\notin x$. So, $q\to d\in F$ and as F is a closed deductive system, $\Diamond q\to\Diamond d\in F\subseteq x$. Taking into account that $q\in y\subseteq\Diamond^{-1}(x)$, we have that $\Diamond d\in x$, which is a contradiction. Thus, $\mu(F)\in\mathcal{C}_{\max}(X)$.

Let $\pi:\mathcal{C}_{\max}(X)\to\mathrm{Ds}_\Diamond(A)$ such that $\pi(Y)=\{a\in A: Y\subseteq\varphi(a)\}$. By proof of Theorem 28, we know that $\pi(Y)$ is a closed deductive system for all $Y\in\mathcal{C}_{\max}(X)$. As μ and π are inverses of each other, we deduce that μ is an anti-isomorphism of lattices. ■

COROLLARY 31. *Let* $\langle A,\Diamond\rangle\in \mathrm{Hil}_\Diamond^\vee$ *and* $\langle X,\mathcal{T}_\mathcal{K},S\rangle$ *its dual* $H_\Diamond^\vee$*-space. Then,*

$$\mathrm{Con}_\Diamond(A)\cong\mathrm{Ds}_\Diamond(A)\cong\mathcal{C}_{\max}(X)^d.$$

5.1. Simples and Subdirectly Irreducible $H_\Diamond^\vee$-Algebras

Let $A=\langle A,\Diamond\rangle\in\mathrm{Hil}_\Diamond^\vee$. Let us recall that A is *subdirectly irreducible* if and only if there exists the smallest non trivial $\Diamond$-congruence relation θ in

A. A particular case are the simple algebras. Recall that an $H_{\Diamond}^{\vee}$-algebra A is *simple* if and only if A has only two $\Diamond$-congruence relations. By Theorem 28 and Proposition 30, we can affirm that an $H_{\Diamond}^{\vee}$-algebra A is subdirectly irreducible iff in its dual $H_{\Diamond}^{\vee}$-space $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ there exists the largest $Y \in \mathcal{C}_{\max}(X)$ distinct from X and $\emptyset$ iff there exists the smallest non-trivial $\Diamond$-deductive system of A. Moreover, A is simple iff $\mathcal{C}_{\max}(X) = \{\emptyset, X\}$ iff $\mathrm{Ds}_{\Diamond}(A) = \{\{1\}, A\}$.

Now, we can characterize the simple and subdirectly irreducible $H_{\Diamond}^{\vee}$-algebras.

THEOREM 32. *Let $A \in \mathrm{Hil}_{\Diamond}^{\vee}$ and $\langle X, \mathcal{T}_{\mathcal{K}}, S\rangle$ its dual $H_{\Diamond}^{\vee}$-space. Then:*

1. *A is simple iff every unit subset of X is S-dense in X, i.e., $\mathrm{cl}_{\max}(x) = X$, for each $x \in X$.*
2. *A is subdirectly irreducible iff $\{x \in X : \mathrm{cl}_{\max}(x) \neq X\} \in \mathcal{C}_{\max}(X) - \{X\}$.*

PROOF. (1) Assume that A is simple. So, $\mathcal{C}_{\max}(X) = \{\emptyset, X\}$. Let $x \in X$. As $x \in \mathrm{cl}_{\max}(x)$, we get $\mathrm{cl}_{\max}(x) \neq \emptyset$ and since $\mathrm{cl}_{\max}(x) \in \mathcal{C}_{\max}(X)$, result $\mathrm{cl}_{\max}(x) = X$. Reciprocally, assume that $\mathrm{cl}_{\max}(x) = X$ for each $x \in X$ and suppose that there exists $Y \in \mathcal{C}_{\max}(X)$ such that $Y \neq \emptyset$. So, there exists $y \in Y$ and consequently, $X = \mathrm{cl}_{\max}(y) \subseteq Y$. Thus, $X = Y$ and so, A is simple.

(2) Consider the set

$$V = \{x \in X : \mathrm{cl}_{\max}(x) \neq X\}.$$

Assume that A is subdirectly irreducible and let Y be the largest element of $\mathcal{C}_{\max}(X) - \{X\}$. We will prove that $Y = V$. Let $x \in Y$. So, $\mathrm{cl}_{\max}(x) \subseteq Y \neq X$ and consequently, $x \in V$. Thus, $Y \subseteq V$. Now, let $x \in V$. So, $\mathrm{cl}_{\max}(x) \in \mathcal{C}_{\max}(X) - \{X\}$ and by assumption, $\mathrm{cl}_{\max}(x) \subseteq Y$. So, $x \in Y$. Thus, $V = Y \in \mathcal{C}_{\max}(X) - \{\emptyset, X\}$. Reciprocally, let $V \in \mathcal{C}_{\max}(X) - \{X\}$. We will prove that V is the largest element of $\mathcal{C}_{\max}(X) - \{X\}$. Suppose that there exists $Y \in \mathcal{C}_{\max}(X)$ such that $Y \nsubseteq V$. So, there exists $x \in Y$ such that $x \notin V$. Hence, $X = \mathrm{cl}_{\max}(x) \subseteq Y$. Thus, $Y = X$, which complete the proof. ■

The subvariety $\mathrm{Hil}_{\Diamond}^{\vee} + \{\Diamond\mathbf{T}, \Diamond\mathbf{5}\}$ is the algebraic counterpart of the $\{\rightarrow, \vee, \Diamond\}$-fragment of the Prior's Intuitionistic Modal logics MIPC (see [1]). For this variety we have found other theory of representation. This representation will be presented in a future paper.

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