Gillman Payette
Peter K. Schotch

Remarks on the Scott–Lindenbaum Theorem

Abstract. In the late 1960s and early 1970s, Dana Scott introduced a kind of generalization (or perhaps simplification would be a better description) of the notion of inference, familiar from Gentzen, in which one may consider multiple conclusions rather than single formulas. Scott used this idea to good effect in a number of projects including the axiomatization of many-valued logics (of various kinds) and a reconsideration of the motivation of C.I. Lewis. Since he left the subject it has been vigorously prosecuted by a number of authors under the heading of *abstract entailment relations* where it has found an important role in both algebra and theoretical computer science. In this essay we go back to the beginnings, as presented by Scott, in order to make some comments about Scott's cut rule, and show how much of Scott's main result may be applied to the case of single-conclusion logic.

Keywords: Dana Scott, Lindenbaum, Abstract entailment relations, Structural rules.

1. Introduction

Ever since the pioneering work of Gentzen, what has come to be called *proof theory* has been identified with his basic method in *sequent logic*.[1] A sequent consists of two sequences of formulas separated by the provability sign $\vdash$, e.g.,

$$\gamma_1, \ldots, \gamma_n \vdash \delta_1, \ldots, \delta_m.$$

The logic in question is a formal system in which a proof consists of a sequence of lines each containing a number of sequents such that later lines follow from earlier ones by means of a specified set of rules.

Scott broke with this tradition. He abandoned sequences of formulas in favor of sets. This effected a simplification by, for example, removing the need for structural rules dealing with reordering the formulas of a sequent and for collapsing two or more iterations of the same formula (in the same cedent–which is to say antecedent or succedent) to a single instance. In what

[1]See Gerhard Gentzen [3].

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follows we use capital Greek letters for sets of formulas and lower case Greek letters for individual formulas from a language $\mathcal{L}$. So sequents look like

$$\Gamma \vdash \Delta.$$

In this paper we will deal with two aspects of Scott's work. First we will examine Scott's comments on the so-called structural rule of *cut*. We show that cut elimination can be applied to what we call Scott Relations. Second, we discuss one of Scott's central results in [7], which he says is a generalization of Lindenbaum's Lemma. His result is proved for multiple-conclusion consequence relations, so one might think it implies an analogous result for single-conclusion consequence relations. We prove such a result, but we expose some complications involved.

2. Some Notational Preliminaries

Scott imposes only three structural rules on his consequence relations. They are versions of those that Gentzen-style proof theorists call *Axiom*, *Thinning*,[2] and *Cut*. We introduce a convention that multiple conclusion relations are indicated by expressions like $\Vdash$ with or without scripts either super or sub. On this convention single-conclusion relations are indicated by expressions like $\vdash$ (with the same remark). These structural rules may then be displayed respectively:

$$\frac{\Gamma \cap \Delta \neq \emptyset}{\Gamma \Vdash \Delta}[\mathrm{R}]$$

$$\frac{\Gamma \Vdash \Delta \;\&\; \Gamma \subseteq \Gamma' \;\&\; \Delta \subseteq \Delta'}{\Gamma' \Vdash \Delta'}[\mathrm{M}]$$

$$\frac{\Gamma, \alpha \Vdash \Delta \;\&\; \Gamma \Vdash \alpha, \Delta}{\Gamma \Vdash \Delta}[\mathrm{T}]$$

Where the rule names are intended to suggest "reflexivity," "monotonicity," and "transitivity," respectively. In these formulations we use the accepted abbreviations, for example, "α, Δ" for "$\{\alpha\} \cup \Delta$" and the like. In other words we abbreviate the unit set of some formula with the formula in question and abbreviate the set operation "union" with comma–where that makes sense.[3] We recognize and sympathize with Scott's motivation in this terminological change but shall stick to the older characterization of the rule names in those

[2] T.J. Smiley has suggested that the German word that gets translated as thinning would be better translated as 'dilution.' We mostly follow Smiley's usage below.

[3] It would definitely not make sense to take the comma in $\{\alpha, \beta\}$ as union.

cases where it seems more pointed to do this. Thus, for example, the move from an expression like $\Gamma \Vdash \Sigma$ to $\Gamma, \beta \Vdash \Sigma$ seems more strikingly described as "diluting on the left," rather than "monotonicity on the left." Similarly it seems to capture the spirit of a proof better to describe the transition from one line to another as "cutting" a certain formula. Transitivity and monotonicity are not action verbs–we cannot transitivity a formula, nor can we monotonicity a set of formulas.

We shall call any relation between sets of formulas[4] which satisfies these rules a *Scott* relation. Typically, and Scott assumes this as well, we take the sets to be finite. Finally, we do not assume that the relations are *structural*, i.e., satisfy the substitution property: If $\Gamma \Vdash \Delta$, and $e : \mathcal{L} \to \mathcal{L}$ is a substitution of atoms to formulas, then $e[\Gamma] \Vdash e[\Delta]$. Making that assumption would have us assume that the language has structure. Although such an assumption is completely reasonable, we don't need it for our purposes.

3. Scott and Cut Elimination

The goal of many proof theorists, some might even say the whole point of doing sequent logic, requires proving what is often called a *normal form* theorem. The proof of the normal form theorem invariably embeds proving that one of the structural rules:

$$\frac{\Gamma, \alpha \Vdash \Delta \quad \Gamma \Vdash \alpha, \Delta}{\Gamma \Vdash \Delta}[\text{Cut}]$$

can be eliminated. Elimination in this sense means showing that every proof that uses Cut can be replaced by a proof with the same last line, which doesn't use the rule. The reason this was thought so central, was that Cut-elimination amounts to a consistency proof. In fact the ability to support a cut-elimination theorem was regarded by Hacking as one of the crucial things which separates logic from non-logic.[5]

Scott reminds us, however, that Cut—or the *rule of transitivity* as he terms it—is not eliminable in any full-blooded sense.[6] It is, after all, the

[4]To call something a formula is to imply that we have in hand a formal language of which the object in question is a well-formed expression. For the most part we suppress the details of that language.

[5]See especially his Hacking [4].

[6]The reason that it isn't eliminable is because, as Scott tells in [6, p. 797], it is required to show that Scott relations are determined by valuations. It is the use of [T] in the argument at the end of Sect. 4.

rule that allows us to chain together individual bits of reasoning to good inferential effect. What does it take for Cut to be eliminable? If there isn't a full-blooded sense, is there a weaker sense? In fact, there is a kind of Cut elimination theorem for Scott relations, in special circumstances. This is what we term T elimination below. We start with the following definition.

DEFINITION 3.1. *(R,M,T, Closure)* If C is a relation between sets of sentences (not necessarily a Scott relation), then its R, M, T-closure, $\Vdash_{rmt}$ is defined as the set of pairs $\langle \Gamma, \Delta \rangle$ such that there is a sequence of pairs, $\{ \langle \Gamma_i, \Delta_i \rangle \}_{i=1}^{n}$ such that

(1) either (i) $\langle \Gamma_1, \Delta_1 \rangle \in C$ or (ii) $\Delta_1 \cap \Gamma_1 \neq \varnothing$,

(2) for each $1 < i \leq n$, either

- (c) either (i) $\langle \Gamma_i, \Delta_i \rangle \in C$ or (ii) $\Delta_i \cap \Gamma_i \neq \varnothing$,
- (t) there are $j, k < i$ such that $\langle \Gamma_j, \Delta_j \rangle = \langle \Gamma_i \cup \{ \beta \}, \Delta_i \rangle$, and $\langle \Gamma_k, \Delta_k \rangle = \langle \Gamma_i, \Delta_i \cup \{ \beta \} \rangle$; or
- (m) there is $j < i$ such that $\Gamma_j \subseteq \Gamma_i$ and $\Delta_j \subseteq \Delta_i$,

(3) and $\langle \Gamma_n, \Delta_n \rangle = \langle \Gamma, \Delta \rangle$.

We call the sequences *construction sequences.*

LEMMA 3.1. *The R,M,T-closure of C obeys* [R]*,* [M]*, and* [T]*.*

PROOF. We will refer to the R,M,T-closure as 'the closure' for simplicity. By (1) of Definition 3.1, we get that for all $\langle \Gamma, \Delta \rangle$ such that $\Gamma \cap \Delta \neq \varnothing$, are in the closure. They will be added by construction sequences of length 1. If $\langle \Gamma', \Delta' \rangle$ is in the closure, then there is a construction sequence of length n with it at the end. So any supersets will be in the closure by using 2.m to form a construction sequence of length $n+1$. If $\langle \Gamma \cup \{ \beta \}, \Delta \rangle$ and $\langle \Gamma, \Delta \cup \{ \beta \} \rangle$ are in the closure, then there are construction sequences with those as pairs at the end, each of respective length m and n. Thus we can form a construction sequence of length $m + n + 1$ using 2.t, and get that $\langle \Gamma, \Delta \rangle$ is in the closure. ■

As we have defined it, the R, M, T-closure of C is the smallest Scott relation that extends C.

LEMMA 3.2. *If $C \subseteq \Vdash$ and $\Vdash$ is a Scott relation, then $\Vdash_{rmt} \subseteq \Vdash$.*

PROOF. By induction on the length of construction sequences. ■

Now we can see under what circumstances [T] is eliminable. What we need is the notion of a relation C being closed under [T]. This notion amounts to: if $\langle \Gamma \cup \{ \beta \}, \Delta \rangle \in C$ and $\langle \Gamma, \Delta \cup \{ \beta \} \rangle \in C$, then $\langle \Gamma, \Delta \rangle \in C$. We will say that [T] is eliminable from the R, M, T-closure of C when: If [T] is used in a construction sequence for $\langle \Gamma, \Delta \rangle$, then there is a construction sequence from C for that pair that doesn't use [T].

LEMMA 3.3. (T-elimination) *C is closed under* [T] *iff* [T] *is eliminable from the R, M, T-closure of C.*

PROOF. Let C be closed under [T]. Suppose that $\langle \Gamma_m, \Delta_m \rangle$ is the first use of T (i.e., 2.t) in the construction sequence of $\langle \Gamma, \Delta \rangle$. Then there are $\langle \Gamma_j, \Delta_j \rangle$ and $\langle \Gamma_k, \Delta_k \rangle$ such that $\Gamma_k = \Gamma_m \cup \{ \beta \}$ and $\Delta_k = \Delta_m$ while $\Delta_j = \Delta_m \cup \{ \beta \}$ and $\Gamma_j = \Gamma_m$. If $\Gamma_m \cap \Delta_m \neq \varnothing$, then we can replace that step with an instance of [R] (i.e., the first disjunct (2.c) in Definition 3.1). So suppose $\Gamma_m \cap \Delta_m = \varnothing$.

If either $\Gamma_m \cap (\Delta_m \cup \{ \beta \}) \neq \varnothing$ or $(\Gamma_m \cup \{ \beta \}) \cap \Delta_m \neq \varnothing$, then one of $\langle \Gamma_m, (\Delta_m \cup \{ \beta \}) \rangle$ or $\langle (\Gamma_m \cup \{ \beta \}), \Delta_m \rangle$ would be identical to $\langle \Gamma_m, \Delta_m \rangle$. That would mean that there would already be a construction sequence of $\langle \Gamma_m, \Delta_m \rangle$ that didn't use [T] since we have assumed that this is the first use of [T].

Hence the construction sequence of $\langle \Gamma_k, \Delta_k \rangle = \langle \Gamma_m \cup \{ \beta \}, \Delta_m \rangle$ doesn't employ [T] (the move to the mth step was the first use of [T]), nor does the construction sequence of $\langle \Gamma_k, \Delta_k \rangle = \langle \Gamma_m, \Delta_m \cup \{ \beta \} \rangle$ nor do they use [R] (2.c (ii)) (there would need to be some sentences in common). So both must have been derived from members of C by uses of [M] (2.m). It follows that there are Γ', Δ' such that $\Gamma' \subseteq \Gamma_m$ and $\Delta' \subseteq \Delta_m$ where $\langle \Gamma' \cup \{ \beta \}, \Delta' \rangle \in C$ and $\langle \Gamma', \Delta' \cup \{ \beta \} \rangle \in C$. Since C is closed under [T], $\langle \Gamma', \Delta' \rangle \in C$; so $\langle \Gamma_m, \Delta_m \rangle$ could have been derived from C using [M]. We can then repeat this process for any other uses of [T] in the construction sequence.

Conversely, let [T] be eliminable from the R, M, T-closure of C. Then whenever we have instances of $\langle \Gamma' \cup \{ \beta \}, \Delta' \rangle \in C$ and $\langle \Gamma', \Delta' \cup \{ \beta \} \rangle \in C$, such that no subsets of pairs of subsets of Γ' and Δ' that are in C, we must have $\langle \Gamma', \Delta' \rangle \in C$, otherwise we couldn't eliminate that use of [T]. ■

We thus see that the eliminability of Cut depends on the starting relation being closed under Cut. If Cut is eliminable from the base relation, then Cut is eliminable from the R, M, T-closure. What does this mean? What if the base relation of what we decide is inference isn't closed under [T], i.e., doesn't have elimination of Cut? Does that mean it isn't inference? Scott's answer was: no. But if we aren't to prove Cut-elimination of some base relation C, then what is the new project? Scott answers this question via another question. What, in its most basic (or even perhaps its purest) form, is that relation that we call inference?

This might not seem to offer much of a challenge to any who subscribe to some particular version of inference, but if we set aside such prejudices for a moment, we can better appreciate Scott's question-challenge. We might put it this way 'How much structure must our formal language have before it might legitimately be said to support something very like an inference relation?' Our first impulse is likely to be that we must need at least some minimal set of connectives and the rules governing their inferential use. When asked about those rules which don't mention connectives, the structural rules as they are usually termed, we might say 'Well of course we shall have those—that goes without saying, but we shall still need at least some connectives.' Scott can and does show us, that when we are talking about the kind of relation between sets of formulas[7] perhaps we don't really need connectives at all.

4. The Scott–Lindenbaum Theorem

We now rehearse the Scott result. Scott defines two notions: that of a Scott relation being *consistent*, and that of a Scott relation being complete. We follow another convention here to the effect that the notation for consistency and completeness for multiple conclusion logics are subscripted with M, while the single-conclusion relations will use the subscript S. Here are Scott's definitions in our notation.

Definition 4.1. A Scott relation $\Vdash$ is said to be *consistent* or $\textsc{con}_M$ provided that there is no formula α such that $\emptyset \Vdash \alpha$ and $\alpha \Vdash \emptyset$. Equivalently, given that $\Vdash$ is a Scott relation, if there is some $\langle \Gamma, \Delta \rangle$ such that $\Gamma \nVdash \Delta$, then $\Vdash$ is $\textsc{con}_M$.[8]

A Scott relation $\Vdash$ is said to be *complete* or $\textsc{com}_M$ provided that for every formula α either $\Vdash \alpha$ or $\alpha \Vdash \emptyset$.

We will use $\textsc{con}_M$ and $\textsc{com}_M$ as predicates, e.g., $\textsc{con}_M(\Vdash)$ means $\Vdash$ is $\textsc{con}_M$. Notice that $\Vdash$ being $\textsc{con}_M$ is also equivalent to $\varnothing \nVdash \varnothing$. Our statement of the result is slightly simpler than his, since we can use the notion of a Scott relation, which modesty forbids Scott.

Theorem 4.1. [7] *Every Scott relation* $\Vdash$ *is the intersection of all complete and consistent Scott relations which contain* $\Vdash$.

[7]It would be begging the question were we to begin calling these 'multiple conclusion relations'.

[8]The subscript 'M' is for multiple-conclusion.

PROOF. We reproduce the original elegant proof from Scott [7], using our notation, since we will refer to it later. "The relation [$\Vdash$] is contained in the intersection of the [$\textsc{com}_M$ and $\textsc{con}_M$ extensions $\Vdash^+$]. Suppose [$\Gamma_0 \nVdash \Delta_0$]. Let [$\Vdash^+$] be a maximal relation such that [$\Vdash \subseteq \Vdash^+$]. Such a relation exists by Zorn's Lemma. The relation [$\Vdash^+$] is clearly consistent. Suppose it is not complete [i.e., not $\textsc{com}_M$]. Let [$\alpha \in \mathcal{L}$] such that [$\nVdash^+ \alpha$] and [$\alpha \nVdash^+$]. Define new relations such that

$$\Phi \Vdash_0 \Psi \textit{ iff } \Phi \Vdash^+ \alpha, \Psi$$

$$\Phi \Vdash_1 \Psi \textit{ iff } \Phi, \alpha \Vdash^+ \Psi$$

For all [$\Phi, \Psi \subseteq \mathcal{L}$]. By choice of [$\alpha$] both [$\Vdash_0$] and [$\Vdash_1$] are proper consistent extensions of [$\Vdash^+$]; and they both satisfy [R], [M] and [T], as is easily checked. It follows that [$\Gamma_0 \Vdash_0 \Delta_0$] and [$\Gamma_0 \Vdash_1 \Delta_0$] both hold by the maximality of [$\Vdash^+$]. But then by [T], we have [$\Gamma_0 \Vdash^+ \Delta_0$], which is a contradiction." (p. 416) ■

The significance of this result is that, as Scott puts it, a complete and consistent relation is a 2-valued valuation in disguise. Suppose $\Vdash$ is a complete and consistent Scott relation, then where $\mathcal{L}$ is the set of formulas of our language we define a valuation $V_{\Vdash} : \mathcal{L} \to \{0, 1\}$:

DEFINITION 4.2. $V_{\Vdash}(\alpha) = 1 \Longleftrightarrow \emptyset \Vdash \alpha$

We shall also require the following abbreviations and definition:

DEFINITION 4.3. "$\Gamma \vDash_V$" *for* "$V(\alpha) = 1$ *for every* $\alpha \in \Gamma$"
"$\vDash_V \Delta$" *for* "$V(\beta) = 1$ *for some* $\beta \in \Delta$"
Next, given any $V : \mathcal{L} \to \{0, 1\}$:
$\Gamma \Vdash_V \Delta \Longleftrightarrow (\Gamma \vDash_V \Longrightarrow \vDash_V \Delta)$

It should be clear that:

$$\Vdash_{V_{\Vdash}} = \Vdash \text{ and that } V_{\Vdash_V} = V$$

which is to say that there is a bijection between consistent and complete Scott relations and 2-valued valuations. But in case it isn't that clear, here is a proof sketch of the first conjunct.

Let $\Vdash$ be any consistent and complete Scott relation.

$$\begin{aligned} \Gamma \Vdash_{V_{\Vdash}} \Delta &\Longleftrightarrow \Gamma \vDash_{V_{\Vdash}} \Longrightarrow \vDash_{V_{\Vdash}} \Delta \text{ by definition} \\ &\Longleftrightarrow \Vdash \gamma \text{ for all } \gamma \in \Gamma \Longrightarrow \Vdash \delta \text{ for some } \delta \in \Delta \end{aligned}$$

At this point we shall first prove that $\Vdash \gamma$ for all $\gamma \in \Gamma \Longrightarrow \Vdash \delta$ for some $\delta \in \Delta$ implies that $\Gamma \Vdash \Delta$. Then we shall prove the converse.

Assume that the conditional is true, which is to say that either (1) $\nVdash \gamma$ for some $\gamma \in \Gamma$ or (2) $\Vdash \delta$ for some $\delta \in \Delta$. If (1) then by the completeness of $\Vdash$, $\gamma \Vdash \emptyset$ for some $\gamma \in \Gamma$. But then by dilution on both the left and right, $\Gamma \Vdash \Delta$. If (2) on the other hand, then the dilution strategy just used may be used again to the same effect.

For the converse, assume that $\Gamma \Vdash \Delta$. We must show that $\Vdash \gamma$ for all $\gamma \in \Gamma$ implies that $\Vdash \delta$ for some $\delta \in \Delta$. Assume the antecedent of this conditional and now assume, for reductio, that $\nVdash \delta$ for all $\delta \in \Delta$. We notice that by the completeness of $\Vdash$, the latter amounts to: $\delta \Vdash \emptyset$ for all $\delta \in \Delta$. By a series of steps[9] of the form:

$$\begin{aligned}
&\text{for } \gamma \in \Gamma, \Gamma \Vdash \gamma, \Delta \text{ by dilution on the left and right}\\
&\Gamma \setminus \{\gamma\} \Vdash \Delta \text{ by Cut, since } \Gamma \Vdash \Delta\\
&\text{for } \gamma' \in \Gamma \setminus \{\gamma\}, \Gamma \setminus \{\gamma\} \Vdash \gamma', \Delta \text{ dilution}\\
&\Gamma \setminus \{\gamma, \gamma'\} \Vdash \Delta \text{ by Cut since } \Gamma \setminus \{\gamma\} \Vdash \Delta\\
&\vdots
\end{aligned}$$

we shall eventually arrive at $\emptyset \Vdash \Delta$, having cut away all of Γ. But now by another, similar, series of steps of the form:

$$\begin{aligned}
&\text{for } \delta \in \Delta, \emptyset, \delta \Vdash \Delta \text{ by dilution on the left and right}\\
&\emptyset \Vdash \Delta \setminus \{\delta\} \text{ by Cut, since } \emptyset \Vdash \Delta\\
&\vdots
\end{aligned}$$

we may likewise cut away all of Δ, leaving us with $\emptyset \Vdash \emptyset$. But to say this is to say that $\Vdash$ is inconsistent, contrary to hypothesis.

5. The Message

We know from the Scott–Lindenbaum result, that every Scott relation is the intersection of all the complete and consistent Scott relations $\Vdash$ which include the one in question. But each of these relations corresponds to a certain valuation, namely $V_{\Vdash}$. So to say that a Scott relation is that intersection is to say that precisely those pairs $\langle \Gamma, \Delta \rangle$ belong to the relation for which

[9] This will not be problematic if we require our sets to be finite. Lacking that requirement we shall need to impose compactness on the relation or allow infinite proof figures. An alternative to these restrictions is presented in Appendix A.

any of the corresponding truth assignments which make all the formulas in Γ true make some of the formulas in Δ true.

So what? Well, a charitable person who has been able to purge herself of prejudices acquired, either consciously or unconsciously, during her logic training, might think of this as the minimal requirement in order for a relation to be a recognizable species of inference. The first thing to note in this connection is that Scott himself is more circumspect than this. In the paper already mentioned, he says, concerning what we have called Scott relations, "I now feel that it would be wrong to call [such a relation] a *consequence relation*."[10] Scott feels that only those Scott relations which support enough structure to admit derivations of conclusions from hypotheses, can comfortably wear the mantle of consequence. We shall have cause to examine this issue, in part at least, below.

Scott says that he would prefer to read the provability symbol in an expression like $\Gamma \Vdash \Delta$ as "gamma entails delta," even though "extraneous meanings" attach to the latter. He finally settles on calling $\Gamma \Vdash \Delta$ a *conditional assertion*, but the difficulty in finding some non-cumbersome way of saying $\Gamma \Vdash \Delta$ using the conditional assertion rubric leads him to stick with entailment. In later developments of Scott's work, authors have continued to use the word "entailment," though it is usually prefixed with the modifier "abstract."

6. Single Conclusion Logic

It will naturally occur to us to wonder if the Scott result carries over to the case in which there is a single formula on the right of the "proves" symbol. Recall that we use $\vdash$ for single-conclusion consequence relations, and $\Vdash$ for multiple conclusion consequence relations. A first instinct is to think that since the single-conclusion case is just a special case of the multiple conclusion one, the Scott result must carry over, trivially. But this isn't quite right, in the sense that it can comfort only those who agree that the general case represents inference, and not those who merely view the multiple conclusion relation with envy.

It is for the latter group that we shall concern ourselves with a separate proof of the Scott–Lindenbaum result for single-conclusion relations. Obviously we will use as much of the Scott proof as we can, but this will turn out

[10] Scott [7, p. 417].

to be less than all of it. To begin with, here are the rules that characterize the single-conclusion relation:

$$\frac{\alpha \in \Gamma}{\Gamma \vdash \alpha}[\mathrm{R}]$$

$$\frac{\Gamma \vdash \alpha \;\&\; \Gamma \subseteq \Gamma'}{\Gamma' \vdash \alpha}[\mathrm{M}]$$

$$\frac{\Gamma, \alpha \vdash \beta \;\&\; \Gamma \vdash \alpha}{\Gamma \vdash \beta}[\mathrm{T}]$$

All of the sets mentioned are finite. We keep the corresponding rule labels and common names letting the context determine when we intend "dilution" or "cut" to refer to the single-conclusion flavor of the rule in question.

We must next modify the definitions of completeness and consistency, now indicated by ($\textsc{com}_S$) ($\textsc{con}_S$) respectively, for the more restricted Scott relations.

Definition 6.1. *(Important Properties of Single Conclusion Scott Relations)* Let φ, ψ be formulas and $\vdash$ a Scott relation.

$\vdash$ is $\textsc{con}_S$ iff there is no φ such that $\varnothing \vdash \varphi$ and $\varphi \vdash \psi$ for all ψ.

$\vdash$ is $\textsc{com}_S$ iff for all φ either $\varnothing \vdash \varphi$ or $\varphi \vdash \psi$ for all ψ.

We should notice that the original Scott result will not be unproblematically applicable to the single-conclusion case. The reason is that the two notions of consistency do not match up in general. To see this, consider how we might get a single-conclusion relation from a multi-conclusion relation.

Definition 6.2. Given $\Vdash$, define $\vdash_d$ by $\Gamma \vdash_d \varphi$ iff $\Gamma \Vdash \{\varphi\}$.

In [8, Ch. 5], the authors call relations that satisfy the condition in Definition 6.2 *counterparts*. They show that $\vdash_d$ is the unique counterpart given $\Vdash$. Now consider the R, M, T-closure of the following set of pairs: $\{\langle\varnothing, \Delta\rangle : \Delta \neq \varnothing\}$. It is easy to see that that relation is closed under [T] since there are no pairs of the form $\langle\Gamma \cup \{\beta\}, \Delta\rangle$. By Theorem 3.3, we get that [T] is eliminable from the R, M, T-closure of that set. Thus we get the following result:

Corollary 6.1. *The R,M,T-closure of* $\{\langle\varnothing, \Delta\rangle : \Delta \neq \varnothing\}$, *is* $\textsc{con}_M$.

Proof. Let $\Vdash$ be the R,M,T-closure of $\{\langle\varnothing, \Delta\rangle : \Delta \neq \varnothing\}$. If $\varnothing \Vdash \varnothing$, then there would have to be a construction sequence of $\langle\varnothing, \varnothing\rangle$ that didn't employ T. But that is impossible. ■

We can conclude that the notions of $\textsc{con}_M$ and $\textsc{con}_S$ come apart. Consider the multi-conclusion Scott $\Vdash$, which is the R,M,T-closure of

$\{\langle\varnothing, \Delta\rangle : \Delta \neq \varnothing\}$. This relation will contain $\Gamma \Vdash \Delta$ for any Γ, and all $\Delta \neq \varnothing$. But it will not contain $\varnothing \Vdash \varnothing$. Thus, this relation is *consistent* according to definition 4.1, i.e., $\text{CON}_M(\Vdash)$. But now consider $\vdash_d$ based on $\Vdash$; it will contain, for any α, $\alpha \vdash_d \beta$ for all β, and $\varnothing \vdash_d \alpha$. So the counterpart single-conclusion relation is inconsistent, i.e., not-$\text{CON}_S(\vdash_d)$.

The notions come apart going from single to multiple conclusion relations as well. Consider the following definitions of multi-conclusion relations given a single-conclusion relation.

Definition 6.3. Let $\vdash$ be a single-conclusion Scott relation. Then define

$$\Gamma \Vdash_{\min} \Delta \iff \exists\delta \in \Delta \text{ s.t. } \Gamma \vdash \delta$$

and

$$\Gamma \Vdash_{\max} \Delta \iff (\forall\alpha)(\forall\Gamma' \supseteq \Gamma)[(\forall\delta \in \Delta, \Gamma', \delta \vdash \alpha) \Rightarrow \Gamma' \vdash \alpha]$$

$\Gamma \Vdash_{\max} \Delta$ holds when every extension of Γ, Γ' is such that if α can be derived from Γ', δ, for each $\delta \in \Delta$, then α already follows from Γ'. $\Gamma \Vdash_{\min} \Delta$ holds when there is some member of Δ that follows from Γ, relative to the single-conclusion relation. Scott offers these two possible ways of specifying a multi-conclusion relation from a Tarski-consequence operator, or what amounts to the same thing, a single-conclusion Scott relation. We offer a brief aside on some other work on these notions. The definitions presented in Definition 6.3 are those from Došen [2]. Došen's paper has an excellent, detailed investigation of which variations of the structural rules of inference are satisfied by the min and max relations that Scott offers. Došen's work was inspired by the very general investigation of multi-conclusion relations in [8]. Shoesmith and Smiley call these two relations $\vdash_\cap$ and $\vdash_\cup$, respectively. They also investigate the circumstances under which the max relation exists. If sequents are permitted to be infinite, and relations are assumed to use different structural rules (which we don't consider here), the max relation may not exist.

One immediate difference between the min and max relations is that we will never have $\varnothing \Vdash_{\min} \varnothing$. The reason being that there are no members on the right hand side for $\varnothing$ to prove relative to the single-conclusion relation. So given a not-CON_S relation $\vdash$, $\Vdash_{\min}$ is always CON_M.

We should pause to notice a commonality in these two cases of where the notions of consistency come apart. The R, M, T-closures of $\{\langle\varnothing, \Delta\rangle : \Delta \neq \varnothing\}$, and $\{\langle\varnothing, \alpha\rangle : \alpha \in \mathcal{L}\}$ are identical. The second is just a special case of the first, and the first is obtained by various uses of [M] from the second. Furthermore, our example of a $\vdash$ that is inconsistent while $\Vdash_{\min}$ is consistent is the single-conclusion R, M, T-closure of $\{\langle\varnothing, \alpha\rangle : \alpha \in \mathcal{L}\}$.

Similarly, the R, M, T-closure of $\{\langle\varnothing,\Delta\rangle : \Delta \neq \varnothing\}$ is really the $\textsc{con}_M$ Scott relation $\Vdash_\top$ such that for all α, $\Vdash \alpha$.

The thought is that the only instance where $\textsc{con}_S$ and $\textsc{con}_M$ come apart is in $\Vdash_\top$. We can apply the notion of $\textsc{con}_S$ to multi-conclusion relations fairly easily giving **a**lternative notions of consistency and completeness for multi-conclusion relations.

Definition 6.4. *(Alternate Properties of Multi-conclusion Relations)* Let $\Gamma, \Delta \subseteq \mathcal{L}$, and $\varphi, \psi \in \mathcal{L}$.

$\Vdash$ is a-consistent iff there is no φ such that $\varnothing \Vdash \varphi$, and for all non-empty Δ, $\varphi \Vdash \Delta$.

$\Vdash$ is a-complete iff for all φ either $\varnothing \Vdash \varphi$ or $\varphi \Vdash \psi$ for all ψ.

We can see that a-consistency is really just $\textsc{con}_S$ in a multi-conclusion guise, similarly for $\textsc{com}_S$. We will refer to the alternative notions of consistency and completeness by $\textsc{con}_S$ and $\textsc{com}_S$ for multi-conclusion relations when we are not also discussing single-conclusion relations. That allows us to show:

Lemma 6.2. *Suppose* $\Vdash \neq \Vdash_\top$. *Then* $\textsc{con}_M(\Vdash)$ *iff* $\textsc{con}_S(\Vdash)$.

Proof. First notice that not-$\textsc{con}_M(\Vdash)$ always implies not-$\textsc{con}_S(\Vdash)$ by [M]. Conversely, suppose $\textsc{con}_M(\Vdash)$, but not-$\textsc{con}_S(\Vdash)$. So, equivalently, there is α such that $\alpha \Vdash \beta$ for all β, and $\Vdash \alpha$. While also $\varnothing \nVdash \varnothing$. But by [T] $\Vdash \beta$ for all β, i.e., $\Vdash = \Vdash_\top$. A contradiction. ■

So the only multi-conclusion Scott relation where $\textsc{con}_M$ and $\textsc{con}_S$ come apart is $\Vdash_\top$. We are trying to use the Scott–Lindenbaum result to derive something similar for single-conclusion relations. So far we have been impeded because the notions of consistency and completeness aren't the same for both relations. But given the last theorem we see where that fails. Now consider the following lemma.

Lemma 6.3. *Suppose that* $\Vdash$ *is a-consistent and a-complete, then* $\vdash_d$ *is* $\textsc{con}_S$ *and* $\textsc{com}_S$.

Proof. Since $\Vdash$ is a-consistent, there is no φ such that $\varnothing \Vdash \varphi$ and $\varphi \Vdash \Delta$ for all non-empty Δ. Suppose $\varnothing \vdash_d \varphi$ for all φ, then $\varnothing \Vdash \varphi$ by definition for all φ. So $\Vdash$ is a-inconsistent. Let β, be any formula. Then either (1) $\beta \Vdash \psi$ for all ψ, or (2) $\varnothing \Vdash \beta$. If (1), then $\beta \vdash_d \psi$ for all ψ. If (2), then $\varnothing \vdash_d \beta$. Therefore, $\textsc{com}_S(\vdash_d)$. ■

We can then show the result that we want.

THEOREM 6.4. (Scott Lindenbaum theorem for single-conclusion Scott relations) *For every Scott relation* $\vdash$, $\Gamma \vdash \varphi$, *iff for all* CON_S *and* COM_S *Scott extensions* $\vdash^+$ *of* $\vdash$, $\Gamma \vdash^+ \varphi$.

PROOF. Suppose that $\vdash$ obeys the structural rules, and $\Gamma \vdash \alpha$. Then for any CON_S and COM_S extension of $\vdash$, $\vdash^+$, $\Gamma \vdash^+ \alpha$.

Now suppose that $\Gamma \nvdash \alpha$. Then $\Gamma \nVdash_{\min} \{\alpha\}$ by definition, and $\Vdash_{\min}$ is clearly CON_M. So there is a CON_M and COM_M extension of $\Vdash_{\min}$, we call it $\Vdash^+_{\min}$, by Scott's theorem such that $\Gamma \nVdash^+_{\min} \{\alpha\}$. But that means $\nVdash^+_{\min} \alpha$, so $\Vdash^+_{\min} \neq \Vdash_\top$. By lemma 6.2, $\Vdash^+_{\min}$ is also a-consistent and a-complete. But then there is a CON_S and COM_S relation $\vdash^+_d$ by lemma 6.3. $\vdash^+_d$ extends $\vdash$ since if $\Sigma \vdash \psi$, then $\Sigma \Vdash_{\min} \{\psi\}$ by definition. So $\Sigma \Vdash^+_{\min} \psi$ and $\Sigma \vdash^+_d \psi$. However, $\Gamma \nvdash^+_d \alpha$. ■

Therefore the Scott theorem can give us what we want, once we are clear about the notions of consistency and completeness. In the appendix we also provide a direct proof of the following theorem. The proof replicates Scott's original proof using the alternative notions of consistency and completeness.

THEOREM 6.5. [7] *For every a-consistent Scott relation* $\Vdash$, $\Gamma \Vdash \Delta$, *iff for all a-consistent and a-complete extensions of* $\Vdash$, $\Vdash^+$, $\Gamma \Vdash^+ \Delta$.

Part of what the result shows is that any Scott relation $\Vdash$ such that $\Gamma \nVdash \alpha$, can be extended to a maximal Scott relation $\Vdash^+$ such that $\Gamma \nVdash^+ \alpha$. So it is determined by all of its a-consistent and a-complete extensions; a-consistency is preserved by taking maximal extensions.

Now we can raise the following question: Do we have everything that we had in the general case? The answer is: not quite. In the general case we had no 2-valued valuation left behind. Every one of them had a corresponding consistent and complete Scott relation, $\Vdash$. And neither were there any complete and consistent Scott relations $\Vdash$ without a 2-valued valuation to which such a relation corresponded. A nice result to be sure, but not one that entirely carries over to the case of single-conclusion Scott relations.

We still have, given that consistent and complete Scott relations *are* 2-valued valuations, that each Scott relation is determined by a class of 2-valued valuations. What we have lost however is that every two-valued valuation has a corresponding consistent and complete Scott relation. There are some which don't enjoy such a correspondence, for instance the valuation $v_\top$, defined by: for every formula α, $v_\top(\alpha) = 1$

The relation that would correspond to $v_\top$ must be such that $\varnothing \vdash \alpha$ for all formulas α, i.e., the single-conclusion version of $\Vdash_\top$. So then take any φ,

$\varnothing \vdash \varphi$ and $\varphi \vdash \alpha$ for all α. Which is to say that $\vdash$ is not a-consistent. Interestingly, the same does not seem to apply to the case of $v_\perp$ which assigns every formula the value 0. Take the single-conclusion relation $\vdash_\perp$ such that for all $\Gamma \neq \varnothing$ and α, $\Gamma \vdash_\perp \alpha$. There are no α such that $\vdash_\perp \alpha$, so it is $\textsc{con}_S$. The obvious next question is to ask whether we can get all of the valuations back?

We lose valuations because the notions of consistency and completeness have changed from those of $\textsc{con}_M$ and $\textsc{com}_M$ to a-consistency and a-completeness. The latter correspond to $\textsc{con}_S$ and $\textsc{com}_S$ directly. Next we will look at how we fix the lack of valuations.

Until now we haven't discussed the structure of the underlying language. As Došen [2] points out, $\Gamma \Vdash_{\max} \Delta$ iff $\Gamma \vdash \bigvee \Delta$, i.e., the disjunction of all of Δ's members. That requires that we have a disjunction and Δ be finite, of course. Inspired by this, and Scott's thoughts about how much structure a language should have in order to be called inference we take up language in the next section.

7. You Want More Structure?

In remarking that a Scott relation might not be robust enough to support an interpretation as a fully-fledged consequence relation, Scott raises the possibility that more structure might be necessary in the way of something like connective rules. We follow up on this idea, in a more general way, in an attempt to repair the broken bijection of Sect. 6.

In [5] we introduce, in logical guise, the notion of a coproduct, the definition of which we now rehearse.

Definition 7.1. Given a Scott relation $\vdash$, we say that the relation *has coproducts* or is *coproductival* iff for every finite set of formulas Γ there is a formula $\coprod(\Gamma)$ which satisfies the following two conditions:

[*Canonical injections*] If there is $\delta \in \Delta$ such that $\Gamma \vdash \delta$, then $\Gamma \vdash \coprod(\Delta)$

[*Universal property*] If for all $\delta \in \Delta$, $\Gamma, \delta \vdash \alpha$ then $\Gamma, \coprod(\Delta) \vdash \alpha$

From this point on we assume that all the Scott relations we mention have coproducts. A special case of this is the coproduct of the finite set $\varnothing$, $\coprod \varnothing$. By the universal property, it has the following property: for all α, $\coprod \varnothing \vdash \alpha$.[11]

[11] Wójcicki [9, p. 334] also considers extending the language with disjunction. His definitions are different from ours, but with the same effect. Czelakowski [1] also has similar definitions of disjunctions to ours, but pursues different lines of inquiry.

We now wish to prove that every single-conclusion Scott relation $\vdash$, is the intersection of all the consistent and complete Scott relations which contain $\vdash$. Here we do this directly using single-conclusion relations, and so invoking consistency and completeness in the sense of $\textsc{con}_S$ and $\textsc{com}_S$, respectively. Evidently $\vdash$ is included in the intersection just mentioned. To show the converse inclusion we argue contrapositively. Let $\vdash$ be a Scott relation with the property that $\Gamma \nvdash \alpha$. We demonstrate the existence of a consistent and complete extension of $\vdash$ which has this same property.

Let $\vdash^+$ be the maximal Scott relation containing $\vdash$ such that $\Gamma \nvdash \alpha$.[12] Clearly this relation extends $\vdash$ and all we need to know is that it is consistent and complete. Consistency follows immediately from the fact that there is some formula that isn't proved by some set of formulas. As for completeness, assume for reductio that $\vdash^+$ is not complete. Thus there must be some formula β such that

(1) $\emptyset \nvdash^+ \beta$ and

(2) there is some formula δ such that $\beta \nvdash^+ \delta$

Define two Scott relations as follows

$$(\vdash_1)\ \Sigma \vdash_1 \sigma \Longleftrightarrow \Sigma, \beta \vdash^+ \sigma$$

$$(\vdash_2)\ \Sigma \vdash_2 \sigma \Longleftrightarrow \Sigma \vdash^+ \coprod(\{\,\beta, \sigma\,\})$$

$\vdash_1$ properly extends $\vdash^+$ since it is clear that since $\beta \vdash^+ \beta$, $\emptyset \vdash_1 \beta$ and if $\Lambda \vdash^+ \lambda$, then $\Lambda, \beta \vdash^+ \lambda$, by dilution. It must also be the case that $\vdash_2$ extends $\vdash^+$ since if $\Lambda \vdash^+ \lambda$ then $\Lambda \vdash^+ \coprod\{\,\beta, \lambda\,\}$ by the existence of canonical injections. Further, $\vdash_2$ must be a proper extension of $\vdash^+$ since from $\beta \vdash^+ \beta$, we must have that $\beta \vdash^+ \coprod\{\,\beta, \delta\,\}$ which is to say that $\beta \vdash_2 \delta$.

It thus follows from the maximality of $\vdash^+$ that $\Gamma \vdash_1 \alpha$ and $\Gamma \vdash_2 \alpha$. By the definitions then, $\Gamma, \beta \vdash^+ \alpha$ and $\Gamma \vdash^+ \coprod\{\,\beta, \alpha\,\}$. By axiom $\alpha \vdash^+ \alpha$ and by dilution $\Gamma, \alpha, \vdash^+ \alpha$. Then, by the universal property of coproducts, $\Gamma, \coprod\{\,\beta, \alpha\,\} \vdash^+ \alpha$. We may then use Cut to derive that $\Gamma \vdash^+ \alpha$ contrary to hypothesis.

The question remains whether we have lost any valuations. Since we are using $\textsc{con}_S$ as the notion of consistency, no Scott relation that satisfies $\textsc{con}_S$ will make every formula true. On the other hand, now that we have given the language more structure, shouldn't we alter the notion of valuation? And if we alter the notion of valuation, how should it be altered?

[12] As Scott remarks in [7], the existence of such a maximal relation is guaranteed by Zorn's lemma.

This takes us into the well explored world of matrix semantics.[13] The idea of a logical matrix is an ordered pair $\langle \mathfrak{A}, D\rangle$ of an algebra[14] of values $\mathfrak{A}$ and a set of special designated values D which is contained in the algebra. The designated values are the values from $\mathfrak{A}$ that we consider to designate *truth*. So if a formula gets a value from D, it is interpreted as true. Given a matrix, we then decide which assignments of formulas to elements of A are acceptable. In essence, we decide which functions $v : \mathcal{L} \to A$ are *models* of $\mathcal{L}$. Of course, what we have been using is the matrix $\langle\langle\{0,1\}, \varnothing\rangle, \{1\}\rangle$, and what Scott's original result shows is that the set of $v \in \{0,1\}^{\mathcal{L}}$ corresponds to all of the $\textsc{con}_M$ and $\textsc{com}_M$ multi-conclusion Scott relations.

But with this new structure in $\mathcal{L}$, viz. coproducts, what should the models be? Inspired by disjunction, we say that a function $v : \mathcal{L} \to \{0,1\}$ is a valuation iff

$$v\left(\coprod(\Delta)\right) = 1 \Longleftrightarrow \exists\delta \in \Delta \text{ s.t. } v(\delta) = 1.$$

As the special case of coproducts for $\varnothing$, it must be $v(\coprod(\varnothing)) = 0$. What happens once we add coproducts, is that the *reasonable* or *acceptable* functions, i.e., valuations, can't make every formula true. That means, in this case *all* of the valuations *are* represented by the $\textsc{con}_S$ and $\textsc{com}_S$ single-conclusion Scott relations. So we face a dilemma: lose valuations, or add structure. But it is a dilemma that is easily resolved: add structure! The moral of these various twists and turns is that it requires (some, but not very much) structure to compress Scott's insight into a single-conclusion framework.

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Appendix

In [2] Kosta Došen suggests axiomatizing multiple conclusion entailment relations by means of

[$\Vdash$ 1] $\Gamma \Vdash \Gamma$

[$\Vdash$ 2.1] for all $\alpha \in \Delta$ $\Gamma \Vdash \{\alpha\} \implies (\Delta \cup \Sigma \Vdash \Lambda \implies \Gamma \cup \Sigma \Vdash \Lambda)$

[$\Vdash$ 2.2] for all $\alpha \in \Gamma$ $\{\alpha\} \Vdash \Delta \implies (\Sigma \Vdash \Lambda \cup \Gamma \implies \Sigma \Vdash \Lambda \cup \Delta)$

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[13] See Wojcicki [9, chapter 3].

[14] An algebra is a pair $\langle A, \tau\rangle$ where A is a set, and τ is a collection of functions $f : A^n \to A$. Note $A^n = \underbrace{A \times \cdots \times A}_{n-\text{times}}$.

It's not difficult to see that since we are using an essentially "set" form of transitivity in this presentation the argument used in Sect. 4 to sketch the proof of bijection, would not need to assume that all the sets involved be finite. This is because we wouldn't, under the alternate structural rules, be forced to cut one formula at a time.

Now we replicate the Scott–Lindenbaum theorem in its new setting. This proof follows the original from Scott [7] almost exactly, hence why we call it Scott's theorem.

Proof. (Proof of Theorem 6.5) Suppose that $\Vdash$ is a-consistent. The if direction is clear. Since $\Vdash$ is a-consistent, it follows that there is a γ such that $\varnothing \nVdash \gamma$, or some δ such that $\gamma \nVdash \delta$. Either way there are some Γ, Δ such that $\Gamma \nVdash \Delta$ and Δ is non-empty. Let C be a chain of a-consistent consequence relations extending $\Vdash$ that obey the structural rules, such that for each $\Vdash' \in C$, $\Gamma \nVdash' \Delta$. Claim: $\Vdash_C =_{df} \cup C$ is a a-consistent consequence relation that obeys the structural rules and Γ doesn't prove Δ, but extends $\Vdash$.

Since C is a chain, $\Vdash \subseteq \Vdash_C$. Now suppose that $\Sigma \cap \Psi \neq \varnothing$, So for $\Sigma \Vdash \Psi$, thus $\Sigma \Vdash_C \Psi$. Suppose that $\Sigma \Vdash_C \Psi$, and $\Sigma \subseteq \Sigma'$, $\Psi \subseteq \Psi'$, so for some $\Vdash' \in C$, $\Sigma \Vdash' \Psi$, thus $\Sigma' \Vdash' \Psi'$, so $\Sigma' \Vdash_C \Psi'$. Now suppose that $\Sigma \Vdash_C \alpha, \Psi$, and $\Sigma, \alpha \Vdash_C \Psi$. So there are $\Vdash'$ and $\Vdash''$ in C such that $\Sigma, \alpha \Vdash' \Psi$ and $\Sigma \Vdash'', \alpha, \Psi$. But since C is a chain, one of the relations is contained in the other, say $\Vdash'$ is the largest, so then $\Sigma \Vdash' \Psi$ by T. Thus, $\Sigma \Vdash_C \Psi$. Finally, suppose that $\Gamma \Vdash_C \Delta$, so there is $\Vdash' \in C$ such that $\Gamma \Vdash' \Delta$, but that is contrary to assumption. So by Zorn's lemma there is a maximal relation $\Vdash^+$ extending $\Vdash$ satisfying the structural rules such that $\Gamma \nVdash^+ \Delta$.

Suppose $\Vdash^+$ isn't a-consistent. So there is φ such that both $\varphi \Vdash^+ \psi$ for all ψ and $\varnothing \Vdash^+ \varphi$. Since Δ isn't empty, there is $\delta \in \Delta$, and $\varphi \Vdash^+ \delta$, but then $\varnothing \Vdash^+ \Delta$ by T. So by M, $\Gamma \Vdash^+ \Delta$. Therefore, $\Vdash^+$ is a-consistent.

Suppose $\Vdash^+$ isn't a-complete. So there is α such that both $\nVdash^+ \alpha$ and $\alpha \nVdash^+ \beta$ for some β. Define new relations

$$\Sigma \Vdash_1 \Psi \Longleftrightarrow \Sigma \Vdash^+ \{\alpha\} \cup \Psi$$
$$\Sigma \Vdash_2 \Psi \Longleftrightarrow \Sigma \cup \{\alpha\} \Vdash^+ \Psi$$

Now, both of these are a-consistent extensions. The first is a-consistent because $\varnothing \nVdash_1 \alpha$. If $\varnothing \Vdash_1 \alpha$, then by definition, $\varnothing \Vdash^+ \{\alpha\} \cup \{\alpha\}$, but that means $\varnothing \Vdash^+ \alpha$. For the second, $\varnothing \nVdash_2 \beta$ since $\alpha \nVdash^+ \beta$. They both obey the structural rules as the reader can check. They are both proper extensions: $\alpha \Vdash_1 \beta$ since $\alpha \Vdash^+ \{\alpha\} \cup \{\beta\}$, and $\varnothing \Vdash_2 \alpha$ since $\alpha \Vdash^+ \alpha$.

Since $\Vdash^+$ is maximal, the only things that could fail for 1 and 2 are that $\Gamma \Vdash_1 \Delta$ and $\Gamma \Vdash_2 \Delta$. But then both $\Gamma \Vdash^+ \alpha, \Delta$ and $\Gamma, \alpha \Vdash^+ \Delta$. So by T, $\Gamma \Vdash^+ \Delta$, a contradiction. ■

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Gillman Payette, Peter K. Schotch
Dalhousie University, Halifax
NS, Canada
gpayette@dal.ca; peter.schotch@dal.ca