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Descriptor Revision

Abstract. A descriptor is a set of sentences that are truth-functional combinations of expressions of the form $\mathfrak{B}p$, where $\mathfrak{B}$ is a metalinguistic belief predicate and p a sentence in the object language in which beliefs are expressed. Descriptor revision (denoted $\circ$) is an operation of belief change that takes us from a belief set K to a new belief set $K \circ \Psi$ where Ψ is a descriptor representing the success condition. Previously studied operations of belief change are special cases of descriptor revision, hence sentential revision can be represented as $\Psi = \{\mathfrak{B}p\}$, contraction as $\Psi = \{\neg\mathfrak{B}p\}$, multiple contraction as $\Psi = \{\neg\mathfrak{B}p_1, \neg\mathfrak{B}p_2, \dots, \neg\mathfrak{B}p_n\}$, replacement as $\Psi = \{\mathfrak{B}p, \neg\mathfrak{B}q\}$, etc. General models of descriptor revision are constructed and axiomatically characterized. The common selection mechanisms of AGM style belief change cannot be used, but they can be replaced by choice functions operating directly on the set of potential outcomes (available belief sets). The restrictions of this construction to sentential revision ($\Psi = \{\mathfrak{B}p\}$) and sentential contraction give rise to operations with plausible properties that are also studied in some some detail.

Keywords: Belief change, Descriptor revision, Expansion, Monoselective choice function, Outcome set, Pure contraction, Revocation, Semirevision.

1. Introduction

In the standard framework for belief change, an individual's beliefs are represented by a belief set, i.e. a logically closed set of sentences. Changes are induced by an input and give rise to an output that is a new belief set [1,4,11,12]. Different types of change are characterized by different requirements on the outcome. In (sentential) revision, a specified sentence should be present in the outcome. In multiple revision, all elements of a specified set of sentences should be present. In (sentential) contraction, a specified sentence should be absent from the outcome. In package contraction, all elements of a specified set of sentences should be absent, whereas in choice contraction at least one element of a specified set of sentences should be absent. In consolidation, falsum should be absent [5,14]. In replacement, one specified sentence should be present and another absent [7], etc. All these are *success conditions* for the respective types of operations.

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In Sect. 2, a general framework for such conditions, *belief descriptors*, is proposed. In Sect. 3, a general operation of *descriptor revision* that covers all operations on belief sets with input-specified conditions on the outcome is introduced. The standard AGM mechanism for choosing what to include in the outcome only works for some descriptors. Therefore, a more general framework is introduced in which the selection takes place on potential outcomes rather than on possible worlds or maximal consistent subsets of the original belief set. Two variants of this construction are axiomatically characterized. In Sect. 4, revision by a single sentence is investigated as a special case, and an ordering-based model is presented that is characterized with plausible axioms. It is shown to have AGM revision as a special case. Section 5 is devoted to the removal of sentences, both in the form of contraction and in the form of a related new type of operation, revocation. All formal proofs are deferred to an Appendix.

Sentences in the object language $\mathcal{L}$ of beliefs will be denoted by lower-case letters (p, q ...) and sets of such sentences by upper-case letters (A,B...). The usual truth-functional operations are denoted $\neg, \&, \vee, \rightarrow$, and $\leftrightarrow$. $\top$ is a tautology and $\perp$ a contradiction. The consequence operator Cn for $\mathcal{L}$ expresses a supraclassical and compact logic satisfying the deduction property ($q \in \text{Cn}(A \cup \{p\})$ if and only if $p \rightarrow q \in \text{Cn}(A)$). $X \vdash p$ is an alternative notation for $p \in \text{Cn}(X)$ and $X + p$ ("expansion") for $\text{Cn}(X \cup \{p\})$. We use K to denote a belief set, i.e. a logically closed subset of the language.

2. Belief Descriptors and Their Interrelations

An *atomic belief descriptor* is sentence $\mathfrak{B}p$ with $p \in \mathcal{L}$. It is *satisfied* by a belief set K if and only if $p \in K$. Atomic descriptors are atomic in the sense of being the smallest building-blocks of descriptors, but they are not logically independent. The symbol $\mathfrak{B}$ is not part of the object language, thus it cannot be used to express an agent's beliefs about her own beliefs. (It cannot either be iterated.)[1]

A *molecular belief descriptor* (denoted by lower-case Greek letters $\alpha, \beta, \ldots$) is a truth-functional combination of atomic descriptors. Conditions of satisfaction for molecular descriptors are defined inductively, hence

[1]The framework can accommodate an operator in the object language that represents the agent's beliefs about her own beliefs. That operator may or may not coincide with the metalinguistic belief operator $\mathfrak{B}$, depending on whether the agent's autoepistemic beliefs accord with her epistemic conduct.

K satisfies $\neg\alpha$ if and only if it does not satisfy α, it satisfies $\alpha \vee \beta$ if and only if it satisfies either α or β, etc.

A *composite belief descriptor* (in short: descriptor; denoted by upper-case Greek letters $\Psi, \Xi, \ldots$) is a set of molecular descriptors. A belief set K satisfies a composite descriptor Ψ if and only if it satisfies all its elements. A descriptor is *satisfiable within* a set of belief sets if and only if it is satisfied by at least one of its elements.

The symbol $\Vdash$ denotes relations of satisfaction. $K \Vdash \Psi$ means that K satisfies Ψ and $\Psi \Vdash \Xi$ that all belief sets satisfying Ψ also satisfy Ξ. The corresponding equivalence relation is written $\dashv\Vdash$, hence $\Psi \dashv\Vdash \Xi$ holds if and only if both $\Psi \Vdash \Xi$ and $\Xi \Vdash \Psi$ hold. $\perp\!\!\!\!\perp$ (descriptor falsum) denotes $\{\mathfrak{B}p, \neg\mathfrak{B}p\}$ for an arbitrary p. $\perp\!\!\!\!\perp$ must be distinguished from the falsum $\perp$ of the object language (that is introducible as $p\&\neg p$ for an arbitrary p). The inconsistent belief set $\mathrm{Cn}(\{\perp\})$ satisfies the condition $K \vdash \perp$, but no belief set satisfies the condition $K \Vdash \perp\!\!\!\!\perp$.

Definition 1. A set $\mathbb{Y}$ of belief sets is *descriptor-definable* if and only if there is some descriptor Ψ such that for all belief sets Y:

$$Y \in \mathbb{Y} \text{ if and only if } Y \Vdash \Psi.$$

Observation 1. (1) Let $\mathbb{Y}$ be a finite set of belief sets. Then $\mathbb{Y}$ is descriptor-definable.

(2) If $\mathcal{L}$ is logically infinite[2] then there are sets of belief sets that are not descriptor-definable.

3. General Descriptor Revision

By descriptor revision will be meant an operation $\circ$ that takes us from a belief set K and an input in the form of a descriptor Ψ to a new belief set $K \circ \Psi$. We want $K \circ \Psi$ to be a belief set satisfying Ψ but – and this is the central problem in belief revision – there is typically more than one possible outcome that satisfies Ψ. The traditional solution to this problem is to select a set of candidate belief sets satisfying Ψ and use their intersection as the outcome [1]. This works for contraction since the intersection of a set of belief sets not containing a sentence p does not contain p. However, the use of intersection to adjudicate between equally plausible options does not work for general descriptors. The reason for this is that the intersection of

[2] A set of sentences is logically infinite if and only if it has infinitely many equivalence classes in terms of logical equivalence.

a set of belief sets satisfying Ψ need not satisfy Ψ. This will for instance be the case for the descriptor $\{\mathfrak{B}p \vee \mathfrak{B}q\}$ that is satisfied by each of the belief sets $\text{Cn}(\{p\})$ and $\text{Cn}(\{q\})$ but not by their intersection $\text{Cn}(\{p \vee q\})$. For general descriptors we therefore need another selection mechanism. We will use a choice function (selection function) that chooses directly among the set of potential outcomes of the operation $\circ$, its *outcome set* [8,9]. More precisely, the selection takes place among those elements of the outcome set that satisfy the descriptor.

Definition 2. A *choice function* for a set $\mathbb{X}$ is a function s such that if $\varnothing \neq \mathbb{Y} \subseteq \mathbb{X}$ then $\varnothing \neq s(\mathbb{Y}) \subseteq \mathbb{Y}$, and otherwise $s(\mathbb{Y})$ is undefined.

A choice function s for $\mathbb{X}$ is *monoselective* if and only if it holds for all $\mathbb{Y}$ that if $\varnothing \neq \mathbb{Y} \subseteq \mathbb{X}$ then $s(\mathbb{Y})$ has exactly one element.

Definition 3. Let s be a choice function on a set $\mathbb{X}$ of belief sets. Then $s(\Psi)$ is an abbreviation of $s(\{X \in \mathbb{X} \mid X \Vdash \Psi\})$.

The following representation theorem introduces what we will take to be the most general form of *selection-based descriptor revision*:

Theorem 1. *Let $\circ$ be an operation on a consistent belief set K, with descriptors as inputs and belief sets as outputs. Then the following two conditions are equivalent:*

(I) $\circ$ is obtainable as $K \circ \Psi = s(\Psi)$ from a set $\mathbb{X}$ of belief sets such that $K \in \mathbb{X}$, and a monoselective choice function s on the descriptor-definable subsets of $\mathbb{X}$, such that for all Ψ: (a) if there is some $X \in \mathbb{X}$ with $X \Vdash \Psi$, then $s(\Psi) \Vdash \Psi$, and (b) otherwise $s(\Psi) = K$.

(II) $\circ$ satisfies the postulates:

$K \circ \Psi = \text{Cn}(K \circ \Psi)$ *(closure)*

$K \circ \Psi \Vdash \Psi$ *or* $K \circ \Psi = K$ *(relative success)*

If $K \circ \Xi \Vdash \Psi$ *then* $K \circ \Psi \Vdash \Psi$ *(regularity)*

Relative success and regularity can be seen as weakened forms of a strong success postulate $K \circ \Psi \Vdash \Psi$ that cannot be used since it is not satisfied by the inconsistent descriptor $\mathrel{\underline{\perp}}$. These two postulates are both generalizations of conditions that have been used for the characterization of semi-revision (not always successful revision) and shielded contraction (not always successful contraction). (Relative success was introduced in [15] and regularity in [10].)

The model of descriptor revision introduced in Theorem 1 includes most plausible patterns of belief change but also some utterly implausible ones. It is for instance compatible with the "absolutely stubborn" pattern such that $\mathbb{X} = \{K\}$ and $K \circ \Psi = K$ for all Ψ.

A more orderly class of operations can be obtained by deriving the selection function from a relation $\leqq$ on $\mathbb{X}$ (with the strict part $<$), in such manner that $K \circ \Psi$ is required to be $\leqq$-minimal among the elements of $\mathbb{X}$ that satisfy Ψ. For this to work, all descriptor-definable subsets of $\mathbb{X}$ must have a minimal element (a weakened form of well-foundedness):

DEFINITION 4. (1) Let $\leqq$ be a relation on $\mathbb{X}$ and let $\mathbb{Y} \subseteq \mathbb{X}$. Then X is $\leqq$-*minimal* in $\mathbb{Y}$ if and only if $X \in \mathbb{Y}$ and $X \leqq Y$ for all $Y \in \mathbb{Y}$.

(2) A relation $\leqq$ on a set $\mathbb{X}$ of belief sets is *descriptor-wellfounded* if and only if each non-empty descriptor-definable subset of $\mathbb{X}$ has a $\leqq$-minimal element.

Relational descriptor revision can be axiomatically characterized as follows:

THEOREM 2. *Let $\circ$ be an operation on a consistent belief set K, with descriptors as inputs and belief sets as outputs. Then the following three conditions are equivalent:*

(I) There is a set $\mathbb{X}$ of belief sets with $K \in \mathbb{X}$, and a relation $\leqq$ on $\mathbb{X}$, such that (i) $K \leqq X$ for all $X \in \mathbb{X}$, and (ii) $K \circ \Psi$ is the unique $\leqq$-minimal element of $\mathbb{X}$ that satisfies Ψ, unless Ψ is unsatisfiable within $\mathbb{X}$, in which case $K \circ \Psi = K$.

(II) There is a set $\mathbb{X}$ of belief sets with $K \in \mathbb{X}$, and a complete, transitive, antisymmetric, and descriptor-wellfounded relation $\leqq$ on $\mathbb{X}$, such that (i) $K \leqq X$ for all $X \in \mathbb{X}$, and (ii) $K \circ \Psi$ is the unique $\leqq$-minimal element of $\mathbb{X}$ that satisfies Ψ, unless Ψ is unsatisfiable within $\mathbb{X}$, in which case $K \circ \Psi = K$.

(III) $\circ$ satisfies the postulates:

If $\Psi \dashv\Vdash \Psi'$ *then* $K \circ \Psi = K \circ \Psi'$ *(extensionality)*

$K \circ \Psi = \mathrm{Cn}(K \circ \Psi)$ *(closure)*

If $K \Vdash \Psi$ *then* $K \circ \Psi = K$ *(confirmation)*

$K \circ \Psi \Vdash \Psi$ *or* $K \circ \Psi = K$ *(relative success)*

If $K \circ \Xi \Vdash \Psi$ *then* $K \circ \Psi \Vdash \Psi$ *(regularity)*

If $K \circ \Psi \Vdash \Xi$ *then* $K \circ \Psi = K \circ (\Psi \cup \Xi)$ *(cumulativity)*

As can be seen from Theorem 2, the distinction between relational and transitively relational operations that is important in AGM [1,16] cannot be transferred to this framework. Here, all relational operations are also transitively relational, i.e. based on a transitive relation. We will refer to the operation characterized in Theorem 2 as *linearly ordered descriptor revision*.

The cumulativity postulate that is used in the theorem can alternatively be replaced by a postulate of reciprocity:

Observation 2. Let $\circ$ be a descriptor revision on a consistent belief set K. If it satisfies regularity and relative success, then it satisfies cumulativity if and only if it satisfies:

If $K \circ \Psi \Vdash \Xi$ and $K \circ \Xi \Vdash \Psi$ then $K \circ \Psi = K \circ \Xi$ (reciprocity)

4. Sentential (Semi)revision Reconstructed

By restricting an operator of descriptor revision to a class of descriptors that only mention a single object language sentence we can construct a *sentential operation* on belief sets, i.e. an operation that takes a single sentence as input and has a new belief set as its output. Most obviously, we can obtain sentential revision by restricting our attention to descriptors of the form $\mathfrak{B}p$, using the definition

$K * p = K \circ \mathfrak{B}p$

We can use the construction in Theorem 1 to obtain a selection-based model of semirevision (revision that may sometimes fail) as follows:

Theorem 3. *Let $*$ be a sentential operation on the consistent belief set K. Then the following two conditions are equivalent:*

(I) $K * p = K \circ \mathfrak{B}p$ *for all* $p \in \mathcal{L}$, *where* $\circ$ *is a selection-based descriptor revision whose choice function* s *satisfies:*

(s0) If $K \Vdash \mathfrak{B}p$ *then* $s(\mathfrak{B}p) = K$

(II) $*$ *satisfies:*

$K * p = \mathrm{Cn}(K * p)$ *(closure)*

$K * p = K * p'$ *whenever* $\vdash p \leftrightarrow p'$ *(extensionality)*

If $p \in K$ *then* $K * p = K$ *(confirmation)*

If $K * q \vdash p$ *then* $K * p \vdash p$ *(regularity)*

Either $K * p \vdash p$ *or* $K * p = K$ *(relative success)*

The operation axiomatized in Theorem 3 is a semirevision, not a revision, since it does not necessarily satisfy the success postulate for sentential revision ($p \in K * p$). Success can be ensured with the additional requirement that $\mathrm{Cn}(\{\perp\}) \in \mathbb{X}$. As the following theorem shows, an extension to the standard ("basic") AGM model is also easily achievable.

Theorem 4. *Let $*$ be a sentential operation on the consistent belief set K. Then the following two conditions are equivalent:*

(I) $K * p = K \circ \mathfrak{B}p$ *for all* $p \in \mathcal{L}$*, where* $\circ$ *is a selection-based descriptor revision such that the following holds for its choice function* s *and its outcome set* $\mathbb{X}$ *:*

($\mathbb{X}1$) *If* $p \nvdash \perp$ *then* p *is satisfiable within* $\mathbb{X} \backslash \{\mathrm{Cn}(\{\perp\})\}$

($\mathbb{X}2$) $\mathrm{Cn}(K \cup \{p\}) \in \mathbb{X}$

($s1$) *If* $K \Vdash \mathfrak{B}p$ *then* $s(\mathfrak{B}p) = K$

($s2$) *If* $\mathfrak{B}p \nVdash \mathfrak{B}\perp$ *then* $s(\mathfrak{B}p) \nVdash \mathfrak{B}\perp$

($s3$) *For all* $X \in \mathbb{X}$*: If* $K \subseteq X \subset s(\mathfrak{B}p)$ *then* $p \notin X$

($s4$) *If* $K \nvdash \neg p$ *then* $K \subseteq s(\mathfrak{B}p)$

(II) $*$ *satisfies the basic AGM postulates, i.e.:*

$K * p = \mathrm{Cn}(K * p)$ *(closure)*

$p \in K * p$ *(success)*

$K * p \subseteq K + p$ *(inclusion)*

If $\neg p \notin K$*, then* $K + p \subseteq K * p$*. (vacuity)*

$K * p$ *is consistent if* p *is consistent. (consistency)*

$K * p = K * p'$ *whenever* $\vdash p \leftrightarrow p'$*. (extensionality)*

($\mathbb{X}2$) derives form the property of AGM revision that if $K \nvdash \neg p$, then $K * p = K + p$. As the following example will show this is a contestable property due to the non-monotonic nature of belief revision. John is a neighbour about whom I know next to nothing. Then I am told that he goes home from work by taxi every day (t). When incorporating this new information into my belief set, I also start to believe that John is a rich man (r). Thus $r \in K * t$. Suppose that $K * t = K + t$. Then equivalently $t \rightarrow r \in K$. Consider an alternative situation in which I receive, together with the information t, also the information that John is a driver by profession (d). This will lead me to incorporate the information $t\&d$ into my belief set. Since $K \nvdash \neg(t\&d)$ we should have $K * (t\&d) = K + (t\&d)$. It then follows from $t \rightarrow r \in K$ that $r \in K * (t\&d)$, i.e. that I will believe in this case as well that John is a rich man. This is implausible, and we can conclude that $K * t \neq K + t$. There does not either seem to be any other input sentence t' such that $K * t' = K + t$. This speaks against the property ($\mathbb{X}2$) that was used in Theorem 4.

We can obtain a *linearly ordered* version of sentential revision from the linearly ordered descriptor revision of Theorem 2:

THEOREM 5. *Let* $*$ *be a sentential operation on the finite-based and consistent belief set* K*. Then the following two conditions are equivalent:*

(I) $K * p = K \circ \mathfrak{B}p$ *for all* $p \in \mathcal{L}$*, where* $\circ$ *is based on a countable set* $\mathbb{X}$ *of finite-based belief sets with* $K \in \mathbb{X}$ *and a complete, transitive, antisymmetric, and wellfounded ordering* $\leqq$ *on* $\mathbb{X}$ *with* $K \leqq X$ *for all* $X \in \mathbb{K}$*.*

$K \circ \Psi$ is the unique $\leqq$-minimal element of $\mathbb{X}$ that satisfies Ψ, unless Ψ is unsatisfiable within $\mathbb{X}$, in which case $K \circ \Psi = K$.

(II) $*$ satisfies the following conditions:

$K * p = \mathrm{Cn}(K * p)$ *(closure)*

If $K * q \vdash p$ then $K * p \vdash p$ (regularity)

Either $K * p \vdash p$ or $K * p = K$ (relative success)

If $p \in K$ then $K * p = K$ (confirmation)

$K * p = K * p'$ whenever $\vdash p \leftrightarrow p'$ (extensionality)

If K is finite-based then so is $K * p$ (finite-based outcome)

$\{X \mid (\exists t)(X = K * (p \vee t))\}$ is finite (finite gradation)

If $q \in K * p$ then $K * p = K * (p\&q)$ (cumulativity)

The operation $*$ introduced in Theorem 5 is a semi-revision. To make it a revision (i.e. make it satisfy the success condition $p \in K * p$) it is necessary and sufficient to ensure that $\mathrm{Cn}(\{\perp\}) \in \mathbb{X}$. The standard consistency postulate of AGM (If $p \nvdash \perp$ then $K * p \nvdash \perp$) is satisfied if and only if $\mathbb{X}$ and $\leqq$ satisfy the following two conditions:

($\mathbb{X}$1) If $p \nvdash \perp$ then p is satisfiable within $\mathbb{X} \setminus \{\mathrm{Cn}(\{\perp\})\}$.

($\leqq$1) If $X \nvdash \perp$ and $Y \vdash \perp$ then $K \leqq X < Y$.

Finally, the construction of Theorem 5 can be further specified to obtain full AGM revision (satisfying both the basic and the supplementary postulates):

Theorem 6. *Let $*$ be a sentential operation on the consistent belief set K and $\mathbb{X}$ its outcome set. Then the following two conditions are equivalent:*

(I) $*$ is based on a transitive, complete, and antisymmetric relation $\leqq$ on $\mathbb{X}$ (with the strict part $<$), such that:

($*\leqq$) *$K * p$ is the unique $\leqq$-minimal p-containing element of $\mathbb{X}$,*

and furthermore:

($\mathbb{X}$0) $K \in \mathbb{X}$

($\mathbb{X}$1) *If $p \nvdash \perp$ then p is satisfiable within $\mathbb{X} \setminus \{\mathrm{Cn}(\{\perp\})\}$.*

($\mathbb{X}$2+) *If $X \in \mathbb{X}$ then $\mathrm{Cn}(X \cup \{p\}) \in \mathbb{X}$.*

($\leqq$1) *If $X \nvdash \perp$ and $Y \vdash \perp$ then $K \leqq X < Y$.*

($\leqq$2) *If $s \in K * t$ and $(K * s) + v \nvdash \perp$, then $(K * s) + v \leqq (K * t) + v$*

(*II*) *$*$ satisfies the complete set of AGM postulates, i.e.:*

$K * p = \mathrm{Cn}(K * p)$ *(closure)*

$p \in K * p$ *(success)*

$K * p \subseteq K + p$ *(inclusion)*

If $\neg p \notin K$, then $K + p \subseteq K * p$. (vacuity)

$K * p$ is consistent if p is consistent. (consistency)
$K * p = K * p'$ whenever $\vdash p \leftrightarrow p'$. (extensionality)
$K * (p\&q) \subseteq (K * p) + q$ (superexpansion)
If $\neg q \notin K * p$, then $(K * p) + q \subseteq K * (p\&q)$ (subexpansion)

The sentential revision $*$ of Theorem 6 is derivable from a partial descriptor revision $\circ$ that is defined for descriptors of the form $\mathfrak{B}p$ (but not necessarily for other descriptors, since the conditions of part II of the theorem do not guarantee descriptor-wellfoundedness for the relation $\leqq$).

The conditions ($\mathbb{X}$2+) and ($\leqq$2) are both quite strong and arguably implausible. ($\mathbb{X}$2+) is a strengthened version of ($\mathbb{X}$2) that was shown above to be problematic. ($\leqq$2) can be seen as a strengthened version of a principle that holds already for basic AGM (as in Theorem 4): In basic AGM, the original belief set K is preferred to all other potential revision outcomes in the sense that if K is a potential outcome (i.e. it includes the input sentence), then K is indeed the outcome, in preference to all other potential outcomes (belief sets containing the input sentence). This preference is *robust under consistent expansion* in the sense that if $K + v$ is consistent then it is the outcome of revising K by v, in preference to all other potential outcomes (beliefs sets containing v). In full AGM (satisfying the supplementary postulates), this robustness under consistent expansion is carried one step further. It extends from the relationship between K and other belief sets to the relation between any two belief sets. Let $K * t$ be a belief set containing s. Then $K * s \leqq K * t$. Provided that v can be consistently added to $K * s$, according to ($\leqq$2) this priority is retained after expansion by v, i.e. it also holds that $(K * s) + v \leqq (K * t) + v$.

5. Revocation and Contraction

By contraction is meant an operation that removes a specified sentence (and whatever needs to go with it) without adding any new beliefs. But in spite of being a standard operation in belief change theory, contraction does not seem to be a fully realistic type of operation. Of course there are belief changes in real life that are driven by a need to give up a certain belief. However, such changes tend to be caused by the acquisition of some new information that is added to the belief set. The only credible examples of pure contraction that have been presented in the literature are hypothetical contractions such as contractions for the sake of argument [3,13]. Therefore it is an interesting option to give up the inclusion postulate ($K \div p \subseteq K$) and search, without this restriction, for an operation satisfying the descriptor $\neg\mathfrak{B}p$. This type of operation will be called *revocation* and denoted by the operation sign $\dot{-}$. Its most basic form can be characterized as follows (in a weak form that does not satisfy the success postulate):

THEOREM 7. *Let $\dot{-}$ be a sentential operator on the consistent belief set K. Then the following two conditions are equivalent:*
(I) $K \dot{-} p = K \circ \neg\mathfrak{B}p$ for all $p \in \mathcal{L}$, where $\circ$ is a descriptor revision that has the outcome set $\mathbb{X}$ with $K \in \mathbb{X}$, and $\circ$ is based on a monoselective choice function s such that for all Ψ: (a) If $K \Vdash \Psi$ then $s(\Psi) = K$, (b) if there is some $X \in \mathbb{X}$ with $X \Vdash \Psi$, then $s(\Psi) \Vdash \Psi$, and (c) if $X \nVdash \Psi$ for all $X \in \mathbb{X}$, then $s(\Psi) = K$.
(II) $\dot{-}$ satisfies:
$K \dot{-} p = \mathrm{Cn}(K \dot{-} p)$ (closure)
If $K \dot{-} p \vdash p$ then $K \dot{-} q \vdash p$ (persistence; [2])
Either $K \dot{-} p \nvdash p$ or $K \dot{-} p = K$ (relative success; [2])
If $p \notin K$ then $K \dot{-} p = K$ (vacuity)
$K \dot{-} p = K \dot{-} p'$ whenever $\vdash p \leftrightarrow p'$ (extensionality)

We can reconstruct contraction as an idealized form of revocation in which we disregard the additions to the belief set that push out the sentence to be contracted:

DEFINITION 5. $K \div p = K \cap (K \dot{-} p)$ (revocation-based contraction)

A property of sentential operations is *inherited* from revocation to contraction if and only if: whenever the property holds for an operation of revocation it also holds for the operator of contraction that is based on it in the manner of Definition 5.

OBSERVATION 3. 1. The following properties are inherited from revocation to contraction: success (If $\nvdash p$ then $p \notin K - p$), failure (If $\vdash p$ then $K - p = K$), vacuity (If $p \notin K$ then $K - p = K$), closure ($K - p = \mathrm{Cn}(K - p)$), extensionality ($K - p = K - p'$ whenever $\vdash p \leftrightarrow p'$), recovery ($K \subseteq (K - p) + p$), finite-based outcome (If K is finite-based, then so is $K - p$), relative success (Either $K - p \nvdash p$ or $K - p = K$), persistence (If $K - p \vdash p$ then $K - q \vdash p$), conjunctive overlap ($(K - p) \cap (K - q) \subseteq K - (p\&q)$), conjunctive factoring (Either $K - (p\&q) = K - p$, $K - (p\&q) = K - q$, or $K - (p\&q) = (K - p) \cap (K - q)$), and conjunctive trisection (If $p \in K - (p\&q)$ then $p \in K - (p\&q\&r)$).

2. If an operation of revocation satisfies vacuity and conjunctive inclusion (If $p \notin K - (p\&q)$ then $K - (p\&q) \subseteq K - p$), then the corresponding revocation-based contraction satisfies conjunctive inclusion.

Observation 3 lists all the basic and supplementary AGM postulates except inclusion ($K - p \subseteq K$) that holds for revocation-based contraction anyhow. The observation also enumerates several other contraction postulates that have been mentioned in the literature (and further such postulates could be added). The observation therefore confirms the credibility of using (pure) contraction as an idealization of the more complex processes involved in giving up beliefs. To the extent that an operation of revocation provides us with a reasonable model of giving up beliefs, a large part of its properties should coincide with those of the corresponding contraction operator.

6. Conclusion

The more general form of belief change introduced through descriptor revision is interesting in its own right. In addition it opens up new ways to construct traditional sentential operations. The linearly ordered sentential revision of Theorem 5 is an interesting alternative to the transitively relational partial meet revision of AGM. It has the more plausible properties of the latter, including cumulativity, but it lacks some of the more implausible ones such as superexpansion. (See [17] for a recent review of the controversial AGM postulates.)

But perhaps the most important advantage of this framework is that it allows for the construction of operations of change that do not fit in with the traditional formal framework. This applies to revocation that was used above in the reconstruction of (pure) contraction. It also applies to the notion of "making up one's mind" (with descriptors of the form $\mathfrak{B}p \vee \mathfrak{B}\neg p$) and many others that remain to investigate.

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Appendix: Proofs

Definition 6. Let X be a finite-based set of sentences. Then $\&X$ is the conjunction of all elements of some finite set X' with $\mathrm{Cn}(X') = \mathrm{Cn}(X)$.

Definition 7. The *descriptor disjunction* $\veebar$ is defined by the relationship
$\Psi \veebar \Xi = \{\alpha \vee \beta \mid (\alpha \in \Psi) \;\&\; (\beta \in \Xi)\}$

Lemma 1. *$K \Vdash \Psi \veebar \Xi$ holds if and only if either $K \Vdash \Psi$ or $K \Vdash \Xi$ holds.*

Proof of lemma 1. From the distribution of disjunction over conjunction. ∎

Lemma 2. *Let $\circ$ be a descriptor revision on a consistent belief set K. Then:*
(1) If $\circ$ satisfies extensionality and cumulativity, then it satisfies:
If $K \circ (\Psi \veebar \Xi) \Vdash \Psi$ then $K \circ (\Psi \veebar \Xi) = K \circ \Psi$ (disjunctive implication)
(2) If $\circ$ satisfies regularity and disjunctive implication then it satisfies reciprocity.

Proof of lemma 2. Part 1: Let $\Psi = \Xi \veebar \Sigma$. By substitution into cumulativity we obtain:

If $K \circ (\Xi \veebar \Sigma) \Vdash \Xi$ then $K \circ (\Xi \veebar \Sigma) = K \circ ((\Xi \veebar \Sigma) \cup \Xi)$,
and since $(\Xi \veebar \Sigma) \cup \Xi \dashv\Vdash \Xi$ extensionality yields the desired result.

Part 2: Let $K \circ \Psi \Vdash \Xi$ and $K \circ \Xi \Vdash \Psi$. Then $K \circ \Psi \Vdash \Psi \veebar \Xi$, and regularity yields $K \circ (\Psi \veebar \Xi) \Vdash \Psi \veebar \Xi$, thus either $K \circ (\Psi \veebar \Xi) \Vdash \Psi$ or $K \circ (\Psi \veebar \Xi) \Vdash \Xi$.

If $K \circ (\Psi \veebar \Xi) \Vdash \Psi$, then disjunctive implication yields $K \circ (\Psi \veebar \Xi) = K \circ \Psi$, thus $K \circ (\Psi \veebar \Xi) \Vdash \Xi$, and disjunctive implication yields $K \circ (\Psi \veebar \Xi) = K \circ \Xi$. Hence, $K \circ \Psi = K \circ \Xi$.

If $K \circ (\Psi \veebar \Xi) \Vdash \Xi$ then $K \circ \Psi = K \circ \Xi$ is obtained in the same way. ∎

Lemma 3. *If $*$ satisfies closure, extensionality and*
If $q \in K * p$ then $K * p \subseteq K * (p\&q)$ (8c, one direction of cumulativity), then it satisfies
If $p \in K * p$ and $K * p = K * (p \vee q \vee r)$, then $K * p = K * (p \vee q)$. (disjunctive interpolation)

Proof of lemma 3. Let $p \in K * p$ and $K * p = K * (p \vee q \vee r)$. Due to closure, $p \vee q \in K * (p \vee q \vee r)$. 8c yields $K * (p \vee q \vee r) \subseteq (K * ((p \vee q \vee r) \& (p \vee q)))$, and due to extensionality $K * (p \vee q \vee r) \subseteq K * (p \vee q)$. Equivalently, $K * p \subseteq K * (p \vee q)$.

Due to $p \in K * p$ we then have $p \in K * (p \vee q)$, 8c yields $K * (p \vee q) \subseteq K * ((p \vee q) \& p)$, and extensionality yields $K * (p \vee q) \subseteq K * p$. We can conclude that $K * p = K * (p \vee q)$ as desired. ∎

Lemma 4. *Let $*$ be a revision operator that satisfies the complete set of AGM postulates (as presented in Theorem 6). Then it satisfies:*

(1) *Either $K * (p \vee q) = K * p$, $K * (p \vee q) = K * q$, or $K * (p \vee q) = (K * p) \cap (K * q)$ (disjunctive factoring)*

(2) $(K * p) \cap (K * q) \subseteq K * (p \vee q)$ *(disjunctive overlap)*

(3) *If $q \in K * p$ and $p \in K * q$ then $K * p = K * q$ (reciprocity)*

Proof of lemma 4. See [4], pp. 211–212 or [6], pp. 270–274. ∎

LEMMA 5. *Let $*$ be a revision operator that satisfies the complete set of AGM postulates (as presented in Theorem 6). Then it satisfies:*

(1) *If $p \in K * q$ then $p \in K * (p \vee q)$.*

(2) *If $p \in K * (p \vee q)$ then $K * (p \vee q) = K * p$.*

(3) *If $K * z = (K * p) \cap (K * q)$, then $K * z = K * (p \vee q)$.*

(4) *If $K * q_1 = K * q_2$ then $K * (p \vee q_1) = (K * p) \cap (K * q_1)$ if and only if $K * (p \vee q_2) = (K * p) \cap (K * q_2)$.*

(5) *If $K * p = K * q$ then $K * p = K * (p \vee q)$*

(6) *If $K * q_1 = K * q_2$ then $K * (p \vee q_1) = K * q_1$ if and only if $K * (p \vee q_2) = K * q_2$.*

(7) *If $K * q_1 = K * q_2$ then $K * (p \vee q_1) = K * (p \vee q_2)$.*

(8) *If $p_1 \in K * p_2$, $p_2 \in K * p_3$, $\ldots$ $p_{n-1} \in K * p_n$, and $p_n \in K * p_1$, then $K * p_1 = K * p_2 = \ldots K * p_n$.*

PROOF OF LEMMA 5. Part 1: It follows from disjunctive factoring (Lemma 4) that $K * (p \vee q)$ is either $K * p$, $K * q$, or $(K * p) \cap (K * q)$. We have $p \in K * q$ and due to success also $p \in K * p$, and it follows in all three cases that $p \in K * (p \vee q)$.

Part 2: Due to success, $p \vee q \in K * p$. We apply reciprocity (Lemma 4) to $p \in K * (p \vee q)$ and $p \vee q \in K * p$.

Part 3: Let $K*z = (K*p)\cap(K*q)$. It follows from success that $p\vee q \in K*z$. Success also yields $z \in (K * p) \cap (K * q)$. Due to disjunctive overlap (Lemma 4), we have $(K * p) \cap (K * q) \subseteq K * (p \vee q)$, thus $z \in K * (p \vee q)$. We apply reciprocity (Lemma 4) to $p \vee q \in K * z$ and $z \in K * (p \vee q)$.

Part 4: $K * (p \vee q_1) = (K * p) \cap (K * q_1)$
$K * (p \vee q_1) = (K * p) \cap (K * q_2)$
$K * (p \vee q_2) = (K * p) \cap (K * q_2)$ (Part 3 of this lemma)

Part 5: Directly from disjunctive factoring (Lemma 4).

Part 6: Let $K * (p \vee q_1) = K * q_1 = K * q_2$. Part 5 yields $K * q_2 = K * (p \vee q_1 \vee q_2)$ and disjunctive interpolation (Lemma 3) yields $K * q_2 = K * (p \vee q_2)$.

Part 7: Let $K * q_1 = K * q_2$. There are three cases:

Case 1, $K * (p \vee q_1) = (K * p) \cap (K * q_1)$: It follows from Part 4 that $K * (p \vee q_2) = (K * p) \cap (K * q_2)$, thus $K * (p \vee q_1) = K * (p \vee q_2)$.

Case 2, $K*(p\vee q_1) = K*q_1$: It follows from Part 6 that $K*(p\vee q_2) = K*q_2$, thus $K * (p \vee q_1) = K * (p \vee q_2)$.

Case 3, $K * (p \vee q_1) \neq (K * p) \cap (K * q_1)$ and $K * (p \vee q_1) \neq K * q_1$: It follows from disjunctive factoring (Lemma 4) that $K * (p \vee q_1) = K * p$. Suppose that $K * (p \vee q_2) = (K * p) \cap (K * q_2)$. Then it follows from Part

4 that $K * (p \vee q_1) = (K * p) \cap (K * q_1)$. This contradiction shows that $K * (p \vee q_2) \neq (K * p) \cap (K * q_2)$. Next, suppose that $K * (p \vee q_2) = K * q_2$. Then it follows from Part 6 that $K * (p \vee q_1) = K * q_1$, again contradicting our assumptions, thus $K * (p \vee q_2) \neq K * q_2$. Due to disjunctive factoring (Lemma 4) it follows from $K * (p \vee q_2) \neq (K * p) \cap (K * q_2)$ and $K * (p \vee q_2) \neq K * q_2$ that $K * (p \vee q_2) = K * p$, thus $K * (p \vee q_1) = K * (p \vee q_2)$ in this case as well.

Part 8: It follows from $p_1 \in K * p_2$ and Parts 1 and 2 that $K * p_1 = K * (p_1 \vee p_2)$, and similarly that $K * p_k = K * (p_k \vee p_{k+1})$ for all k with $1 \leq k < n$ and $K * p_n = K * (p_n \vee p_1)$. From $K * p_1 = K * (p_1 \vee p_2)$ and $K * p_2 = K * (p_2 \vee p_3)$ we obtain via Part 7 that $K * p_1 = K * (p_1 \vee p_2 \vee p_3)$. Repeating this step we obtain $K * p_1 = K * (p_1 \vee p_2 \vee \cdots \vee p_n)$. In the same way we obtain $K * p_k = K * (p_1 \vee p_2 \vee \cdots \vee p_n)$ for all k with $1 \leq k \leq n$. ■

LEMMA 6. *Let $\leqq$ be a relation on a set $\mathbb{X}$ of belief sets. Then the following three conditions are equivalent:*

(A) *For all descriptors Ψ that are satisfiable within $\mathbb{X}$ there is a unique $\leqq$-minimal Ψ-satisfying element X of $\mathbb{X}$, i.e. a unique element X such that $X \leqq Y$ for all $Y \in \mathbb{X}$ with $Y \Vdash \Psi$.*

(B) *$\leqq$ is antisymmetric, complete, transitive and descriptor-wellfounded.*

(C) *$\leqq$ is antisymmetric and descriptor-wellfounded.*

PROOF OF LEMMA 6. From (A) to (B): Antisymmetry: Suppose to the contrary that $X \leqq Y \leqq X$ and $X \neq Y$. Then both X and Y are $\leqq$-minimal elements for the descriptor $\Pi_{\{X,Y\}}$ as defined in the proof of Observation 1. This contradicts (A).

Completeness: Let $X, Y \in \mathbb{X}$. Apply (A) to the descriptor $\Pi_{\{X,Y\}}$.

Transitivity: Let $X \leqq Y \leqq Z$ and suppose to the contrary that $X \nleqq Z$. Since $\leqq$ is complete it is reflexive, thus $X \neq Z$. It also follows from completeness and $X \nleqq Z$ that $Z \leqq X$.

If $X = Y$ then $Y \leqq Z$ would yield $X \leqq Z$, contrary to what we have assumed. Thus $X \neq Y$. If $Y = Z$, then $X \nleqq Z$ would yield $X \nleqq Y$, also contradicting our assumptions. Thus $Y \neq Z$. We therefore have the cycle $X \leqq Y \leqq Z \leqq X$ of three distinct elements, which means that there is no unique $\leqq$-minimal element for $\Pi_{\{X,Y,Z\}}$, contrary to (A). We can conclude that $X \leqq Z$.

Descriptor-wellfoundedness follows directly from (A).

From (C) to (A): Let Ψ be a descriptor that is satisfiable within $\mathbb{X}$. Since $\leqq$ is descriptor-wellfounded there is some $\leqq$-minimal Ψ-satisfying element X of $\mathbb{X}$. Suppose that there is some other such element Y. Then $X \leqq Y$

and $Y \leqq X$, and antisymmetry yields $X = Y$. This proves the uniqueness of X. ∎

LEMMA 7. *If $*$ satisfies closure, confirmation, finite-based outcome, cumulativity, relative success, and extensionality, then it satisfies: $K * p = K * \mathcal{E}(K * p)$.*

PROOF OF LEMMA 7. It follows from closure and finite-based outcome that $\mathcal{E}(K*p) \in K*p$. If $K*p = K$ then confirmation yields $K = K*\mathcal{E}(K*p)$. If $K*p \neq K$ we use cumulativity to obtain $K*(p\,\&\,\mathcal{E}(K*p)) = K*p$. Relative success yields $\mathcal{E}(K*p) \vdash p$ and consequently $\vdash p\,\&\,\mathcal{E}(K*p) \leftrightarrow \mathcal{E}(K*p)$, and then extensionality yields $K * \mathcal{E}(K*p) = K*p$. ∎

DEFINITION 8. Let $\star$ be a revision operator on the belief set K and let $p, q \in \mathcal{L}$. Then q *weakens* p if and only if: $p \vdash q$, $K*p \vdash p$, $K*q \vdash q$, and $K*p \neq K*q$.

Let K' and K'' be outcomes of revisions of K. Then K'' is a *weakening* of K' if and only if there are p and q such that $K' = K*p$, $K'' = K*q$, and q weakens p.

LEMMA 8. *Let $\star$ be a revision operator on the belief set K that satisfies extensionality and disjunctive interpolation. If p_2 weakens p_1 and p_3 weakens p_2, then p_3 weakens p_1.*

PROOF OF LEMMA 8. Since $p_1 \vdash p_2$ and $p_2 \vdash p_3$ we have $p_1 \vdash p_3$. Suppose that $K*p_1 = K*p_3$. Due to extensionality we have $K*p_2 = K*(p_1 \vee p_2)$ and $K*p_3 = K*(p_1 \vee p_2 \vee p_3)$. It then follows from our assumption $K*p_1 = K*p_3$ that $K*p_1 = K*(p_1 \vee p_2 \vee p_3)$, but this is impossible due to disjunctive interpolation since $K*p_1 \neq K*p_2$, i.e. $K*p_1 \neq K*(p_1 \vee p_2)$. We can conclude from this contradiction that $K*p_1 \neq K*p_3$, thus p_3 weakens p_1. ∎

LEMMA 9. *If $*$ satisfies closure, confirmation, finite-based outcome, cumulativity, relative success, and extensionality, and K'' is a weakening of K', then there is some t such that*

$K' = K * \mathcal{E}K' \neq K * (\mathcal{E}K' \vee t) = K''$ *and* $K'' \vdash \mathcal{E}K' \vee t$

PROOF OF LEMMA 9. Since K'' is a weakening of K' there are r and s such that

$K' = K*r \neq K*(r \vee s) = K''$, $K' \vdash r$, and $K'' \vdash r \vee s$.

Using finite-based outcome we note that $\mathcal{E}(K*r) \vdash r$ and consequently $\vdash \mathcal{E}(K*r) \vee r \vee s \leftrightarrow r \vee s$. Due to extensionality, $K*(r \vee s) = K*(\mathcal{E}(K*r) \vee r \vee s)$. We know from Lemma 7 that $K' = K*\mathcal{E}K'$. Letting $t = r \vee s$ we obtain directly that

$K' = K * \mathcal{E}K' \neq K * (\mathcal{E}K' \vee t) = K''$
as desired. It follows from $K'' \vdash r \vee s$ that $K'' \vdash \mathcal{E}K' \vee t$. ■

Proof of Observation 1. Part 1: For each belief set Y, let $\Pi_Y = \{\mathfrak{B}p \mid p \in Y\} \cup \{\neg\mathfrak{B}p \mid p \notin Y\}$. Then it holds for all belief sets X that $X \Vdash \Pi_Y$ if and only if $X = Y$. Next, for $\mathbb{Y} = \{Y_1, \ldots, Y_n\}$ let $\Pi_{\mathbb{Y}} = \Pi_{Y_1} \veebar \cdots \veebar \Pi_{Y_n}$. Then it holds for all belief sets X that $X \in \mathbb{Y}$ iff $X \Vdash \Pi_{\mathbb{Y}}$.

Part 2: The set of possible worlds ($\mathcal{W}$) has cardinality $2^{\aleph_0}$. Since possible worlds are belief sets, the set of belief sets has at least cardinality $2^{\aleph_0}$. The set of sets of belief sets is its power set and therefore has higher cardinality than $2^{\aleph_0}$. Descriptors are sets of sentences in a denumerable language and therefore the cardinality of the set of descriptors cannot be higher than $2^{\aleph_0}$.■

Proof of Theorem 1. From (I) to (II): Left to the reader.

From (II) to (I): Let $\mathbb{X} = \{X \mid (\exists\Psi)(X = K \circ \Psi)\}$ and let s be such that $s(\Psi) = K \circ \Psi$. To verify the construction we need to show that (1) all elements of $\mathbb{X}$ are logically closed, (2) $K \in \mathbb{X}$, (3) $s(\Psi) \in \mathbb{X}$ for all Ψ, (4) If $X \in \mathbb{X}$ and $X \Vdash \Psi$ then $s(\Psi) \Vdash \Psi$, and (5) If $X \nVdash \Psi$ for all $X \in \mathbb{X}$, then $s(\Psi) = K$.

(1) follows from closure.

(2) Let Ψ be inconsistent, i.e. $\Psi \dashv\Vdash \perp$. Then $K \circ \Psi \nVdash \Psi$ and it follows from relative success that $K \circ \Psi = K$.

(3) follows directly from the construction.

(4) Let $X \in \mathbb{X}$ and $X \Vdash \Psi$. Due to our construction of $\mathbb{X}$, there is some Ξ with $X = K \circ \Xi$. It follows from $K \circ \Xi \Vdash \Psi$ and regularity that $K \circ \Psi \Vdash \Psi$.

(5) Let $X \nVdash \Psi$ for all $X \in \mathbb{X}$. Then $K \circ \Psi \nVdash \Psi$, and relative success yields $K \circ \Psi = K$. ■

Proof of Theorem 2. The equivalence of (I) and (II) follows from Lemma 6. The direction from (II) to (III) is left to the reader. For the direction from (III) to (I) we define the set $\mathbb{X} = \{X \mid (\exists\Psi)(X = K \circ \Psi)\}$ and the relation $\leqq$ on $\mathbb{X}$ such that for all Ψ and Ξ:

$K \circ \Psi \leqq K \circ \Xi$ if and only if $K \circ \Psi = K \circ (\Psi \veebar \Xi)$.

We have to prove that (1) that $\mathbb{X}$ is a set of belief sets, (2) that it contains K, (3) that K is the $\leqq$-minimal element of $\mathbb{X}$, (4) that if there is some $X \in \mathbb{X}$ with $X \Vdash \Psi$, then $K \circ \Psi$ is the unique $\leqq$-minimal element of $\mathbb{X}$ that satisfies Ψ, and (5) if $X \nVdash \Psi$ for all $X \in \mathbb{X}$, then $K \circ \Psi = K$.

(1) follows from closure.

(2) Since K is a belief set we have $K \Vdash \mathfrak{B}\top$, and it follows from confirmation that $K \circ \mathfrak{B}\top = K$.

(3) Let $X \in \mathbb{X}$. Due to Observation 1 there is some descriptor Ψ such that X is the only Ψ-satisfying element of $\mathbb{X}$. It follows from regularity that

$X = K \circ \Psi$. Due to extensionality $K \circ \mathfrak{B}\top = K \circ (\mathfrak{B}\top \veebar \Psi)$, and it follows from our definition of $\leqq$ that $K \circ \mathfrak{B}\top \leqq K \circ \Psi$. It was shown in (ii) that $K \circ \mathfrak{B}\top = K$, thus $K \leqq K \circ \Psi$, i.e. $K \leqq X$.

(4) Let $K \circ \Xi \Vdash \Psi$. It follows from regularity that $K \circ \Psi \Vdash \Psi$. To prove the unique $\leqq$-minimality of $K \circ \Psi$ among Ψ-satisfying elements of $\mathbb{X}$, we first prove minimality and then uniqueness.

For minimality, suppose to the contrary that $K \circ \Xi \Vdash \Psi$ and $K \circ \Psi \nleqq K \circ \Xi$, i.e. $K \circ \Psi \neq K \circ (\Psi \veebar \Xi)$. It follows from disjunctive implication (that holds due to Lemma 2) that $K \circ (\Psi \veebar \Xi) \nVdash \Psi$.

It follows from $K \circ \Psi \Vdash \Psi$ that $K \circ \Psi \Vdash \Psi \veebar \Xi$, and regularity yields $K \circ (\Psi \veebar \Xi) \Vdash \Psi \veebar \Xi$, thus either $K \circ (\Psi \veebar \Xi) \Vdash \Psi$ or $K \circ (\Psi \veebar \Xi) \Vdash \Xi$. Thus $K \circ (\Psi \veebar \Xi) \Vdash \Xi$, and disjunctive implication (Lemma 2) yields $K \circ \Xi = K \circ (\Psi \veebar \Xi)$, thus $K \circ (\Psi \veebar \Xi) \Vdash \Psi$, contrary to what was just shown.

For uniqueness, suppose to the contrary that there is some $X \in \mathbb{X}$ such that $X \Vdash \Psi$ and $X \leqq K \circ \Psi \neq X$. It follows from our definition of $\mathbb{X}$ that $X = K \circ \Xi$ for some Ξ. Our definition of $\leqq$ yields $K \circ \Xi = K \circ (\Psi \veebar \Xi) \neq K \circ \Psi$. It follows from $K \circ (\Psi \veebar \Xi) \Vdash \Psi$ and disjunctive implication (Lemma 2) that $K \circ (\Psi \veebar \Xi) = K \circ \Psi$. Contradiction. Since this holds for all $X \in \mathbb{X}$ with $X \Vdash \Psi$ we can conclude that $K \circ \Psi$ is the unique $\leqq$-minimal Ψ-satisfying element of $\mathbb{X}$.

(5) Let $K \circ \Xi \nVdash \Psi$ for all $K \circ \Xi \in \mathbb{X}$. Then $K \circ \Psi \nVdash \Psi$, and relative success yields $K \circ \Psi = K$. ■

Proof of Observation 2. From cumulativity to reciprocity: Let $K \circ \Psi \Vdash \Xi$ and $K \circ \Xi \Vdash \Psi$. Then cumulativity yields $K \circ \Psi = K \circ (\Psi \cup \Xi) = K \circ \Xi$.

From reciprocity to cumulativity: Let $K \circ \Psi \Vdash \Xi$. There are two cases.

Case (i), $K \circ \Psi \nVdash \Psi$: Regularity yields $K \circ (\Psi \cup \Xi) \nVdash \Psi$, thus $K \circ (\Psi \cup \Xi) \nVdash \Psi \cup \Xi$. Relative success yields $K \circ \Psi = K = K \circ (\Psi \cup \Xi)$.

Case (ii), $K \circ \Psi \Vdash \Psi$: Then $K \circ \Psi \Vdash \Psi \cup \Xi$. Regularity yields $K \circ (\Psi \cup \Xi) \Vdash \Psi \cup \Xi$. We thus have $K \circ \Psi \Vdash \Psi \cup \Xi$ and $K \circ (\Psi \cup \Xi) \Vdash \Psi$, and reciprocity yields $K \circ \Psi = K \circ (\Psi \cup \Xi)$. ■

Proof of Theorem 3. From I to II: Left to the reader.

From II to I: Let $\mathbb{X} = \{X \mid (\exists p)(K * p = X)\}$ and let s be a function from descriptors to $\mathbb{X}$ such that for all Ψ:

(i) If there is some p such that $\Psi \dashv\Vdash \mathfrak{B}p$, then $s(\Psi) = K * p$ (which is possible due to extensionality)

(ii) Otherwise: (a) if $K \Vdash \Psi$ then $s(\Psi) = K$, (b) if there is some $X \in \mathbb{X}$ with $X \Vdash \Psi$, then $s(\Psi) \Vdash \Psi$, and (c) if $X \nVdash \Psi$ for all $X \in \mathbb{X}$, then $s(\Psi) = K$.

To verify the construction we need to show (1) that $\mathbb{X}$ is a set of belief sets, (2) that $K \in \mathbb{X}$, (3) that $s(\Psi) \in \mathbb{X}$ for all Ψ, (4) that if there is some $X \in \mathbb{X}$ with $X \Vdash \Psi$, then $s(\Psi) \Vdash \Psi$, and (5) that otherwise $s(\Psi) = K$. Finally we need to show (6) that that $\circ$ satisfies (s0) and (7) that $*$ is based on $\circ$ in the way specified above.

(1) follows from closure.

For (2) note that due to closure, $\top \in K$ and thus due to confirmation we have $K * \top = K$ and consequently $K \in \mathbb{X}$.

(3) follows directly from the construction.

(4) For $\Psi \dashv\Vdash \mathfrak{B}p$, note that due to the construction of $\mathbb{X}$, if $X \in \mathbb{X}$ then $X = K * q$ for some q. Thus we have $K * q \vdash p$. Regularity yields $p \in K * p$, i.e. $K * p \Vdash \Psi$, and our construction yields $s(\Psi) = K * p$, thus $s(\Psi) \Vdash \Psi$.

For other descriptors, this follows from the construction.

(5) Let Ψ be such that $K \circ \Xi \nVdash \Psi$ for all Ξ.

For $\Psi \dashv\Vdash \mathfrak{B}p$: It follows that $K \circ \mathfrak{B}p \nVdash \mathfrak{B}p$, thus $K * p \nvdash p$. It follows from relative success that $K * p = K$.

For other descriptors, this follows from the construction.

(6) For $\Psi \dashv\Vdash \mathfrak{B}p$ this follows from confirmation and for other descriptors it follows from the construction.

(7) follows directly from the construction. ■

Proof of Theorem 4. From (I) to (II): Closure follows from the definition of a selection-based descriptor revision in Sect. 3.

Success: It follows from ($\mathbb{X}$2) that for all $p \in \mathcal{L}$ there is some $X \in \mathbb{X}$ such that $p \in X$. It then follows from the definition of descriptor revision that success holds.

Inclusion: If $K \vdash \neg p$ then $K + p = \text{Cn}(\{\perp\})$ and we are done. If $K \nvdash \neg p$ then (s4) yields $K \subseteq s(\mathfrak{B}p)$ and success yields $p \in s(\mathfrak{B}p)$. Due to (s3), $s(\mathfrak{B}p)$ is the smallest superset of K that contains p, i.e. $K * p = K + p$.

Vacuity: Let $\neg p \notin K$. Then $K \subseteq K * p$ follows from (s4) and $p \in K * p$ from success. Closure yields $K + p \subseteq K * p$.

Consistency follows from (s2).

Extensionality: Let $\vdash p \leftrightarrow p'$ and use Definition 3 to obtain $s(\mathfrak{B}p) = s(\{X \in \mathbb{X} \mid X \Vdash \mathfrak{B}p\}) = s(\{X \in \mathbb{X} \mid X \Vdash \mathfrak{B}p'\}) = s(\mathfrak{B}p')$.

From (II) to (I): Let $\mathbb{X} = \{X \mid (\exists p)(K * p = X)\}$ and let s be a function from descriptors to $\mathbb{X}$ such that for all Ψ:

(i) If there is some p such that $\Psi \dashv\Vdash \mathfrak{B}p$, then $s(\Psi) = K * p$ (which is possible due to extensionality)

(ii) Otherwise: (a) if $K \Vdash \Psi$ then $s(\Psi) = K$, (b) if there is some $X \in \mathbb{X}$ with $X \Vdash \Psi$, then $s(\Psi) \Vdash \Psi$, and (c) if $X \nVdash \Psi$ for all $X \in \mathbb{X}$, then $s(\Psi) = K$.

To verify the construction we need to show (1) that $\mathbb{X}$ is a set of belief sets, (2) that $s(\Psi) \in \mathbb{X}$ for all Ψ, (3) that if there is some $X \in \mathbb{X}$ with $X \Vdash \Psi$, then $s(\Psi) \Vdash \Psi$, and (4) that otherwise $s(\Psi) = K$. We also need to show that ($\mathbb{X}$1), ($\mathbb{X}$2), (s1), (s2), (s3), and (s4) hold and (5) that $*$ is based on $\circ$ in the way specified above. For (1), (2) (3), (4), and (5) the same proofs can be used as for the corresponding parts of Theorem 3, but (4) is simplified by use of success.

($\mathbb{X}$1) follows from success and consistency.

($\mathbb{X}$2): When $K \nvdash \neg p$ then this follows from inclusion and vacuity. When $K \vdash \neg p$ then success and closure yield $K + p = \text{Cn}(\{\bot\})$ and we are done.

(s1): Let $p \in K$. It follows from inclusion and vacuity that $s(\mathfrak{B}p) = K * p = K$.

(s2) follows from consistency.

(s3) Let $K \subseteq X \subset s(\mathfrak{B}p)$.

If $\vdash \neg p$: It follows from success that $s(\mathfrak{B}p) = \text{Cn}(\{\bot\})$, and since $X \subset s(\mathfrak{B}p)$ we have $X \nvdash \bot$, thus $p \notin X$.

If $\nvdash \neg p$: It follows from consistency that $s(\mathfrak{B}p) \nvdash \bot$, and then from success and $K \subseteq s(\mathfrak{B}p)$ that $K + p \subseteq s(\mathfrak{B}p)$, thus $K + p \nvdash \bot$. Inclusion and vacuity yield $K * p = K + p$. Since X is logically closed and $K + p$ is the inclusion-minimal logically closed set containing both p and K we can conclude from $K \subseteq X \subset K + p$ that $p \notin X$.

(s4) Let $K \nvdash \neg p$. It follows from vacuity that $K \subseteq K * p = s(\mathfrak{B}p)$. ■

Proof of Theorem 5. From (I) to (II): Left to the reader.

From (II) to (I): We are going to construct a set $\mathbb{X}$ and a relation $\leqq$ on $\mathbb{X}$, and define $\circ$ as indicated in the theorem. It then needs to be verified that (1) $\mathbb{X}$ is a countable set of finite-based belief sets, (2) $K \in \mathbb{X}$, (3) $\leqq$ is a complete, transitive, antisymmetric, and wellfounded ordering on $\mathbb{X}$, (4) $K \leqq X$ for all $X \in \mathbb{X}$, (5) if Ψ is satisfied by some element of $\mathbb{X}$ then $K \circ \Psi$ is the $\leqq$-minimal element of $\mathbb{X}$ that satisfies Ψ, (6) if Ψ is not satisfied any element of $\mathbb{K}$ then $K \circ \Psi = K$, and (7) $K * p = K \circ \mathfrak{B}p$ for all p.

The construction: Let $\mathbb{X} = \{X \mid (\exists p)(X = K * p)\}$. We are going to construct inductively a relation $\leqq$ on $\mathbb{X}$, numbering its elements $K_0, K_1, \ldots,$. This series will also inductively be shown to have the following property:

If X is a weakening of K_m, then $X \in \{K_0, K_1, \ldots, K_{m-1}\}$ (the tightness condition)

We begin by setting $K_0 = K$. Clearly, since K_0 is the first element of the series the tightness condition is satisfied vacuously at this stage. For the inductive construction we use a list containing all sentences $p \in \mathcal{L}$ such that $K * p \vdash p$. In each step, we assume that we already have a series $K_0, \ldots, K_m$ of belief sets and that this series satisfies the tightness condition.

Let p be the first sentence on our list such that $K * p \vdash p$ and $p \notin K_0 \cup \cdots \cup K_m$.

If it holds for all sentences q that weaken $\mathscr{E}(K * p)$ that $K * q \in \{K_0, \ldots, K_m\}$, then let $K_{m+1} = K * p$. If not, then let K_{m+1} be a weakening of $K * p$ such that all weakenings of it are in $\{K_0, \ldots, K_m\}$. This is possible due to finite gradation and Lemma 8. Clearly there is one less weakening of $K * p$ not included in the series $K_0, \ldots, K_{m+1}$ than not included in the series $K_0, \ldots, K_m$. We repeat this process, finding a sentence q that weakens p and such that $K * q$ has no weakening in the series $K_0, \ldots, K_{m+1}$, etc., until we arrive at some belief set K_{m+k} that is identical to $K * p$. After this the process is repeated with the next sentence after p on the list of sentences, etc. Due to Lemma 9, the tightness condition is still satisfied after each addition to the series $K_0, K_1, \ldots$.

To finish the construction we define $\leqq$ such that $K_s \leqq K_t$ if and only if $s \leq t$. We define $\circ$ such that $K \circ \Psi$ is the unique $\leqq$-minimal element of $\mathbb{K}$ that satisfies Ψ, unless Ψ is not satisfied by any element of $\mathbb{K}$, in which case $K \circ \Psi = K$. Furthermore, we let $*$ be a sentential operator such that for all $p \in \mathcal{L}$, $K * p = K \circ \mathfrak{B}p$.

Verification of the construction: (1), (2), (3), (4), (5), and (6) follow directly from the construction. For (7), first consider the case when there is some $X \in \mathbb{X}$ with $r \in X$. We are going to show inductively that $K * r$ is equal to the $\leqq$-minimal element of $\mathbb{X}$ that contains r. Due to regularity and our construction of $\mathbb{X}$ it follows from $r \in X \in \mathbb{X}$ that $K * r \vdash r$.

It follows from $K_0 = K$ and confirmation that for all $r \in \mathcal{L}$: If $r \in K_0$ then $K * r = K_0$ iff K_0 is the lowest ranked set containing r. For the inductive step, we assume that for all r, if $r \in K_0 \cup \cdots \cup K_m$, then $K * r$ is equal to the lowest $\leqq$-ranked of the sets $K_0, \ldots, K_m$. In order to show that the same holds for all sentences $r \in K_0 \cup \cdots \cup K_{m+1}$, let $r \in K_{m+1} \setminus (K_0 \cup \cdots \cup K_m)$. Due to Lemma 7 we have $K_{m+1} = K * \mathscr{E}(K_{m+1})$. Since $\mathscr{E}(K_{m+1}) \vdash r$ we have $\vdash r \leftrightarrow \mathscr{E}(K_{m+1}) \vee r$, and extensionality yields $K * r = K * (\mathscr{E}(K_{m+1}) \vee r)$. Since our construction satisfies the tightness condition $K * \mathscr{E}(K_{m+1})$ has no weakening outside of the list $K_0, \ldots, K_m$. We can conclude that $K * r = K * \mathscr{E}(K_{m+1}) = K_{m+1}$, as desired.

In the other case, $X \nvdash r$ for all $X \in \mathbb{X}$. Then $K * r \nvdash r$, and relative success yields $K * r = K$ as desired. ∎

PROOF OF THEOREM 6. From (I) to (II): Closure: Due to the above definition of sentential operations (at the beginning of Sect. 4), $\mathbb{X}$ is a collection of belief sets.

Success: It follows from ($\mathbb{X}$2+) that for all $p \in \mathcal{L}$ there is some X with $p \in X \in \mathbb{X}$. The rest follows from ($*\leqq$).

Inclusion: There are three cases. 1. If $K \vdash \neg p$ then $K + p = \text{Cn}(\{\perp\})$, and we are done.

2. If $K \nvdash \neg p$ and $p \in K$, we note that due to ($\leqq$1), $K \leqq X$ for all $X \in \mathbb{X}$. Therefore ($*\leqq$) yields $K * p = K$, i.e. $K * p = K + p$.

3. If $K \nvdash \neg p$ and $p \notin K$: We obtain $K + p \in \mathbb{X}$ from ($\mathbb{X}$0) and ($\mathbb{X}$2+). Let Z be an element of $\mathbb{X}$ that contains p. It follows from ($\leqq$2), $K = K * \top$ and $\top \in Z$ that $K + p \leqq Z$. (Substitute $\top$ for s, Z for $K * t$, and p for v.) Thus $K + p$ is the $\leqq$-minimal p-containing element of $\mathbb{X}$, and due to our construction $K * p = K + p$.

Vacuity follows from parts 2-3 of the proof of inclusion.

Consistency follows from ($\mathbb{X}$1), ($\leqq$1), and ($*\leqq$).

Extensionality follows from ($*\leqq$) since if p and p' are equivalent, then they are included in the same elements of $\mathbb{X}$.

Superexpansion and subexpansion: If $\neg q \in K * p$ then $(K * p) + q = \text{Cn}(\{\perp\})$ and we are done.

If $\neg q \notin K * p$ we need to show that $(K * p) + q$ is the $\leqq$-minimal element of $\mathbb{X}$ that contains $p\&q$. It clearly contains $p\&q$. Let Z be some $p\&q$-containing element of $\mathbb{X}$. We can use ($\leqq$2) (substituting p for s, q for v, and Z for $K * t$) to obtain $(K * p) + q \leqq Z + q$, i.e. equivalently $(K * p) + q \leqq Z$.

From (II) to (I): We will use the following construction: Let $\mathbb{X} = \{X \mid (\exists p)(K * p = X)\}$. Due to closure, $\mathbb{X}$ is a set of belief sets. We will construct the relation $\leqq$ inductively. The initial step will result in a relation $\leqq_0$ that covers only part of $\mathbb{X} \times \mathbb{X}$. Since $\mathcal{L}$ is denumerable, so is $\mathbb{X}$, and then so is $\mathbb{X} \times \mathbb{X}$. We can therefore have a denumerable list of all pairs of non-identical elements of $\mathbb{X}$. For each $\leqq_k$ we take the first pair $\langle X, Y \rangle$ on that list such that $X \nleqq_k Y \nleqq_k X$ and extend the relation to that pair in order to construct $\leqq_{k+1}$.

Initial step, preliminary part: As a preliminary step in the construction of $\leqq_0$, we introduce the relation $\leqq'$ on $\mathbb{X}$ such that:

For all $X, Y \in \mathbb{X}$: $X \leqq' Y$ if and only if there is some $p \in Y$ such that $X = K * p$.

We need to show that $\leqq'$ satisfies ($*\leqq$), ($\mathbb{X}$0), ($\mathbb{X}$1), ($\mathbb{X}$2+), ($\leqq$1), and ($\leqq$2).

($*\leqq$): It follows from our definition of $\leqq'$ that $K * p$ is $\leqq'$-minimal among the p-containing elements of $\mathbb{X}$. Suppose there is another belief set $K * s$ that is also $\leqq'$-minimal among the p-containing elements of $\mathbb{X}$. Then we have both $K * p \leqq' K * s$ and $K * s \leqq' K * p$, thus $s \in K * p$ and $p \in K * s$. Reciprocity (Lemma 4) yields $K * p = K * s$. Thus $K * p$ is the unique $\leqq'$-minimal p-containing element of $\mathbb{X}$.

($\mathbb{X}0$) Since K is consistent it follows from inclusion and vacuity that $K * \top = K$.

($\mathbb{X}1$) follows from success and consistency.

($\mathbb{X}2+$) Let $X = K * t$. If $(K * t) + p \vdash \bot$, then $(K * t) + p = \text{Cn}(\{\bot\})$, and due to success and closure we have $\text{Cn}(\{\bot\}) = K * \bot$, thus $\text{Cn}(\{\bot\}) \in \mathbb{X}$. If $(K * t) + p \nvdash \bot$, i.e. $(K * t) \nvdash \neg p$, then subexpansion and superexpansion yield $(K * t) + p = K * (t \& p)$, thus $(K * t) + p \in \mathbb{X}$.

($\leqq 1$) Since K is consistent it follows from inclusion and vacuity that $K * \top = K$. Due to closure, $\top \in X$ for all $X \in \mathbb{X}$, thus $K \leqq' X$.

Next, let $X \nvdash \bot$ and $Y \vdash \bot$. Then $X = K * p$ for some p. Due to closure, $Y = \text{Cn}(\{\bot\})$. We can conclude from $p \in Y$ that $K * p \leqq' Y$, thus $X \leqq' Y$. Suppose that $Y \leqq' X$. Then there is some $q \in X$ such that $K * q = \text{Cn}(\{\bot\})$. Due to consistency, $q \vdash \bot$, thus $X \vdash \bot$, contrary to the conditions. Thus $Y \nleqq' X$, thus $X <' Y$.

($\leqq 2$): Let $s \in K * t$ and $K * s \nvdash \neg v$. It follows from superexpansion and subexpansion that $K * (s \& v) = (K * s) + v$. Since $(K * t) + v$ contains $s \& v$, our definition of $\leqq'$ yields $K * (s \& v) \leqq' (K * t) + v$, i.e. $(K * s) + v \leqq' (K * t) + v$.

Concluding the initial step: Next we define $\leqq_0$ as the transitive closure of $\leqq'$. We are going to show that $\leqq_0$ satisfies antisymmetry, ($* \leqq$), ($\leqq 1$), and ($\leqq 2$).

Antisymmetry: Let $X \leqq_0 Y \leqq_0 X$. Then there is a cycle $Z_1 \leqq' Z_2 \leqq' \cdots \leqq' Z_n \leqq' Z_1$, two of whose elements are equal to X, respectively Y. Let $Z_1 = K * p_1, \dots\ Z_n = K * p_n$. Due to the construction of $\leqq'$, we have $p_k \in K * p_{k+1}$ for all k with $1 \leq k \leq n-1$ and $p_n \in K * p_1$. If follows from Lemma 5, Part 8, that $K * p_1 = K * p_2 = \cdots = K * p_n$, thus $X = Y$.

($* \leqq$): That $K * p$ is $\leqq_0$-minimal among the p-containing elements of $\mathbb{X}$ follows from $\leqq' \subseteq \leqq_0$. For uniqueness, let $K * s$ be $\leqq_0$-minimal among the p-containing elements of $\mathbb{X}$. We then have both $K * s \leqq_0 K * p$ and $K * p \leqq_0 K * s$, and antisymmetry that we have just proved yields $K * p = K * s$.

($\leqq 1$) Let $X \nvdash \bot$ and $Y \vdash \bot$. Since ($\leqq 1$) holds for $\leqq'$ as shown above, and $\leqq' \subseteq \leqq_0$ we have $K \leqq_0 X \leqq_0 Y$. To show that $Y \nleqq_0 X$, suppose to the contrary that $Y \leqq_0 X$. Since $\leqq_0$ is the transitive closure of $\leqq'$ there must then be $Z_1, \dots, Z_n \in \mathbb{X}$ that are all non-identical to Y and such that $Y \leqq' Z_1 \leqq' \dots, Z_n \leqq' X$. But $Y \leqq' Z_1$ contradicts ($\leqq 1$) for $\leqq'$. We can conclude that $Y \nleqq_0 X$, thus ($\leqq 1$) holds for $\leqq_0$.

($\leqq 2$) follows from $\leqq' \subseteq \leqq_0$.

The inductive step: We assume that $\leqq_0 \subseteq \leqq_1 \subseteq \cdots \subseteq \leqq_k$ and that $\leqq_k$ satisfies transitivity, antisymmetry, $(*\leqq)$, $(\leqq 1)$, and $(\leqq 2)$. Let $\langle X, Y\rangle$ be the first element of $\mathbb{X} \times \mathbb{X}$ on the list referred to above such that $X \nleqq_k Y \nleqq_k X$. Let $\leqq_{k+1}$ be the transitive closure of $\leqq_k \cup \{\langle X, Y\rangle\}$. We are going to prove that it satisfies antisymmetry, $(*\leqq)$, $(\leqq 1)$, and $(\leqq 2)$.

Antisymmetry: Suppose to the contrary that there are Z_1 and Z_2 such that $Z_1 \leqq_{k+1} Z_2 \leqq_{k+1} Z_1$ and $Z_1 \neq Z_2$. Since $\leqq_k$ is antisymmetric, $\langle X, Y\rangle$ appears in at least one of the two chains between Z_1 and Z_2 in the underlying $\leqq_k \cup \{\langle X, Y\rangle\}$.

First case, it occurs only in one of these chains: Without loss of generality, we assume that it appears in the chain from Z_2 to Z_1. If it appears more than once, then the chain can be contracted, so we can assume that it appears exactly once. We then have:

$Z_1 \leqq_k Z_2$, $Z_2 \leqq_k X$, and $Y \leqq_k Z_1$

Since $\leqq_k$ is transitive it follows immediately that $Y \leqq_k X$, contrary to the assumptions of our choice of $\langle X, Y\rangle$.

Second case, it occurs in both chains: We then have:

$Z_1 \leqq_k X$, $Y \leqq_k Z_2$, $Z_2 \leqq_k X$, and $Y \leqq_k Z_1$

In this case as well it follows from the transitivity of $\leqq_k$ that $Y \leqq_k X$, contrary to the assumptions of our choice of $\langle X, Y\rangle$. We can conclude that $\leqq_{k+1}$ is antisymmetric.

$(*\leqq)$: It follows from $\leqq_k \subseteq \leqq_{k+1}$ that $K * p$ is $\leqq_{k+1}$-minimal among the p-containing elements of $\mathbb{X}$. To show that it is unique, let $K * s$ be $\leqq_{k+1}$minimal among the p-containing elements of $\mathbb{X}$. Then $K*p \leqq_{k+1} K*s$ and $K * s \leqq_{k+1} K * p$, and antisymmetry that we have just proved yields $K * p = K * s$.

$(\leqq 1)$ Let $Z \nvdash \perp$. Since $(\leqq 1)$ holds for $\leqq_k$ and $\leqq_k \subseteq \leqq_{k+1}$ we have $K \leqq_{k+1} Z \leqq_{k+1} \mathrm{Cn}(\{\perp\})$. To show that $Z <_{k+1} \mathrm{Cn}(\{\perp\})$, suppose to the contrary that $\mathrm{Cn}(\{\perp\}) \leqq_{k+1} Z$. Then the relation $\leqq'' = \leqq_k \cup \{\langle X, Y\rangle\}$ forms a chain $\mathrm{Cn}(\{\perp\}) \leqq'' V_1 \leqq'' \cdots \leqq'' V_n \leqq'' Z$, where $V_1, \ldots, V_n$ of elements of $\mathbb{X}$ and each of them is non-identical to $\mathrm{Cn}(\{\perp\})$. Due to $(\leqq 1)$ for $\leqq_k$, $\mathrm{Cn}(\{\perp\}) \nleqq_k V_1$, thus $\mathrm{Cn}(\{\perp\}) = X$. Due to $(\leqq 1)$ for $\leqq_k$, it follows from $X = \mathrm{Cn}(\{\perp\})$ that $Y \leqq_k X$, contrary to the above selection criterion for $\langle X, Y\rangle$, namely that $X \nleqq_k Y \nleqq_k X$. We can conclude from this contradiction that $\mathrm{Cn}(\{\perp\}) \nleqq_{k+1} Z$, thus $(\leqq 1)$ holds for $\leqq_{k+1}$.

$(\leqq 2)$ follows from $\leqq_k \subseteq \leqq_{k+1}$.

Conclusion: Finally, let $\leqq = \leqq_0 \cup \leqq_1 \cup \ldots$. It follows directly that $\leqq$ is transitive and complete and that it satisfies the conditions inductively shown, namely antisymmetry, $(*\leqq)$, $(\leqq 1)$, and $(\leqq 2)$. ■

Proof of Theorem 7. From I to II: Left to the reader.

From II to I: Let $\mathbb{X} = \{X \mid (\exists p)(K \mathbin{\dot{-}} p = X)\}$ and let s be a choice function such that: For all Ψ, if $\Psi \dashv\Vdash \neg\mathfrak{B}p$ then $s(\Psi) = K \mathbin{\dot{-}} p$. (This is possible due to extensionality.) For all other descriptors: (a) if $K \Vdash \Psi$ then $s(\Psi) = K$, (b) if there is some $X \in \mathbb{X}$ with $X \Vdash \Psi$, then $s(\Psi) \Vdash \Psi$, and (c) if $X \nVdash \Psi$ for all $X \in \mathbb{X}$, then $s(\Psi) = K$.

To verify the construction we need to show that (1) $\mathbb{X}$ is a set of belief sets, (2) $K \in \mathbb{X}$, (3) $s(\Psi) \in \mathbb{X}$ for all Ψ, (4) if $K \Vdash \Psi$ then $s(\Psi) = K$, (5) if there is some Ξ with $K \circ \Xi \Vdash \Psi$, then $s(\Psi) \Vdash \Psi$, and (6) otherwise $s(\Psi) = K$. Finally we need to show (7) that $K \mathbin{\dot{-}} p = K \circ \neg\mathfrak{B}p$ for all $p \in \mathcal{L}$.

(1) follows from closure.

For (2) note that since K is consistent, vacuity yields $K \mathbin{\dot{-}} \bot = K$ and consequently $K \in \mathbb{X}$.

(3) follows directly from the construction.

(4) follows from vacuity and the construction.

(5) For $\Psi \dashv\Vdash \neg\mathfrak{B}p$, note that due to the construction of $\mathbb{X}$, $K \circ \Xi = K \mathbin{\dot{-}} q$ for some q. Thus we have $K \mathbin{\dot{-}} q \nvdash p$. Persistence yields $K \mathbin{\dot{-}} p \nvdash p$, i.e. $K \mathbin{\dot{-}} p \Vdash \Psi$, and our construction yields $s(\Psi) = K \mathbin{\dot{-}} p$, thus $s(\Psi) \Vdash \Psi$. For other descriptors, (4) follows directly from the construction.

(6) Let Ψ be such that $K \circ \Xi \nVdash \Psi$ for all Ξ. For $\Psi \dashv\Vdash \neg\mathfrak{B}p$ it follows that $K \circ \neg\mathfrak{B}p \nVdash \Psi$, thus $K \mathbin{\dot{-}} p \vdash p$. It follows from relative success that $K \mathbin{\dot{-}} p = K$, thus $K \circ \Psi = K$. For other descriptors, (5) follows directly from the construction.

(7) follows directly from the construction. ■

Proof of Observation 3. Part 1: All these proofs are straight-forward. The following are given as examples:

Recovery: $K \subseteq (K \mathbin{\dot{-}} p) + p$

If $r \in K$ then $p \rightarrow r \in K \mathbin{\dot{-}} p$

If $r \in K$ then $p \rightarrow r \in (K \mathbin{\dot{-}} p) \cap K$

If $r \in K$ then $p \rightarrow r \in K \div p$

$K \subseteq (K \div p) + p$

Finite-based outcome: If K is finite-based, then there is a sentence $\&K$ such that $\mathrm{Cn}(\{\&K\}) = K$, and since $\mathbin{\dot{-}}$ satisfies the postulate there is for each p a sentence $\&(K \mathbin{\dot{-}} p)$ such that $\mathrm{Cn}(\{\&(K \mathbin{\dot{-}} p)\}) = K \mathbin{\dot{-}} p$. Then $K \div p = K \cap (K \mathbin{\dot{-}} p) = \mathrm{Cn}(\{(\&K) \vee (\&(K \mathbin{\dot{-}} p))\})$.

Persistence: Let $\mathbin{\dot{-}}$ satisfy persistence, and let $K \div p \vdash p$. Then:

$K \vdash p$ and $K \mathbin{\dot{-}} p \vdash p$

$K \vdash p$ and $K \mathbin{\dot{-}} q \vdash p$ (persistence for $\mathbin{\dot{-}}$)

$K \div q \vdash p$

Part 2: Let $\dot{-}$ satisfy vacuity and conjunctive inclusion, and let $p \notin K \div (p\&q)$. Then either $p \notin K \mathbin{\dot{-}} (p\&q)$ or $p \notin K$.

Case i, $p \notin K \mathbin{\dot{-}} (p\&q)$:

$K \mathbin{\dot{-}} (p\&q) \subseteq K \mathbin{\dot{-}} p$ (conjunctive inclusion for $\dot{-}$)

$K \cap (K \mathbin{\dot{-}} (p\&q)) \subseteq K \cap (K \mathbin{\dot{-}} p)$

$K \div (p\&q) \subseteq K \div p$

Case ii, $p \notin K$: Then $K \mathbin{\dot{-}} p = K$ due to vacuity for $\dot{-}$. It follows that $K \div p = K$, hence $K \div (p\&q) \subseteq K \div p$. ■

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