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# Computability Issues for Adaptive Logics in Multi-Consequence Standard Format

**Abstract.** In a rather general setting, we prove a number of basic theorems concerning computational complexity of derivability in adaptive logics. For that setting, the so-called standard format of adaptive logics is suitably adopted, and the corresponding completeness results are established in a very uniform way.

*Keywords*: Adaptive logics, Dynamic reasoning, Standard format, Reliability strategy, Minimal abnormality strategy, Computational complexity, Expressiveness.

## Introduction and overview

Adaptive logic is a well-developed approach to non-monotonic (and, in effect, dynamic) reasoning, which may be viewed as a unifying tool for capturing the idea of default reasoning (cf. [1]). Naturally, since the logics under consideration are non-monotonic, their consequence relations tend to be rather complicated—this, of course, raises the task of measuring their computational complexity. Surprisingly, there still exist just a few works devoted to issues of computability in adaptive logics. But an even more surprising fact is that one of the first articles in this direction [7] (restricting attention to inconsistency-adaptive logics) was aimed at criticizing the importance of this branch of logic for applications.

In their work [7], L. Horsten and P. Welsh were interested in the complexity of the consequences in the adaptive logic $CLuN^{\mathbf{r}}$ (with the propositional weak paraconsistent logic $CLuN$ being its lower limit logic, and supplied with the reliability strategy). They proved that the set of all its $CLuN^{\mathbf{r}}$-consequences is $\Sigma^0_3$ for every computable set $\Gamma$ of premisses. Moreover, they established that this estimation is exact by constructing an infinite computable set $\Gamma$ of premisses s. t. the set of all its $CLuN^{\mathbf{r}}$-consequences is $\Sigma^0_3$-hard (a simple proof that $\Sigma^0_3$ is an upper bound even in case $\Gamma$ is computably enumerable can be found in [8, Section 3]). Later, P. Verdée [13]

Special Issue: Advances in Philosophical Logic
*Edited by* **Heinrich Wansing, Roberto Ciuni and Caroline Willkommen**

*Studia Logica* (2013) 101: 1237–1262
DOI: 10.1007/s11225-013-9531-2 

showed that the adaptive logic $CLuN^{\mathbf{m}}$ (based on the same logic $CLuN$, but supplied with the minimal abnormality strategy) is significantly more complex—he presents a computable set of premisses for which the set of $CLuN^{\mathbf{m}}$-consequences is $\Pi^1_1$-hard and, thus, not even arithmetical (note that such a complexity is typical for different kinds of model-theoretic logics). The fact that $\Pi^1_1$ can be achieved at the propositional level is quite remarkable. The expressive power of $CLuN^{\mathbf{m}}$ and other propositional adaptive logics based on the minimal abnormality strategy deserves special attention in the subsequent research.

Previously, the algorithmic properties of the inconsistency-adaptive logics $CLuN^{\mathbf{r}}$ and $CLuN^{\mathbf{m}}$ were investigated in [8]. Namely, we provided very simple (and straightforward) proofs for decidability of the finitary consequence relations of both logics, and established general connections between the complexity of a set of premisses and that of its $CLuN^{\mathbf{r}}$- and $CLuN^{\mathbf{m}}$-consequences. E. g., we proved that, whenever there are only finitely many formulas unreliable with respect to (w. r. t.) a set of premisses $\Gamma$, then one can essentially reduce the complexity bound for its $CLuN^{\mathbf{m}}$-consequences—viz. they form a set computably enumerable relative to $\Gamma$ (as an oracle).

The main goal of what follows is to obtain similar results in a much broader context. To this end, we suggest a modified version of the standard format for the adaptive logics. The latter was introduced by D. Batens to provide a uniform presentation for a very general class of adaptive logics (cf. [2]); this approach is well developed in the Ghent school of logic and philosophy of science. Any adaptive logic is characterized by a lower limit logic (**LLL**), a set of abnormalities, and an adaptive strategy. Though the adaptive consequence relations are defined in the object language of **LLL**, sometimes the expressive power of that language is not enough to describe the abnormalities. Due to this reason, the standard format presupposes having an expanded language, in which all classical connectives are included. To avoid the complications caused by the necessity of working with two languages, it is sometimes assumed (see, e.g., [4]) that the language of **LLL** itself includes all these classical connectives. At the same time, many adaptive logics can be treated solely in the object language of **LLL**, without presupposing the availability of the classical connectives. Particularly, this is true of the adaptive logics $CLuN^{\mathbf{r}}$ and $CLuN^{\mathbf{m}}$ (cf. below), which served as the basic examples in the process of developing our version of the standard format. We do not impose any restrictions on the language of **LLL** and work solely with that language—it simplifies the definition of the adaptive logics and allows to track whether the presence of the classical connectives is indeed necessary. All adaptive logics from [4] can be treated in this way.

The characteristic feature of our version of the standard format consists in using multi-conclusion consequence relations, i. e., relations between the sets of formulae. For this reason, our version will be called the *multi-consequence standard format* (of adaptive logics).

The rest of the paper is organized as follows. Section 1 contains preliminary material on computability, including a description of the arithmetical hierarchy and its connections to the first-order definability. In Section 2, we present a multi-consequence version of the standard format for adaptive logics. Here, starting with a lower limit logic **LLL** that is defined syntactically via a multi-conclusion consequence relation and satisfies the strong completeness property, we establish the completeness theorems for the adaptive logics $\mathbf{LLL^r}$ and $\mathbf{LLL^m}$, i. e., **LLL** supplied with the reliability strategy and the minimal abnormality strategy, respectively. Section 3 is devoted to providing complexity upper bounds for derivability in adaptive logics and related notions. For instance, we prove that finitary consequence relations of $\mathbf{LLL^r}$ and $\mathbf{LLL^m}$ are both decidable, whenever the finitary consequence relation of **LLL** is decidable and satisfies the so-called property of local abnormalities. Also, letting the finitary consequence relation of **LLL** be enumerable and assuming various restrictions on a set of premisses $\Gamma$, the estimations for computational complexity of $\mathbf{LLL^r}$- and $\mathbf{LLL^m}$-consequences of $\Gamma$ are given. In particular, we answer the question of Christian Straßer about the complexity of $\mathbf{LLL^m}$-consequences for $\Gamma$ with $\Phi(\Gamma)$ consisting of finite sets only (see Section 2). We conclude with a survey of results on complexity lower bounds in Section 4—there it is shown that the principal estimations obtained earlier in Section 3 are exact and, indeed, can be achieved by considering $CLuN^{\mathbf{r}}$ and $CLuN^{\mathbf{m}}$, probably the most basic and the simplest (inconsistency-)adaptive logics.

## 1. Preliminaries on computability

We assume the reader is familiar with the basics of computability theory—cf. [6, 10]. Still, it is reasonable to outline the definition of the *arithmetical hierarchy*. Say that an $n$-ary relation $R$ on the set of natural numbers $\omega$ is $\Sigma^0_0$ (or $\Pi^0_0$) iff $R$ is computable. Further, we say that an $n$-ary relation $R$ on the set of natural numbers $\omega$ is $\Sigma^0_1$ (or is *in* $\Sigma^0_1$) iff it can be obtained as a projection of a $(n+1)$-ary computable relation, i. e.,

$$R \;=\; \{(m_1, \ldots, m_n) \mid \exists x\,((m_1, \ldots, m_n, x) \in S)\}$$

for some $S \subseteq \omega^{n+1}$ with $S \in \Pi^0_0$. Then, $R \subseteq \omega^n$ is (in) $\Pi^0_1$ iff its complement $\overline{R} := \omega^n \setminus R$ is $\Sigma^0_1$. Next, $\Sigma^0_{k+1}$ consists precisely of all projections of $\Pi^0_k$-relations, and the elements of $\Pi^0_{k+1}$ are just complements of those in $\Sigma^0_{k+1}$. In fact, taking into account that the class of $\Sigma^0_k$-relations is closed under projections, one can easily prove that any relation

$$\{\overline{m} \mid (\exists x_1 \ldots \exists x_{n_1})\,(\forall y_1 \ldots \forall y_{n_2}) \ldots R\,(x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}, \ldots, \overline{m})\}$$

defined via a computable $R$ preceded by the prefix with $k$ alternations of (blocks of) quantifiers, and starting with an existential quantifier ($\exists$), is in $\Sigma^0_{k+1}$. Clearly, in a similar situation when the prefix starts with a universal quantifier ($\forall$), we arrive at $\Pi^0_{k+1}$-relations. The two collections

$$\left\{\Sigma^0_k \mid k \in \omega\right\} \quad \text{and} \quad \left\{\Pi^0_k \mid k \in \omega\right\},$$

form the so-called arithmetical hierarchy. Let us denote $\Sigma^0_k \cap \Pi^0_k$ by $\Delta^0_k$. Remark that $\Sigma^0_1$ coincides with the class of all *computably enumerable* (*c. e.*, for short) relations, while $\Delta^0_1 = \Sigma^0_0 = \Pi^0_0$ (due to Post's theorem).

If one defines $\Sigma^{0,X}_0 = \Pi^{0,X}_0$ as the class of all relations computable w. r. t. a fixed *oracle* $X \subseteq \omega$ (e. g., see [6, Chapter 10] or [10, Chapter 9]), and then the classes $\Sigma^{0,X}_n$ and $\Pi^{0,X}_n$ following the same line as above, it results in the definition of the *relativised w. r. t. X arithmetical hierarchy* consisting of the collections

$$\left\{\Sigma^{0,X}_k \mid k \in \omega\right\} \quad \text{and} \quad \left\{\Pi^{0,X}_k \mid k \in \omega\right\}.$$

A relation which belongs to one of the classes in the (relativised w. r. t. $X$) arithmetical hierarchy is called *arithmetical* (*w. r. t. X*).

Here is a well-known presentation of arithmetical relations: $R \subseteq \omega^n$ is $\Sigma^0_k$ ($\Pi^0_k$) iff there is an arithmetical $\Sigma_k(\Pi_k)$-formula $\Psi\,(x_1, \ldots, x_n)$ s. t.

$$R \;=\; \{(m_1, \ldots, m_n) \mid \mathfrak{N} \Vdash_{\mathrm{FOL}} \Psi\,(m_1, \ldots, m_n)\}\,,$$

where $\mathfrak{N} := \langle \omega; +, \times, 0, 1, \leqslant \rangle$ is the *standard model of arithmetic.* Thus, the arithmetical relations are those definable via arithmetical first-order formulas.

A relation $R \subseteq \omega^n$ is called $\Pi^1_1$ iff there is a second-order arithmetical formula $\Psi\,(x_1, \ldots, x_n, P)$ with the only predicate variable $P$ (which is free, and no set quantifiers occur in $\Psi$) s. t.

$$R \;=\; \{(m_1, \ldots, m_n) \mid \mathfrak{N} \Vdash_{\mathrm{SOL}} \forall P\, \Psi\,(m_1, \ldots, m_n, P)\}$$

(here $P$ ranges over all subsets of natural numbers).

Similarly, $R$ is $\Pi^{1,X}_1$ (i. e., is $\Pi^1_1$-definable w. r. t. $X$) iff

$$R \;=\; \{(m_1, \ldots, m_n) \mid \mathfrak{N} \Vdash_{\mathrm{SOL}} \forall P\, \Psi\,(m_1, \ldots, m_n, P, Q)\,[Q/X]\}\,,$$

where $\Psi(x_1, \ldots, x_n, P, Q)$ is a second-order arithmetical formula with only two predicate variables $P$ and $Q$, and $Q$ is interpreted by $X$ in $\mathfrak{N}$ (so $X$ plays the role of a second-order parameter).

Henceforth by *bounded quantifies* we mean all expressions of the sorts

$$\exists x \leqslant y\,, \quad \forall x \leqslant y\,, \quad \exists x < y \quad \text{and} \quad \forall x < y\,.$$

For $\alpha(x, y) \in \{x \leqslant y\,,\, x < y\}$ and an arithmetical (first- or second-order) formula $\Psi$, let $(\exists \alpha(x, y))\,\Psi$ and $(\forall \alpha(x, y))\,\Psi$ abbreviate the formulas

$$\exists x\,(\alpha(x, y) \wedge \Psi) \quad \text{and} \quad \forall x\,(\neg \alpha(x, y) \vee \Psi)\,,$$

respectively.[1] Recall that the classes in the (relativised w. r. t. $X$) arithmetical hierarchy are closed under bounded quantification, and even computable terms may be used in place of '$y$'. Namely, for every arithmetical $\Sigma_k(\Pi_k)$-formula $\Psi(x, \overline{y}, \overline{z})$, where $\overline{y} := (y_1, \ldots, y_n)$ and $\overline{z} := (z_1, \ldots, z_l)$, and every computable function $f$ of $n$ arguments, the sets

$$\{(\overline{m}, \overline{s}) \mid \mathfrak{N} \Vdash_{\mathrm{FOL}} (\exists \alpha(x, f(\overline{m})))\,\Psi(x, \overline{m}, \overline{s})\}\,,$$
$$\{(\overline{m}, \overline{s}) \mid \mathfrak{N} \Vdash_{\mathrm{FOL}} (\forall \alpha(x, f(\overline{m})))\,\Psi(x, \overline{m}, \overline{s})\}\,,$$

belong to the same class $\Sigma_k^0$ $(\Pi_k^0)$ in the arithmetical hierarchy; and the analogous result holds for the relativised arithmetical hierarchy.

## 2. Multi-consequence standard format

We begin with presenting the *multi-consequence standard format* of adaptive logics (which is a modification of the standard format from [2, 4]). Here, we suppose that the *lower limit logic* is characterized in terms of a multi-consequence relation, i. e., a binary relation between the sets of formulas. Thus, one may avoid mentioning the disjunction connective ($\vee$) when describing various adaptive logics. Still, if $\vee$ is already in the language, then, using a (one-)consequence relation $\vdash$ (viz. between the sets of formulas and the formulas), the multi-consequence can be defined as:

$$\Gamma \vdash \Delta \quad \text{iff} \quad \Gamma \vdash A_1 \vee \ldots \vee A_n \text{ for some } \{A_1, \ldots, A_n\} \subseteq \Delta\,,$$

where $\Gamma$ and $\Delta$ are some sets of formulas. In this definition, the collection of consequences $\Delta$ is required to be non-empty. Hereafter, we assume that all considered multi-consequence relations satisfy this restriction.

[1] Note that $x < y$ is, in turn, a shorthand for $(x \leqslant y \wedge x \neq y)$.

Fix a language $\mathcal{L}$, with the set of $\mathcal{L}$-formulas denoted by $For_{\mathcal{L}}$. Let **LLL** be a *lower limit logic* in $\mathcal{L}$, which is a monotonic logic (in $\mathcal{L}$) supplied with a multi-consequence relation $\vdash_{\mathbf{LLL}}$ (between the sets of $\mathcal{L}$-formulas), a suitable class of **LLL**-models $\mathcal{K}_{\mathbf{LLL}}$, and a satisfiability relation $\Vdash_{\mathbf{LLL}}$ (between the **LLL**-models and the $\mathcal{L}$-formulas). Here we require that $\vdash_{\mathbf{LLL}}$ satisfies certain standard properties, namely

- $\Gamma \vdash_{\mathbf{LLL}} \Delta \implies \Delta \neq \varnothing$ (*non-empty consequence*)
- $A \in \Gamma \implies \Gamma \vdash_{\mathbf{LLL}} \{A\}$ (*reflexivity*);
- $\Gamma \vdash_{\mathbf{LLL}} \{A\} \cup \Delta$ for all $A \in \Gamma'$, and $\Gamma' \vdash_{\mathbf{LLL}} \Delta' \implies \Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Delta'$ (*transitivity*);
- $\Gamma \vdash_{\mathbf{LLL}} \Delta$, $\Gamma \subseteq \Gamma'$, and $\Delta \subseteq \Delta' \implies \Gamma' \vdash_{\mathbf{LLL}} \Delta'$ (*monotonicity*);
- $\Gamma \vdash_{\mathbf{LLL}} \Delta \implies \Gamma' \vdash_{\mathbf{LLL}} \Delta$ for some finite $\Gamma' \subseteq \Gamma$ (*left-compactness*);
- $\Gamma \vdash_{\mathbf{LLL}} \Delta \implies \Gamma \vdash_{\mathbf{LLL}} \Delta'$ for some finite $\Delta' \subseteq \Delta$ (*right-compactness*).

Further, define the *semantical consequence relation* $\vDash_{\mathbf{LLL}}$: $\Gamma \vDash_{\mathbf{LLL}} \Delta$ holds iff for every $\mathcal{M} \in \mathcal{K}_{\mathbf{LLL}}$, if $\mathcal{M} \Vdash_{\mathbf{LLL}} A$ for all $A \in \Gamma$, then $\mathcal{M} \Vdash_{\mathbf{LLL}} B$ for some $B \in \Delta$.

Finally, assume that the above two (syntactical and semantical) consequence relations coincide, i. e., we have the (strong) completeness theorem: for any $\Gamma \cup \Delta \subseteq For_{\mathcal{L}}$ with $\Delta \neq \varnothing$,

$$\Gamma \vdash_{\mathbf{LLL}} \Delta \quad \Longleftrightarrow \quad \Gamma \vDash_{\mathbf{LLL}} \Delta\,.$$ [2]

Remark that there is nothing extraordinary in this property, since, whenever the completeness result for a single-consequence relation is established by means of the canonical model method, very often the whole construction may be easily extended to the associated multi-consequence relation. Many basic examples of completeness results of this kind for monotonic non-classical logics, both modal and non-modal ones, can be found in Part I of [5]. In what follows, we will write $\Gamma \vdash_{\mathbf{LLL}} A$ and $\Gamma \vDash_{\mathbf{LLL}} A$ instead of $\Gamma \vdash_{\mathbf{LLL}} \{A\}$ and $\Gamma \vDash_{\mathbf{LLL}} \{A\}$, correspondingly.

Let us fix a set $\Omega \subseteq For_{\mathcal{L}}$ the elements of which will be called *abnormalities*. Since it is commonly assumed that these are distinguished by their syntactical form (for instance, $\Omega$ may consist of all $\mathcal{L}$-formulas of the sort $A \wedge \neg A$), $\Omega$ is supposed to be decidable.

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[2] In effect, $\vdash_{\mathbf{LLL}} = \vDash_{\mathbf{LLL}}$ will be a subrelation of the associated adaptive consequence relations we are aiming to define.

For $\Delta \cup \Gamma \subseteq For_{\mathcal{L}}$, let $\Delta \subseteq_{fin} \Gamma$ be a shorthand for '$\Delta$ is a finite subset of $\Gamma$'. A non-empty $\Delta \subseteq_{fin} \Omega$ is a *minimal Ab-consequence of* $\Gamma$ iff $\Gamma \vdash_{\mathbf{LLL}} \Delta$ and there is no $\Delta' \subset \Delta$ s. t. $\Gamma \vdash_{\mathbf{LLL}} \Delta'$.[3] We employ the following notation:

$$\Sigma(\Gamma) := \{\Delta \mid \Delta \text{ is a minimal } Ab\text{-consequence of } \Gamma\},$$
$$U(\Gamma) := \{A \in For_{\mathcal{L}} \mid A \in \Delta \text{ for some } \Delta \in \Sigma(\Gamma)\}$$

(the elements of the latter are said to be *unreliable with respect to* $\Gamma$).

Take $Ab(\mathcal{M})$ to be $\{A \in \Omega \mid \mathcal{M} \Vdash_{\mathbf{LLL}} A\}$, for each **LLL**-model $\mathcal{M}$. An **LLL**-model $\mathcal{M}$ of $\Gamma$ (viz. $\mathcal{M} \Vdash_{\mathbf{LLL}} A$ for all $A \in \Gamma$ holds) is *reliable* iff $Ab(\mathcal{M}) \subseteq U(\Gamma)$, and is *minimally abnormal* iff there is no (other) **LLL**-model $\mathcal{M}'$ of this $\Gamma$ with $Ab(\mathcal{M}') \subset Ab(\mathcal{M})$. Now we are ready to define semantically the two associated adaptive multi-consequence relations: for $\Gamma \cup \Delta \subseteq For_{\mathcal{L}}$ with $\Delta \neq \varnothing$, $\Gamma \vDash_{\mathbf{LLL^r}} \Delta$ ($\Gamma \vDash_{\mathbf{LLL^m}} \Delta$) iff for every reliable (minimally abnormal, respectively) model $\mathcal{M}$ of $\Gamma$, there exists $A \in \Delta$ s. t. $\mathcal{M} \Vdash_{\mathbf{LLL}} A$.

In this way, $\vDash_{\mathbf{LLL^r}}$ provides the semantics for the adaptive logic $\mathbf{LLL^r}$ based on the lower limit logic **LLL**, the set of abnormalities $\Omega$, and augmented by the *reliability strategy*. Similarly, $\vDash_{\mathbf{LLL^m}}$ corresponds to the adaptive logic $\mathbf{LLL^m}$ which is based on the same lower limit logic and abnormalities, but exploits a different strategy of handling the abnormalities involved, namely the *minimal abnormality strategy*.

The next criterion for the semantical $\mathbf{LLL^r}$-consequence is a version of [4, Theorem 7] adopted to our modified form of the standard format. Note that the proof from [4] essentially exploits the presence of classical negation, whereas our proof does not presuppose it. Actually, Theorem 7 of [4] can also be proved without referring to the properties of classical negation.

Theorem 2.1. *For any* $\Gamma \cup \Delta \subseteq For_{\mathcal{L}}$ *with* $\Delta \neq \varnothing$, $\Gamma \vDash_{\mathbf{LLL^r}} \Delta$ *iff there exists* $\Theta \subseteq_{fin} \Omega$ *s. t.* $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ *and* $\Theta \cap U(\Gamma) = \varnothing$.

Proof. $\boxed{\Leftarrow}$ Assume that $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ for $\Theta \subseteq_{fin} \Omega \setminus U(\Gamma)$. The strong completeness implies $\Gamma \vDash_{\mathbf{LLL}} \Delta \cup \Theta$. Let $\mathcal{M}$ be a reliable model of $\Gamma$. Then $Ab(\mathcal{M}) \subseteq U(\Gamma)$ and $\mathcal{M} \Vdash_{\mathbf{LLL}} A$, where $A \in \Delta \cup \Theta$. Since $\Theta \cap U(\Gamma) = \varnothing$, we have $\Theta \cap Ab(\mathcal{M}) = \varnothing$. Consequently, $\mathcal{M} \Vdash_{\mathbf{LLL}} A$ for some $A \in \Delta$. We have thus proved $\Gamma \vDash_{\mathbf{LLL^r}} \Delta$.

$\boxed{\Rightarrow}$ Let $\Gamma \nvdash_{\mathbf{LLL}} \Delta \cup \Theta$ for all $\Theta \subseteq_{fin} \Omega \setminus U(\Gamma)$. The right compactness and the monotonicity of $\vdash_{\mathbf{LLL}}$ imply that $\Gamma \nvdash_{\mathbf{LLL}} \Delta \cup (\Omega \setminus U(\Gamma))$. By strong completeness we have $\Gamma \nvDash_{\mathbf{LLL}} \Delta \cup (\Omega \setminus U(\Gamma))$. Consequently, there

[3]Here and below '$S_1 \subset S_2$' always stands for '$S_1 \subseteq S_2$ and $S_1 \neq S_2$'.

is a model $\mathcal{M}$ of $\Gamma$ such that $\mathcal{M} \not\Vdash_{\mathbf{LLL}} A$ for all $A \in \Delta \cup (\Omega \setminus U(\Gamma))$. This implies, in particular, that $Ab(\mathcal{M}) \subseteq U(\Gamma)$. We have thus found out a reliable model of $\Gamma$ that refutes all $A \in \Delta$, i. e., $\Gamma \nvDash_{\mathbf{LLL}^{\mathbf{r}}} \Delta$. ■

Before moving on, to a semantical criterion for the minimal abnormality strategy, we need to say a few words about so-called 'choice sets'. Taking $\Sigma$ to be a collection[4] of sets, a set $\Delta$ is a *choice set for* $\Sigma$ iff for any $\varphi \in \Sigma$, $\Delta \cap \varphi \neq \varnothing$. Further, such a choice set $\Delta$ is *minimal* (for $\Sigma$) iff there is no other choice set $\Delta'$ for $\Sigma$ with $\Delta' \subset \Delta$.

It is well-known that for an arbitrary collection of finite sets, there exists a minimal choice set for it—see, e.g., [1, Fact 5.1.2]. Moreover, the following simple but important fact was established in [12].

Proposition 2.2. *Let $\Sigma$ be a collection of sets. A choice set $\Delta$ for $\Sigma$ is minimal iff for each $a \in \Delta$, there exists $\varphi \in \Sigma$ s. t. $\Delta \cap \varphi = \{a\}$.*

For every $\Gamma \subseteq For_{\mathcal{L}}$, we denote the collection of all minimal choice sets for $\Sigma(\Gamma)$ by $\Phi(\Gamma)$. It turns out that the elements of $\Phi(\Gamma)$ are precisely those sets that can be represented as a set of abnormalities true in some minimally abnormal model of $\Gamma$. The next statement is an analog of Lemma 4 in [4] and again its proof does not need the classical negation.

Proposition 2.3. *Let $\Gamma \subseteq For_{\mathcal{L}}$. Then*

$$\Phi(\Gamma) \ = \ \{Ab(\mathcal{M}) \mid \mathcal{M} \ \textit{is a minimally abnormal}\ \mathbf{LLL}\textit{-model of}\ \Gamma\}\,.$$

Proof. First we notice that $Ab(\mathcal{M})$ is a choice set for $\Sigma(\Gamma)$ for every **LLL**-model $\mathcal{M}$ of $\Gamma$. Indeed, if $\Delta \in \Sigma(\Gamma)$, then $\Gamma \vdash_{\mathbf{LLL}} \Delta$ and by strong completeness $\Gamma \vDash_{\mathbf{LLL}} \Delta$. Consequently, $\mathcal{M} \Vdash_{\mathbf{LLL}} A$ for some $A \in \Delta$, i. e., $Ab(\mathcal{M}) \cap \Delta \neq \varnothing$.

Let $\varphi \in \Phi(\Gamma)$. We show that there is a minimally abnormal model $\mathcal{M}$ of $\Gamma$ with $Ab(\mathcal{M}) = \varphi$.

Prove that $\Gamma \nvdash_{\mathbf{LLL}} \Omega \setminus \varphi$. If it does not hold, then by the right compactness there is a non-empty $\Delta \subseteq_{fin} \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} \Delta$ and $\Delta \cap \varphi = \varnothing$. Then there is a $\Delta' \subseteq \Delta$ such that $\Gamma \vdash_{\mathbf{LLL}} \Delta'$ and $\Delta' \in \Sigma(\Gamma)$. In this case $\Delta' \cap \varphi = \varnothing$, which conflicts with the fact that $\varphi$ is a choice set for $\Sigma(\Gamma)$. The obtained contradiction proves $\Gamma \nvdash_{\mathbf{LLL}} \Omega \setminus \varphi$.

By strong completeness there is a model $\mathcal{M}$ of $\Gamma$ such that $\mathcal{M} \not\Vdash_{\mathbf{LLL}} A$ for all $A \in \Omega \setminus \varphi$. Consequently, $Ab(\mathcal{M}) \subseteq \varphi$. Since $Ab(\mathcal{M})$ is a choice set for $\Sigma(\Gamma)$ and $\varphi$ is a minimal choice set, we conclude that $\varphi = Ab(\mathcal{M})$ and $\mathcal{M}$ is a minimally abnormal model of $\Gamma$.

[4]We use the word "collection" as a synonym of "set".

Now we take a minimally abnormal model $\mathcal{M}$ of $\Gamma$ and prove that $Ab(\mathcal{M}) \in \Phi(\Gamma)$. We know that $Ab(\mathcal{M})$ is a choice set for $\Sigma(\Gamma)$. If this choice set is not minimal, then there is $\varphi \in \Sigma(\Gamma)$ such that $\varphi \subset Ab(\mathcal{M})$. It was proved above that $\varphi = Ab(\mathcal{M}')$ for some minimally abnormal model $\mathcal{M}'$ of $\Gamma$, which contradicts to the minimal abnormality of $\mathcal{M}$. ■

COROLLARY 2.4. *Every minimally abnormal model of* $\Gamma \subseteq For_{\mathcal{L}}$ *is reliable.*

PROOF. If $\mathcal{M}$ is a minimally abnormal model of $\Gamma$, then by the previous proposition $Ab(\mathcal{M})$ is a minimal choice set for $\Sigma(\Gamma)$. In particular, $Ab(\mathcal{M}) \subseteq \bigcup \Sigma(\Gamma)$. By definition $U(\Gamma) = \bigcup \Sigma(\Gamma)$, consequently, $\mathcal{M}$ is a reliable model of $\Gamma$. ■

Finally, let us provide the semantical criterion for $\mathbf{LLL^m}$-consequence. Notice, a similar fact for the adaptive logics in the (usual) standard format was established in the proof of Theorem 9 in [4].

THEOREM 2.5. *For any* $\Gamma \cup \Delta \subseteq For_{\mathcal{L}}$ *with* $\Delta \neq \varnothing$, $\Gamma \vDash_{\mathbf{LLL^m}} \Delta$ *iff for each* $\varphi \in \Phi(\Gamma)$, *there exists* $\Theta \subseteq_{fin} \Omega$ *s. t.* $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ *and* $\Theta \cap \varphi = \varnothing$.

PROOF. $\boxed{\Rightarrow}$ Suppose there exists $\varphi \in \Phi(\Gamma)$ such that for every $\Theta \subseteq_{fin} \Omega \setminus \varphi$, we have $\Gamma \nvdash_{\mathbf{LLL}} \Delta \cup \Theta$. By the right compactness of $\vdash_{\mathbf{LLL}}$ we conclude that $\Gamma \nvdash_{\mathbf{LLL}} \Delta \cup (\Omega \setminus \varphi)$. Hence, due to the strong completeness, there is a model $\mathcal{M}$ of $\Gamma$ that refutes all elements of $\Delta \cup (\Omega \setminus \varphi)$. Particularly, $Ab(\mathcal{M}) \subseteq \varphi$. From Proposition 2.3 it follows that $Ab(\mathcal{M}) = \varphi$ and $\mathcal{M}$ is a minimally abnormal model of $\Gamma$. Since $\mathcal{M} \nVdash_{\mathbf{LLL}} A$ for all $A \in \Delta$ we proved that $\Gamma \nvDash_{\mathbf{LLL^m}} \Delta$.

$\boxed{\Leftarrow}$ Assume that for every $\varphi \in \Phi(\Gamma)$, there exists $\Theta \subseteq_{fin} \Omega$ with the property: $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ and $\Theta \cap \varphi = \varnothing$. If there is a minimally abnormal model $\mathcal{M}$ of $\Gamma$ such that $\mathcal{M} \nVdash_{\mathbf{LLL}} A$ for all $A \in \Delta$, then $Ab(\mathcal{M}) \in \Phi(\Gamma)$ and so $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ for some $\Theta \subseteq_{fin} \Omega$ with $\Theta \cap Ab(\mathcal{M}) = \varnothing$. By the strong completeness we have $\Gamma \vDash_{\mathbf{LLL}} \Delta \cup \Theta$. Since $\mathcal{M} \nVdash_{\mathbf{LLL}} A$ for all $A \in \Delta$, we obtain $\mathcal{M} \Vdash_{\mathbf{LLL}} B$ for $B \in \Theta$ which conflicts with $\Theta \cap Ab(\mathcal{M}) = \varnothing$. ■

Further, we describe proof procedures for the adaptive logics $\mathbf{LLL^r}$ and $\mathbf{LLL^m}$. A crucial notion here is that of a 'stage of a proof' (from a given set of premisses). Namely, for every $\Gamma \subseteq For_{\mathcal{L}}$, a *stage of a proof from* $\Gamma$ is represented by a sequence $s$ (finite or infinite) of lines, each of which is a tuple consisting of the *five components*:

i. its number (that is, a natural number);

ii. head (a non-empty finite set of $\mathcal{L}$-formulas);

iii. line numbers of local premises (a string of natural numbers);
iv. name of adaptive rule (PREM, RU, or RC);
v. condition (a finite subset of abnormalities—those from $\Omega$), —

and, moreover, any such line must be of one of the following *three types*:

$$\text{(i) } n\,,\ \ \text{(ii) } \{A\}\,,\ \ \text{(iii) —},\ \ \text{(iv) PREM}\,,\ \ \text{(v) } \varnothing \qquad \text{(PREM)}$$

where $n$ is its number in the sequence $s$, and $A$ belongs to $\Gamma$;

$$\text{(i) } n\,,\ \ \text{(ii) } \Delta,\ \ \text{(iii) } i_1, \dots, i_k\,,\ \ \text{(iv) RU}\,,\ \ \text{(v) } \Theta_1 \cup \dots \cup \Theta_k \qquad \text{(RU)}$$

where $n$ is its number in $s$, the heads and conditions of lines numbered by $i_1, \dots, i_k < n$ in $s$ are $\{A_1\}, \dots, \{A_k\}$ and $\Theta_1, \dots, \Theta_k$, respectively, and $\{A_1, \dots, A_k\} \vdash_{\mathbf{LLL}} \Delta$, with $\varnothing \neq \Delta \subseteq_{fin} For_{\mathcal{L}}$;

$$\text{(i) } n\,,\ \ \text{(ii) } \Delta,\ \ \text{(iii) } i_1, \dots, i_k\,,\ \ \text{(iv) RC}\,,\ \ \text{(v) } \Theta_1 \cup \dots \cup \Theta_k \cup \Theta \qquad \text{(RC)}$$

where $n$ is its number in $s$, the heads and conditions of lines numbered by $i_1, \dots, i_k < n$ in $s$ are $\{A_1\}, \dots, \{A_k\}$ and $\Theta_1, \dots, \Theta_k$, respectively, and $\{A_1, \dots, A_k\} \vdash_{\mathbf{LLL}} \Delta \cup \Theta$, with $\varnothing \neq \Delta \subseteq_{fin} For_{\mathcal{L}}$ and $\Theta \subseteq_{fin} \Omega$.[5]

In case a stage of a proof $s$ (for $\Gamma$ fixed) contains a line numbered $i$ with a head $\Delta$ and a condition $\Theta$, we say that $\Delta$ *is derived in $s$ at line $i$ under condition* $\Theta$. A stage of a proof $s'$ is called an *extension of $s$* iff the sequence of lines of $s$ forms a subsequence of that of $s'$, i.e., all lines of $s$ occur in the same order in $s'$ (whenever all the (i)-st and (iii)-rd components of lines in $s$ are appropriately renumbered).

Proposition 2.6. *Let $\Gamma \subseteq For_{\mathcal{L}}$, $\varnothing \neq \Delta \subseteq_{fin} For_{\mathcal{L}}$, and $\Theta \subseteq_{fin} \Omega$. Then $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ iff there exists a finite stage of a proof from $\Gamma$ s. t. $\Delta$ is derived in this stage at some line under condition $\Theta$.*

Proof. Suppose $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$. Since the relation $\vdash_{\mathbf{LLL}}$ is left-compact there are formulas $\{A_1, \dots, A_n\} \subseteq \Gamma$ such that $\{A_1, \dots, A_n\} \vdash_{\mathbf{LLL}} \Delta \cup \Theta$. We may start a stage of a proof with lines: $i\,,\ \{A_i\}\,,\ —,\ \text{PREM}\,,\ \varnothing\,$; where $i \in \{1, \dots, n\}$. Then we add the following line:

$$n+1\,,\ \Delta\,,\ \langle 1, \dots, n\rangle\,,\ \text{RC}\,,\ \Theta\,.$$

We have thus constructed the stage $s$ of a proof from $\Gamma$ s. t. $\Delta$ is derived at line $n+1$ of $s$ under condition $\Theta$.

Now we assume that $s$ is a stage of a proof from $\Gamma$ s. t. $\Delta$ is derived at line $i$ of this stage under condition $\Theta$. The fact that $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ can be proved by induction on $i$ using the reflexivity and the transitivity of $\vdash_{\mathbf{LLL}}$. ■

[5] Remark that $k \in \omega$, so the tuple $(i_1, \dots, i_k)$ may be empty.

Remark that the notion of a stage of a proof does not depend on the strategy of handling abnormalities. Rather, the two strategies are involved (in the adaptive proof theory) in the form of 'marking definitions'.

We start with the reliability strategy. Suppose $s$ is a stage of a proof from $\Gamma \subseteq For_{\mathcal{L}}$. A non-empty $\Delta \subseteq_{fin} \Omega$ is called a *minimal Ab-consequence at s* iff it is derived at some line in $s$ under the empty condition, and no proper subset $\Delta' \subset \Delta$ has that property (i. e., is derived at some line in $s$ under the empty condition). Take

$$\Sigma_s := \{\Delta \mid \Delta \text{ is a minimal } Ab\text{-consequence at } s\}\,;$$
$$U_s := \{A \in For_{\mathcal{L}} \mid A \in \Delta \text{ for some } \Delta \in \Sigma_s\}$$

(the $\mathcal{L}$-formulas from $U_s$ are said to be *unreliable at s*).[6] Henceforth, in a context where no confusion may arise, lines (of a given stage $s$ of a proof) are named by their numbers, at times.

DEFINITION 2.7. An $i$-th line of a finite stage $s$ of a proof from $\Gamma$ is $\mathbf{r}$-*marked* (or *marked according to the reliability strategy*) *in s* iff $\Delta \cap U_s \neq \varnothing$, where $\Delta$ is the condition for the $i$-th line. Whenever $i$ is not $\mathbf{r}$-marked in $s$, the term $\mathbf{r}$-*unmarked* will also be used, for convenience.

DEFINITION 2.8. A non-empty $\Delta \subseteq_{fin} For_{\mathcal{L}}$ is *finally* $\mathbf{LLL^r}$-*derived in a finite stage s of a proof from* $\Gamma$ iff $\Delta$ is derived at some line $i$ of $s$, and the following requirements are satisfied:

- the $i$-th line (or simply 'the line $i$') is not $\mathbf{r}$-marked in $s$;
- any finite extension of $s$, in which $i$ becomes $\mathbf{r}$-marked, may be further finitely extended so that this line will turn out to be $\mathbf{r}$-unmarked again.

DEFINITION 2.9. A non-empty $\Delta \subseteq_{fin} For_{\mathcal{L}}$ is *finally* $\mathbf{LLL^r}$-*derivable from* $\Gamma$ (we denote this by $\Gamma \vdash_{\mathbf{LLL^r}} \Delta$) iff it can be finally $\mathbf{LLL^r}$-derived in some finite stage $s$ of a proof from $\Gamma$.

Now a syntactical variant of Theorem 2.1 (i. e., a criterion for the final $\mathbf{LLL^r}$-derivability) can be established.

THEOREM 2.10. *For any* $\Gamma \subseteq For_{\mathcal{L}}$ *and non-empty* $\Delta \subseteq_{fin} For_{\mathcal{L}}$, *we have* $\Gamma \vdash_{\mathbf{LLL^r}} \Delta$ *iff there exists* $\Theta \subseteq_{fin} \Omega$ *s. t.* $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ *and* $\Theta \cap U(\Gamma) = \varnothing$.

---

[6]Here, we write $U_s$ instead of $U_s(\Gamma)$, which is more widely used, to emphasize that this set is determined solely by the stage $s$, while the full set of premisses $\Gamma$ is not necessarily required. Similarly, we employ the notation $\Phi_s$ instead of $\Phi_s(\Gamma)$ below.

PROOF. $\boxed{\Leftarrow}$ Assume that $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ and $\Theta \cap U(\Gamma) = \varnothing$. Naturally, we may assume $\Delta \cap \Theta = \varnothing$. Otherwise, we take $\Theta \setminus \Delta$ instead of $\Theta$. By compactness we have $\{A_1, \ldots, A_n\} \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ for $\{A_1, \ldots, A_n\} \subseteq \Gamma$. As in the proof of Proposition 2.6 we construct a stage of a proof consisting of $n+1$ lines such that $\Delta$ is derived under the condition $\Theta$ at line $n+1$ of this stage. If this line is **r**-marked at this stage, this means that some formula $A_i$ is an abnormality and belongs to $\Theta$. That is impossible, since in this case $A_i \in U(\Gamma)$, where as $\Theta \cap U(\Gamma) = \varnothing$. We have thus proved that line $n+1$ is not **r**-marked.

Assume that some extension $t$ of $s$ is such that the line, where $\Delta$ is derived under the condition $\Theta$, becomes **r**-marked. Let $\Theta_1, \ldots, \Theta_k$ be all elements of $\Sigma_t$ such that $\Theta \cap \Theta_i \neq \varnothing$. Since $\Theta \cap U(\Gamma) = \varnothing$, for each $\Theta_i$ there is $\Theta'_i \in \Sigma(\Gamma)$ such that $\Theta'_i \subset \Theta_i$. Acting as in the proof of Proposition 2.6 we extend $t$ to the stage $u$ such that all $\Theta'_i$ are derived under the empty conditions at some lines of $u$. It is obvious that $(\Sigma_t \setminus \{\Theta_1, \ldots, \Theta_k\}) \cup \{\Theta'_1, \ldots, \Theta'_k\} \subseteq \Sigma_u$. Additionally, $\Sigma_u$ may contain singletons $\{A\}$, in which case $A \in U(\Gamma)$ and $A \notin \Theta$. In this way, $\Delta$ is derived in $u$ at a line, which is not **r**-marked. We have thus proved that $\Gamma \vdash_{\mathbf{LLL}^{\mathbf{r}}} \Delta$.

$\boxed{\Rightarrow}$ Now we assume that $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ implies $\Theta \cap U(\Gamma) \neq \varnothing$. Let $s$ be a stage of a proof from $\Gamma$ such that $\Delta$ is derived at some line of this stage under a condition $\Theta$. By Proposition 2.6 we have $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ and so $\Theta \cap U(\Gamma) \neq \varnothing$. Let $\Theta_1 \in \Sigma(\Gamma)$ be such that $\Theta \cap \Theta_1 \neq \varnothing$. If we extend $s$ to a stage $t$ such that $\Theta_1$ is derived at some line of $t$ under the empty condition, then $\Theta_1 \subseteq U_t$ and $\Theta_1 \subseteq U_v$ for any extension $v$ of $t$. Thus, the line, where $\Delta$ is derived under the condition $\Theta$, received an **r**-mark in $t$, which can not be removed in any further finite extension of $t$. We have proved that $\Gamma \nvdash_{\mathbf{LLL}^{\mathbf{r}}} \Delta$. ■

Thus, by combining Theorems 2.1 and 2.10, we immediately obtain the (strong) completeness for the adaptive logic $\mathbf{LLL}^{\mathbf{r}}$.

COROLLARY 2.11. *For any* $\Gamma \subseteq For_{\mathcal{L}}$ *and non-empty* $\Delta \subseteq_{fin} For_{\mathcal{L}}$,

$$\Gamma \vdash_{\mathbf{LLL}^{\mathbf{r}}} \Delta \iff \Gamma \vDash_{\mathbf{LLL}^{\mathbf{r}}} \Delta\,.$$

Next, let us consider the minimal abnormality strategy, where infinite stages of proofs play an important role. Suppose $s$ is a stage of a proof from $\Gamma \subseteq For_{\mathcal{L}}$. Let $\Phi_s$ denote the collection of all minimal choice sets for $\Sigma_s$ (introduced above).

DEFINITION 2.12. An $i$-th line of a stage $s$ of a proof from $\Gamma$ is **m**-*marked* (or *marked according to the minimal abnormality strategy*) *in* $s$ iff for the head

$\Delta$ and the condition $\Theta$ of the $i$-th line, one of the following requirements is satisfied:

- there is no $\varphi \in \Phi_s$ with $\varphi \cap \Theta = \varnothing$;
- for some $\varphi \in \Phi_s$, there is no line in $s$ at which $\Delta$ is derived under a condition $\Theta'$ with $\varphi \cap \Theta' = \varnothing$.

At times, the phrase 'not **m**-marked' will be replaced by '**m**-*unmarked*' (cf. also Definition 2.7), for convenience.

DEFINITION 2.13. A non-empty $\Delta \subseteq_{fin} For_{\mathcal{L}}$ is *finally* $\mathbf{LLL^m}$*-derived in a stage* $s$ *of a proof from* $\Gamma$ iff $\Delta$ is derived at some line $i$ of $s$, and the following requirements are satisfied:

- the $i$-th line is not **m**-marked in $s$;
- any extension of $s$, in which $i$ becomes **m**-marked, may be further extended so that this line will turn out to be **m**-unmarked again.

DEFINITION 2.14. A non-empty $\Delta \subseteq_{fin} For_{\mathcal{L}}$ is *finally* $\mathbf{LLL^m}$*-derivable from* $\Gamma$ (we denote this by $\Gamma \vdash_{\mathbf{LLL^m}} \Delta$) iff it can be finally $\mathbf{LLL^m}$-derived in some finite stage $s$ of a proof from $\Gamma$.

Traditionally, the final $\mathbf{LLL^m}$-derivability from $\Gamma$ is defined as a final $\mathbf{LLL^m}$-derivability in some finite stage $s$ of a proof, although to check whether the final derivability holds we have to consider all infinite extensions of $s$. It is not hard to modify the last definition so that it involves only one (infinite in the general case) stage of a proof.

PROPOSITION 2.15. *A non-empty* $\Delta \subseteq_{fin} For_{\mathcal{L}}$ *is finally* $\mathbf{LLL^m}$*-derivable from* $\Gamma$ *iff there exists a stage* $s$ *of a proof from* $\Gamma$ *possessing the properties*[7]*:*

- $\Sigma_s$ *coincides with* $\Sigma(\Gamma)$*;*
- *for every* $\varphi \in \Phi(\Gamma)$*, there is some line* $i$ *in* $s$ *s. t.* $\Delta$ *is derived at this line under a condition* $\Theta_i$ *with* $\varphi \cap \Theta_i = \varnothing$*.*

PROOF. $\boxed{\Leftarrow}$ Assume that $s$ is a stage of a proof from $\Gamma$ such that $\Sigma_s = \Sigma(\Gamma)$ and for every $\varphi \in \Phi(\Gamma)$, there is a line $i$ of $s$ such that $\Delta$ is derived at this

[7] It is easy to see that every line of $s$, where $\Delta$ is derived at a condition $\Theta$ with $\Theta \cap \varphi = \varnothing$ for some $\varphi \in \Phi(\Gamma)$, is not **m**-marked. Moreover, such lines remain **m**-unmarked in every extension of $s$. Such stages of a proof were called stable proofs in [1, Sec. 4.4.]. More exactly, a stage $s$ of a proof is a *stable proof* for $\Delta$, if $\Delta$ is proved at an **m**-unmarked line of $s$ such that this line remains **m**-unmarked in every extension of $s$.

line under a condition $\Theta_i$ with $\varphi \cap \Theta_i = \varnothing$. Choose some $\varphi_0 \in \Phi(\Gamma)$ and the respective line $i_0$ such that $\Delta$ is derived at this line under a condition $\Theta_0$. By Proposition 2.6 we have $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta_0$, and exactly as in the proof of this proposition we construct a stage $s_0$ of a proof from $\Gamma$ consisting of $n+1$ lines such that $\Delta$ is derived under the condition $\Theta_0$ at line $n+1$, and a singleton $\{A_i\}$ with $A_i \in \Gamma$ is derived at line $i \leq n$ under the empty condition. Thus, $\Sigma_{s_0}$ may include only singletons $\{A_i\}$ such that $A_i \in \Omega$. This means that $\Phi_{s_0}$ contains only one element $\psi = \{A_i \mid \{A_i\} \in \Sigma_{s_0}\}$. Moreover, $\psi$ is such that $\psi \subseteq \varphi$ for all $\varphi \in \Phi(\Gamma)$. In particular, $\psi \subseteq \varphi_0$, whence $\Theta_0 \cap \psi = \varnothing$. We have thus proved that the line $n+1$ of $s_0$ is not **m**-marked.

Let $t$ be an extension of $s_0$ such that the line of $s_0$ with the head $\Delta$ is **m**-marked in $t$. We consider then an extension $u$ of $t$ which extends $s$ as well. It is clear that $\Sigma_u = \Sigma_s = \Sigma(\Gamma)$ and that the line of $s_0$ with the head $\Delta$ becomes **m**-unmarked in $u$. Thus, the stage $s_0$ proves that $\Delta$ is finally $\mathbf{LLL^m}$-derivable from $\Gamma$.

$\boxed{\Rightarrow}$ Let $\Gamma \vdash_{\mathbf{LLL^m}} \Delta$ and a finite stage $s$ of a proof from $\Gamma$ confirm this fact. We can extend $s$ to a stage $t$ of a proof such that $\Sigma_t = \Sigma(\Gamma)$. If $\Delta$ is not derived in $t$ at an **m**-unmarked line, then there is an extension $u$ of $t$, where $\Delta$ is derived at an **m**-unmarked line. Since $\Sigma_t = \Sigma(\Gamma)$, we also have $\Sigma_u = \Sigma(\Gamma)$ and $\Phi_u = \Phi(\Gamma)$. Since $\Delta$ is derived at an **m**-unmarked line of $u$, for every $\varphi \in \Phi(\Gamma)$, there is a line $i$ of $u$ such that $\Delta$ is derived at this line under a condition $\Theta_i$ with $\varphi \cap \Theta_i = \varnothing$. ■

Using this observation, a syntactical variant of Theorem 2.5 (i. e., a criterion for the final $\mathbf{LLL^m}$-derivability) is obtained.

THEOREM 2.16. *For any $\Gamma \subseteq For_{\mathcal{L}}$ and non-empty $\Delta \subseteq_{fin} For_{\mathcal{L}}$, we have $\Gamma \vdash_{\mathbf{LLL^m}} \Delta$ iff for each $\varphi \in \Phi(\Gamma)$, there exists $\Theta \subseteq_{fin} \Omega$ s. t. $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ and $\Theta \cap \varphi = \varnothing$.*

PROOF. The implication from left to right immediately follows from Propositions 2.6 and 2.15.

Assume that for each $\varphi \in \Phi(\Gamma)$, there exists $\Theta \subseteq_{fin} \Omega$ such that $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ and $\Theta \cap \varphi = \varnothing$. We can construct a stage $s$ of a proof from $\Gamma$ such that for every $\Theta \in \Sigma(\Gamma)$, $\Theta$ is derived at some line of $s$ under the empty condition, and for every $\varphi \in \Phi(\Gamma)$, $\Delta$ is derived at some line of $s$ under the condition $\Theta$ with $\Theta \cap \varphi = \varnothing$. It is clear that $s$ satisfies the right-hand side condition of Proposition 2.15. ■

In this way, it is straightforward that Theorems 2.5 and 2.16 together imply the (strong) completeness theorem for the adaptive logic $\mathbf{LLL^m}$, which is a version of Theorem 8 from [4] adopted to multi-consequence standard format.

COROLLARY 2.17. *For any* $\Gamma \subseteq For_{\mathcal{L}}$ *and non-empty* $\Delta \subseteq_{fin} For_{\mathcal{L}}$,

$$\Gamma \vdash_{\mathbf{LLL^m}} \Delta \quad \Longleftrightarrow \quad \Gamma \vDash_{\mathbf{LLL^m}} \Delta\,.$$

Concluding this section we recall that both relations $\Gamma \vdash_{\mathbf{LLL^x}} \Delta$ and $\Gamma \vDash_{\mathbf{LLL^x}} \Delta$, where $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$, were defined for arbitrary set $\Gamma$ of premises and for a finite set $\Delta$ of conclusions. We can extend in a natural way the semantic adaptive consequences to infinite sets of conclusions by putting, e.g., $\Gamma \vDash_{\mathbf{LLL^m}} \Delta$ iff at least one formula from $\Delta$ holds in every minimally abnormal model of $\Gamma$. But it is not clear how to do it for the syntactic relations $\vdash_{\mathbf{LLL^r}}$ and $\vdash_{\mathbf{LLL^m}}$. At least the most obvious extensions of these relations to infinite sets of conclusions lead to the failure of the strong completeness. Let us define the relation $\Gamma \vdash_{\mathbf{LLL^m}} \Delta$ for an infinite $\Delta$ using the right compactness: $\Gamma \vdash_{\mathbf{LLL^m}} \Delta$ holds iff $\Gamma \vdash_{\mathbf{LLL^m}} \Delta'$ for some $\Delta' \subseteq_{fin} \Delta$. But the semantic adaptive consequence $\vDash_{\mathbf{LLL^m}}$ is not right compact in general case. We illustrate it with the example suggested by the anonymous referee.

Let $CLuN_{\bot}$ denote $CLuN$ with the classical falsity constant $\bot$ (see Section 4). Take $\Gamma = \{(p_i \wedge \neg p_i) \vee (p_j \wedge \neg p_j) \mid i, j \in \omega,\ i \neq j\} \cup \{p_i \mid i \in \omega\}$. We have then $\Gamma \vDash_{CLuN_{\bot}^{\mathbf{m}}} \{\neg p_i \to \bot \mid i \in \omega\}$, but $\Gamma \nvDash_{CLuN_{\bot}^{\mathbf{m}}} \{\neg p_i \to \bot \mid i \in J\}$ for $J \subset \omega$.

However, it is not a big surprise that a non-monotonic consequence relation is not compact as well.

## 3. Complexity upper bounds

Fix a Gödel numbering $\gamma$ for $For_{\mathcal{L}}$, i. e., $\gamma$ is an effective one-to-one mapping from $For_{\mathcal{L}}$ onto $\omega$ that additionally satisfies the condition:

$$A \text{ is a proper subformula of } B \quad \Longrightarrow \quad \gamma(A) < \gamma(B)\,.$$

Having such a numbering (for $\mathcal{L}$-formulas) allows us to provide an effective coding for more complex syntactical objects (cf. [9]), like finite sequences of $\mathcal{L}$-formulas, lines of stages of proofs, finite stages of proofs, finite sets of $\mathcal{L}$-formulas, finite sets of finite sets of $\mathcal{L}$-formulas, etc. In this way, one may speak, e. g., of Gödel numbers $\gamma(\Gamma)$ for $\Gamma \subseteq_{fin} For_{\mathcal{L}}$ (and, intuitively, even identify these with their codes).[8]

[8]Notice that all such numberings are presupposed to satisfy the corresponding natural analogs of the above monotonicity requirement.

The lower limit logic **LLL** is called *f-decidable* (*f-enumerable*) iff the set of pairs

$$\{(\gamma(\Gamma), \gamma(\Delta)) \mid \Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}} \text{ and } \Gamma \vdash_{\mathbf{LLL}} \Delta\}$$

is computable (computably enumerable, respectively). Due to the above remarks, we will not mention Gödel numbers explicitly in this and similar situations, and so (instead) will speak, e. g., of decidability or enumerability of the set

$$\{(\Gamma, \Delta) \mid \Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}} \text{ and } \Gamma \vdash_{\mathbf{LLL}} \Delta\}.$$

Further, **LLL** has the *property of local abnormalities* iff for any $\Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}}$, we can effectively construct $\Omega_{\Gamma,\Delta} \subseteq_{fin} \Omega$ s. t. for every $\Theta \subseteq \Omega$,

$$\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta \quad \Longrightarrow \quad \Gamma \vdash_{\mathbf{LLL}} \Delta \cup (\Theta \cap \Omega_{\Gamma,\Delta}).$$

Here the expression 'effectively constructed' means that there is a (total) computable function $f$ of two arguments transforming each $(\gamma(\Gamma), \gamma(\Delta))$ into $\gamma(\Omega_{\Gamma,\Delta})$. In this context, we write $\Omega_\Gamma$ instead of $\Omega_{\Gamma,\varnothing}$. For instance, it was shown in [8] that in case of $CLuN$ the set $\Omega_{\Gamma,\Delta}$ can be defined as

$$\Omega_{\Gamma,\Delta} := \{A \wedge \neg A \mid \neg A \in Subf(\Gamma \cup \Delta)\},$$

where $Subf(\Gamma \cup \Delta)$ is the set of all subformulas of formulas in $\Gamma \cup \Delta$.

Thus, from Theorems 2.10 and 2.16, we immediately obtain

PROPOSITION 3.1. *Suppose* **LLL** *has the property of local abnormalities. Then, for any* $\Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}}$ *with* $\Delta \neq \varnothing$*, the following hold:*

1. $\Gamma \vdash_{\mathbf{LLL^r}} \Delta$ *iff there exists* $\Theta \subseteq \Omega_{\Gamma,\Delta}$ *s. t.* $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ *and* $\Theta \cap U(\Gamma) = \varnothing$*;*
2. $\Gamma \vdash_{\mathbf{LLL^m}} \Delta$ *iff for each* $\varphi \in \Phi(\Gamma)$*, there exists* $\Theta \subseteq_{fin} \Omega_{\Gamma,\Delta}$ *s. t.* $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ *and* $\Theta \cap \varphi = \varnothing$*.*

Also, restricting attention to finite sets of $\mathcal{L}$-formulas, we get

PROPOSITION 3.2. *Suppose* **LLL** *has the property of local abnormalities, and let* $\Gamma \subseteq_{fin} For_{\mathcal{L}}$*. Then the following hold:*

1. $U(\Gamma)$ *is a finite set of* $\mathcal{L}$*-formulas,* $\Sigma(\Gamma)$ *and* $\Phi(\Gamma)$ *are both finite collections of finite sets of* $\mathcal{L}$*-formulas;*
2. *if* **LLL** *is f-decidable, then the functions*

$$\lambda_U : \Gamma \subseteq_{fin} For_{\mathcal{L}} \mapsto U(\Gamma), \quad \lambda_\Sigma : \Gamma \subseteq_{fin} For_{\mathcal{L}} \mapsto \Sigma(\Gamma), \quad \text{and} \quad \lambda_\Phi : \Gamma \subseteq_{fin} For_{\mathcal{L}} \mapsto \Phi(\Gamma)$$

*are all computable.*

PROOF. $\boxed{1}$ If $\Gamma \vdash_{\mathbf{LLL}} \Theta$ and $\Theta \subseteq_{fin} \Omega$, then $\Gamma \vdash_{\mathbf{LLL}} \Theta \cap \Omega_\Gamma$ by the property of local abnormalities. Thus, since $\Sigma(\Gamma)$ consists precisely of minimal *Ab*-consequences of $\Gamma$, $\Theta \in \Sigma(\Gamma)$ implies $\Theta \subseteq \Omega_\Gamma$. But $\Omega_\Gamma$ is finite, whence $\Sigma(\Gamma)$ is a finite collection of finite sets. Next, $U(\Gamma)$ is the union of $\Sigma(\Gamma)$ and so, too, is finite. Finally, $\Phi(\Gamma)$ is the collection of all minimal choice sets for $\Sigma(\Gamma)$, and $\varphi \in \Phi(\Gamma)$ entails $\varphi \subseteq U(\Gamma)$. Therefore, $\Phi(\Gamma)$ itself, as well as all its members, are finite.

$\boxed{2}$ Assume **LLL** is f-decidable. It means, particularly, that for any $\Gamma \subseteq_{fin} For_{\mathcal{L}}$ and $\Theta \subseteq \Omega_\Gamma$, we can computably check whether $\Gamma \vdash_{\mathbf{LLL}} \Theta$. This allows us to effectively (and uniformly in $\Gamma$) construct both $\Sigma(\Gamma)$ and $U(\Gamma)$. Given these two, for each $\varphi \subseteq U(\Gamma)$, we can also decide (again, effectively) whether $\varphi$ is a choice set for $\Sigma(\Gamma)$—as a result, we distinguish all the minimal choice sets for $\Sigma(\Gamma)$, and eventually obtain $\Phi(\Gamma)$. ■

THEOREM 3.3. *If* **LLL** *has the property of local abnormalities and is f-decidable, then the adaptive consequence relations* $\vdash_{\mathbf{LLL^r}}$ *and* $\vdash_{\mathbf{LLL^m}}$ *are also f-decidable, i. e., the two sets*

$$\{(\Gamma, \Delta) \mid \Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}} \text{ and } \Gamma \vdash_{\mathbf{LLL^r}} \Delta\},$$
$$\{(\Gamma, \Delta) \mid \Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}} \text{ and } \Gamma \vdash_{\mathbf{LLL^m}} \Delta\}$$

*are decidable.*

PROOF. Suppose that $\Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}}$ and $\Delta \neq \varnothing$. According to Item 1 of Proposition 3.1, to decide whether $\Gamma \vdash_{\mathbf{LLL^r}} \Delta$ or not, we need to check if $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ for all $\Theta \subseteq \Omega_{\Gamma,\Delta} \setminus U(\Gamma)$. While $\Omega_{\Gamma,\Delta}$ (which is finite) can be computed from $\Gamma$ and $\Delta$ by the property of local abnormalities, $U(\Gamma)$ is finite and may be effectively constructed from $\Gamma$ by Proposition 3.2. So we only have to check whether $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ or not for the finite number of known $\Theta$. Since **LLL** is f-decidable, the latter can be carried out, again, in a computable way.

For the second condition (namely $\Gamma \vdash_{\mathbf{LLL^m}} \Delta$), the proof is analogous (and employs Item 2 of Proposition 3.1). ■

Therefore, both adaptive consequence relations are f-decidable, provided that **LLL** is f-decidable and has the property of local abnormalities. How does the situation change when **LLL** is f-enumerable? Let us start with considering the relation 'to be a minimal *Ab*-consequence' between finite sets of $\mathcal{L}$-formulas (for a f-enumerable **LLL**).

PROPOSITION 3.4. *If* **LLL** *is f-enumerable, then the set*

$$\{(\Gamma, \Delta) \mid \Gamma \subseteq_{fin} For_{\mathcal{L}} \text{ and } \Delta \in \Sigma(\Gamma)\}$$

*is* $\Delta^0_2$*. And if, in addition,* **LLL** *has the property of local abnormalities, then the sets*

$$\{(\Gamma, A) \mid \Gamma \subseteq_{fin} For_{\mathcal{L}} \ \textit{and} \ A \in U(\Gamma)\},$$
$$\{(\Gamma, \varphi) \mid \Gamma \subseteq_{fin} For_{\mathcal{L}}, \ \varphi \subseteq_{fin} \Omega \ \textit{and} \ \varphi \in \Phi(\Gamma)\}$$

*are* $\Delta^0_2$ *as well.*[9]

PROOF. Assume henceforth that $\Gamma$ and $\Delta$ range over finite subsets of $For_{\mathcal{L}}$.

Clearly, one is able to check effectively whether a given finite set of $\mathcal{L}$-formulas consists of abnormalities, i. e., that it is a subset of $\Omega$. The relation $\Delta \in \Sigma(\Gamma)$ may be expressed as

$$(\varnothing \neq \Delta \subseteq \Omega) \wedge (\Gamma \vdash_{\mathbf{LLL}} \Delta) \wedge \left(\forall \Delta' \subset \Delta\right) \left(\Gamma \nvdash_{\mathbf{LLL}} \Delta'\right).$$

Since **LLL** is f-enumerable, the condition $\Gamma \vdash_{\mathbf{LLL}} \Delta$ is definable (in $\mathfrak{N}$) by an arithmetical $\Sigma_1$-formula, and so $\Gamma \nvdash \Delta'$ is, in turn, definable via a $\Pi_1$-formula. Moreover, $(\forall \Delta' \subset \Delta)(\Gamma \nvdash_{\mathbf{LLL}} \Delta')$ is, too, expressible by a $\Pi_1$-formula, because '$\forall \Delta' \subset \Delta$' can be viewed as a kind of bounded quantifier (remember that all Gödel numberings employed are monotone, in a natural sense). Thus, the (binary) relation $\Delta \in \Sigma(\Gamma)$ is definable by means of a conjunction of a $\Sigma^0_1$- and a $\Pi^0_1$-formula, whence it is at worst $\Delta^0_2$.

Suppose **LLL** also has the property of local abnormalities. As we've already mentioned in the proof of Proposition 3.2, $\Delta \in \Sigma(\Gamma)$ implies $\Delta \subseteq \Omega_\Gamma$, so $A \in U(\Gamma)$ may be expressed as

$$(\exists \Delta \subseteq \Omega_\Gamma)(\Delta \in \Sigma(\Gamma) \wedge A \in \Delta),$$

which is a $\Delta^0_2$-relation preceded by a bounded quantifier (here $\Omega_\Gamma$ should be replaced by an effective mapping sending each $\Gamma \subseteq_{fin} For_{\mathcal{L}}$ to $\Omega_\Gamma$). Consequently, the (binary) relation $A \in U(\Gamma)$ is $\Delta^0_2$ as well.

Recall that $\Phi(\Gamma)$ is the collection of all minimal choice sets for $\Sigma(\Gamma)$. Thus, taking into account Proposition 2.2, we obtain the following presentation for the condition $\varphi \in \Phi(\Gamma)$:

$$(\forall \Delta \subseteq \Omega_\Gamma)(\Delta \notin \Sigma(\Gamma) \vee (\varphi \cap \Delta \neq \varnothing)) \wedge$$
$$(\forall A \in \varphi)(\exists \Delta \subseteq \Omega_\Gamma)(\Delta \in \Sigma(\Gamma) \wedge (\varphi \cap \Delta = \{A\})).$$

[9] If $\Gamma$ is fixed, then by the property of local abnormalities the set $U(\Gamma)$ is finite and $A \in U(\Gamma)$ is decidable as a unary relation. Unfortunately, we can not prove that $U(\Gamma)$ is decidable uniformly in $\Gamma$. Due to this reason the binary relation $A \in U(\Gamma)$ can be estimated only as $\Delta^0_2$. Of course, we did not proved yet that this estimation is exact.

Clearly, since $\Delta \in \Sigma(\Gamma)$ is $\Delta^0_2$, its negation is also $\Delta^0_2$. In this way, the (binary) relation $\varphi \in \Phi(\Gamma)$ may be eventually defined (notice, we used only bounded quantifiers) as an intersection of two $\Delta^0_2$-sets, whence it is $\Delta^0_2$. ■

PROPOSITION 3.5. *If* **LLL** *has the property of local abnormalities and is f-enumerable, then the two sets*

$$\{(\Gamma,\Delta) \mid \Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}} \text{ and } \Gamma \vdash_{\mathbf{LLL^r}} \Delta\},$$
$$\{(\Gamma,\Delta) \mid \Gamma \cup \Delta \subseteq_{fin} For_{\mathcal{L}} \text{ and } \Gamma \vdash_{\mathbf{LLL^m}} \Delta\}$$

*are* $\Delta^0_2$.

PROOF. Here assume that $\Gamma$ and $\Delta$ range over finite subsets of $For_{\mathcal{L}}$.

Due to Item 1 of Proposition 3.1, we may express $\Gamma \vdash_{\mathbf{LLL^r}} \Delta$ as

$$(\Delta \neq \varnothing) \wedge (\exists \Theta \subseteq \Omega_{\Gamma,\Delta})\,(\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta \wedge (\forall A \in \Theta)\,(A \notin U(\Gamma)))\,.$$

By the previous proposition, the relation $A \in U(\Gamma)$ is $\Delta^0_2$, whence its negation and also $(\forall A \in \Theta)\,(A \notin U(\Gamma))$ are $\Delta^0_2$. On the other hand, the condition $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ is definable via a $\Sigma_1$-formula. Since an intersection of a $\Sigma^0_1$- and a $\Delta^0_2$-set gives a $\Delta^0_2$-set, while adding a bounded quantifier does not change the complexity class, we eventually get $\Delta^0_2$, as desired.

As we have already seen, $\varphi \in \Phi(\Gamma)$ entails $\varphi \subseteq U(\Gamma)$, and $U(\Gamma) \subseteq \Omega_\Gamma$. Thus, due to Item 2 of Proposition 3.1, $\Gamma \vdash_{\mathbf{LLL^m}} \Delta$ can be presented as

$$(\forall \varphi \subseteq \Omega_\Gamma)\,(\exists \Theta \subseteq \Omega_{\Gamma,\Delta})\,(\varphi \notin \Phi(\Gamma) \vee (\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta \wedge (\varphi \cap \Theta = \varnothing)))$$

Again, by the previous proposition, the relations $\varphi \in \Phi(\Gamma)$ and $A \in U(\Gamma)$ are $\Delta^0_2$ (while $\Gamma \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ is even $\Sigma_1$). Consequently, since only bounded quantifiers are involved, we arrive at $\Delta^0_2$ again. ■

Now we turn to (adaptive) consequences of infinite sets of $\mathcal{L}$-formulas. For an arbitrary $\Gamma \subseteq For_{\mathcal{L}}$, denote

$$Cn_{\mathbf{LLL^r}}(\Gamma) := \{\Delta \subseteq_{fin} For_{\mathcal{L}} \mid \Gamma \vdash_{\mathbf{LLL^r}} \Delta\},$$
$$Cn_{\mathbf{LLL^m}}(\Gamma) := \{\Delta \subseteq_{fin} For_{\mathcal{L}} \mid \Gamma \vdash_{\mathbf{LLL^m}} \Delta\}.$$

In the sequel, the subscripts $\mathbf{LLL^r}$ and $\mathbf{LLL^m}$ will be replaced with **r** and **m**, respectively, when there is no risk of confusion.

Remark that for any $\Gamma \subseteq For_{\mathcal{L}}$, $\Sigma(\Gamma)$ is a collection of finite sets of abnormalities, and $U(\Gamma)$ is a set of abnormalities. Therefore, we can always view their elements as appropriately encoded by natural numbers, and it makes sense to talk about algorithmic complexity of $\Sigma(\Gamma)$ and $U(\Gamma)$ in a general case. However, $\Phi(\Gamma)$ may easily contain infinite sets (of abnormalities), so the situation is more difficult and certain restrictions are needed.

Proposition 3.6. *Let* **LLL** *be f-enumerable. For an arbitrary set of $\mathcal{L}$-formulas $\Gamma$, $\Sigma(\Gamma)$ is $\Delta_2^{0,\Gamma}$, while $U(\Gamma)$ is $\Sigma_2^{0,\Gamma}$. And if, in addition, all elements of $\Phi(\Gamma)$ turn out to be finite, then $\Phi(\Gamma)$ is $\Delta_3^{0,\Gamma}$.*

Proof. In what follows, all unbounded quantifiers are assumed to range over finite subsets of $For_{\mathcal{L}}$ (or, rather, over their Gödel codes).

Trivially, the unary relation $\Theta \subseteq \Gamma$ on the finite subsets $\Theta$ of $For_{\mathcal{L}}$ is presented by $(\forall A \in \Theta)(A \in \Gamma)$, and hence is computable w. r. t. (the oracle) $\Gamma$. Then, using the left-compactness of $\vdash_{\mathbf{LLL}}$, we may express the condition $\Delta \in \Sigma(\Gamma)$ (for $\Delta$ finite, like before) as

$$(\varnothing \neq \Delta \subseteq \Omega) \wedge \exists\Theta\,(\Theta \subseteq \Gamma \wedge \Theta \vdash_{\mathbf{LLL}} \Delta) \wedge \\ \forall\Theta'\left(\Theta' \not\subseteq \Gamma \vee \left(\forall\Delta' \subset \Delta\right)\left(\Theta' \nvdash_{\mathbf{LLL}} \Delta'\right)\right).$$

Due to the finite enumerability of $\vdash_{\mathbf{LLL}}$, $\Theta \vdash_{\mathbf{LLL}} \Delta$ and $\Theta' \nvdash_{\mathbf{LLL}} \Delta'$ are definable (in $\mathfrak{N}$) via a $\Sigma_1$- and a $\Pi_1$-formula, respectively. Thus, employing some standard transformations, it is straightforward to get both $\Sigma_2$- and $\Pi_2$-definability w. r. t. $\Gamma$, whence $\Sigma(\Gamma)$ is $\Delta_2^{0,\Gamma}$. The condition $A \in U(\Gamma)$ means that $\exists\Delta\,(\Delta \in \Sigma(\Gamma) \wedge A \in \Delta)$, so $U(\Gamma)$ is $\Sigma_2^{0,\Gamma}$.

Suppose all elements of $\Phi(\Gamma)$ are finite. Now $\varphi \in \Phi(\Gamma)$ is presented by

$$\forall\Delta\,(\Delta \notin \Sigma(\Gamma) \vee (\varphi \cap \Delta \neq \varnothing)) \wedge \\ \forall A \in \varphi\, \exists\Delta\,(\Delta \in \Sigma(\Gamma) \wedge (\varphi \cap \Delta = \{A\})).$$

Since $\Delta \in \Sigma(\Gamma)$ is $\Sigma_2^{0,\Gamma}$ and, therefore, $\Delta \notin \Sigma(\Gamma)$ is $\Pi_2^{0,\Gamma}$, we arrive at an intersection of a $\Sigma_2^{0,\Gamma}$- and a $\Pi_2^{0,\Gamma}$-relation, hence $\Phi(\Gamma)$ is $\Delta_3^{0,\Gamma}$. ∎

Notice, the quantifiers over finite sets in the above proof, unlike in the proof of Proposition 3.4, are not bounded. However, the advantage here is that we no longer need the property of local abnormalities for **LLL**.

In many situations, we already know (the upper bound for) the complexity of $\Gamma$—that allows us to improve the estimations given.

Corollary 3.7. *Let* **LLL** *be f-enumerable. For each $\Gamma \subseteq For_{\mathcal{L}}$, if $\Gamma$ is $\Sigma_{m+1}^0$, then $\Sigma(\Gamma)$ is $\Delta_{m+2}^0$, and $U(\Gamma)$ is $\Sigma_{m+2}^0$. And if, in addition, all elements of $\Phi(\Gamma)$ turn out to be finite, then $\Phi(\Gamma)$ is $\Delta_{m+3}^0$.*

PROOF. Remark that if $\Gamma$ is $\Sigma^0_{m+1}$, the condition $\Theta \subseteq \Gamma$ (from the proof of Proposition 3.4) is $\Sigma^0_{m+1}$, and $\Theta' \not\subseteq \Gamma$ is $\Pi^0_{m+1}$. The rest is straightforward. ■

THEOREM 3.8. *Suppose that* **LLL** *is f-enumerable, and* $\Gamma \subseteq For_{\mathcal{L}}$. *Then* $Cn^{\mathbf{r}}(\Gamma)$ *is* $\Sigma^{0,\Gamma}_3$. *Moreover, the following implications hold:*

1. *if all elements of* $\Phi(\Gamma)$ *are finite, then* $Cn^{\mathbf{m}}(\Gamma)$ *is* $\Pi^{0,\Gamma}_3$;[10]
2. *if* $U(\Gamma)$ *is finite, then both* $Cn^{\mathbf{r}}(\Gamma)$ *and* $Cn^{\mathbf{m}}(\Gamma)$ *are* $\Sigma^{0,\Gamma}_1$.

PROOF. Just as before, all unbounded quantifiers are assumed to range over finite subsets of $For_{\mathcal{L}}$. Fix some set $\Gamma$ of premisses.

According to Theorem 2.10, the unary relation $\Gamma \vdash_{\mathbf{LLL}^{\mathbf{r}}} \Delta$ on the finite subsets $\Delta$ of $For_{\mathcal{L}}$ can be specified by

$$(\Delta \neq \varnothing) \wedge \exists\Theta\, \exists\Gamma' \big(\Theta \subseteq \Omega \wedge \Gamma' \subseteq \Gamma \wedge \\ \Gamma' \vdash_{\mathbf{LLL}} \Delta \cup \Theta \wedge (\forall A \in \Theta)\,(A \notin U(\Gamma))\big) \quad (\star)$$

Since **LLL** is f-enumerable, the condition $\Gamma' \vdash_{\mathbf{LLL}} \Delta \cup \Theta$ is $\Sigma_1$-definable (in $\mathfrak{N}$), while $U(\Gamma)$ is $\Sigma^{0,\Gamma}_2$ by Proposition 3.6 and, consequently, its complement is $\Pi^{0,\Gamma}_2$. As a result, we obtain a $\Pi^{0,\Gamma}_2$-relation preceded by two existential quantifiers. In this way, $Cn^{\mathbf{r}}(\Gamma)$ turns out to be $\Sigma^{0,\Gamma}_3$.

[1] Assume that $\Phi(\Gamma)$ consists of finite sets of $\mathcal{L}$-formulas. So $\Phi(\Gamma)$ is $\Delta^{0,\Gamma}_3$ by Proposition 3.6. At the same time, due to Theorem 2.16, the unary relation $\Gamma \vdash_{\mathbf{LLL}^{\mathbf{m}}} \Delta$ (for $\Delta \subseteq_{fin} For_{\mathcal{L}}$) may be presented as

$$(\Delta \neq \varnothing) \wedge \forall\varphi\,(\varphi \notin \Phi(\Gamma) \vee \\ \exists\Theta\, \exists\Gamma' \big(\Theta \subseteq \Omega \wedge \Gamma' \subseteq \Gamma \wedge \Gamma' \vdash_{\mathbf{LLL}} \Delta \cup \Theta \wedge (\varphi \cap \Theta = \varnothing)\big)) \quad (\dagger)$$

Here, the condition $\varphi \notin \Phi(\Gamma)$ is $\Delta^{0,\Gamma}_3$, and so we arrive at a $\Pi^{0,\Gamma}_3$-relation (because $\Delta^{0,\Gamma}_3 \subseteq \Pi^{0,\Gamma}_3$) preceded by a $\forall$-quantifier, which eventually entails (after gluing together universal quantifiers) that $Cn^{\mathbf{m}}(\Gamma)$ is $\Pi^{0,\Gamma}_3$.

[2] If $U(\Gamma)$ is finite, then the unary relation $\Theta \cap U(\Gamma) = \varnothing$, that is, $(\forall A \in \Theta)\,(A \notin U(\Gamma))$ (for $\Theta$ finite), is computable. In such a case, the relation represented by $(\star)$ is easily seen to be $\Sigma_1$-definable w. r. t. (the oracle) $\Gamma$, whence $Cn^{\mathbf{r}}(\Gamma)$ is $\Sigma^{0,\Gamma}_1$.

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[10] It was proved in [11] that in presence of classical connectives the condition "all $\varphi \in \Phi(\Gamma)$ are finite" is equivalent to the finiteless of $U(\Gamma)$. So the conditions of both items of this theorem are equivalent in this case. It is not clear however, whether this equivalence takes place for arbitrary lower limit logic and a set of abnormalities in case of multi-consequence standard format.

Since all elements of $\Phi(\Gamma)$ are subsets of $U(\Gamma)$, $\Phi(\Gamma)$ is a finite collection of finite sets, and hence computable. The expression (†) can then be rewritten as

$$(\Delta \neq \varnothing) \wedge (\forall \varphi \subseteq U(\Gamma))\, (\exists \Theta)\, (\exists \Gamma')\, (\varphi \not\subseteq \Phi(\Gamma) \vee \\ \left(\Theta \subseteq \Omega \wedge \Gamma' \subseteq \Gamma \wedge \Gamma' \vdash_{\mathbf{LLL}} \Delta \cup \Theta \wedge (\varphi \cap \Theta = \varnothing)\right)) \quad (\ddagger)$$

which clearly defines a $\Sigma_1^{0,\Gamma}$-set, namely $Cn^{\mathbf{m}}(\Gamma)$. ■

Corollary 3.9. *Suppose* **LLL** *is f-enumerable, and* $\Gamma \subseteq For_{\mathcal{L}}$ *is* $\Sigma^0_{m+1}$. *Then* $Cn^{\mathbf{r}}(\Gamma)$ *is* $\Sigma^0_{m+3}$. *Moreover, the following implications hold:*

1. *if all elements of* $\Phi(\Gamma)$ *are finite, then* $Cn^{\mathbf{m}}(\Gamma)$ *is* $\Pi^0_{m+3}$*;*
2. *if* $U(\Gamma)$ *is finite, then both* $Cn^{\mathbf{r}}(\Gamma)$ *and* $Cn^{\mathbf{m}}(\Gamma)$ *are* $\Sigma^0_{m+1}$.

Proof. Again, if $\Gamma$ is $\Sigma^0_{m+1}$, the condition $\Gamma' \subseteq \Gamma$ (from the proof of Theorem 3.8) has the complexity $\Sigma^0_{m+1}$ as well. Thus, using Corollary 3.7, it is easy to show that ($\star$) is equivalent to a $\Sigma^0_{m+3}$-formula.

[1] Analogously, by transforming (†) into a $\Pi^0_{m+3}$-form.

[2] By a similar argument, (‡) is reduced to a $\Sigma^0_{m+1}$-form. ■

Note that the above statement can be reformulated in a uniform way for certain classes of premiss sets—cf. [8, Corollary 3.11], for example.

Finally, we consider the algorithmic complexity for $\mathbf{LLL^m}$-consequences in the general case—which, involving infinite stages of proofs, forces us to pass to a more abstract framework of second-order arithmetic.

Theorem 3.10. *Suppose that* **LLL** *is f-enumerable, and* $\Gamma \subseteq For_{\mathcal{L}}$. *Then* $Cn^{\mathbf{m}}(\Gamma)$ *is* $\Pi^{1,\Gamma}_1$.

Proof. As usual, (first-order) variables in expressions are: $\Delta$, $\Theta$ and $\Gamma'$ ranging over finite subsets of $For_{\mathcal{L}}$, and $A$ ranging over $\mathcal{L}$-formulas.

Let $P$ be an unary predicate variable (that, intuitively, will range over arbitrary subsets of $\omega$—or rather over all subsets of $For_{\mathcal{L}}$, modulo a Gödel numbering in hand). Fix some $\Gamma$. In view of Proposition 2.2, the property '$P \in \Phi(\Gamma)$', i.e., '$P$ is a minimal choice set for $\Sigma(\Gamma)$' may be expressed as

$$\Phi^{\Gamma}(P) \;:=\; \forall \Delta\, (\Delta \notin \Sigma(\Gamma) \vee (P \cap \Delta \neq \varnothing)) \wedge \\ \forall A\, (A \notin P \vee \exists \Delta\, (\Delta \in \Sigma(\Gamma) \wedge (P \cap \Delta = \{A\})))$$

Taking into account Proposition 3.6, this can be specified by a second-order arithmetical formula without set quantifiers (viz. over predicates), with parameter $\Gamma$ (as an oracle), and the only free variable $P$.

Next, by employing Theorem 2.16, $\Gamma \vdash_{\mathbf{LLL^m}} \Delta$ is presented as

$$(\Delta \neq \varnothing) \wedge \forall P \left( \neg \Phi^{\Gamma}(P) \vee \exists \Theta \, \exists \Gamma' \left( \Theta \subseteq \Omega \wedge \Gamma' \subseteq \Gamma \wedge \Gamma' \vdash_{\mathbf{LLL}} \Delta \cup \Theta \wedge (P \cap \Theta = \varnothing) \right) \right).$$

Obviously, it implies that $Cn^{\mathbf{m}}(\Gamma)$ is definable (in $\mathfrak{N}$) by means of a $\Pi^1_1$-formula with parameter $\Gamma$, as desired. ■

COROLLARY 3.11. *Suppose* **LLL** *is f-enumerable, and* $\Gamma \subseteq For_{\mathcal{L}}$. *If* $\Gamma$ *is arithmetical, then* $Cn^{\mathbf{m}}(\Gamma)$ *is* $\Pi^1_1$.

## 4. Complexity lower bounds: examples

In effect, many of the estimations provided in the previous section turn out to be exact for particular adaptive logics. Possibly the most prominent examples of inconsistency-adaptive logics are $CLuN^{\mathbf{r}}$ and $CLuN^{\mathbf{m}}$ [3], so we've chosen them to be our 'model logics'.

Let $For_{CL}$ be the set of all propositional formulas build up from the propositional symbols $Prop$ using logical connectives $\wedge$, $\vee$, $\rightarrow$ and $\neg$. In our setting, the lower limit logic **LLL** is the propositional weak paraconsistent logic $CLuN$—which may be viewed as the smallest subset of $For_{CL}$ containing the axioms of propositional classical positive logic, plus $p \vee \neg p$, and closed under the rules of substitution and 'modus ponens'.

The consequence relation $\vdash_{CLuN}$ (associated with $CLuN$) is defined as follows: for $\Gamma \cup \Delta \subseteq For_{CL}$, $\Gamma \vdash_{CLuN} \Delta$ iff there exist $\{A_1, \ldots, A_n\} \subseteq \Delta$ s. t. $A_1 \vee \cdots \vee A_n$ can be derived (in a finite number of steps) from the elements of $CLuN \cup \Gamma$ by means of 'modus ponens' only. Clearly, $\vdash_{CLuN}$ satisfies all the requirements (on $\vdash_{\mathbf{LLL}}$) from Section 2.

The models for $CLuN$ are simply valuations $v : For_{CL} \rightarrow \{0, 1\}$ possessing the properties:

1. $v(A \wedge B) = 1 \iff v(A) = 1$ and $v(B) = 1$;
2. $v(A \vee B) = 1 \iff v(A) = 1$ or $v(B) = 1$;
3. $v(A \rightarrow B) = 1 \iff v(A) = 0$ or $v(B) = 1$;
4. $v(A) = 0 \implies v(\neg A) = 1$.

If $\varepsilon \in \{0,1\}$, then $v(\Gamma) = \varepsilon$ abbreviates '$v(A) = \varepsilon$ for all $A \in \Gamma$'. Now, assuming $\Gamma \cup \Delta \subseteq For_{CL}$, $\Gamma \vDash_{CLuN} \Delta$ means that for every $CLuN$-valuation $v$, either $v(\Delta) \neq 0$, or $v(\Gamma) \neq 1$.[11]

It is well-known that $CLuN$ is strongly complete w. r. t. the semantics just described, i. e.,

$$\Gamma \vdash_{CLuN} \Delta \quad \Longleftrightarrow \quad \Gamma \vDash_{CLuN} \Delta\,.$$

In addition, since any $v(\Gamma)$ is completely determined by how $v$ acts on the subformulas of formulas in $\Gamma$, the $\vdash_{CLuN}$-relation restricted to finite sets (for both premisses and conclusions) is decidable.

Now, taking

$$\Omega \;:=\; \{A \wedge \neg A \mid A \in For_{CL}\}\,,$$

it is straightforward to define the adaptive logics $CLuN^{\mathbf{r}}$ and $CLuN^{\mathbf{m}}$ (viz. $CLuN$ supplied with the reliability strategy and the minimal abnormality strategy, respectively), according to the presentation from Section 2.

To avoid confusion with the general case, for each $\Gamma \subseteq For_{CL}$, let

$$\mathtt{Cn}^{\mathbf{r}}(\Gamma) \;:=\; Cn_{CLuN^{\mathbf{r}}}(\Gamma)\,.$$

Then, the $\Sigma^0_3$ lower bound proof (for $\mathtt{Cn}^{\mathbf{r}}(\Gamma)$, with $\Gamma$ computable) from [7] can be easily adapted to derive

Proposition 4.1 (see [8]). *For every $m \in \omega$, there exists a $\Pi^0_m(\Sigma^0_{m+1})$-set $\Gamma \subseteq For_{CL}$ s. t. $\mathtt{Cn}^{\mathbf{r}}(\Gamma)$ is $\Sigma^0_{m+3}$-hard.*

Together with Corollary 3.9, it implies $\Sigma^0_{m+3}$-completeness of $\mathtt{Cn}^{\mathbf{r}}(\Gamma)$ for certain $\Pi^0_m(\Sigma^0_{m+1})$-sets of premisses $\Gamma$.

Next, let us write $\mathtt{U}(\Gamma)$ (where $\Gamma \subseteq For_{CL}$) for $U(\Gamma)$ in case of $CLuN$ with the abnormalities $\Omega$ as above.

Proposition 4.2. *For every $m \in \omega$, there is a $\Sigma^0_{m+1}$-set $\Gamma \subseteq For_{CL}$ s. t. $\mathtt{U}(\Gamma)$ is $\Sigma^0_{m+2}$-hard.*

Proof. Take a $\Sigma^0_{m+2}$-complete subset $S$ of natural numbers. Certainly, it is definable in $\mathfrak{N}$ by an arithmetical formula of the sort $\exists i\, \Psi(i,n)$, for some $\Pi^0_{m+1}$-formula $\Psi(i,n)$. Assume $\Gamma$ consists of:

- $(p_n \wedge \neg p_n) \vee \left(q^i_n \wedge \neg q^i_n\right)$ for all $i$ and $n$ in $\mathbb{N}$;

[11] Imagine that $\Gamma$ and $\Delta$ stand for the (possibly infinite) conjunction and disjunction of their elements, correspondingly.

- $\left(q^i_n \wedge \neg q^i_n\right)$ for any $i$ and $n$ with $\neg\Psi\,(i,n)$.

The resulting $\Gamma$ is obviously $\Sigma^0_{m+1}$, and it is not hard to show that

$$p_n \wedge \neg p_n \in \mathtt{U}\,(\Gamma) \quad \Longleftrightarrow \quad \mathfrak{N} \Vdash_{\mathrm{FOL}} \exists i\, \Psi\,(i,n)\,,$$

whence $\mathtt{U}\,(\Gamma)$ is at least $\Sigma^0_{m+2}$-hard. ■

Thus, the estimation from Corollary 3.7 also turns out to be exact for certain $\Sigma^0_{m+1}$-sets of premisses.

Moreover, at times there is room for further improvements. Particularly, though $\mathtt{U}\,(\Gamma)$ ($\mathtt{Cn}^{\mathbf{r}}\,(\Gamma)$) is $\Sigma^0_2(\Sigma^0_3)$-complete for some c. e. set $\Gamma$ (due to the above), even a computable set will suffice here, since $\Gamma$ is, in fact, **LLL**-equivalent to a computable $\Gamma'$—see [8, Proposition 3.12].

Take $CLuN_\bot$ to be $CLuN$ augmented with the constant $\bot$ interpreted as 'always false formula' (consequently, the classical negation of $A$ is definable as $A \to \bot$), here $\mathcal{L} := \{\wedge, \vee, \to, \neg, \bot\}$. Then, the upper bound from Corollary 3.11 is exact, because, as was shown earlier in [13], there exists a computable $\Gamma \subseteq For_{\mathcal{L}}$ s. t. $Cn_{CLuN^{\mathbf{m}}_\bot}\,(\Gamma)$ is $\Pi^1_1$-hard.

Finally, remark that the $\Delta$-like bounds from Section 4 cannot be indeed totally precise w. r. t. the $m$-reducibility, since no $\Delta^0_k$-universal sets (here $k \in \omega$) are possible in the arithmetical hierarchy, and it may well be that a thiner classification is needed for this situation. However, $\Sigma^0_i$-, $\Pi^0_i$- and $\Delta^0_i$-sets with $i \in \{0,1,2,3\}$ are the most studied in the hierarchy (cf. [6, Section 10.5] and [10, § 14.8] for the details) and, e. g., there is an interesting intuition behind $\Delta^0_2$-sets based on computable approximations.

**Acknowledgements.** We would like to thank Diderik Batens, Peter Verdée, Christian Straßer, and Frederik Van De Putte for useful and encouraging discussions. We are grateful to the anonymous referee for his very detailed and helpful comments. Both authors acknowledge the support of the Russian Foundation for Basic Research (projects No. 12-01-00168-a and No. 11-07-00560-a). Also, the first author was supported by the Siberian Branch of the Russian Academy of Sciences (interdisciplinary integration project for fundamental research No. 3), and the second author was supported by the Ministry of Education and Science of Russian Federation (project No. 8227).

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