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Coherence and Computational Complexity of Quantifier-free Dependence Logic Formulas

Abstract. We study the computational complexity of the model checking problem for quantifier-free dependence logic ($\mathcal{D}$) formulas. We characterize three thresholds in the complexity: logarithmic space (LOGSPACE), non-deterministic logarithmic space (NL) and non-deterministic polynomial time (NP).

Keywords: Dependence, Logic, Team, Complexity, Coherence.

1. Introduction

Dependence logic $\mathcal{D}$ [6] incorporates explicit dependence relations between terms into first-order logic (FO). The dependence relations between terms are expressed by means of dependence atoms

$$=(t_1, \ldots, t_n) \tag{1}$$

which are taken as atomic formulas. The intuitive meaning of (1) is that the values of the terms $t_1, \ldots, t_{n-1}$ determine the value of the term t_n. The expressive power of $\mathcal{D}$ equals that of existential second order logic Σ^1_1 [6].

We are interested in characterizing the computational complexity of the model checking problem for quantifier-free $\mathcal{D}$-formulas. The problem of charting fragments of logics which fall under specific computational classes is a widely studied question in descriptive complexity theory. The classic result in this field by Fagin [1] establishes a perfect match between Σ^1_1-formulas and languages in NP. When we combine Fagin's result with the result that $\mathcal{D}$ is equally expressive to Σ^1_1 [6], we get that the classes of finite structures definable in $\mathcal{D}$ are exactly the ones recognized in NP.

The semantics of $\mathcal{D}$-formulas can be defined in terms of semantic games and in terms of sets of assignments, which we call teams. We focus in this paper only on the Team-semantics in which a formula is evaluated with respect to a set of assignments.

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We study the fragment of k-coherent formulas of dependence logic. A formula ϕ is k-coherent, $k \in \mathbb{N}$, if for all teams $\mathcal{X}$ it holds that ϕ is satisfied by $\mathcal{X}$ if and only if all k-element subsets of the team $\mathcal{X}$ satisfy ϕ. Coherence allows us to evaluate the satisfiability of the formula in finite fixed size subteams, which is very useful as the teams are potentially very large.

We will give a syntactic characterization for the k-coherence of a $\mathcal{D}$-formula. We will show that there are k-coherent formulas for every $k \in \mathbb{N}$ and that a disjunction of two distinct dependence atoms is not k-coherent for any $k \in \mathbb{N}$. We will also show that the set of k-coherent $\mathcal{D}$-formulas is a proper subset of FO.

The main results of the paper are about the computational complexity of the model checking for $\mathcal{D}$-formulas. We will establish that the model checking problem for all k-coherent formulas is in LOGSPACE. When we allow one disjunction in the formula, the model checking can be done in NL. Furthermore, we will show that the model checking problem of two distinct dependence atoms, $=\!(x, y) \vee =\!(z, u)$, is complete for NL. Last we will show that the model checking problem for the formula $=\!(x, y) \vee =\!(z, u) \vee =\!(z, u)$ is NP-complete.

2. Preliminaries

Definition 2.1. The syntax of $\mathcal{D}$ extends FO, defined in terms of $\vee$, $\wedge$, $\neg$, $\forall$, $\exists$, by new atomic formulas of the form

$$=\!(t_1, \ldots, t_n) \tag{2}$$

where $t_1, \ldots, t_n$ are terms. We will denote the set of free variables of ϕ by $Fr(\phi)$. We write $\approx t_1 t_2$ for identity between terms t_1 and t_2.

The semantics of $\mathcal{D}$ is given in terms of sets of assignments, teams.

Definition 2.2. Let $V = \{x_i \mid i \in I\}$, $I \subseteq \mathbb{N}$, be a set of variables and $\mathcal{M}$ a structure with domain M. Then, *an assignment s with domain V and range M* is a function $s : V \to M$. *A team $\mathcal{X}$ with a domain V and range M* is any set of assignments with domain V and range M. We use the following notation when the team is given as a relation:

$$Rel(X) = \{(s(x_1), \ldots, s(x_n)) \mid s \in \mathcal{X}\}.$$

Notice that all the assignments in a team have the same domain.

Definition 2.3. Suppose $\mathcal{X}$ is a team of domain V and range M, $x_n \in V$ and $F : V \to M$ is a function. Let $\mathcal{X}(F, x_n)$ denote the *supplement team*

$$\{s(F(s)/x_n) \mid s \in \mathcal{X}\},$$

where $s(F(s)/x_n)$ is the assignment obtained by replacing $(x_n, s(x_n))$ in s with $(x_n, F(s))$.

Definition 2.4. Suppose $\mathcal{X}$ is a team of domain V and range M and $x_n \in V$. Let $\mathcal{X}(M, x_n)$ denote the *duplicated team*

$$\{s(a/x_n) \mid a \in M,\ s \in \mathcal{X}\},$$

where $s(a/x_n)$ is the tuple obtained by replacing $(x_n, s(x_n))$ in s with (x_n, a).

Definition 2.5. (*Semantics*) Suppose τ is a vocabulary, $\mathcal{X}$ is a team of domain V and range M, $\mathcal{M}$ a τ-structure and ϕ and θ formulas of $\mathcal{D}(\tau)$. The semantics of $\mathcal{D}$-formulas is defined in the following way:

1. $\mathcal{M} \models_{\mathcal{X}} \; =\!(t_1, \ldots, t_n)$, $n > 1$, iff for all $s, s' \in \mathcal{X}$ it holds that, if $s(t_i) = s'(t_i)$ for $i \leq n-1$, then $s(t_n) = s'(t_n)$.
2. $\mathcal{M} \models_{\mathcal{X}} \neg =\!(t_1, \ldots, t_n)$ iff $\mathcal{X} = \emptyset$.
3. $\mathcal{M} \models_{\mathcal{X}} \; \approx t_1 t_2$, iff for every $s \in \mathcal{X}$, $s(t_1) = s(t_2)$.
4. $\mathcal{M} \models_{\mathcal{X}} \neg \approx t_1 t_2$, iff for every $s \in \mathcal{X}$, $s(t_1) \neq s(t_2)$.
5. $\mathcal{M} \models_{\mathcal{X}} R(t_1, \ldots, t_n)$, iff for every $s \in \mathcal{X}$, $(s(t_1), \ldots, s(t_n)) \in R^{\mathcal{M}}$.
6. $\mathcal{M} \models_{\mathcal{X}} \neg R(t_1, \ldots, t_n)$, iff for every $s \in \mathcal{X}$, $(s(t_1), \ldots, s(t_n)) \notin R^{\mathcal{M}}$.
7. $\mathcal{M} \models_{\mathcal{X}} \phi \wedge \theta$, iff $\mathcal{M} \models_{\mathcal{X}} \phi$ and $\mathcal{M} \models_{\mathcal{X}} \theta$.
8. $\mathcal{M} \models_{\mathcal{X}} \phi \vee \theta$, iff there exists $\mathcal{Y}$ and $\mathcal{Z}$, such that $\mathcal{Y} \cup \mathcal{Z} = \mathcal{X}$, $\mathcal{M} \models_{\mathcal{Y}} \phi$ and $\mathcal{M} \models_{\mathcal{Z}} \theta$.
9. $\mathcal{M} \models_{\mathcal{X}} \exists x \phi(x)$, iff there is $F : V \to M$, such that $\mathcal{M} \models_{\mathcal{X}(F,x)} \phi(x)$.
10. $\mathcal{M} \models_{\mathcal{X}} \forall x \phi(x)$, iff $\mathcal{M} \models_{\mathcal{X}(M,x)} \phi(x)$.

Finally, a sentence ϕ is true in a structure $\mathcal{M}$ if $\mathcal{M} \models_{\{\emptyset\}} \phi$. When ϕ is quantifier-free, we use $\mathcal{X} \models \phi$ as a shorthand for $\mathcal{M} \models_{\mathcal{X}} \phi$ and we say that the team $\mathcal{X}$ *is of type* ϕ.

Theorem 2.6. (Downwards closure [6]) *Suppose* $\phi \in \mathcal{D}$ *and* $\mathcal{X}$ *and* $\mathcal{Y}$ *are teams, such that* $\mathcal{Y} \subseteq \mathcal{X}$. *Then the following holds:*

$$\mathcal{M} \models_{\mathcal{X}} \phi \Rightarrow \mathcal{M} \models_{\mathcal{Y}} \phi.$$

It is know that dependence logic and Σ^1_1 are equivalent in the level of sentences [6]. In the level of formulas the setup of the question is slightly different, since dependence logic formulas are verified with respect to sets of assignments. From Theorems 2.7 and 2.8 it follows that the dependence logic formulas have the same expressive power as the fragment of Σ^1_1 in which the relation symbol interpreting the team appears only negatively.

THEOREM 2.7. [6] *For every formula* $\phi(x_1, \ldots, x_k) \in \mathcal{D}(\tau)$, *there is a sentence* $\phi(R)^* \in \Sigma^1_1(\tau \cup \{R\})$, *where* R *is* k*-ary and in which* R *appears only negatively, such that for all* τ*-structures* $\mathcal{M}$ *and teams* $\mathcal{X}$ *with domain* $\{x_1 \ldots, x_k\}$:

$$\mathcal{M} \models_{\mathcal{X}} \phi(x_1, \ldots, x_k) \Leftrightarrow (\mathcal{M}, Rel(X)) \models \phi(R)^*$$

THEOREM 2.8. [4] *For every sentence* $\phi \in \Sigma^1_1(\tau \cup \{R\})$, *where* R *is* k*-ary and in which* R *appears only negatively, there is a formula* $\phi(x_1, \ldots, x_k)^* \in \mathcal{D}(\tau)$, *such that for all* τ*-structures* $\mathcal{M}$ *and teams* $\mathcal{X}$ *with domain* $\{x_1, \ldots, x_k\}$:

$$(\mathcal{M}, Rel(\mathcal{X})) \models \phi(R) \Leftrightarrow \mathcal{M} \models_{\mathcal{X}} \phi(x_1, \ldots x_k)^*.$$

3. Coherence

In this section we study the fragment of coherent $\mathcal{D}$-formulas. We first define k-coherence and then investigate which formulas are coherent and which are not. We will also show that there are formulas which are incoherent.

DEFINITION 3.1. Suppose $\phi(x_1 \ldots, x_n) \in \mathcal{D}$ is quantifier-free. Then ϕ is *k-coherent* if and only if for all structures $\mathcal{M}$ and teams $\mathcal{X}$ of range M, such that $Fr(\phi) \subseteq dom(\mathcal{X})$ the following are equivalent:

1. $\mathcal{M} \models_{\mathcal{X}} \phi$.
2. For all k-element sub-teams $\mathcal{Y} \subseteq \mathcal{X}$ it holds that $\mathcal{M} \models_{\mathcal{Y}} \phi$.

The *coherence-level* of ϕ, is the least natural number k, such that ϕ is k-coherent.

We will observe that the satisfiability of first-order atomic formula is determined by the singleton sub-teams, whereas the satisfiability of the dependence atom is determined by the two-element sub-teams. Furthermore, we will show that conjunction preserve coherence, whereas disjunction does not.

The next proposition follows directly from the definition of semantics for the atomic formulas.

Proposition 3.2. *First-order atomic formulas are* 1*-coherent.*

Proposition 3.3. *Dependence atoms are* 2*-coherent.*

Proof. Suppose $\mathcal{X} \models =(x_1, \ldots, x_n)$. Then by Downwards closure 2.6 it holds that for all 2-assignment subsets $\mathcal{Y} \subseteq \mathcal{X}$ holds $\mathcal{Y} \models =(x_1, \ldots, x_n)$.

Suppose that for all $\{s, s'\} \subseteq \mathcal{X}$ holds $\{s, s'\} \models =(x_1, \ldots, x_n)$. Then by Definition 2.5, if $s(x_1) = s'(x_1)$, then also $s(x_2) = s'(x_2)$. On the other hand, if $s = s'$, then $s(x_2) = s'(x_2)$. Thus for all $s, s' \in \mathcal{X}$ holds that if $s(x_1) = s'(x_1)$, then $s(x_2) = s'(x_2)$. This is exactly the condition of $\mathcal{X} \models =(x_1, \ldots, x_n)$. ■

Proposition 3.4. *Suppose ϕ and ψ are quantifier-free formulas such that ϕ is k-coherent and ψ is l-coherent for $l, k \in \mathbb{N}$, $l \leq k$. Then $\phi \wedge \psi$ is k-coherent.*

Proof. Suppose $\mathcal{X} \models \phi \wedge \psi$. Then by Downwards closure 2.6 all the k-element subsets satisfy $\phi \wedge \psi$.

Suppose all k-element subsets $\mathcal{Y} \subseteq \mathcal{X}$ satisfy $\phi \wedge \psi$. Then $\mathcal{Y}$ satisfies ϕ and ψ for all k-element $\mathcal{Y} \subseteq \mathcal{X}$. Then, by k-coherence of ϕ it follows that $\mathcal{X} \models \phi$. By Downward closure 2.6 and the fact that all l-element subsets of $\mathcal{X}$ are contained in some k-element subset, we conclude that all l-element subsets of $\mathcal{X}$ satisfy ψ. Then by l-coherence of ψ holds $\mathcal{X} \models \psi$. Thus $\mathcal{X} \models \phi \wedge \psi$. ■

Disjunction does not preserve coherence in general. In some cases, however, we can show that disjunction does preserve it.

Proposition 3.5. *Suppose ϕ and ψ are quantifier-free $\mathcal{D}$-formulas, such that ϕ is* 1*-coherent and ψ is k-coherent, $k \in \mathbb{N}$. Then $\phi \vee \psi$ is k-coherent.*

Proof. Suppose $\mathcal{X} \models \phi \vee \psi$ holds. Then by Theorem 2.6 $\mathcal{X}_k \models \phi \vee \psi$ holds for all k-element subsets $\mathcal{X}_k \subseteq \mathcal{X}$.

The other direction: Suppose that $\mathcal{X}_k \models \phi \vee \psi$ holds for all k-element subsets $\mathcal{X}_k \subseteq \mathcal{X}$. Now the division of $\mathcal{X}$ into $\mathcal{Y}$ and $\mathcal{Z}$, such that $\mathcal{Y} \models \phi$ and $\mathcal{Z} \models \psi$ is obtained in the following way for all $s \in \mathcal{X}$:

- $s \in \mathcal{Y}$ iff $\{s\} \models \phi$ and $s \in \mathcal{Z}$ otherwise.

Clearly, it holds that $\mathcal{Y} \models \phi$. Let us show that $\mathcal{Z} \models \psi$. By k-coherence of ψ we have to check that for all k-element subsets $\mathcal{Z}_k \subseteq \mathcal{Z}$, it holds that $\mathcal{Z}_k \models \psi$. Suppose $\mathcal{Z}_k \subseteq \mathcal{Z}$, such that $\mathcal{Z}_k \not\models \psi$. Since all the singletons $s \in \mathcal{Z}_k$ fail ϕ it holds that $\mathcal{Z}_k \not\models \phi \vee \psi$, which is a contradiction with the assumption. Thus all the k-element subsets of $\mathcal{Z}$ satisfy ψ. Then by k-coherence of ψ holds $\mathcal{Z} \models \psi$. Thus $\mathcal{Y} \cup \mathcal{Z} \models \phi \vee \psi$. ■

As established in the previous proposition, combining a k-coherent formula with a 1-coherent formula does not increase the coherence-level. Thus, to obtain formulas with higher coherence-level, we have to take disjunctions over dependence atoms.

We denote the disjunction of size k over a single dependence atom $=(x_1,\ldots,x_n)$, by $\bigvee_k =(x_1,\ldots,x_n)$. We will next show that disjunctions over the same dependence atom increases the coherence-level, i.e. the coherence-level is increased by 1 for each disjunct. We will first define some notions we need in the proof:

Definition 3.6. Suppose $\mathcal{M}$ is a τ-structure and $\mathcal{X}$ is a team of domain V and range M. Suppose $(a_1,\ldots,a_{n-1}) \in M^{n-1}$ and $x_1,\ldots,x_n \in V$. Let $S(a_1,\ldots,a_{n-1})$ is now defined in the following way:

$$S(a_1,\ldots,a_{n-1}) = \{s \in \mathcal{X} \mid (s(x_1),\ldots,s(x_{n-1})) = (a_1,\ldots,a_{n-1})\}.$$

Let $|S(a_1,\ldots,a_{n-1})|^*$ be the number of different values of x_n under the assignments in $S(a_1,\ldots,a_{n-1})$.

Lemma 3.7. *Suppose $\mathcal{M}$ is a τ-structure and $\mathcal{X}$ is a team of domain V and range M. Suppose $(a_1,\ldots,a_{n-1}) \in M^{n-1}$ and $x_1,\ldots,x_n \in V$. Then, the following are equivalent:*

1. $\mathcal{X} \models \bigvee_k =(x_1,\ldots,x_n)$.
2. $|S(\bar{a})|^* \leq k+1$ *for each* $\bar{a} \in M^{n-1}$.

Proof. Suppose (2) holds. Then each $S(a_1,\ldots,a_{n-1})$ can be divided into sets $S(a_1,\ldots,a_{n-1})^i$, $1 \leq i \leq k+1$, where x_n gets a constant value. Now the following partition of $\mathcal{X}$ into sets $\mathcal{X}_i$, $1 \leq i \leq k+1$, is what we are looking for:

$$\mathcal{X}_i = \bigcup_{\bar{a}\in M^{n-1}} S(a_1,\ldots,a_{n-1})^i.$$

Next we will show that $\mathcal{X}_i \models =(x_1,\ldots,x_n)$ for each $\mathcal{X}_i$, $1 \leq i \leq k+1$.

Suppose $s, s' \in \mathcal{X}_i$ such that s and s' are from the same $S(a_1,\ldots,a_{n-1})^i$ for some $(a_1,\ldots,a_{n-1}) \in M^{n-1}$. Now s and s' will agree on x_n since x_n is constant in each $S(a_1,\ldots,a_{n-1})^i$. Thus $\{s,s'\} \models =(x_1,\ldots,x_n)$ holds. Suppose s and s' are from different sets, say s from $S(a_1,\ldots,a_{n-1})$ and s' from $S(a'_1,\ldots,a'_{n-1})$. Then s and s' will disagree on the sequence $(x_1,\ldots,x_{n-1})$. Thus $\{s,s'\} \models =(x_1,\ldots,x_n)$ holds. Now $\mathcal{X}_i \models =(x_1,\ldots,x_n)$ holds for each $\mathcal{X}_i$, $1 \leq i \leq k+1$ by 2-coherence of dependence atoms.

Suppose (2) does not hold. Then there exists $(a_1, \ldots, a_{n-1}) \in M^n$, such that $|S(a_1, \ldots, a_{n-1})|^* > k + 1$. By *pigeon hole principle*[1] it is not possible to divide $S(a_1, \ldots, a_n)$ into $k + 1$ subsets $S(a_1, \ldots, a_{n-1})^i$, $1 \leq i \leq k + 1$, so that in each set $S(a_1, \ldots, a_{n-1})^i$ the value of x_n would be constant. Since all the tuples in $S(a_1, \ldots, a_n)$ agree on sequence $x_1, \ldots, x_{n-1}$ it follows that the dependence atom $=(x_1, \ldots, x_n)$ will be dissatisfied in some subset independent of the division of $S(a_1, \ldots, a_n)$. Thus $\mathcal{X}$ does not satisfy $\bigvee_k =(x_1, \ldots, x_n)$. ■

PROPOSITION 3.8. *Suppose $k \in \mathbb{N}$ and ϕ is a dependence atom. Then $\bigvee_k \phi$ is $k + 1$-coherent.*

PROOF. Suppose ϕ is the dependence atom $=(x_1, \ldots, x_n)$ and $\mathcal{X}$ is a team of type $\bigvee_k =(x_1, \ldots, x_n)$. Then, by downwards closure property all $k + 1$-element subsets of $\mathcal{X}$ satisfy $\bigvee_k =(x_1, \ldots, x_n)$.

Suppose all $k+1$-element subsets of $\mathcal{X}$ satisfy $\bigvee_k =(x_1, \ldots, x_n)$. Thus it holds that there are no such $k+1$-element subsets in $\mathcal{X}$ where the assignments agree on the first $n - 1$ terms and all disagree on the last term x_n. Thus for every $(a_1, \ldots, a_{n-1}) \in M^{n-1}$ it holds that $|S(a_1, \ldots, a_{n-1})|^* \leq k + 1$. Now the claim follows from Lemma 3.7. ■

3.1. Incoherence

We will show that disjunction does not preserve coherence. Given a team $\mathcal{X}$ and a formula $\bigvee_{i \in I} =(\bar{x}_i, y_i)$, where $\bar{x}_i = (x_{i_i}, \ldots, x_{i_n})$, we interpret the team as a multigraph in such a way that the $|I|$-colorability of the multigraph corresponds to $\mathcal{X}$ satisfying $\bigvee_{i \in I} =(\bar{x}_i, y_i)$. The interpretation is such that each assignment corresponds to a vertex in the graph. Furthermore, each dependence atom $=(\bar{x}_i, y_i)$, $i \in I$, induces edges between the vertices in such a way that if two assignments dissatisfy the dependence atom, then the corresponding vertices share the corresponding edge E_i.

DEFINITION 3.9. Suppose $\mathcal{X} = \{s_1, \ldots, s_n\}$ is a team of domain V and range M and $\phi \in \mathcal{D}$ is of the form $\bigvee_{i \in I} =(\bar{x}_i, y_i)$. Then the multigraph $\mathcal{G}^{\phi}_{\mathcal{X}} = (V, \{E_i | \ i \in I\})$ is defined in the following way:

1. $V = \{v_j \mid s_j \in \mathcal{X}\}$.
2. For each $i \in I$, if $\{s_j, s_l\} \not\models =(\bar{x}_i, y_i)$, then $(v_j, v_l) \in E_i$.

[1]Formally it states that there does not exist an injective function on finite sets whose codomain is smaller than its domain.

The k-colorability of a multigraph is defined as an existence of a coloring function $\sigma : V \to k$, such that if two nodes share an edge E_i then they cannot be colored both with the same color i, $i \leq k$. The existence of a coloring function for $\mathcal{G}^{\phi}_{\mathcal{X}}$ matches exactly with the existence of the division of the team $\mathcal{X}$ in the Team-semantics.

PROPOSITION 3.10. *Suppose $\mathcal{G}^{\phi}_{\mathcal{X}}$ is a multigraph defined as in Definition 3.9 for a team $\mathcal{X}$ and formula $\phi =: \bigvee_{i\in I} =\!(\bar{x}_i, y_i)$. Then the following two conditions are equivalent:*

1. *There exists a $|I|$-coloring of the multigraph $\mathcal{G}^{\phi}_{\mathcal{X}}$.*
2. *$\mathcal{X} \models \bigvee_{i\in I} =\!(\bar{x}_i, y_i)$.*

PROOF. Suppose $\sigma : V \to I$ is a function, such that if $\sigma(v_i) = \sigma(v_j) = m$, $m \in I$, then $(v_i, v_j) \notin E_m$. Let $\mathcal{X}_i$, $i \in I$, be defined the following way:

$$\mathcal{X}_i = \{s_j \mid s_j \in \mathcal{X} \wedge \sigma(v_j) = i\}.$$

Since σ is defined on the domain of $\mathcal{G}^{\phi}_{\mathcal{X}}$, it holds that $\mathcal{X} = \bigcup_{i\in I} \mathcal{X}_i$. We will show next that $\mathcal{X}_i \models =\!(\bar{x}_i, y_i)$, holds for each $i \in I$:

Suppose $s_l, s_k \in \mathcal{X}_i$. Then, the corresponding vertices v_l and v_k are assigned the value i under σ. Then, by assumption on σ, it follows that $(v_l, v_k) \notin E_i$. Thus by Definition 3.9, it follows that $\{s_l, s_k\} \models =\!(\bar{x}_i, y_i)$. By 2-coherence of the dependence atoms, it holds that $\mathcal{X}_i \models =\!(\bar{x}_i, y_i)$. Thus $\mathcal{X} \models \bigvee_{i\in I} =\!(\bar{x}_i, y_i)$ holds.

The other direction: Suppose $\mathcal{X} \models \bigvee_{i\in I} =\!(\bar{x}_i, y_i)$ holds. Then, there is a partition of $\mathcal{X}$ into sets $\mathcal{X}_i$, such that $\mathcal{X}_i \models =\!(\bar{x}_i, y_i)$ for each $i \in I$, and $\mathcal{X} = \bigcup_{i\in I} \mathcal{X}_i$. Let $\sigma : V \to |I|$ be defined the following way:

- $\sigma(v_j) = m$, if $s_j \in \mathcal{X}_m$.

Clearly, σ is well defined and it holds that if $\sigma(v_i) = \sigma(v_j) = m$, then $(v_i, v_j) \notin E_m$. ■

The next lemma will show that disjunction does not preserve coherence. An important detail to notice is that the two dependence atoms do not use the same variables.

THEOREM 3.11. *$=\!(x, y) \vee =\!(z, v)$ is not k-coherent for any $k \in \mathbb{N}$.*

PROOF. We will actually show that a stronger claim holds, namely that $=\!(x, y) \vee =\!(z, v)$ is not $f(n)$-coherent for any function $f : \mathbb{N} \to \mathbb{N}$, such that $f(n) < n$, for all n. Here the meaning of $f(n)$-coherence is that a formula

ϕ is $f(n)$-coherent, if for all teams $\mathcal{X}$, such that $|\mathcal{X}| = n$, it holds that $\mathcal{X} \models \phi \Leftrightarrow \mathcal{Y} \models \phi$ for every $\mathcal{Y} \subseteq \mathcal{X}$, such that $|\mathcal{Y}| = f(n)$.

We will construct a team $\mathcal{X}$ such that every proper subset of $\mathcal{X}$ satisfies $=(x, y) \vee =(z, v)$, but the whole team dissatisfies $=(x, y) \vee =(z, v)$. We represent the team as a multigraph as in Definition 3.9. Each of the vertices corresponds to an assignment of the team. Suppose $s_v, s_w \in \mathcal{X}$. There are two type of edges we assign between vertices in the following way.

- If $\{s_v, s_w\} \not\models =(x, y)$, then we draw a straight edge between the vertices v and w.
- If $\{s_v, s_w\} \not\models =(z, v)$, then we draw a wavy edge between the corresponding vertices v and w.

A $|2|$-coloring of the multigraph will be a partition of the universe into two sets, black and white vertices, such that the black vertices do not share any wavy edges and the white vertices do not share any straight edges. The graph in Figure 1 is such that every proper subgraph is 2-colorable, but the whole multigraph is not.

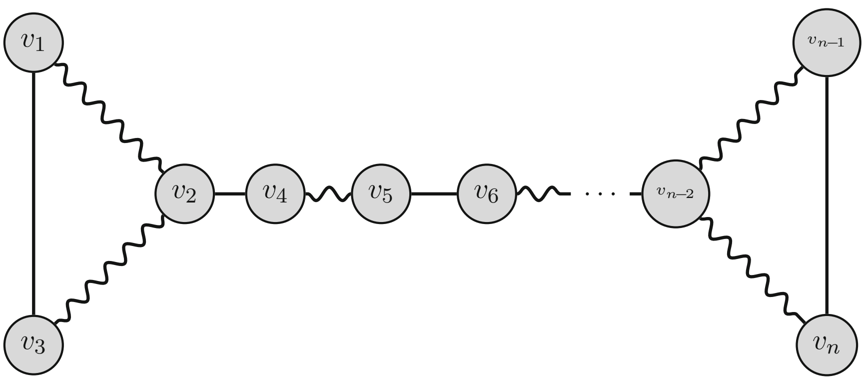

Figure 1. Multigraph $\mathcal{G}_{\mathcal{X}}$

$\mathcal{G}_{\mathcal{X}}$ **is not 2-colorable**: Suppose v_2 is colored black. Then vertices v_1, v_3 should be colored white as they both share a wavy edge with v_2. But since there is a straight edge between v_1 and v_3 and the fact that white color does not allow straight edges, this cannot be a proper coloring. Thus the only way to properly color the triangle is to color v_2 white. The colors of v_1, v_3 can be chosen black or white as long as v_1, v_3 are not both white. Similar reasoning shows that v_{n-2} has to be colored also white and the colors of v_{n-1} and v_n cannot be both white.

The triangles $\{v_1, v_2, v_3\}$ and $\{v_{n-2}, v_{n-1}, v_n\}$ are connected with a path of even length (even number of vertices). The path is such that the edges alternates between straight and wavy, which forces the proper coloring also to alternate between black and white for the vertices on the path. Since the length of the path is even, there cannot be a coloring for the whole graph as the color of v_2 determines the coloring of the whole path, in the same way as the color of v_{n-2}. They both force different colors on the path, thus making the proper coloring impossible. Thus the whole graph is not 2-colorable.

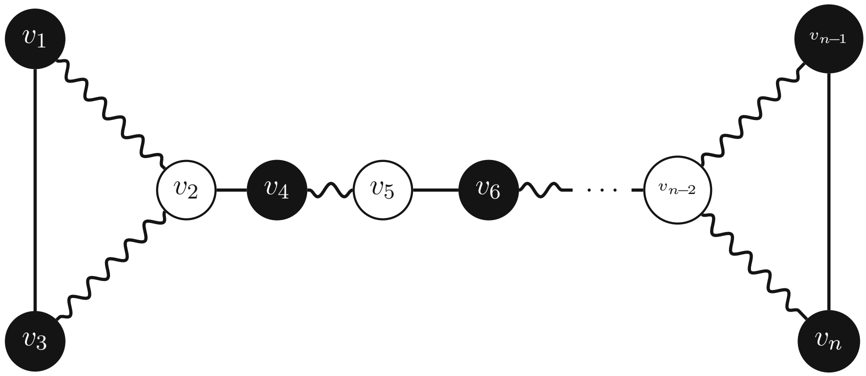

Figure 2. Coloring of the multigraph $\mathcal{G}_\mathcal{X}$

Every proper subgraph of $\mathcal{G}_\mathcal{X}$ is 2-colorable: We will show that if we remove a vertex from either of the triangles, then the coloring of the vertex v_2 (or v_{n-2}), which is connected to the path, can be chosen either black or white. Suppose v_1 is removed. Then we can choose so that v_2 is colored black and v_3 is colored white. The vertex v_{n-2} has to be still colored with white. Now, since v_2 and v_{n-2} are colored with different colors and the path connecting them is even, it holds that the whole graph can be colored. The cases where we remove any other vertex from the two triangles are analogous to this one.

On the other hand, suppose one of the vertices from the path connecting the two triangles is removed. Then we have two components of the graph that are not connected by edges. The coloring of the whole graph reduces to the coloring of the two subgraphs for which there is a trivial coloring induced by the coloring of two nodes v_2 and v_{n-2}.

The team that corresponds to the graph $\mathcal{G}_\mathcal{X}$ is given in the Table 1.

As one can observe, the values of the whole path are not explicitly given in the picture. If two vertices share a straight edge, the corresponding assignments of the team in Table 1 are assigned the same value for x and

assignment	x	y	z	v
s_1	0	0	0	0
s_2	1	2	0	1
s_3	0	2	0	0
s_4	1	1	1	2
s_5	2	3	1	3
s_6	2	4	2	4
.				
.				
.				
s_{n-2}	n	$n+2$	n	$n+1$
s_{n-1}	$n+1$	$n+3$	n	$n+2$
s_n	$n+1$	$n+4$	n	$n+2$

Table 1. Team corresponding to multigraph $\mathcal{G}_{\mathcal{X}}$

different ones for y. Similarly, if two vertices share a wavy edge the corresponding assignments assign the same value for z and different one for v. When we choose the values for the assignments that correspond to a vertex in the path, we always use new values for the variables if possible. This way we ensure that there will be no unintended edges between the vertices in the triangle and the vertices in the path, just the ones that appear in the picture.

For example, $s_4(x)$ is assigned the same as $s_2(x)$ and $s_4(y)$ is assigned different to $s_2(y)$, but $s_4(z)$ and $s_4(v)$ are chosen new values. With the next vertex on the path, which is v_5, we can already assign new values for $s_5(x)$ and $s_5(v)$. We also have to take care that the values of $s_5(z)$ and $s_5(v)$ are assigned such that the dependence $=\!(z, v)$ is dissatisfied. At this point of the path the values that the assignments s_1, s_2 and s_3 assign to variables x, y, z, v are no more assigned when we go left in the path. Thus, the ranges of the variables under the assignments corresponding to the vertices of the triangles are disjoint with ranges of variables under the assignment that correspond to the vertices on the path (excluding the endpoints of the path).

Let us show that the team in Table 1 indeed translates into the graph in Figure 1. Recall that straight edges are drawn when two assignments dissatisfy the dependence atom $=(x, y)$ and wavy edges when $=(z, v)$ is dissatisfied;

Straight edges from vertex v_1: It holds that $s_1(x) = 0 = s_3(x)$, but $s_3(y) \neq s_1(y)$. Thus there is a straight edge (v_1, v_3). All the other assignments assign value other than 0 for x. Thus there cannot be straight edges from a to other vertices.

Wavy edges from vertex v_1: $s_1(z) = s_2(z) = 0$ and $s_1(v) = 0 \neq 2 = s_2(v)$, thus there is a wavy edge (v_1, v_2). Indeed $s_3(z) = 0$ but $s_3(v) \neq s_1(v)$, thus there is no wavy edge (v_1, v_3). All the other assignments assign a value different to 0 for z, thus there are no wavy edges between v_1 and other vertices.

Straight edges from vertex v_3: The only straight edge from v_3 is the one that is shared with v_1. All other assignments assign a value different from 0 to x, thus there are no other straight edges from v_3.

Wavy edges from vertex v_3: $s_3(z) = s_2(z) = 0$ and $s_3(v) = 0 \neq 2 = s_2(v)$, thus there is a wavy edge (v_3, v_2). Again, $s_3(v) = s_1(v)$, thus there is no wavy edge between them. All the other assignments assign a value different from 0 to z. Thus there are no other wavy edges from v_3.

Straight edges from vertex v_2: The only assignment that assigns x as 1 is s_4, but they disagree on y. Thus there is straight edge (v_2, v_4). As we earlier noted, after the next vertex on the path, the values that are assigned by the assignments that correspond to nodes that appear in the triangle do not appear in the ranges of the assignments that correspond to the vertices that come later in the path. Thus all the other assignments disagree with s_2 on x. Thus there are no other straight edges from the node v_2.

Wavy edges from vertex v_2: The edges (v_2, v_1) and (v_2, v_3) have been already established. The value $s_2(z)$ does not appear as a value for z under the assignment corresponding the nodes that appear later in the path. Thus there are no other wavy edges from v_2.

The other triangle $\{v_{n-2}, v_{n-1}, v_n\}$ is isomorphic to that of $\{v_1, v_2, v_3\}$. It can be checked analogously that exactly the edges that appear in the graph will be drawn under the translation given in Definition 3.9.

We have given a construction of collection of graphs like in Figure 1,which are not 2-colorable, but for which hold that every proper sub-graph is. Now by Proposition 3.10 the following are equivalent:

1. $\mathcal{G}_{\mathcal{X}}$ is 2-colorable.
2. $\mathcal{X} \models =\!(x, y) \vee =\!(z, v)$.

Thus the whole team $\mathcal{X}$ as in Table 1 does not satisfy $=\!(x, y) \vee =\!(z, v)$, but every proper sub-team of $\mathcal{X}$ satisfies $=\!(x, y) \vee =\!(z, v)$. By increasing the number of vertices on the path that connects the two triangles in Figure 1 we get the same counter example for different cardinalities. ∎

4. Computational complexity of quantifier-free $\mathcal{D}$-formulas

Model checking is one of the central problems considered in finite model theory. Given a structure $\mathcal{M}$ and a formula ϕ it is to decide whether $\mathcal{M}$ is a model of ϕ. When we fix the formula ϕ and let the structure vary, we talk about the model checking problem for a formula ϕ.

Each $\phi \in \mathcal{D}(\tau)$ defines a collection of pairs $(\mathcal{M}, \mathcal{X})$, where $\mathcal{M}$ is a τ-structure and $\mathcal{X}$ is a team of range M, such that $Fr(\phi) = dom(\mathcal{X}) = \{x_1, \ldots, x_n\}$. Given a team $\mathcal{X}$ we define a relation $Rel(\mathcal{X})$ in the following way:

$$Rel(\mathcal{X}) = \{(s(x_1), \ldots, s(x_n)) \mid s \in \mathcal{X}\}.$$

Definition 4.1. Suppose $\phi \in \mathcal{D}(\tau)$. *The Model checking problem for a formula ϕ, MC(ϕ)*, is to decide whether it holds that

$$\mathcal{M} \models_{\mathcal{X}} \phi,$$

where $\mathcal{M}$ is a τ-structure and $\mathcal{X}$ is a team, such that $Fr(\phi) = dom(\mathcal{X})$.

We will use the following versions of Boolean satisfiability problem to show the NL- and NP-completeness of $MC(\phi)$ for certain $\phi \in \mathcal{D}$.

Definition 4.2. *Boolean satisfiability problem*(SAT) is a problem to determine whether a given quantifier-free first order formula is satisfiable. The variables are boolean and may occur positively or negatively in the formula. The formulas are assumed to be in the conjunctive normal form. The problem is to determine, whether there is an assignment, that satisfies the given formula. There are several variations of SAT from which we consider the following two:

- 2-SAT: At most 2 disjuncts in each clause.
- 3-SAT: At most 3 disjuncts in each clause.

It is known that 2-SAT is NL-complete [5] and 3-SAT is NP-complete [3].

4.1. Logarithmic space

We will start by showing that the model checking problem for k-coherent formulas is in LOGSPACE. We will establish this by showing that for every k-coherent τ-formula there is an equivalent FO-sentence over vocabulary $\tau \cup \{R\}$, where R is a $|\, dom(\mathcal{X})|$-ary relation symbol interpreting the team.

We say that the formula $\phi_{\mathcal{M}}$ characterizes the τ-structure $\mathcal{M}$ up to isomorphism if $\mathcal{M} \models \phi_{\mathcal{M}}$, and the following equivalence holds for all τ-structures $\mathcal{N}$:

$$\mathcal{M} \cong \mathcal{N} \Leftrightarrow \mathcal{N} \models \phi_{\mathcal{M}}.$$

We want to characterize finite fixed size teams up to isomorphism. For this, we introduce the notion of team-structure, which is essentially the substructure induced by the team. First we give a definition of substructure:

Definition 4.3. Suppose $\mathcal{M}$ is a relational τ-structure and $S \subseteq M$. Then let the *substructure induced by* S, denoted by $\mathcal{M}|S$, be the S endowed with a k-ary relation $R_i^{\mathcal{M}} \cap S^k$, for each k-ary $R_i \in \tau$.

Definition 4.4. Suppose $\mathcal{M}$ is a τ-structure and $\mathcal{X}$ is a team of domain V and range M. Let $A_{\mathcal{X}} = \{a \in M \mid s(x_i) = a$, for $s \in \mathcal{X}$ and $x_i \in V\}$.

Definition 4.5. Suppose $\mathcal{M}$ is a τ-structure and $\mathcal{X}$ is a team of domain $\{x_1, \dots, x_n\}$ and range M. Then the *team-structure induced by* $\mathcal{X}$ is the $\tau \cup \{R\}$-structure $(\mathcal{M}|A_{\mathcal{X}}, Rel(\mathcal{X}))$, which we denote by $\mathcal{M}_{\mathcal{X}}$.

Let $\mathcal{K}(k, n)$ be the class of all team-structures $\mathcal{M}_{\mathcal{X}}$ where $|\mathcal{X}| = k$ and $\mathrm{dom}(\mathcal{X}) = \{x_1, \dots, x_n\}$. We will characterize the isomorphism type of the team-structures inside the class $\mathcal{K}(k, n)$, $k, n \in \mathbb{N}$. This can be done with a first-order quantifier-free formula. For this, we need the notion of *k-τ-type.*

Definition 4.6. Suppose τ is a relational vocabulary. Then a *k-τ-type* $t_\tau^k(x_1, \dots, x_k)$ is a maximal consistent set of τ-atomic-, negated τ-atomic-, identity- and negated identity formulas over $\{x_1, \dots, x_k\}$.

Suppose $\mathcal{M}$ is a τ-structure and $\bar{a} = (a_1, \dots, a_k) \in M^k$. We say that the tuple $(a_1, \dots, a_k)$ realizes the k-τ-type $t_\tau^k(x_1, \dots, x_k)$ in $\mathcal{M}$ if

$$(\mathcal{M}, \bar{a}) \models \bigwedge_{\phi \in t_\tau^k} \phi(x_1, \dots, x_k).$$

We denote the k-τ-type realized by the tuple $\bar{a}$ in $\mathcal{M}$ by $t_{\bar{a}}^{\mathcal{M}}$.

Suppose $\mathrm{Rel}(\mathcal{X}) \subseteq M^n$, such that $|Rel(\mathcal{X})| = k$ and $\Pi(\mathcal{X})$ is the set of all orderings of the tuples of $\mathrm{Rel}(\mathcal{X})$. Let $\pi \in \Pi(\mathcal{X})$. Then $\bar{a}_x^\pi$ is the concatenation of the tuples of $Rel(\mathcal{X})$ in the order π. Let

$$\Phi_{\Pi(\mathcal{X})}^{\mathcal{M},\tau}(x_1, \dots, x_{kn}) =: \bigvee_{\pi \in \Pi(\mathcal{X})} \bigwedge_{\psi \in t_{\bar{a}_x^\pi}^{\mathcal{M},\tau}} \psi(x_1, \dots, x_{kn}).$$

The next lemma shows that we can characterize the isomorphism type of a team-structure $\mathcal{M}_{\mathcal{X}} \in \mathcal{K}(k,n)$ with a quantifier-free formula. Since all the elements of the domain are in one of the tuples of $Rel(\mathcal{X})$ we do not have to explicitly state the cardinality of structure. With the assumption that both of the teams have the same domain and that they are of same size, it is enough to state all the identities between the elements occurring in the tuples of Rel($\mathcal{X}$). This determines the cardinality of the domain. We also state all the relational atomic formulas the elements in the tuples of Rel($\mathcal{X}$) satisfy.

Since the tuples of Rel($\mathcal{X}$) are not in order, we have to consider all the possible orders of the tuples. Thus, the formula characterizing $\mathcal{M}_{\mathcal{X}}$ is a disjunction over all orders of the tuples of Rel($\mathcal{X}$), where each disjunct is a conjunction over all the formulas in the type the concatenated tuple realizes.

LEMMA 4.7. *Suppose $\mathcal{M}_{\mathcal{X}}, \mathcal{N}_{\mathcal{Y}} \in \mathcal{K}(k,n)$ are team-structures. Then*

$$\mathcal{M}_{\mathcal{X}} \cong \mathcal{N}_{\mathcal{Y}} \Leftrightarrow \mathcal{N}_{\mathcal{Y}} \models \Phi_{\Pi(\mathcal{X})}^{\mathcal{M}_{\mathcal{X}},\tau\cup\{R\}}(x_1,\ldots,x_{kn}).$$

PROOF. Suppose $f : A_{\mathcal{X}} \to A_{\mathcal{Y}}$ is an isomorphism. Suppose π_1 is an ordering of the tuples of $Rel(\mathcal{X})$. Then let $\pi_2 : Rel(\mathcal{Y}) \to Rel(\mathcal{Y})$, such that for all $\bar{a}_i, \bar{a}_j \in Rel(\mathcal{Y})$: $(\bar{a}_i, \bar{a}_j) \in \pi_2 \Leftrightarrow (f^{-1}(\bar{a}_i), f^{-1}(\bar{a}_j)) \in \pi_1$. Here $f^{-1}(\bar{a}_i)$ is a shorthand for $(f^{-1}(a_{i_1}),\ldots,f^{-1}(a_{i_n}))$. Since f is isomorphism π_2 is well defined and it is an ordering of the tuples of Rel($\mathcal{Y}$).

Let us show that $t_{\bar{a}_x^{\pi_1}}^{\mathcal{M}_{\mathcal{X}},\tau\cup\{R\}} = t_{\bar{a}_y^{\pi_2}}^{\mathcal{N}_{\mathcal{Y}},\tau\cup\{R\}}$. Since f is isomorphism it holds $(a_{i_1},\ldots,a_{i_m}) \in R_i^{\mathcal{M}_{\mathcal{X}}} \Leftrightarrow (f(a_{i_1}),\ldots,f(a_{i_m})) \in R_i^{\mathcal{N}_{\mathcal{Y}}}$ for all m-ary $R_i \in \tau \cup \{R\}$.

Since f is a bijection, it holds that $a_i = a_j \Leftrightarrow f(a_i) = f(a_j)$ for all $i, j \leq kn$. Thus $(\mathcal{M}_{\mathcal{X}}, a_i, a_j) \models (x_i = x_j) \Leftrightarrow (\mathcal{N}_{\mathcal{Y}}, b_i, b_j) \models (x_i = x_j)$. Thus $t_{\bar{a}_x^{\pi_1}}^{\mathcal{M}_{\mathcal{X}},\tau\cup\{R\}} = t_{\bar{a}_y^{\pi_2}}^{\mathcal{N}_{\mathcal{Y}},\tau\cup\{R\}}$ holds.

The other direction: Suppose $\mathcal{N}_{\mathcal{Y}} \models \Phi_{\Pi(\mathcal{X})}^{\mathcal{M}_{\mathcal{X}},\tau\cup\{R\}}(x_1,\ldots,x_{kn})$. Then, there is kn-tuple $(b_1,\ldots,b_{kn}) \in (A_{\mathcal{Y}})^{nk}$, such that

$$(\mathcal{N}_{\mathcal{Y}}, b_1,\ldots,b_{kn}) \models \Phi_{\Pi(\mathcal{X})}^{\mathcal{M}_{\mathcal{X}},\tau\cup\{R\}}(x_1,\ldots,x_{kn}).$$

Let $t_{\bar{b}}^{\mathcal{N}_{\mathcal{Y}},\tau\cup\{R\}}$ be the kn-$\tau \cup \{R\}$-type the tuple $\bar{b}$ realizes in $\mathcal{N}_{\mathcal{Y}}$.

On the other hand, by definition of $\Phi_{\Pi(\mathcal{X})}^{\mathcal{M}_{\mathcal{X}},\tau\cup\{R\}}(x_1,\ldots,x_{kn})$ there is an order π of the tuples of Rel($\mathcal{X}$), such that $t_{\bar{a}_{\mathcal{X}}^{\pi}}^{\mathcal{M}_{\mathcal{X}},\tau\cup\{R\}} = t_{\bar{b}}^{\mathcal{N}_{\mathcal{Y}},\tau\cup\{R\}}$. Let $\bar{a}_{\mathcal{X}}^{\pi} = (a_1,\ldots,a_{kn})$.

Let $f = \bigcup_{i \leq kn}(a_i, b_i)$. We will show that $f : A_{\mathcal{X}} \to A_{\mathcal{Y}}$ is an isomorphism.

f is a function: Clearly f is defined in the whole $A_{\mathcal{X}}$. Suppose $a_i = a_j$, $i \neq j$, $i, j \leq kn$. Then $(\mathcal{M}_{\mathcal{X}}, a_i, a_j) \models (x_i = x_j)$. Since $t_{\bar{a}^\pi_{\mathcal{X}}}^{\mathcal{M}_{\mathcal{X}}, \tau\cup\{R\}} = t_{\bar{b}}^{\mathcal{N}_{\mathcal{Y}}, \tau\cup\{R\}}$, it holds that $(\mathcal{N}_{\mathcal{X}}, b_i, b_j) \models (x_i = x_j)$. Thus it holds that $f(a_i) = f(a_j)$.

f is surjection: We need to show that the tuple $\bar{b}$ contains all the elements of the domain $A_{\mathcal{Y}}$. For each order π, the type $t_{\bar{a}^\pi_{\mathcal{X}}}^{\mathcal{M}_{\mathcal{X}}, \tau\cup\{R\}}$ contains the formulas $R(x_1, \ldots, x_n)$,..., $R(x_{(k-1)n+1}, \ldots, x_{kn})$. Since $t_{\bar{a}^\pi_{\mathcal{X}}}^{\mathcal{M}_{\mathcal{X}}, \tau\cup\{R\}} = t_{\bar{b}}^{\mathcal{N}_{\mathcal{Y}}, \tau\cup\{R\}}$ it also holds that $R(x_1, \ldots, x_n)$,..., $R(x_{(k-1)n+1}, \ldots, x_{kn})$ are in $t_{\bar{a}}^{\mathcal{N}_{\mathcal{Y}}, \tau\cup\{R\}}$. Thus all the n-tuples $(b_1, \ldots, b_n)$,..., $(b_{n(k-1)+1}, \ldots, b_{nk}) \in Rel(\mathcal{Y})$. Furthermore, since all the tuples in Rel($\mathcal{X}$) are pairwise distinct and since the both of the types $t_{\bar{a}^\pi_{\mathcal{X}}}^{\mathcal{M}_{\mathcal{X}}, \tau\cup\{R\}}$ and $t_{\bar{b}}^{\mathcal{N}_{\mathcal{Y}}, \tau\cup\{R\}}$ contain all the identities between the elements of the tuples, it also hold that all the tuples $(b_1, \ldots, b_n)$,..., $(b_{n(k-1)+1}, \ldots, b_{nk})$ are pairwise distinct. Thus it holds $\{(b_1, \ldots, b_n),\ldots, (b_{n(k-1)+1}, \ldots, b_{nk})\} = Rel(\mathcal{Y})$.

Suppose $b_j \in A_{\mathcal{Y}}$, $j \leq kn$. Then b_j is in the tuple $\bar{b}$. Then by definition of f it holds that $(a_j, b_j) \in f$.

f is injection: Suppose $a_i, a_j \in A_{\mathcal{X}}$, $a_i \neq a_j$. Then $(\mathcal{M}_{\mathcal{X}}, a_i, a_j) \models \neg(x_i = x_j)$. Since $t_{\bar{a}^\pi_{\mathcal{X}}}^{\mathcal{M}_{\mathcal{X}}, \tau\cup\{R\}} = t_{\bar{b}}^{\mathcal{N}_{\mathcal{Y}}, \tau\cup\{R\}}$, it holds $(\mathcal{N}_{\mathcal{Y}}, b_i, b_j) \models \neg(x_i = x_j)$. Thus it holds that $f(a_i) \neq f(a_j)$.

f is homomorphism: Suppose $R_i \in \tau \cup \{R\}$ and $(a_{i_1}, \ldots, a_{i_k}) \in R_i^{\mathcal{M}_{\mathcal{X}}}$. Since $t_{\bar{a}^\pi_{\mathcal{X}}}^{\mathcal{M}_{\mathcal{X}}, \tau\cup\{R\}} = t_{\bar{b}}^{\mathcal{N}_{\mathcal{Y}}, \tau\cup\{R\}}$, it holds $(b_{i_1}, \ldots, b_{i_k}) \in R_i^{\mathcal{N}_{\mathcal{Y}}}$. ■

We use the following lemma in the proof of Theorem 4.9. We will omit the proof, which is a straightforward induction on the structure of formula.

LEMMA 4.8. *Suppose $\mathcal{M}$ is a τ-structure and $\mathcal{X}$ a team of range $\{x_1, \ldots, x_n\}$ and of domain M. Then the following equivalence holds for all quantifier-free $\phi(x_1, \ldots, x_n) \in \mathcal{D}(\tau)$:*

$$\mathcal{M} \models_{\mathcal{X}} \phi \Leftrightarrow \mathcal{M}|A_{\mathcal{X}} \models_{\mathcal{X}} \phi.$$

THEOREM 4.9. *Suppose $\phi(x_1, \ldots, x_n)$ is a quantifier-free k-coherent $\mathcal{D}(\tau)$-formula. Then there is a sentence $\phi^* \in FO(\tau \cup \{R\})$, where R is n-ary, such that for all τ-structures $\mathcal{M}$ and for all teams $\mathcal{X}$ of domain $\{x_1, \ldots, x_n\}$ the following holds:*

$$\mathcal{M} \models_{\mathcal{X}} \phi(x_1, \ldots, x_n) \Leftrightarrow (\mathcal{M}, Rel(\mathcal{X})) \models \phi^*(R).$$

PROOF. Suppose $\phi(x_1,\dots,x_n)$ is a k-coherent $\mathcal{D}(\tau)$-formula. Then for all teams $\mathcal{X}$ of domain $\{x_1,\dots,x_n\}$ holds:

$$\mathcal{M} \models_{\mathcal{X}} \phi \Leftrightarrow \text{for all } \mathcal{Y} \subseteq \mathcal{X}(\text{ if } |\mathcal{Y}| = k, \text{ then } \mathcal{M} \models_{\mathcal{Y}} \phi).$$

By Lemma 4.8 it holds that

$$\mathcal{M} \models_{\mathcal{Y}} \phi \Leftrightarrow \mathcal{M}|A_{\mathcal{Y}} \models_{\mathcal{Y}} \phi.$$

Let $\mathcal{Y} \subseteq \mathcal{X}$, such that $|\mathcal{Y}| = k$ and $\mathcal{M}_{\mathcal{Y}}$ is the team-structure induced by $\mathcal{Y}$. The size of the domain of $\mathcal{M}_{\mathcal{Y}}$ is bounded by kn. Since also the vocabulary is finite, there are only finitely many different τ-isomorphism types of $\mathcal{M}_{\mathcal{Y}}$. By lemma 4.7 the isomorphism type of a team-structure $\mathcal{M}_{\mathcal{Y}} \in \mathcal{K}(k,n)$ can be characterized in FO with a quantifier-free formula. Let I^{ϕ} be the set of all different isomorphism types $\psi_{\mathcal{N}_{\mathcal{Z}}}$ of $\mathcal{N}_{\mathcal{Z}} \in \mathcal{K}(k,n)$, such that $\psi_{\mathcal{N}_{\mathcal{Z}}} \in I^{\phi} \Leftrightarrow \mathcal{N} \models_{\mathcal{Z}} \phi$. Notice that for a given pair $(\mathcal{N}, \mathcal{Z})$, the structure $\mathcal{N}_{\mathcal{Z}}$ is unique. Now $\phi^*(R)$ can be written in the following way:

$$\phi^* =: \forall \bar{x}_1 \dots \forall \bar{x}_k((\bigwedge_{i\le k} \bar{x}_i \in R \bigwedge_{i\ne j} \bar{x}_i \ne \bar{x}_j) \to \bigvee_{\psi\in I^{\phi}} \psi(\bar{x}_1,\dots,\bar{x}_k)),$$

where $\bar{x}_i$ is a shorthand for a tuple $(x_{i_1},\dots,x_{in})$, $i \le k$. Let us show that the claimed equivalence holds.

Suppose $\mathcal{M} \models_{\mathcal{X}} \phi$. Let $\bar{a}_1,\dots,\bar{a}_k \in R^{\mathcal{M}}$ such that $\bar{a}_i \ne \bar{a}_j$ for $i \ne j$, $i,j \le k$. Then $\{\bar{a}_1,\dots,\bar{a}_k\} = Rel(\mathcal{Y})$ for a $\mathcal{Y} \subseteq \mathcal{X}$, $|\mathcal{Y}| = k$. By assumption it holds that $\mathcal{M} \models_{\mathcal{X}} \phi$. Then by k-coherence of ϕ it holds that $\mathcal{M} \models_{\mathcal{Y}} \phi$. Thus the isomorphism type of the team-structure $\mathcal{M}_{\mathcal{Y}}$ is in I^{ϕ}. Thus it holds that $(\mathcal{M}, Rel(\mathcal{X})) \models \phi^*(R)$.

Suppose $(\mathcal{M}, Rel(\mathcal{X})) \models \phi^*(R)$. Suppose $\mathcal{Y} \subseteq \mathcal{X}$ such that $|\mathcal{Y}| = k$. Then, the isomorphism type of the team-structure $\mathcal{M}_{\mathcal{Y}}$ is in I^{ϕ}. Thus it holds that $\mathcal{M} \models_{\mathcal{Y}} \phi$. Then $\mathcal{M} \models_{\mathcal{X}} \phi$ follows by k-coherence of ϕ. ■

The model checking problem of FO-formulas is in LOGSPACE [2]. This yields the following corollary for the computational complexity of k-coherent formulas.

COROLLARY 4.10. *Suppose $\phi \in \mathcal{D}$ is a k-coherent formula. Then $MC(\phi) \in$ LOGSPACE.*

We have shown that the fragment of k-coherent $\mathcal{D}$-formulas is contained in FO and thus the model checking problem for k-coherent formulas is in

LOGSPACE. Natural question arises whether coherent fragment of $\mathcal{D}$ is as expressive as FO?

This question can be answered by using the Downwards Closure property 2.6. It states that if a team $\mathcal{X}$ is of type ϕ, then all subsets $\mathcal{Y} \subseteq \mathcal{X}$ are also of type ϕ. Thus any property of teams that is not downwards monotone is not expressible in $\mathcal{D}$. This follows from the result of Juha Kontinen and Jouko Väänänen [4].

Proposition 4.11. *Let Coh($\mathcal{D}$) be the set of all k-coherent $\mathcal{D}$-formulas, for all $k \in \mathbb{N}$. Then Coh($\mathcal{D}$) $\subsetneq$ FO.*

Proof. Coh($\mathcal{D}$) $\subseteq$FO by Theorem 4.9. Let $\mathcal{M}_k$ be the class of $\tau \cup \{R\}$-structures, where R is n-ary and interprets the team $\mathcal{X}$, such that:

$$(\mathcal{M}, Rel(\mathcal{X})) \in \mathcal{M}_k^n \Leftrightarrow |\mathcal{X}| = k.$$

Let us assume that $\mathcal{M}_k$ is definable in $\mathcal{D}$: Suppose $\phi_k(x_1, \ldots, x_n) \in \mathcal{D}$ is such that for all $\tau \cup \{R\}$-structures $(\mathcal{M}, Rel(\mathcal{X}))$:

$$\mathcal{M} \models_{\mathcal{X}} \phi_k(x_1, \ldots, x_n) \Leftrightarrow (\mathcal{M}, Rel(\mathcal{X})) \in \mathcal{M}_k.$$

Suppose $(\mathcal{M}, Rel(\mathcal{X})) \in \mathcal{M}_k$. Thus $\mathcal{M} \models_{\mathcal{X}} \phi_k(x_1, \ldots, x_n)$. Then it holds $\mathcal{M} \models_{\mathcal{Y}} \phi_k(x_1, \ldots, x_n)$ for all $\mathcal{Y} \subsetneq \mathcal{X}$ by Downwards Closure 2.6. Thus $(\mathcal{M}, Rel(\mathcal{Y})) \in \mathcal{M}_k$. But $|\mathcal{Y}| \neq k$ since $\mathcal{Y} \subsetneq \mathcal{X}$, which is a contradiction. Thus there is no such $\phi_k(x_1, \ldots, x_n) \in \mathcal{D}$. It follows that $\mathcal{M}_k$ is not definable in $\mathcal{D}$.

On the other hand, we can define $\mathcal{M}_k$ in first order logic for all $k \in \mathbb{N}$.

$$\exists \bar{x}_1 \ldots \exists \bar{x}_k \forall \bar{x}_{k+1} (\bigwedge_{i \leq k} R(\bar{x}_i) \bigwedge_{i \neq j, i, j \leq k} (\bar{x}_i \neq \bar{x}_j) \wedge (R(\bar{x}_{k+1}) \to \bigvee_{i \leq k} (\bar{x}_i = \bar{x}_{k+1}))) \tag{3}$$

$\bar{x}_i$ is a shorthand for the tuple $(x_{i_1}, \ldots, x_{i_n})$ and $\forall \bar{x}_i$ is a shorthand for the sequence $\forall x_{i_1} \ldots \forall x_{i_n}$. The formula 3 is true in a structure $(\mathcal{M}, Rel(\mathcal{X}))$ if and only if $|Rel(\mathcal{X}))| = k$. ■

4.2. Non-deterministic logarithmic space

In the previous section we established that all quantifier-free formulas without disjunction are coherent. Furthermore, we showed that with the use of one disjunction one obtains already formulas which are incoherent. In this section we will show that the model checking problem for all quantifier-free formulas with at most one disjunction is in NL. We will also show that the model checking problem of the formula $=(x, y) \vee =(z, v)$ is complete for NL.

We write $P_1 \leq_{LOGSPACE} P_2$, if there is a LOGSPACE-reduction from problem P_1 to problem P_2. Notice that in the following theorem we do not restrict the number of disjunctions in the formula, but rather the coherence-level of the disjuncts. The coherence-level of formulas without disjunctions is at most 2.

THEOREM 4.12. *Suppose ϕ and ψ are 2-coherent quantifier-free $\mathcal{D}$-formulas. Then $MC(\phi \vee \psi) \leq_{LOGSPACE}$ 2-SAT.*

PROOF. Suppose we are given a team $\mathcal{X} = \{s_1, \ldots, x_k\}$. We will go through all the two-element subsets $\{s_i, s_j\} \subseteq \mathcal{X}$, and construct an instance $\Theta_{\mathcal{X}}$ of 2-SAT in the following way:

- If $\{s_i, s_j\} \not\models \phi$, then $(x_i \vee x_j) \in C$.
- If $\{s_i, s_j\} \not\models \psi$, then $(\neg x_i \vee \neg x_j) \in C$.

We let $\Theta_{\mathcal{X}} = \bigwedge_{\psi \in C} \psi$. By the construction, it holds that$\Theta_{\mathcal{X}}$ is a proper instance of 2-SAT. We will next show that there is an assignment S that satisfies $\Theta_{\mathcal{X}}$, if and only if $\mathcal{X} \models \phi \vee \psi$ holds: Suppose there is an assignment $S : Var(\Theta_{\mathcal{X}}) \to \{0, 1\}$ that satisfies $\Theta_{\mathcal{X}}$. Let us define the partition of $\mathcal{X}$ in the following way:

- $\mathcal{Z} = \{s_i \in \mathcal{X} \mid S(x_i) = 1\}$.
- $\mathcal{Y} = \mathcal{X} \setminus \mathcal{Z}$.

Clearly it holds that $\mathcal{X} = \mathcal{Z} \cup \mathcal{Y}$. Let us show that $\mathcal{Z} \models \psi$ and $\mathcal{Y} \models \phi$ hold:

Suppose $s_i, s_j \in \mathcal{Z}$. Since S satisfies $\Theta_{\mathcal{X}}$, $(\neg x_i \vee \neg x_j)$ cannot be a clause in $\Theta_{\mathcal{X}}$. By the construction above, it follows that $\{s_i, s_j\} \models \psi$ holds. Now, by 2-coherence of ψ it follows that $\mathcal{Z} \models \psi$.

Suppose $s_i, s_j \in \mathcal{Y}$. Since S was assumed to satisfy $\Theta_{\mathcal{X}}$, $(x_i \vee x_j)$ cannot be a clause in $\Theta_{\mathcal{X}}$. It follows by the construction above that $\{s_i, s_j\} \models \phi$ holds. Again, from 2-coherence of ϕ it follows that $\mathcal{Y} \models \phi$ holds.

The other direction: Suppose $\mathcal{X} \models \phi \vee \psi$ holds. Then, by Definition 2.5 it holds that there is a division of $\mathcal{X}$ into two sets $\mathcal{Z}$ and $\mathcal{Y}$, such that $\mathcal{X} = \mathcal{Z} \cup \mathcal{Y}$, $\mathcal{Z} \cap \mathcal{Y} = \emptyset$, $\mathcal{Y} \models \phi$ and $\mathcal{Z} \models \psi$. Let S be defined the following way:

- $S(x_i) = 1$, if $s_i \in \mathcal{Z}$.
- $S(x_i) = 0$, if $s_i \in \mathcal{Y}$.

Clearly it holds that $S : Var(\Theta_{\mathcal{X}}) \to \{0, 1\}$ is a function. Let us show that S satisfies $\Theta_{\mathcal{X}}$: Suppose $\theta \in \Theta_{\mathcal{X}}$ of form $(x_i \vee x_j)$. Then $\{s_i, s_j\}$ dissatisfies ϕ by the construction of $\Theta_{\mathcal{X}}$. Then s_i and s_j cannot be both in $\mathcal{Y}$, since $\mathcal{Y}$ was supposed to satisfy ϕ. Thus, either s_i or s_j must be in $\mathcal{Z}$. Then, it holds that $S(x_i) = 1$ or $S(x_j) = 1$, which implies that $S(x_i \vee x_j) = 1$.

Suppose θ is $(\neg x_i \vee \neg x_j)$. Then, by the construction of $\Theta_{\mathcal{X}}$, it holds that $\{s_i, s_j\}$ fails ψ. Then, s_i and s_j cannot be both in $\mathcal{Z}$, since $\mathcal{Z}$ was supposed to satisfy ψ. Thus either s_i or s_j must be in $\mathcal{Y}$. Then, it holds that $S(x_i) = 0$ or $S(x_j) = 0$, which implies that $S(\neg x_i \vee \neg x_j) = 1$.

Last, the complexity of this reduction is in LOGSPACE: We need to go through the 2-element subsets of the team $\mathcal{X}$ and check if they dissatisfy ϕ or ψ. All the 2 assignment sub-teams of $\mathcal{X}$ can be generated in LOGSPACE when $\mathcal{X}$ is given. Since ϕ and ψ were coherent, the model checking for both of these formulas can be done in LOGSPACE. ■

Corollary 4.13. *Suppose ϕ and ψ are 2-coherent $\mathcal{D}$-formulas. Then*

$$MC(\phi \vee \psi) \in NL.$$

Next we will show that the model checking of the formula $=(x, y)\vee =(z, v)$ is complete for NL. We will reduce 2-SAT to the model checking problem of the formula $=(x, y)\vee =(z, v)$.

Theorem 4.14. *2-SAT* $\leq_{LOGSPACE} MC(=(x, y)\vee =(z, v))$.

Proof. Suppose $\theta(p_0, ..., p_{m-1})$ is an instance of 2-SAT of the form $\bigwedge_{i\in I} E_i$, where each conjunct $E_i = (A_{i_1} \vee A_{i_2})$, $i \in I$, where A_{i_j}, $j \leq 1$, are positive or negative boolean variables.

We will construct a team $\mathcal{X}$, such that the following are equivalent:

1. $\mathcal{X} \models =(x, y)\vee =(z, v)$.
2. $\theta(p_0, \ldots, p_{m-1})$ is satisfiable.

For each conjunct E_i, $i \in I$, we create a team $\mathcal{X}_{E_i}$ with two assignments s_{i_1} and s_{i_2} of domain $\{x, y, z, v\}$ where we encode both of the boolean variables $A_j \in E_i$ and the truth values of the variables which satisfy the clause E_i. The clause $E_i : (A_{i_1} \vee A_{i2})$ will be satisfied if one of the boolean variables A_{i_1} or A_{i2} will be assigned value 1.

Thus, variable x ranges over the set boolean variables in $\theta(p_0, \ldots, p_{m-1})$, variable y ranges over the truth values $\{0, 1\}$. The variable z denotes the clause E_i. Thus z ranges over the indices $i \in I$. Variable v ranges over values $\{0, 1\}$. Variables z and v together make sure we have to choose at least one of the assignments form each $\mathcal{X}_{E_i}$ into the subset of $\mathcal{X}$, that eventually encodes the assignment that satisfies θ.

x	y	z	v
p_k	1	i	1
p_j	1	i	2

Table 2. Team for $(p_k \vee p_j)$.

Each boolean variable A_{i_j} in E_i, $i \in I$, gives rise to one assignment. For example, the team $\mathcal{X}_{E_i}$ for a clause $(p_k \vee p_j)$ is the one in Table 2.

The team for the whole instance of 2-SAT, $\bigwedge_{i \in I}(A_{i_1} \vee A_{i_2})$, is the one in Table 3, where $t(A_i) = 1$ if A_i is a unnegated variable and $t(A_i) = 0$, if A_i is a negated variable. Now $\mathcal{X}$ is the union $\bigcup_{i \in I} \mathcal{X}_{Ei}$.

x	y	z	v
A_{0_1}	$t(A_{0_1})$	0	1
A_{0_2}	$t(A_{0_2})$	0	2
A_{1_1}	$t(A_{1_1})$	1	1
A_{1_2}	$t(A_{1_2})$	1	2
A_{2_1}	$t(A_{2_1})$	2	1
A_{2_2}	$t(A_{2_2})$	2	2
.	.	.	.
.	.	.	.
.	.	.	.
A_{I_1}	$t(A_{I_1})$	n	1
A_{I_2}	$t(A_{I_2})$	n	2

Table 3. Team $\bigcup_{i \in I} \mathcal{X}_{E_i}$.

Suppose $\theta(p_0, \ldots, p_{m-1})$ is satisfiable. Then there exists an assignment $F : \{p_0, \ldots, p_{m-1}\} \to \{0, 1\}$, that satisfies $\theta(p_0, \ldots, p_{m-1})$. We define the partition of the team $\mathcal{X}$ into two sets in the following way:

$$\mathcal{X}_1 = \{s \in \mathcal{X} \mid F(s(x)) = s(y)\},$$

$$\mathcal{X}_2 = \mathcal{X} \setminus \mathcal{X}_1.$$

The assignments in $\mathcal{X}$ that agree with the assignment F are chosen to $\mathcal{X}_1$. Since F evaluates $\bigwedge_{i \in I} E_i$ true, it satisfies every conjunct E_i. Thus $\mathcal{X}_1$ contains at least one assignment from each $\mathcal{X}_{E_i}$. Thus there will be at most one tuple from each $\mathcal{X}_{E_i}$ left to $\mathcal{X}_2$. Thus $\mathcal{X}_2$ trivially satisfies $=\!(z, v)$ since all tuples in $\mathcal{X}_2$ disagree on z.

Next we will show that $\mathcal{X}_1$ satisfies $=(x, y)$: Let $s, s' \in \mathcal{X}_1$, such that $s(x) = s'(x) = p_i$. Then by the definition of $\mathcal{X}_1$ it holds that $s(y) = F(p_i) = s'(y)$ holds. Thus $\mathcal{X}_1 \models\ =(x, y)$.

The other direction: Suppose $\mathcal{X} \models\ =(x, y)\vee =(z, v)$. Then there is a partition of $\mathcal{X}$ into $\mathcal{X}_1$ and $\mathcal{X}_2$, such that $\mathcal{X}_1 \models =(x, y)$ and $\mathcal{X}_2 \models =(z, v)$. We will define the assignment $F : \{p_0, \ldots, p_m\} \to \{0, 1\}$ in the following way:

- If $\exists s \in \mathcal{X}_1$, such that $s(x) = p_i$, then $F(p_i) = s(y)$.
- If $\forall s \in \mathcal{X}_1$ it holds $s(x) \neq p_i$, then $F(p_i) = 1$.[2]

Let us show that $F : \{p_0, \ldots, p_{m-1}\} \to \{0, 1\}$ is a function, which satisfies $\theta(p_0, \ldots, p_{m-1})$:

1. Clearly, $Dom(F) = \{p_0, \ldots, p_{m-1}\}$ and $Range(F) = \{0, 1\}$.
2. F is a function: Let $p_i \in \{p_0, \ldots, p_m\}$. Suppose there exists $s, s' \in \mathcal{X}_1$, such that $s(x) = s'(x) = p_i$ holds. Since $\mathcal{X}_1 \models =(x, y)$ holds, it follows that $s(y) = s'(y)$ holds. Suppose there are no $s \in \mathcal{X}_1$, such that $s(x) = p_i$. Then by definition of F it holds that $F(p_i) = 1$.
3. F satisfies $\theta(p_0, \ldots, p_{m-1})$: Note that z is constant and v is assigned different value by each tuple in each $\mathcal{X}_{E_i}$. Thus $\mathcal{X}_1$ contains at least one of the tuples from each $\mathcal{X}_{E_i}$. Let $s \in \mathcal{X}_{E_i}$, such that $s \in \mathcal{X}_1$. Recall, that in every assignment the value $s(y)$ encoded the truth value for $s(x)$ such that E_i is satisfied. Since s agrees with F, it holds that $F(A_{i_j}) = s(y)$, which implies that $F(E_i) = 1$.

Each conjunct E_i, $i \in I$, of θ gives rise to a constant size team of two assignments with domain $\{x, y, z, v\}$. Thus the team $\mathcal{X}$ can be constructed in LOGSPACE for each θ. ∎

The problem 2−SAT is known to be complete for NL [5]. Now we have the following corollary:

COROLLARY 4.15. *$MC(=(x, y)\vee =(z, v)$ is complete for NL.*

Next we will show that when we consider formulas with two disjunctions, the model checking becomes NP-complete for certain formulas.

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[2]If for all the assignments $s \in X_1$ holds $s(x) \neq p_i$, then the value of p_i is not relevant to the satisfiability of Θ. Thus the value of p_i can be chosen 0 or 1.

4.3. Non-deterministic polynomial time

We will reduce 3-SAT to MC($=(x, y)\vee =(z, v)\vee =(z, v)$).

Recall that an instance $\theta \in$ 3-SAT is a first-order formula in conjunctive normal form, where each conjunct has at most three variables: $\bigwedge_{i\in I} E_i$, where I is finite. Each E_i is of form $(A_{i_0} \vee A_{i_2} \vee A_{i_3})$, where A_i is either a positive or a negated boolean variable. Formula θ is accepted if there is an assignment, that satisfies θ. The reduction is analogous to the reduction given in Theorem 4.14.

THEOREM 4.16. *3-SAT* $\leq_{LOGSPACE}$ *MC*($=(x, y)\vee =(z, v)\vee =(z, v)$).

PROOF. Suppose $\theta(p_0, \ldots, p_{m-1})$ is an instance of 3-SAT with conjuncts E_i, $i \in I$. We will construct a team $\mathcal{X}$, such that the following are equivalent:

- $\mathcal{X} \models =(x, y)\vee =(z, v)\vee =(z, v)$.
- $\theta(p_0, \ldots, p_{m-1})$ is satisfiable.

For each conjunct E_i, $i \in I$, we create a team $\mathcal{X}_{E_i}$ of three assignments with domain $\{x, y, z, v\}$, which encodes all boolean variables of E_i as well as the truth value for the variable such that E_i is satisfied. Thus, the variable x ranges over the boolean variables in $\theta(p_0, \ldots, p_{m-1})$, variable y ranges over the truth values $\{0, 1\}$. The variable z denotes the clause E_i. Thus z ranges over the indices $i \in I$. Variable v ranges over values $\{0, 1, 2\}$.

For example, a clause $E_i = (p_l \vee \neg p_j \vee \neg p_k)$ will be satisfied if $p_l = 1$ or $p_j = 0$ or $p_k = 0$. The team for $(p_l \vee \neg p_j \vee \neg p_k)$ is the one in Table 4.

x	y	z	v
p_l	1	1	0
p_j	0	1	1
p_k	0	1	2

Table 4. A team for $(p_l \vee \neg p_j \vee \neg p_k)$.

The team $\mathcal{X}$ for the whole instance $\theta(p_0, \ldots, p_{m-1})$ is then $\mathcal{X} = \bigcup_{i\in I} \mathcal{X}_{E_i}$.

Suppose $\theta(p_0, \ldots, p_{m-1})$ is satisfiable. Then there exists an assignment $F : \{p_0, \ldots, p_{m-1}\} \to \{0, 1\}$, such that F satisfies $\theta(p_0, \ldots, p_{m-1})$. We define $\mathcal{X}_1 \subset \mathcal{X}$ in the following way:

$$\mathcal{X}_1 = \{s \in \mathcal{X} \mid F(s(x)) = s(y)\},$$

Since F satisfies $\bigwedge_{i\in n} E_i$, it satisfies every conjunct E_i. Then by the definition of $\mathcal{X}_{E_i}$ one of the tuples $s \in \mathcal{X}_{E_i}$ agrees with F, which means $s(x) = p_{i_j}$,

$j \leq 3$, and $F(p_{i_j}) = s(y)$. Thus. by the definition of $\mathcal{X}_1$ it holds that $s \in \mathcal{X}_1$. Thus the two "leftover"-assignment form each $\mathcal{X}_{E_i}$ can be easily divided into $\mathcal{X}_2$ and $\mathcal{X}_3$ in such a way that $=(z, v)$ holds in both of them. We just place one of the assignments into $\mathcal{X}_2$ and one into $\mathcal{X}_3$.

Let us show that $\mathcal{X}_1 \models =(x, y)$. Suppose $s, s' \in \mathcal{X}_1$, such that $s(x) = s'(x) = p_i$. Then, by definition of $\mathcal{X}_1$, it follows that $s(y) = s'(y) = F(p_i)$. Thus $\mathcal{X}_1 \models =(x, y)$.

The other direction: Suppose $\mathcal{X} \models =(x, y) \vee =(z, v) \vee =(z, v)$ holds. Then by the truth definition of the disjunction, it follows that $\mathcal{X}$ can be partitioned into three sets $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$, such that $\mathcal{X}_1 \models =(x, y)$, $\mathcal{X}_2 \models =(z, v)$ and $\mathcal{X}_3 \models =(z, v)$ hold. Let F be defined in the following way for each variable p_i.

- If $\exists s \in \mathcal{X}_1$, such that $s(x) = p_i$, then $F(p_i) = s(y)$.
- If $\forall s \in \mathcal{X}_1$ it holds $s(x) \neq p_i$, then $F(p_i) = 1$.

Let us show that $F : \{p_0, \ldots, p_m\} \to \{0, 1\}$ is a function, which satisfies $\theta(p_0, \ldots, p_{m-1})$.

1. Clearly, F is well defined and the domain of F is $\{p_0, \ldots, p_{m-1}\}$ and the range is $\{0, 1\}$.
2. F is a function: Let $p_i \in \{p_0, \ldots, p_m\}$. Suppose there exists $s, s' \in \mathcal{X}_1$, such that $s(x) = s'(x) = p_i$ holds. Since $\mathcal{X}_1 \models =(x, y)$ holds, it follows that $s(y) = s'(y) = F(p_i)$ holds. If there exists no $s \in \mathcal{X}_1$, such that $s(x) = p_i$, then it holds by the definition of F, that $F(p_i) = 1$.
3. F satisfies $\theta(p_0, \ldots, p_{m-1})$: Note that z is constant and v is assigned different value by each tuple in each $\mathcal{X}_{E_i}$. Thus $\mathcal{X}_1$ contains at least one of the tuples from each $\mathcal{X}_{E_i}$. Let $s \in \mathcal{X}_{E_i}$, such that $s \in \mathcal{X}_1$. Recall, that in every assignment the value $s(y)$ encoded the truth value for $s(x)$ such that E_i is satisfied. Since s agrees with F, it holds that $F(A_{i_j}) = s(y)$, which implies that $F(E_i) = 1$.

Each conjunct E_i, $i \in I$, in θ gives rise to a constant size team of three assignments with domain $\{x, y, z, v\}$. Thus team $\mathcal{X}$ can be constructed in LOGSPACE for each θ. ■

3-SAT is complete for NP [3]. We have the following corollary:

Corollary 4.17. *$MC(=(x, y) \vee =(z, v) \vee =(z, v))$ is complete for NP.*

Notice that $=(x, y)$ is 2-coherent and $=(z, v) \vee =(z, v)$ is 3-coherent. Thus in the light of Theorem 4.12, this is the best possible result.

5. Directions for future work and open problems

As we have shown the quantifier-free fragment of dependence logic is computationally as hard as the whole logic. The model checking problem becomes NP-complete already for relatively simple quantifier-free formulas. The coherent fragment contains the essential building blocks of dependence logic, that is the dependence atoms, and it is relatively low in complexity, thus it gives us a good starting point to look for more expressive, but yet tractable fragments or extensions of dependence logic. The notion of coherence also gives us a mean to evaluate the complexity of connectives in the Team-semantics.

1. Is there is some natural extension of coherent fragment of $\mathcal{D}$ that coincides with LOGSPACE?
   In Theorem 3.11 we generalized the notion of coherence. We say that ϕ is $f(n)$-coherent, if for all teams $\mathcal{X}$, such that $|\mathcal{X}| = n$ holds
   $\mathcal{X} \models \phi \Leftrightarrow$ for all $\mathcal{Y} \subseteq \mathcal{X}$, (if $|\mathcal{Y}| = f(n)$, then $\mathcal{Y} \models \phi$).
2. Are there $\phi \in \mathcal{D}$, such that the coherence-level of ϕ is not a constant-function, e.g. $f(n) = \sqrt{n}$?

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