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Bi-facial Truth: a Case for Generalized Truth Values

Abstract. We explore a possibility of generalization of classical truth values by distinguishing between their ontological and epistemic aspects and combining these aspects within a joint semantical framework. The outcome is four generalized classical truth values implemented by Cartesian product of two sets of classical truth values, where each generalized value comprises both ontological and epistemic components. This allows one to define two unary twin connectives that can be called "semi-classical negations". Each of these negations deals only with one of the above mentioned components, and they may be of use for a logical reconstruction of argumentative reasoning.

Keywords: Generalized truth values, Semi-Boolean complementation, Semi-classical negation, Super-classical logic.

1. Preliminaries: Generalized Truth Values

The conception of *generalized truth values* as subsets of a given basic set was first introduced in [23, p. 763] as a generalization of some fundamental ideas by J.M. Dunn and N. Belnap concerning interrelations between truth and falsehood.

Initially Dunn in [10, 11] was aiming at representing incomplete or/and inconsistent pieces of information by allowing sentences to be sometimes neither (classically) true nor (classically) false, as well as true and false simultaneously. To this effect it was proposed in [11, p. 156] to consider valuation to be a function from sentences into subsets of the set of classical truth values $\mathbf{2} = \{T, F\}$. Then a valuation of a sentence can sometimes be under-determined (if one ascribes to the sentence the empty set) or over-determined (if one ascribes to the sentences the whole set $\mathbf{2}$).

Belnap [5, 6] employed this idea to develop his much celebrated four-valued logic for a computer-based reasoning. To be able to take account of incomplete or inconsistent data, Belnap's computer operates with truth values interpreted as information that has been "told" to it, namely **T** (plain truth), **F** (plain falsehood), **B** (both truth and falsehood), **N** (neither truth nor falsehood). These four truth values can also be interpreted as the subsets

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of **2**, and one arrives thus at the set **4** being power-set of **2**: $\mathbf{4} = \mathcal{P}(\mathbf{2}) = \{\{t\}, \{f\}, \{t, f\}, \{\}\} = \{\mathbf{T}, \mathbf{F}, \mathbf{B}, \mathbf{N}\}$.

This approach has been summarized and developed further in [24, 25, 29, 30] and expounded in a systematic way in [27]. The key notions of a generalized truth value and a generalized truth value function can be specified as follows:

DEFINITION 1.1.
Let X be a (basic) set of initial truth values, and let $\mathcal{P}(X)$ be the power-set of X. Then the elements of $\mathcal{P}(X)$ are called **generalized truth values** defined on the basis of X.[1]

DEFINITION 1.2.
Let X be a (basic) set of initial truth values, $\mathcal{P}(X)$ the set of generalized truth values defined on the basis of X, and $\mathcal{L}$ a given language. Then a **generalized truth value function** (defined on the basis of X) is a function from the set of sentences of $\mathcal{L}$ into $\mathcal{P}(X)$.

According to Definition 1.1, generalized truth values are constructed on the basis of some set of primitive truth values of an underlying low-level logic (or several such logics with their truth values "lumped together"). In the present paper we rather take as a starting point *two* basic sets of classical truth values representing *two different types* of truth and falsity.

The idea is to expose ontological and epistemic aspects of classical truth values, and to consider then new generalized truth values which may comprise *both* of these aspects. It turns out that the resulting construction allows one to define two different negation-like connectives which can be regarded as "semi-classical". These connectives possess some (but not all) properties of classical negation, and their superposition gives us the negation operator of classical logic.

2. Ontological and epistemological interpretations of truth values

G. Frege [16, 17] introduced *the True* and *the False* as special kinds of objects serving as denotations of sentences[2]. A. Church specifies this further by

[1]Typically the sets of generalized truth values constitute specific algebraic structures, which can be called *multilattices*, see [27, p. 51]. Multilattices are just lattices with several orderings defined on them, such as bilattices [18, 3, 15], trilattices [23, 24, 22], tetralattices [33], etc. These structures can be used for generating various logical systems.

[2]Note that Fregean approach to truth values is based on two important presuppositions embodying the fundamental principles of classical logic, namely, that there are but two

saying that truth values are "abstract" objects [8, p. 25]. J. Łukasiewicz explains that "truth has its analogue in being, and falsehood, in non-being" [20, p. 90]. Such an understanding reveals an explicitly *ontological* view on truth values.

By contrast, Belnap claims that the truth values of his four-valued logic are "unabashedly epistemic" [2, p. 521]. "Epistemic" means here that truth value of a sentence is identified with information concerning this sentence, i.e. with a certain information state. In classical cases the respective information may just be that the given sentence is true or false.

This situation parallels the situation with the notion of truth as a whole. It is sometimes observed that there are "two major strands of truth theories, ontological and epistemological ones" [31, p. 194]. Whereas the correspondence theory of truth "stays entirely on the ontological side" and is "completely silent about the epistemological side of truth", some of its rivals, such as pragmatic and coherence conceptions, can be called "the epistemological truth theories" [31, p. 197].

Moreover, it is also claimed that truth by itself "is really a hybrid notion", and this "hybrid nature of truth" embodies "the issues of epistemology and ontology":

> The truth operator certainly has the role of expressing epistemic endorsement. But there is a separate metaphysical question to be asked. Do the statements that we would endorse tell us how the world really is? [13, p. 20]

Taking into account that subject matter of logic is generally determined by the notion of truth which plays crucial role in defining central logical notions (such as logical entailment), these different interpretations may fit into a wider philosophical landscape concerning logic and its explications as "formal epistemology" *versus* "formal ontology". According to J. Corcoran, there are two different logical "paradigms" associated with the work of Aristotle and Boole respectively:

> Aristotle was the founder of logic as formal epistemology and ... Boole was the founder of logic as formal ontology. Aristotle laid down the groundwork for a science of determining validity and invalidity of arguments. Boole laid down the groundwork for a science of formal laws of being, in Tarskis words "general laws governing the concepts common to all sciences" or "the most general laws of thinkables" [9, p. 286].

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such values—*the True* and *the False* (bi-valence) and that any sentence may denote only one of these values at once (unique-valence). The very idea of generalized truth values has emerged resulting from questioning exactly these classical presuppositions.

Turning back to truth values one might convey the above distinction as follows: from an epistemological viewpoint logic can be delineated by the role truth values play in valid arguments, whereas ontologically logic can be construed as exploring truth values as such (and their relations to other entities, such as propositions, etc.).

Let us call the truth values considered from an ontological perspective "referential values", and let "inferential values" be the truth values treated as characteristics (information) of sentences involved in reasoning. If we confine ourselves just to two standard truth values—truth and falsity, then their referential understanding will be that some sentence *is* (objectively) true or false, and their inferential interpretation means that a sentence *is taken as* (i.e., considered) true (and thus accepted) or false (and thus rejected)[3].

R. Suszko in [32] put his famous *thesis* (named after him) dismissing the very idea of a many-valued logic by discriminating between "algebraic values", standing for admissible referents of sentences, and "logical values" used for defining an inference relation. The distinction between referential and inferential values evidently conforms with this idea. Suszko himself claimed that there can only be two logical values, true and false, but G. Malinowski in [21] argues persuasively in favor of inferential three-valuedness, and [30] defends the idea of an "inferential many-valuedness" by generalizing the very concept of a logical system.

Belnap observes an interesting tendency that frequently reveals in logical considerations, namely a tendency "to vacillate between reading the various values as epistemic on the one hand, and ontological on the other" [2, p. 521]. Such vacillations are especially typical in many-valued logics, where the precise meaning of the multiple truth values often remains unspecified:

> Does Łukasiewicz's middle value, $\frac{1}{2}$, mean "doesn't have a proper truth value," or does it mean "truth value unknown"? In informal explanations of what is going on, logicians sometimes move from one of these readings to the other in order to save the interest of the enterprise [2, p. 521].

Belnap considers also a possibility of combining classical *the True* and *the False* with his four "told values", obtaining in this way eight hybrid values which "have a mixed status: they are in part epistemological and in part ontological" [2, p. 521]. The resulting eight-valued logic turns out to be coincident with Belnap's four-valued logic which is not surprising, since ontological values play in this construction merely the role of a peculiar "make-weight" to his epistemological values. However, the situation might

[3] Cf. also the distinction by Frege between "Wahrsein" and "Fürwahrhalten".

be not as stable, if the ontological values were involved in the calculation of logical operations.

Note also, that even within a two-valued framework (dealing with just truth and falsity) it is not always clear in exactly which sense—referential or inferential—one uses the employed truth values. While Fregean approach is entirely ontological, classical logic can also be developed in terms of inferential values, granted that certain presuppositions (idealizations) are taken—indeed, all one needs to do is just to secure for these values the principles of bi- and unique-valence.

Moreover, in some cases an informal understanding of classical truth values may include (and combine) both ontological (referential) and epistemic (inferential) attitudes. For example, consider a rational and omniscient agent who knows *all* the "objectively existing" truths and never accepts anything contrary to what he knows. Assume further that the world by itself is entirely complete and non-contradictory. Then, when describing reasoning of this agent about such a world we can well confine ourselves with just two truth values, where one value means "a sentence is true (objectively) and accepted (by the agent)", and the other—"a sentence is false (objectively) and rejected (by the agent)". These two truth values also are of a mixed—onto-epistemic—character, nevertheless, the logic of such an agent would be perfectly classical.

But a real reasoner can hardly ever be such an ideal agent. In "real life" people can and do sometimes accept objectively false sentences or reject objectively true ones either by ignorance, or simply being incorrectly informed or pragmatically motivated. Even if we assume the completeness and non-contradictoriness of the world as such, and rationality of an arguing agent in the sense that he never accepts and rejects anything simultaneously, as well as has something to say on any sentence, we still may need some additional values for the situations when a sentence can be objectively false but still accepted by an agent or objectively true but nevertheless rejected.

3. Generalized Classical Truth Values

Let us mark the set of classical truth values interpreted referentially by $\mathbf{2}^t = \{T, F\}$. The elements of this set can be ordered to form a two-element Boolean lattice TWO^t with T as the top and F as the bottom (see Figure 1). The relation $\leq_t$ naturally orders the elements of $\mathbf{2}^t$ by their "truth-content" where truth is considered to be "more true" than falsehood.

Consider a propositional language $\mathcal{L}$ with classical $\wedge$, $\vee$, $\supset$, $\neg$. Let v_t^2 be a valuation mapping the set of variables of $\mathcal{L}$ into $\mathbf{2}^t$. Then we can define:

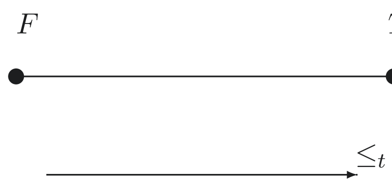

Figure 1. Lattice TWO^t

Definition 3.1.
(1) $v_t^2(A \wedge B) = v_t^2(A) \sqcap_t v_t^2(B)$;
(2) $v_t^2(A \vee B) = v_t^2(A) \sqcup_t v_t^2(B)$;
(3) $v_t^2(A \supset B) = -_t v_t^2(A) \sqcup_t v_t^2(B)$;
(4) $v_t^2(\neg A) = -_t v_t^2(A)$,

where $\sqcap_t$, $\sqcup_t$, $-_t$ are the operations of meet, join and complement defined through the lattice ordering on TWO^t in a standard way. Classical entailment relation is introduced for arbitrary A and B of $\mathcal{L}$ as follows:

Definition 3.2.
$A \models_t^2 B \Leftrightarrow \forall v_t^2(v_t^2(A) \leq_t v_t^2(B))$.

Thus, entailment relation is defined here through a lattice ordering, whereby the latter can be called a *logical order*[4].

If one marks the set of classical truth values interpreted inferentially by $\mathbf{2}^1 = \{1, 0\}$, then classical logic can be reconstructed in the same language with Definitions 3.1 and 3.2 where subscript t is uniformly substituted by 1. And of course we obtain a two element Boolean lattice TWO^1 which is strongly analogous to TWO^t.

Now we can return to the idea of interpreting truth values as hybrid entities embodying both an ontological and an epistemological sides. We wish these values to remain essentially classical, i.e. any sentence should be either true or false ontologically as well as epistemically, and it is impossible for a sentence to be simultaneously both true and false ontologically or both true and false epistemically. At the same time the principles of bi-unique-valence

[4]In bilattices and trilattices a logical order is also defined through the truth and/or falsity "content" of truth values and is also used for determining an entailment relation of the corresponding logic, see [26].

hold only within each dimension (ontological and epistemic) *separately*, and thus, when we combine these dimensions inside one and the same truth value, it may well happen that some sentence is, e.g., ontologically true whereas epistemically false.

To implement this idea we can pursue the way of combining initial truth values which is essentially the method of constructing generalized truth values as described in Definition 1.1. However, as distinct from this definition we now will accomplish a generalization procedure not on the basis of *one* set of truth values, but taking *two* such sets simultaneously, $\mathbf{2}^t$ and $\mathbf{2}^1$. Correspondingly, the generalization itself will be performed not by means of a power-setting formation, but by taking a Cartesian product of the basic sets.

We thus introduce what can be called *Cartesian classical truth values* by constructing a direct product of the above two sets: $\mathbf{4}^{t,1} = \mathbf{2}^t \times \mathbf{2}^1$. In this way we arrive at the following four values which can be marked and explained as follows:

$\mathbf{T1} = \langle T, 1\rangle$: a sentence is both ontologically and epistemically true, i.e. it is objectively true and accepted;
$\mathbf{T0} = \langle T, 0\rangle$: a sentence is ontologically true and epistemically false, i.e. it is objectively true but rejected;
$\mathbf{F1} = \langle F, 1\rangle$: a sentence is ontologically false and epistemically true, i.e. it is objectively false but accepted;
$\mathbf{F0} = \langle F, 0\rangle$: a sentence is both ontologically and epistemically false, i.e. it is objectively false and rejected.

Note again the difference between these four values and Belnap's values for computer-based reasoning. Whereas the truth values in Belnap's logic are purely epistemic, the four values above have a twofold onto-epistemic nature. This duality not only allows one to express some important features of the "real reasoning", but retains also its essentially classical character in the sense that the principles of bi-unique-valence concerning truth and falsity (if taken in one and the same respect) continue to hold.

By taking a direct product of TWO^t and TWO^1 we obtain a distributive lattice $FOUR^{t,1} = TWO^t \times TWO^1$ as presented on Figure 2.

Let for every $a \in \mathbf{4}^{t,1}$ a^t means the corresponding element from TWO^t and a^1—the one from TWO^1, and let $\leq$, $\sqcap$ and $\sqcup$ be the lattice order and the operations of meet and join in $FOUR^{t,1}$. We have then the following proposition:

Proposition 3.3.
For any $a = \langle a^t, a^1\rangle$, $b = \langle b^t, b^1\rangle$:

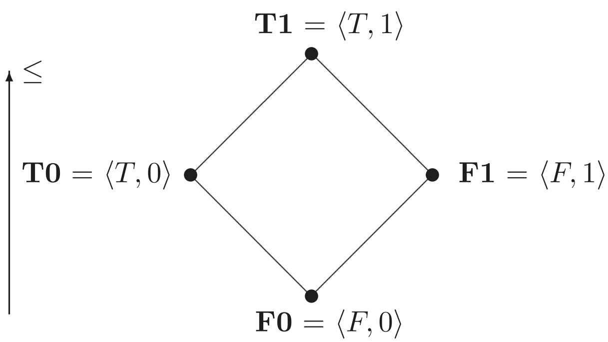

Figure 2. Lattice $FOUR^{t,1}$

(1) $a \leq b \Leftrightarrow a^t \leq_t b^t$ *and* $a^1 \leq_1 b^1$;
(2) $a \sqcap b = \langle a^t \sqcap_t b^t, a^1 \sqcap_1 b^1 \rangle$;
(3) $a \sqcup b = \langle a^1 \sqcup_t b^1, a^1 \sqcup_1 b^1 \rangle$.

Define a valuation v^4 as a map from the set of propositional variables into $\mathbf{4}^{t,1}$. Having in mind a possibility of extending this valuation to an arbitrary compound formula, we can generally introduce an entailment relation between formulas A and B of our language through the given lattice order:

Definition 3.4.
$A \models B \Leftrightarrow \forall v^4(v^4(A) \leq v^4(B))$.

Again, as in TWO^t and TWO^1 the lattice order represents here a logical order[5]. Concerning logical connectives, it is straightforward to define conjunction and disjunction through the operations of meet and join in $FOUR^{t,1}$:

Definition 3.5.
(1) $v^4(A \wedge B) = v^4(A) \sqcap v^4(B)$;
(2) $v^4(A \vee B) = v^4(A) \sqcup v^4(B)$.

It is interesting to observe that behavior of the elements from $FOUR^{t,1}$ in accordance to this definition seems much more natural than their lattice-theoretical "prototypes" from Belnap's logic. The later has sometimes been

[5]Another way of defining entailment relation consists in selecting among truth values some designated (and maybe also anti-designated) ones and to introduce entailment as a relation which preserves designated values from premises to conclusion and/or preserves anti-designated values from conclusion to premises.

criticized for the counter-intuitive outcomes of certain combinations of "intermediate values", when a disjunction of *Both* and *None* should be true, and a conjunction of them should be false. In contrast to this, the underlying onto-epistemic interpretation makes the analogous combinations of Cartesian truth values completely plausible[6].

To see this, it is instructive to look more closely at the cases when con(dis)juncts take the values **T0** and **F1**. Indeed, if we have two sentences, one of which is valuated as **T0** (being ontologically true but epistemically false), and another as **F1** (being ontologically false but epistemically true), we have to assess their conjunction as overall false, both ontologically (because of the second conjunct) and epistemically (because of the first conjunct). That is, it *must* take the truth value **F0**—as it really is according to Definition 3.5! And analogously one can argue for **T1** as the natural truth value of their disjunction.

For the sake of illustration consider a simple example. Imagine Fred, who is looking for his cat and thinks it is somewhere in the kitchen, whereas it is actually in the garden. Then, by employing values from $\mathbf{4}^{t,1}$ sentence "The cat is in the garden" will get value **T0** and sentence "The cat is in the kitchen" value **F1**. Next, if we consider disjunction of the two sentences, "The cat is in the garden *or* the cat is in the kitchen", it should be both ontologically and epistemically true (since the first disjunct is ontologically and the second epistemically true), and thus, its value can be nothing but **T1**. As to conjunction "The cat is in the garden *and* it is in the kitchen", we have no other option but to evaluate it by **F0**, and here intuitive considerations are also completely on a par with formal requirements of Definition 3.5.

Now, to get a full-fledged logical structure, we have to introduce some kind of negation operator. A common semantic way to do this is to interpret negation through a suitable operation of involution, i.e., an operation of period 2. It can generally (and naturally) be expected from such an operation to turn truth into falsity and *vice versa*.

In $FOUR^{t,1}$ one can immediately propose two operations of this kind. The first one interchanges simultaneously between truth and falsity of the same nature, and the second one connects ontological and epistemic dimensions of the truth values interchanging between ontological truth and epistemic falsity, as well as between ontological falsity and epistemic truth. The corresponding operations which we label by '$\sim_b$' and '$\sim_d$' can be defined as presented in Table 1.

[6]We are grateful to an anonymous referee for bringing forward this observation, as well as for the example with Fred and his cat below.

a	$\sim_b a$	$\sim_d a$
T1	**F0**	**F0**
T0	**F1**	**T0**
F1	**T0**	**F1**
F0	**T1**	**T1**

Table 1. Complementations in $FOUR^{t,1}$

The following facts about $\sim_b$ and $\sim_d$ can be easily verified (for any $a, b \in FOUR^{t,1}$):

PROPOSITION 3.6.
(1) $a \sqcup \sim_b a = \mathbf{T1}$;
(2) $a \sqcap \sim_b a = \mathbf{F0}$.

PROPOSITION 3.7.
(1) $\sim_d \sim_d a = a$;
(2) $a \leq \sim_d b \Rightarrow b \leq \sim_d a$.

Proposition 3.6 means that $\sim_b$ is just Boolean (classical) complementation, whereas Proposition 3.7 says that $\sim_d$ is exactly De Morgan complementation. Hence, $FOUR^{t,1}$ with $\sim_b$ constitutes Boolean algebra, and with $\sim_d$ it forms De Morgan algebra. The following definition determines valuations for classical negation ($\neg_b$) and De Morgan negation ($\neg_d$) correspondingly:

DEFINITION 3.8.
(1) $v^4(\neg_b A) = \sim_b v^4(A)$;
(2) $v^4(\neg_d A) = \sim_d v^4(A)$.

As a result, if v^4 is extended to negative formulas by Definition 3.8 (1), then Definition 3.4 determines classical consequence, and if we take Definition 3.8 (2), then the relation introduced by Definition 3.4 is axiomatized by a consequence system of *first degree entailment* (**FDE**) formulated by Anderson and Belnap in [1, p. 158][7].

Let us visualize the difference between $\neg_b$ and $\neg_d$ more clearly. By way of example, consider again the situation with Fred and his cat as described

[7]Anderson and Belnap conceived this system as the first degree fragment of their calculus **E** (of entailment) and marked it by $\mathbf{E}_{fde}$. Dunn in [12] calls the same system $\mathbf{R}_{fde}$ stressing thus the fact that it is also a fragment of system **R** (of relevant implication). We here use for the first degree consequence systems a generic label **FDE**—taken as such it denotes the original system by Anderson and Belnap, and suitable subscripts can be added for denoting other systems of this kind.

above. Taking into account the truth value of the sentence "The cat is in the kitchen" (**F1**), what truth value should be ascribed to its negation—"The cat *is not* in the kitchen"? Since the cat in fact is not in the kitchen, the latter sentence must be ontologically true. At the same time it must be epistemically false, because it says something opposite to what Fred thinks. In other words, the value of the negation is **T0**. Note, that each component (ontological and epistemic) of a truth value was ascribed here independently of each other by employing a classical (correspondence) approach, i.e. by looking at what is really the case in the corresponding sphere (the world and Fred's mind). It is therefore no wonder that according to Definition 3.8 (1) the negation so evaluated turns out to be just Boolean (classical) negation.

But what if to take the ontological and epistemic dimensions of a truth value as somehow interconnected, and not in isolation of each other? For example, think of an ontology as it is conceived in computer and information science, i.e., as of certain "taxonomy of entities" constituting a framework for knowledge representation used by information systems [28, p. 158]. In this sense ontology can be defined as "an explicit specification of a conceptualization", the latter being represented by "the objects, concepts, and other entities that are assumed to exist in some area of interest and the relationships that hold among them" [19, p. 907]. Moreover, an information system can operate with a set of data (a database) divergent in some respects from its underlying ontology. It could be desirable to provide some mechanism of interaction between ontology and database to enable their mutual adjustments. Such adjustments can go both ways: not only databases can be "calibrated" in view of the given ontology (cf. [28, p. 159]), but also ontology can be "acquired" (learned, extracted, generated, etc.) from this or that database (see, e.g. [7]). It seems then quite natural for such systems to evaluate sentences by means of Cartesian truth values which allow certain logical interplay between their ontological and epistemic components.

Suitably interpreted negation operator can play its role in securing interconnection of this kind. Namely, let us stipulate that not-A should be included in our database, and thus, taken as epistemically true if and only if the given ontology induces falsity of A, and in turn, not-A should be rendered as ontologically true if and only if A is not present in our database, i.e., there is some *external evidence* that A is not accepted.

For example, let an ontology of a computer (a computer program) includes a cat, a garden and a kitchen, and also relation "to be in" holding between the cat and the garden only. Assume further that Fred creates on this computer some database to which he for some reason (maybe by ignorance) includes the sentence "The cat is in the kitchen" (referring to the same cat and kitchen as specified in the underlying ontology). Clearly, our computer, taking onto account both ontology and the database, evaluates this sentence by **F1**. But what about the sentence “The cat is *not* in the kitchen”? According to the above stipulations, the computer should modify Fred’s database with the latter sentence, but also the given ontology should be adjusted to ensure the ontological falsity of the sentence. Thus, the negative sentence obtains the truth value **F1** as well.

In this way the operator determined by Definition 3.8 (2) behaves exactly like De Morgan negation of relevant logic with **F1** and **T0** as the fixed points. Interestingly, it does so by specifying that the negation of a sentence is accepted if and only if there is some (objective) reason for the falsity of the sentence, and a sentence is rejected if and only if there is some (objective) reason for taking its negation as true. It is noteworthy that under such interpretation it may well happen that both A and $\neg_d A$ belong to some database; and it is also possible that neither A nor $\neg_d A$ are present in a database. But this is exactly as it should be in a semantics of relevant logic, which may allow simultaneous truth (falsity) of some formulas together with their negations.

In this way one obtains an interesting non-standard explanation of what the difference between classical and relevant logics might consist in. Namely, whereas in classical logic ontological and epistemic components of Cartesian truth values work in a complete isolation of each other (although in a perfectly “parallel way”), in relevant logic—under the proposed interpretation of truth values—these components become closely interconnected (and indeed mutually relevant).

4. Semi-classical Negations

The operation of Boolean complementation is subject both to the conditions (1) and (2) from Proposition 3.6. We now consider another kind of an involutive operation that can naturally be labeled as “semi-Boolean” (or “semi-classical”) complementation. Namely, an element of a lattice is said to be *semi-classically complemented* exactly in the case when it can obey only one of the above mentioned conditions at once.

Definition 4.1.
Let L be a lattice with $\top$ as the top and $\bot$ as the bottom, and let $'$ be an unary operation defined on L. Then this operation is called a ***semi-Boolean complementation*** iff for any $a \in L$ the following condition holds:

$$a \sqcup a' = \top \Leftrightarrow a \sqcap a' \neq \bot.$$

a	a'^t	a'^1
T1	**F1**	**T0**
T0	**F0**	**T1**
F1	**T1**	**F0**
F0	**T0**	**F1**

Table 2. Semi-Boolean complementations in $FOUR^{t,1}$

The operation of semi-Boolean complementation so defined can be used for determining a specific negation-like connective which can be called a *semi-classical negation*.

Definition 4.2.
Let $\mathcal{L}$ be a propositional language with a unary connective $\neg$, and L be a lattice of truth values with an operation of semi-Boolean complementation $'$ defined on it. Let v be a map from the set of atomic sentences of $\mathcal{L}$ into L. Then $\neg$ is called a ***semi-classical negation*** (determined by $'$ in L) iff $v(\neg A) = v(A)'$.

It turns out that splitting between ontological and epistemic components of truth values allows one to define in $FOUR^{t,1}$ two natural semi-Boolean complementations. Indeed, whereas Boolean and De Morgan complementations deal with both aspects of the elements from $FOUR^{t,1}$, it can sometimes be useful to operate only with one of these components (ontological or epistemic) leaving the other untouched.

We arrive thus at the idea of two operations, one of which fixes the inferential constituent of a truth value and inverts referential truth into referential falsity and back, and the other interchanges only between inferential truth and falsity completely ignoring the referential component. We mark these operations by a'^t and a'^1 correspondingly (for an arbitrary $a \in FOUR^{t,1}$), and define them as presented in Table 2. A direct check shows that these operations are indeed semi-Boolean complementations.

The following proposition is easy to establish (for any $a \in FOUR^{t,1}$):

Proposition 4.3.
(1) $a'^t{}'^t = a'^1{}'^1 = a$;
(2) $a'^t{}'^1 = a'^1{}'^t = \sim_b a$.

Thus, both semi-Boolean complementations defined on $FOUR^{t,1}$ are involutions and their superposition gives classical complementation. In full conformity with Definition 4.2 we introduce then two unary connectives determined by semi-Boolean complementations in $FOUR^{t,1}$:

DEFINITION 4.4.
(1) $v^4(\neg_t A) = v^4(A)'^t$;
(2) $v^4(\neg_1 A) = v^4(A)'^1$.

In this way we have got two different unary operators (semi-classical negations) which can be called *referential negation* and *inferential negation* correspondingly. These negations may be of use to construct logics for argumentative reasoning. For example, inferential negation can be employed if we wish to deny someone's statements while "objective reality" remains unchanged. On the other side referential negation may well come in handy when we are going to stick out for our beliefs, no matter how radically the reality is changing.

Let us be a little bit more explicit at this point. Since a semi-classical negation is in fact performed only on one half of a Cartesian truth value, it turns out to be of a complex character, managing to embody some compound content by a sole unary operator. Namely, for a sentence A its inferential negation $\neg_1 A$ can be informally explicated as "although it is the case that A, an agent denies A", whereas referential negation $\neg_t A$ stands for a composite construction "although it is not the case that A, an agent accepts A". Thus, employing semi-classical negations allows one to "enucleate" ontological or epistemic component of a sentence and make it manifest. For example, if A represents the sentence "The cat is in the kitchen" with the value **F1** (in virtue of the intended interpretation that the cat is in fact in the garden, but Fred thinks, it is in the kitchen), then $\neg_1 A$ would mean "Although the cat is in the kitchen, Fred denies this", and obtains thus the value **F0**.

Some combinations of referential and inferential negations might be of special interest. E.g., a sentence of the form $\neg_t A \wedge \neg_1 A$ should be interpreted as "it is not the case that A, and an agent accepts A, and it is the case that A, and an agent denies A". A simple rearrangement gives us: "it is not the case that A, and it is the case that A, and an agent accepts A, and an agent denies A", which clearly demonstrates that such sentence should be logically false. By contrast, a sentence of the form $\neg_t \neg_1 A$ should have a different meaning, since here we deal with a superposition of two operations. The resulting meaning would be something like "it is not the case that it is the case that A, and an agent denies that he/she accepts A". An obvious simplification gives us just "it is not the case that A, and an agent denies A", which amounts to the Boolean negation of A.

In what follows we will suggest an idea of two logical systems each dealing with only one of the above semi-classical negations. We will formulate these logics semantically, elucidate some of their essential properties, and then point out a possibility of combining the negations under consideration within a joint logical framework.

5. Logics for Semi-classical Negations

Let us define three (nonimplicational) propositional languages in Backus-Naur form as follows:

$\mathcal{L}_t \quad A ::= p \mid A \wedge A \mid A \vee A \mid \neg_t A;$
$\mathcal{L}_1 \quad A ::= p \mid A \wedge A \mid A \vee A \mid \neg_1 A;$
$\mathcal{L}_{t,1} \quad A ::= p \mid A \wedge A \mid A \vee A \mid \neg_t A \mid \neg_1 A.$

Thus, $\mathcal{L}_{t,1} = \mathcal{L}_t \cup \mathcal{L}_1$. We can now conceive three logics, each corresponding to one of these languages.

Namely, let valuation v^4 for $\wedge$, $\vee$ and $\neg_t$ be determined by Definitions 3.5 and 4.4 (1), and let $\models_t$ be determined by Definition 3.4 with the restriction that $A, B \in \mathcal{L}_t$. We obtain then a consequence logic $\mathbf{FDE}_{ref}$ defined semantically for the formulas of language $\mathcal{L}_t$. This logic consists of all the pairs of formulas (A, B), where $A, B \in \mathcal{L}_t$, such that $A \models_t B$ constitutes a valid consequence according to Definition 3.4.

It is not difficult to see that the set of all the valid consequences $A \models_t B$, where neither A nor B contains $\neg_t$, is determined by the "positive part" of the first degree entailment system $\mathbf{E}_{fde}$ from [1, § 15], namely the postulates for *Entailment*, *Conjunction*, *Disjunction* and *Distribution* presented in [1, p. 158]. As to formulas with $\neg_t$, the following propositions are easily verifiable:

Proposition 5.1.
The following statements are valid consequences of $\boldsymbol{FDE}_{ref}$*:*

1. $A \models_t \neg_t\neg_t A$
2. $\neg_t\neg_t A \models_t A$
3. $\neg_t A \wedge \neg_t B \models_t \neg_t(A \vee B)$
4. $\neg_t(A \wedge B) \models_t \neg_t A \vee \neg_t B$
5. $\neg_t A \wedge \neg_t B \models_t \neg_t(A \wedge B)$
6. $\neg_t(A \vee B) \models_t \neg_t A \vee \neg_t B$
7. $\neg_t A \wedge B \models_t \neg_t(A \wedge B)$

8. $A \wedge \neg_t A \models_t \neg_t(A \vee B)$
9. $\neg_t(A \wedge B) \models_t A \vee \neg_t A$
10. $(A \vee \neg_t A) \wedge (B \vee \neg_t B) \models_t (A \wedge B) \vee \neg_t(A \wedge B)$
11. $(A \vee B) \wedge \neg_t(A \vee B) \models_t (A \wedge \neg_t A) \vee (B \wedge \neg_t B)$
12. $\neg_t A \models_t (A \vee B) \vee \neg_t(A \vee B)$
13. $(A \wedge B) \wedge \neg_t(A \wedge B) \models_t \neg_t A$
14. $A \wedge \neg_t A \wedge B \models_t \neg_t B \wedge B$.

Proposition 5.2.
*The following statements are **not** valid in $\boldsymbol{FDE}_{ref}$:*

1. $\neg_t(A \vee B) \models_t \neg_t A \wedge \neg_t B$
2. $\neg_t A \vee \neg_t B \models_t \neg_t(A \wedge B)$
3. $A \wedge \neg_t A \models_t B$
4. $B \models_t A \vee \neg_t A$
5. $\dfrac{A \models_t B}{\neg_t B \models_t \neg_t A}$

Thus, although semi-classical negation is an involution (an operation of period 2), it is conformed with only two of De Morgan laws and is not contrapositive.

The consequence logic $\mathbf{FDE}_{inf}$ for formulas of language $\mathcal{L}_1$ is obtained semantically by considering valuation for $\neg_1$ as subject to condition (2) from Definition 4.2, and by determining relation $\models_1$ by Definition 3.4 restricted to the formulas of $\mathcal{L}_1$. Clearly, the analogues to Propositions 5.1 and 5.2 hold with respect to $\mathbf{FDE}_{inf}$ by changing uniformly the subscript t to 1.

Remarkably, although $\neg_t$ and $\neg_1$ should have analogues (actually, the same) deductive characterizations, they are still different connectives, i.e. they are not *unique* and can play different "inferential role" in the sense of *plonk* and *plink* treated by Belnap in [4, p. 133] (cf. also [14]). Indeed, if we consider a logic resulting from the union of $\mathbf{FDE}_{ref}$ and $\mathbf{FDE}_{inf}$, with a united semantics by taking together Definitions 3.5 and 4.4 in full generality and entailment relation $\models$ determined by Definition 3.4 for the united language $\mathcal{L}_t \cup \mathcal{L}_1$, we can take note of the fact that neither $\neg_t A \models \neg_1 A$ nor $\neg_1 A \models \neg_t A$ are valid in this new logic.

The latter observation opens the way to logical systems which can comprise in a non-trivial way *both* semi-classical negations. It should be possible to introduce within these systems the classical negation as a derivative connective by means of a superposition of two different semi-classical negations, cf. Proposition 4.3 (2). In this way one can obtain logical systems that not only incorporate the whole classical logic, but contain some additional principles which result from the splitting of classical negation. In view of this such logics can be dubbed as the *super-classical* ones.

We thus label the union of $\mathbf{FDE}_{ref}$ and $\mathbf{FDE}_{inf}$ described above as the consequence logic $\mathbf{FDE}_{scl}$ (the first degree fragment of super-classical logic) with both semi-classical negations. Propositions 5.1 and 5.2, as well as their 1-analogues hold for $\mathbf{FDE}_{scl}$ (with $\models$ as the entailment relation). We have additionally the following statements which clarify the interrelations between $\neg_t$ and $\neg_1$ within $\mathbf{FDE}_{scl}$ ($=\!\mid\models$ means entailment in both directions, and $\sim$ means $\neg_t\neg_1$ or $\neg_1\neg_t$ *ad lib*):

Proposition 5.3.
The following statements are valid consequences of $\boldsymbol{FDE}_{scl}$*:*

1. $\neg_t\neg_1 A =\!\mid\models \neg_1\neg_t A$
2. $\neg_1 A =\!\mid\models \neg_t\neg_1\neg_t A$
3. $\neg_t A =\!\mid\models \neg_1\neg_t\neg_1 A$
4. $\sim(A \wedge B) =\!\mid\models \sim A \vee \sim B$
5. $\sim(A \vee B) =\!\mid\models \sim A \wedge \sim B$
6. $A =\!\mid\models \sim\sim A$
7. $\neg_t A \wedge \neg_1 B \models A \vee B$
8. $B \models \neg_t A \vee \neg_1 A$
9. $\neg_t A \wedge \neg_1 A \models B$
10. $B \models A \vee \sim A$
11. $A \wedge \sim A \models B$
12. $\neg_t A \wedge \sim B \models \neg_t(A \vee B)$
13. $\neg_1 A \wedge \sim B \models \neg_1(A \vee B)$
14. $\dfrac{\neg_t A \models \neg_t B}{\neg_1 B \models \neg_1 A}$
15. $\dfrac{\neg_1 A \models \neg_1 B}{\neg_t B \models \neg_t A}$
16. $\dfrac{A \models B}{\sim B \models \sim A}$

We leave for future work the task of providing an adequate deductive formalization of the logics $\mathbf{FDE}_{ref}$, $\mathbf{FDE}_{inf}$ and $\mathbf{FDE}_{scl}$.

6. Concluding remarks

In the present paper we have developed a method of generalizing classical truth values by taking their Cartesian product. In this way we were able to represent ontological and epistemic aspects of these values within a united semantic framework.

Essentially the same effect can be achieved by employing a generalization procedure specified in Definition 1.1 by a power setting formation on the basis of some initial set of truth values. Namely, why not to take as such a starting point a basic set consisting exclusively of values representing just two *different types of truth*? In furtherance of this idea *generalized classical truth values* can be constructed on the basis of the set **II**, which includes two basic values standing for truth from different spheres, i.e. $\mathbf{II} = \{T, 1\}$, and $\mathcal{P}(\mathbf{II}) = \{\{T, 1\}, \{T\}, \{1\}, \varnothing\}$. In full conformity with the underlying classical principles falsity (ontological or epistemic) means here simply an absence of (the corresponding) truth.

We obtain thus another interpretation of our four truth values: $\mathbf{T1} = \{T, 1\}$, $\mathbf{T0} = \{T\}$, $\mathbf{F1} = \{1\}$, $\mathbf{F0} = \varnothing$. The lattice $FOUR^{t,1}$ can be reconstructed then by ordering the elements from $\mathcal{P}(\mathbf{II})$ by the standard set-inclusion. This ordering amalgamates then an information ordering and a truth ordering into a single logical order, which can be used for definition of an entailment relation. It is also quite natural to define conjunction and disjunction through the standard set-theoretic operations which are also the operations of meet and join in $FOUR^{t,1}$ so reconstructed. Actually, the whole semantic framework developed in the present paper can be uniformly reconstructed on this basis.

However, as we believe, the very method of Cartesian truth values can open interesting prospects in investigating new logical systems arising from various combinations of logics of different types.

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