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Varieties of Commutative Integral Bounded Residuated Lattices Admitting a Boolean Retraction Term

Abstract. Let $\mathbb{BRL}$ denote the variety of commutative integral bounded residuated lattices (bounded residuated lattices for short). A *Boolean retraction term* for a subvariety $\mathbb{V}$ of $\mathbb{BRL}$ is a unary term t in the language of bounded residuated lattices such that for every $\boldsymbol{A} \in \mathbb{V}$, $t^{\boldsymbol{A}}$, the interpretation of the term on $\boldsymbol{A}$, defines a *retraction* from $\boldsymbol{A}$ onto its Boolean skeleton $\boldsymbol{B}(\boldsymbol{A})$. It is shown that Boolean retraction terms are equationally definable, in the sense that there is a variety $\mathbb{V}_t \subsetneq \mathbb{BRL}$ such that a variety $\mathbb{V} \subsetneq \mathbb{BRL}$ admits the unary term t as a Boolean retraction term if and only if $\mathbb{V} \subseteq \mathbb{V}_t$. Moreover, the equation $s(x) = t(x)$ holds in $\mathbb{V}_s \cap \mathbb{V}_t$.

The radical of $\boldsymbol{A} \in \mathbb{BRL}$, with the structure of an unbounded residuated lattice with the operations inherited from $\boldsymbol{A}$ expanded with a unary operation corresponding to double negation and a a binary operation defined in terms of the monoid product and the negation, is called the *radical algebra* of $\boldsymbol{A}$. To each involutive variety $\mathbb{V} \subseteq \mathbb{V}_t$ is associated a variety $\mathbb{V}^r$ formed by the isomorphic copies of the radical algebras of the directly indecomposable algebras in $\mathbb{V}$. Each free algebra in such $\mathbb{V}$ is representable as a weak Boolean product of directly indecomposable algebras over the Stone space of the free Boolean algebra with the same number of free generators, and the radical algebra of each directly indecomposable factor is a free algebra in the associated variety $\mathbb{V}^r$, also with the same number of free generators.

A hierarchy of subvarieties of $\mathbb{BRL}$ admitting Boolean retraction terms is exhibited.

Keywords: Residuated lattices, Boolean products, Boolean retraction terms, Free algebras.

Introduction

By a *residuated lattice* we mean a *pointed integral residuated lattice-ordered commutative monoid*, and by a *bounded residuated lattice* we mean a residuated lattice with a smallest element (see §1 for the definitions). We shall denote by $\mathbb{RL}$ the variety of residuated lattices and by $\mathbb{BRL}$ the subvariety of $\mathbb{RL}$ formed by bounded residuated lattices.

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Each algebra in a subvariety $\mathbb{V}$ of $\mathbb{BRL}$ can be represented as a weak Boolean product of directly indecomposable algebras in $\mathbb{V}$ over the Stone space of its Boolean skeleton [17]. In particular, free algebras are weak Boolean products of directly indecomposable algebras. To obtain such representation the first step is to characterize the Boolean skeleton of a free algebra. If the Boolean skeleton is isomorphic to the two-element Boolean algebra, then the free algebra is directly indecomposable and no further information we can get from this method. This is the case, for instance, of free residuated lattices, free involutive residuated lattices, free MV-algebras and free BL-algebras (see [5] and [9]). If the Boolean skeleton has more than two elements, then we have to characterize the directly indecomposable factors of the weak Boolean product representation.

The Boolean skeleton of free algebras in $\mathbb{V}$ can be easily characterized if there is a unary term t in the language of residuated lattices such that the evaluation of t on each algebra $\boldsymbol{A} \in \mathbb{V}$ defines a retraction from $\boldsymbol{A}$ onto its Boolean skeleton. Indeed, in this case the Boolean skeleton of a free algebra in $\mathbb{V}$ is the free Boolean algebra with the *same number* of free generators.

Hence in the presence of a Boolean retraction term for $\mathbb{V}$, the main problem to describe the representation of free algebras in $\mathbb{V}$ as weak Boolean products is to obtain a description of the directly indecomposable factors.

This problem was considered in [9, 11] for subvarieties of BL-algebras and of MTL-algebras, and in [6] for varieties of bounded residuated lattices admitting the double negation as a Boolean retraction term.

The aim of this paper is to continue this line of research, investigating subvarieties of bounded residuated lattices admitting a Boolean retraction term.

We show that Boolean retraction terms are equationally definable, in the sense that there is a variety $\mathbb{V}_t \subsetneq \mathbb{BRL}$ such that a variety $\mathbb{V} \subsetneq \mathbb{BRL}$ admits the unary term t as a Boolean retraction term if and only if $\mathbb{V} \subseteq \mathbb{V}_t$ (Corollary 2.5). Moreover, the equation $s(x) = t(x)$ holds in $\mathbb{V}_s \cap \mathbb{V}_t$ (Corollary 2.8). It is also shown that for all $\boldsymbol{A} \in \mathbb{V}_t$ the radical $Rad(\boldsymbol{A})$ of $\boldsymbol{A}$, i.e., the intersection of all maximal implicative filters of $\boldsymbol{A}$, is characterized by the equation $t(x) = \top$ (Theorem 2.7). This allows us to characterize the directly indecomposable algebras in $\mathbb{V}_t$ as those which are the disjoint union of the radical and the coradical (Lemma 3.5).

The radical of a bounded residuated lattice $\boldsymbol{A}$, being a proper implicative filter, can be considered as an unbounded residuated lattice with the operations inherited from $\boldsymbol{A}$. It is also closed under the double negation of its elements, and under a binary operation $\oplus$ defined in terms of the monoid product and the negation (see (1.5)). The radical, with the structure of a residuated lattice expanded with the binary operation $\oplus$ and a unary operation corresponding to double negation is called *the radical algebra of* $\boldsymbol{A}$. For each variety $\mathbb{V}$ of bounded residuated lattices, we denote by $\mathbb{V}^r$ the class of isomorphic copies of the radical algebras of elements of $\mathbb{V}$. We show that $\mathbb{V}_t^r$ is closed under homomorphic images and direct products. In case that $\mathbb{V}_t$ is an involutive variety, i.e., in case that the double negation coincides with the identity for all algebras in $\mathbb{V}_t$, then $\mathbb{V}_t^r$ is also closed under subalgebras. Hence it is a variety, the *radical variety associated with* $\mathbb{V}$. In proving this result the binary operation $\oplus$ plays a crucial role (see the proof of Theorem 3.9).

Radical algebras were introduced in [9, 11] for varieties of MTL-algebras, and were defined as residuated lattices expanded with just one unary operation, corresponding to the double negation. As a matter of fact the operation $\oplus$ was hidden, because it follows from Lemma 3.1 that when the operation $\oplus$ is applied to elements of the radical of an MTL-algebra, it gives always the top element.

The techniques developed in [9, 11] allow us to get information for free algebras in a variety $\mathbb{V}$ with a Boolean retraction term t in two cases: when $\mathbb{V}$ is involutive, and when for all $\boldsymbol{A} \in \mathbb{V}$, the double negation of all elements on the radical of $\boldsymbol{A}$ is the top element. It follows from Corollary 2.11 that in this last case, the term t coincides with the double negation. Since free algebras in varieties of bounded residuated lattices admitting the double negation as a Boolean retraction term are described in [6], in the present paper we focus our attention in *involutive* subvarieties of bounded residuated lattices admitting a Boolean retraction term. We show that in the Boolean product representation of free algebras in such varieties, *the radical algebras of the directly indecomposable factors are free in the associated radical varieties* (Theorem 4.8).

We construct a sequence $\{\nabla_n\}_{n\in\mathbb{N}}$ of unary terms and a hierarchy of subvarieties of $\mathbb{BRL}$,

$$\mathbb{WL}_1 \subsetneq \mathbb{WL}_2 \subsetneq \cdots \subsetneq \mathbb{WL}_n \subsetneq \mathbb{WL}_{n+1} \subsetneq \cdots$$

such that ∇_n is the unique admissible Boolean retraction term for subvarieties of $\mathbb{WL}_n$, for $n > 0$. Moreover, for each $n > 0$ the existence of a Boolean retraction term for a subvariety $\mathbb{V}$ of $\mathbb{WL}_n$ is guaranteed by the existence of a homomorphism from each directly indecomposable algebra in $\mathbb{V}$ onto its Boolean skeleton, and the Boolean retraction term is ∇_n (Theorem 5.7, Corollary 5.9). $\mathbb{WL}_1$ is the variety of Stonean residuated lattices and ∇_1 is the double negation. Since $\mathbb{WL}_2$ contains the variety of MTL-algebras and

∇_2 coincides with the Boolean retraction term defined in [11], these results generalize those proved for varieties of MTL-algebras ([11, Remark 3.3]). We also construct an involutive subvariety $\mathbb{V}_n$ of each $\mathbb{WL}_n$ admitting ∇_n as a Boolean retraction term and such that $\mathbb{V}_n^r$ is the variety generated by the n-element totally ordered Wajsberg hoop expanded with a new binary operation (Lemma 5.15).

Although we assume familiarity with residuated lattices, in §1 we recall the main definitions and properties that we shall need in the following sections. We also define and establish properties of the binary operation $\oplus$, and introduce the varieties $\mathbb{WL}_n$ for $n > 0$. In §2 we establish the main properties of subvarieties of $\mathbb{BRL}$ admitting a Boolean retraction term; radical algebras and radical varieties are considered in §3. In §4 we study the factors of the Boolean product representation of free algebras in involutive varieties of bounded residuated lattices having a Boolean retraction term. Finally in §5 we introduce the mentioned family ∇_n, $n > 0$ of Boolean retraction terms and the corresponding varieties.

1. Preliminaries

1.1. Residuated lattices

Following the nomenclature of [13, Chapter 3], we define a *pointed integral residuated lattice-ordered commutative monoid*, or *residuated lattice* for short, as an algebra $\boldsymbol{A} = \langle A; *, \to, \vee, \wedge, \bot, \top \rangle$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ such that $\langle A; *, \top \rangle$ is a commutative monoid, $\boldsymbol{L}(\boldsymbol{A}) = \langle A; \vee, \wedge, \top \rangle$ is a lattice with greatest element $\top$, and the following residuation condition holds:

$$x * y \leq z, \text{ iff } x \leq y \to z \tag{1.1}$$

where x, y, z denote arbitrary elements of A and $\leq$ is the order given by the lattice structure, which is called *the natural order of* $\boldsymbol{A}$.

It is well known that residuated lattices form a variety, that we shall denote $\mathbb{RL}$. Indeed, the residuation condition can be replaced by the following equations:

$$x = x \wedge (y \to (x * y) \vee z)), \tag{1.2}$$

$$z = (y * (x \wedge (y \to z))) \vee z. \tag{1.3}$$

Following [18], residuated lattices satisfying the equation $x * y = x \wedge y$ will be called *generalized Heyting algebras*. They were called *Brouwerian algebras* in [12].

In the next lemma we list, for further reference, some well known consequences of (1.1) that will be used through this paper.

LEMMA 1.1. *The following properties hold true in any residuated lattice* $\boldsymbol{A}$*, where* a, b, c *denote arbitrary elements of* A*:*

(*i*) $a \leq b$ *if and only if* $a \to b = \top$,

(*ii*) $\top \to a = a$,

(*iii*) $(a \to b) \to ((b \to c) \to (a \to c)) = \top$,

(*iv*) $(a * b) \to c = a \to (b \to c)$,

(*v*) $(a \vee b) \to c = (a \to c) \wedge (b \to c))$.

On a residuated lattice $\boldsymbol{A}$ we consider the unary operation:

$$\neg x =: x \to \bot, \text{ for all } x \in A. \tag{1.4}$$

Observe that if $\bot = \top$, then for any $a \in A$, $\neg a = \top \to \top = \top$. By taking into account that the $\{\to, \top\}$-reduct of a residuated lattice is a BCK-algebra we have:

LEMMA 1.2. *The following identities and quasi-identities hold true in any residuated lattice* $\boldsymbol{A}$*:*

a) $x \leq y \Rightarrow \neg y \leq \neg x$.

b) $x \leq \neg\neg x$.

c) $\neg x = \neg\neg\neg x$.

d) $x \to \neg y = y \to \neg x$.

e) $x \to \neg y = \neg\neg x \to \neg y$.

f) $\neg\neg(x \to \neg y) = x \to \neg y$.

PROOF. Items *a*), *b*) and *d*) follow from Lemma 1.1. *c*) follows from *a*) and *b*), and *e*) follows from *c*) and *d*). To prove *f*), let $a, b \in A$. By *b*), $(a \to \neg b) \leq \neg\neg(a \to \neg b)$. On the other hand, taking into account (*iv*) in Lemma 1.1 and *e*) one has:

$$\begin{aligned}
\neg\neg(a \to \neg b) \to (a \to \neg b) &= a \to (\neg\neg(a \to \neg b) \to \neg b) \\
&= a \to ((a \to \neg b) \to \neg b) \\
&= (a \to \neg b) \to (a \to \neg b) = \top,
\end{aligned}$$

hence $\neg\neg(a \to \neg b) \leq a \to \neg b$. ■

On each residuated lattice $\boldsymbol{A}$ we consider the term

$$x \oplus y =: \neg(\neg x * \neg y) \tag{1.5}$$

It is easy to see that $\langle A, \oplus\rangle$ is a commutative semigroup. We define the terms x^n and $n.x$ for any non negative integer n recursively:

- $x^0 = \top$ and $x^{n+1} = x * x^n$
- $0.x = \bot$ and $(n+1).x = x \oplus n.x$

LEMMA 1.3. *Let $\boldsymbol{A} \in \mathbb{RL}$. For every $a, b \in A$ and for every non negative integers n, m, the following properties hold:*

1) $a \oplus b = \neg a \to \neg\neg b$,

2) $1.a = \neg\neg a$,

3) $n.a = \neg(\neg a)^n$,

4) $n.a = \neg\neg(n.a) = n.(\neg\neg a)$,

5) $(n+m).a = (n.a) \oplus (m.a)$,

6) $(mn).a = m.(n.a)$,

7) *if* $n \leq m$ *then* $n.a \leq m.a$ *and* $a^m \leq a^n$,

8) $\neg((n.a)^m) = m.(\neg a)^n$,

9) *if* $a \leq b$, *then* $n.a^m \leq n.b^m$.

PROOF. The proofs of these properties follow from Lemma 1.2. As example we will prove 8): By 3), $\neg((n.a)^m) = \neg((\neg(\neg a)^n)^m) = m.(\neg a)^n$. ■

By an *implicative filter* or *i-filter* of a residuated lattice $\boldsymbol{A}$ we mean a subset $F \subseteq A$ satisfying the following conditions:

F1) $\top \in F$.

F2) For all $a, b \in A$, if $b \in F$ and $a \leq b$, then $b \in F$.

F3) If a, b are in F, then $a * b \in F$.

Alternatively, i-filters may be defined as subsets F of A satisfying F1) and

F4) If $a, a \to b$ are in F, then $b \in F$.

Each i-filter F is the universe of a subalgebra of $\boldsymbol{A}^- = \langle A; *, \to, \vee, \wedge, \bot^-, \top\rangle$ (with $\bot^- = \top$), which we shall denote $\boldsymbol{F}$. It is easy to prove that for each $X \subseteq A$

$$\langle X\rangle = \{a \in A : x_1^{n_1} * \cdots * x_k^{n_k} \leq a,\ k, n_1, \ldots, n_k \geq 0, x_1, \ldots, x_k \in X\} \quad (1.6)$$

is the smallest i-filter containing X, i.e., the intersection of all i-filters containing X. For each $x \in A$, we shall write $\langle x\rangle$ instead of $\langle\{x\}\rangle$.

Given an i-filter F of a residuated lattice $\boldsymbol{A}$, the binary relation

$$\theta(F) := \{(x, y) \in A \times A : x \to y \in F \text{ and } y \to x \in F\}$$

is a congruence on $\boldsymbol{A}$ such that $F = \top/\theta(F)$, the equivalence class of $\top$. As a matter of fact, the correspondence $F \mapsto \theta(F)$ is an order isomorphism from the set of filters of $\boldsymbol{A}$ onto the set of congruences of $\boldsymbol{A}$, both sets ordered by inclusion, whose inverse is given by the map $\theta \mapsto \top/\theta$. We will write simply $\boldsymbol{A}/F$ instead of $\boldsymbol{A}/\theta(F)$, and a/F instead of $a/\theta(F)$, the equivalence class determined by $a \in A$.

An i-filter F of $\boldsymbol{A}$ is *proper* provided $F \neq A$. A *maximal i-filter* is a proper i-filter F of $\boldsymbol{A}$ such that for each i-filter G of $\boldsymbol{A}$, $F \subsetneq G$ implies $G = A$. We recall:

LEMMA 1.4. *An i-filter F of $\boldsymbol{A} \in \mathbb{RL}$ is maximal if and only if for any $a \in A$,*

- *$a \notin F$ iff for every $b \in A$ there is $n > 0$ such that $a^n \to b \in F$.*

It follows that for each $\boldsymbol{A} \in \mathbb{RL}$

$$Rad(\boldsymbol{A}) = \{a \in A : \forall b \in A, \forall\, n > 0,\ \exists\, k_{n,b} \text{ such that } (a^n \to b)^{k_{n,b}} \to b = \top\}$$

is the intersection of all maximal i-filters of $\boldsymbol{A}$ (see, for instance [15, 17]).

By a *residuated chain* we mean a residuated lattice $\boldsymbol{A}$ whose natural order is total, i.e., given a, b in A, $a \leq b$ or $b \leq a$.

Given a class $\mathbb{K}$ of algebras we represent by $\mathbb{K}_{si}$, the class of its subdirectly irreducible members. Recall that every variety is generated by its subdirectly irreducible members. We also recall that after [17, Proposition 1.4], $\top$ *is join irreducible in any member of* $\mathbb{RL}_{si}$.

1.2. Bounded Residuated lattices

By a *bounded residuated lattice* we mean a residuated lattice in which the following equation holds:

$$\bot \to x = \top \tag{1.7}$$

or equivalently the equation:

$$\neg x \to (x \to y) = \top \tag{1.8}$$

that is, $\bot$ is the smallest element of the lattice $\boldsymbol{L}(\boldsymbol{A})$. By definition the class $\mathbb{BRL}$ of all bounded residuated lattices is a variety. Notice that bounded generalized Heyting algebras are precisely the *Heyting algebras*, i.e., the algebras of intuitionistic logic (see, for instance [18]).

An *involutive residuated lattice*(or *integral, commutative Girard monoid* [15]) is a bounded residuated lattice satisfying the *double negation equation*:

$$\neg\neg x = x. \tag{1.9}$$

It follows from (*iv*) of Lemma 1.1 that in an involutive residuated lattice the operations $*$ and $\to$ are related as follows:

$$x * y = \neg(x \to \neg y), \tag{1.10}$$

$$x \to y = \neg(x * \neg y). \tag{1.11}$$

When $\boldsymbol{A}$ is a bounded residuated lattice, an i-filter F is proper if and only if $\bot \notin F$. Then

Lemma 1.5. *An i-filter F of a bounded residuated lattice $\boldsymbol{A}$ is a maximal i-filter if and only if for any $a \in A$,*

- *$a \notin F$ if and only if there is $n > 0$ such that $\neg(a^n) \in F$.*

Then if $\boldsymbol{A}$ is a bounded residuated lattice we have:

$$Rad(\boldsymbol{A}) = \{a \in A : \forall n > 0, \exists k_n > 0, \text{ such that } k_n.a^n = \top\}. \tag{1.12}$$

Remark 1.6. *Let $\mathbb{V}$ be a subvariety of $\mathbb{BRL}$. Each $\boldsymbol{A} \in \mathbb{V}$ is a subdirect product of a family $\{\boldsymbol{A}_i\}_{i\in I}$ of $\mathbb{V}_{si}$, and $Rad(\boldsymbol{A})$ is embedded in $\prod_{i\in I} Rad(\boldsymbol{A}_i)$.*

In what follows we give some examples of subvarieties of $\mathbb{BRL}$, which shall be used through this paper.

We denote by $\mathbb{SRL}$ the variety of *stonean residuated lattices*, i.e., the variety of bounded residuated lattices determined by the Stone equation

$$\neg\, x \vee \neg\neg\, x = \top. \tag{1.13}$$

LEMMA 1.7. *For each* $\boldsymbol{A} \in \mathbb{SRL}$, $Rad(\boldsymbol{A}) = \{a \in A : 1.a = \top\}$.

PROOF. Let $\boldsymbol{A} \in \mathbb{SRL}$ be subdirectly irreducible. Since $\top$ is join irreducible, for all $x \in A$, $\neg\, x = \top$ or $\neg\neg\, x = \top$. If there is an integer $n \geq 1$ such that $\neg\, x^n = \top$, then $x \notin Rad(\boldsymbol{A})$. Hence $Rad(\boldsymbol{A}) = \{x \in A : 1.x = \neg\neg x = \top\}$. The result follows from Remark 1.6. ■

For each $k > 0$, we denote by $\mathbb{WL}_k$ the subvariety of $\mathbb{BRL}$ determined by the following equation:

$$k.x \vee k.(\neg x) = \top. \tag{1.14}$$

It follows from 7) of Lemma 1.3 that for any $k > 0$, $\mathbb{WL}_k \subseteq \mathbb{WL}_{k+1}$. Observe that for $k = 1$, the equation (1.14) is the Stone equation (1.13); and for $k = 2$, (1.14) can be written as the *weak prelinearity equation*:

$$(\neg x \to \neg\neg x) \vee (\neg\neg x \to \neg x) = \top, \tag{1.15}$$

which, by *e*) in Lemma 1.2, is equivalent to

$$(\neg x \to \neg\neg x) \vee (x \to \neg x) = \top. \tag{1.16}$$

LEMMA 1.8. *If* $k > 0$, *then for all* $\boldsymbol{A} \in \mathbb{WL}_k$, $Rad(\boldsymbol{A}) = \{a \in A : k.a^n = \top, \textit{for all } n > 0\}$.

PROOF. Suppose that $\boldsymbol{A} \in \mathbb{WL}_k$ is subdirectly irreducible. Since $\top$ is join irreducible, we have that for each $a \in A$, $k.a = \top$ or $k.(\neg a) = \top$. Take $a \in Rad(\boldsymbol{A})$. For any $n \geq 1$, $a^n \in Rad(\boldsymbol{A})$ and for any $r \geq 0$ $(\neg\neg(a^n))^r \in Rad(\boldsymbol{A})$. If $\top = k.(\neg a^n) = (\neg\neg a^n)^{k-1} \to (\neg a^n)$, then we would have $\neg a^n \in Rad(\boldsymbol{A})$, and this would imply that $\perp \in Rad(\boldsymbol{A})$, which is impossible. Hence $Rad(\boldsymbol{A}) = \{a \in A : k.a^n = \top$ for each $n > 0\}$. Now the result follows from Remark 1.6. ■

COROLLARY 1.9. *Let* $\boldsymbol{A} \in \mathbb{WL}_k$. *If* $\boldsymbol{B}$ *is a Boolean algebra and* $h\colon \boldsymbol{A} \to \boldsymbol{B}$ *is a homomorphism, then* $h(a) = \top$ *for all* $a \in Rad(\boldsymbol{A})$. *Moreover, if* $\boldsymbol{A}$ *is subdirectly irreducible, then* $h(a) = \perp$ *for all* $a \in A \setminus Rad(\boldsymbol{A})$.

PROOF. If $a \in Rad(\boldsymbol{A})$ then $\top = h(\top) = h(k.a) = h(a)$. Suppose now that $\boldsymbol{A}$ is subdirectly irreducible and let $a \in A \setminus Rad(\boldsymbol{A})$. Then there exists $p > 0$ such that $k.a^p < \top$, and since $\top$ is join irreducible, (1.14) implies that $k.\neg(a^p) = \top$. Therefore $\top = h(k.\neg(a^p)) = \neg h(a)$, and $h(a) = \perp$. ■

Let $\mathbb{MTL}$ denote the variety of *MTL-algebras*, i.e., the variety of bounded residuated lattices characterized by the prelinearity equation

$$(x \to y) \vee (y \to x) = \top. \tag{1.17}$$

It is easy to see that $\mathbb{MTL}$ is the variety generated by the class of all bounded residuated chains. It is clear that $\mathbb{MTL} \subseteq \mathbb{WL}_2$. Since the variety $\mathbb{MV}$ of MV-algebras is a subvariety of $\mathbb{MTL}$, we have that $\mathbb{MV} \subseteq \mathbb{WL}_2$.

A *pseudocomplemented residuated lattice* is a bounded residuated lattice that satisfies the equation

$$x \wedge \neg x = \bot. \tag{1.18}$$

Or equivalently the equation:

$$\neg x \wedge \neg\neg x = \bot. \tag{1.19}$$

We denote by $\mathbb{PRL}$ the variety of pseudocomplemented residuated lattices.

LEMMA 1.10. *For each $k > 1$, $\mathbb{WL}_k \cap \mathbb{PRL} = \mathbb{WL}_{k-1} \cap \mathbb{PRL}$.*

PROOF. Let $\boldsymbol{A}$ be a subdirectly irreducible in $\mathbb{WL}_k \cap \mathbb{PRL}$, and let $a \in A$. Since $\top$ is join irreducible, we have that $k.a = \top$ or $k.(\neg a) = \top$. Suppose that $k.a = \top$. By (i) in Lemma 1.1 and 1) in Lemma 1.3, $\neg(k-1).a \leq \neg\neg a$. Moreover, since $a \leq (k-1).a$, by a) in Lemma 1.2 we have that $\neg(k-1).a \leq \neg\neg a \wedge \neg a = \bot$, and by 4) in Lemma 1.3, $(k-1).a = \top$. Analogous arguments show that $k.(\neg a) = \top$ implies $(k-1).(\neg a) = \top$. Hence $\boldsymbol{A} \in \mathbb{WL}_{k-1} \cap \mathbb{PRL}$ and this implies that $\mathbb{WL}_k \cap \mathbb{PRL} \supseteq \mathbb{WL}_{k-1} \cap \mathbb{PRL}$. Since the other inclusion is obvious, we have completed the proof. ■

Since stonean residuated lattices are pseudocomplemented [7, Lemma 1.5] and $\mathbb{WL}_1 = \mathbb{SRL}$, from the above lemma we have:

$$\mathbb{WL}_k \cap \mathbb{PRL} = \mathbb{SRL}, \text{ for any } k > 0. \tag{1.20}$$

Since Heyting algebras are pseucomplemented, we have that *a Heyting algebra $\boldsymbol{A}$ satisfies* (1.15) *if and only if $\boldsymbol{A}$ satisfies the Stone equation* (1.13).

Given $\boldsymbol{A}$ a bounded residuated lattice, an element a of A is called *Boolean* if it is complemented in $\langle A, \vee, \wedge, \bot, \top\rangle$. The set of all Boolean elements of $\boldsymbol{A}$ is denoted by $B(\boldsymbol{A})$, it is well known that if $a \in B(\boldsymbol{A})$, then its complement is $\neg a$. Moreover, $B(\boldsymbol{A})$ is universe of a subalgebra $\boldsymbol{B}(\boldsymbol{A})$ of $\boldsymbol{A}$, which is Boolean algebra. In the particular case that $\boldsymbol{A}$ is subdirectly irreducible $B(\boldsymbol{A}) = \{\bot, \top\}$. Recall also that for any $a, b \in B(\boldsymbol{A})$,

$$\neg\neg a = a, \; a * b = a \wedge b, \text{ and } a \oplus b = a \vee b \tag{1.21}$$

In general, an algebra $\boldsymbol{A}$ is called *directly indecomposable* iff A has more than one element and whenever it is isomorphic to a direct product of two algebras $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, then either $\boldsymbol{A}_1$ or $\boldsymbol{A}_2$ is the trivial algebra with just one element.

The next result is proved in [17, Proposition 1.5].

LEMMA 1.11. *A bounded residuated lattice $\boldsymbol{A}$ is directly indecomposable if and only if $\boldsymbol{B}(\boldsymbol{A})$ is the two-element Boolean algebra.*

REMARK 1.12. *Notice that all members in $\mathbb{BRL}_{si}$ are directly indecomposable. Hence, if a bounded residuated lattice $\boldsymbol{A}$ is a subdirect product of a family $(\boldsymbol{A}_i)_{i\in I}$ of subdirectly irreducible, then $a \in B(\boldsymbol{A})$ if and only if, for any $i \in I$ $a(i) \in \{\top, \bot\}$.*

By a *Boolean space* we mean a totally disconnected compact Hausdorff space X. As usual, a subset of X that is simultaneously open and closed will be called *clopen*.

We recall that a weak Boolean product of a family $(A_x : x \in X)$ of algebras over a Boolean space X is a subdirect product $\boldsymbol{A}$ of the given family such that the following conditions hold:

(i) if $a, b \in A$, then $[\![a = b]\!] = \{x \in X : a(x) = b(x)\}$ is open,

(ii) if $a, b \in A$ and Z is a clopen in X, then $a \restriction_Z \cup\, b \restriction_{X \smallsetminus Z} \in A$.

An algebra $\boldsymbol{A}$ is *representable as weak Boolean product* when it is isomorphic to a weak Boolean product. As explained in [4], weak Boolean products are the global sections of (not necessarily Hausdorff) sheaves of algebras over Boolean spaces.

Since bounded residuated lattices form a congruence distributive variety, they have the Boolean Factor Congruence property (see [1]). Moreover, since $\mathbb{RL}$ is a congruence permutable variety with the class of directly indecomposable is closed under subalgebras, it follows from the results given [19] that each nontrivial bounded residuated lattice can be represented as a weak Boolean product of directly indecomposable bounded residuated lattices.

An explicit description of this representation is given in [11] and [6]. We shall recall it later.

2. Varieties of $\mathbb{BRL}$ having a Boolean retraction term

Let $\mathbb{V}$ be a variety of bounded residuated lattices. A *Boolean retraction term for* $\mathbb{V}$ is a unary term t in the language of bounded residuated lattices such that for every $\boldsymbol{A} \in \mathbb{V}$, $t^{\boldsymbol{A}}$, the interpretation of the term on $\boldsymbol{A}$, defines a *retraction* from $\boldsymbol{A}$ onto $\boldsymbol{B}(\boldsymbol{A})$, i.e., $t^{\boldsymbol{A}} \colon a \mapsto t^{\boldsymbol{A}}(a)$ is a homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{B}(\boldsymbol{A})$ such that $t^{\boldsymbol{A}}(z) = z$ for all $z \in B(\boldsymbol{A})$.

We will characterize the varieties of bounded residuated lattices admitting a Boolean retraction term. In what follows $\mathbb{V}$ shall represent a variety of bounded residuated lattices, and t a unary term.

LEMMA 2.1. *If $\mathbb{V}$ admits t as a Boolean retraction term, then $\mathbb{V}$ satisfies following equations and quasiequation:*

$$t(x) \vee t(\neg x) = \top \tag{2.22}$$

$$\neg t(x) = t(\neg x) \tag{2.23}$$

$$t(x * y) = t(x) * t(y) \tag{2.24}$$

$$x \to y = \top \Rightarrow t(x) \to t(y) = \top \tag{2.25}$$

PROOF. It suffices to see the equations hold true in $\mathbb{V}_{si}$. Take $\boldsymbol{A} \in \mathbb{V}_{si}$, since by assumption $t^{\boldsymbol{A}}$ defines and homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{B}(\boldsymbol{A})$, then (2.23), (2.24) and (2.25) hold in $\boldsymbol{A}$. Moreover, since $B(\boldsymbol{A}) = \{\top, \bot\}$, using (2.23), we deduce that (2.22) hold in $\boldsymbol{A}$. ■

LEMMA 2.2. *Let $\boldsymbol{A} \in \mathbb{BRL}_{si}$, and let t be a unary term. If* (2.22), (2.23), (2.24) *and* (2.25) *hold in $\boldsymbol{A}$, then we have:*

1) *for each $a \in A$, $t^{\boldsymbol{A}}(a) \in \{\top, \bot\}$*
2) *for each $a \in A$, $t^{\boldsymbol{A}}(a) = t^{\boldsymbol{A}}(\neg\neg a)$.*
3) $A = \{a \in A : t^{\boldsymbol{A}}(a) = \top\} \cup \{a \in A : t^{\boldsymbol{A}}(\neg a) = \top\}$
$= \{a \in A : t^{\boldsymbol{A}}(a) = \top\} \cup \{a \in A : t^{\boldsymbol{A}}(a) = \bot\}$ *(disjoint union).*
4) $Rad(\boldsymbol{A}) = \{a : t^{\boldsymbol{A}}(a) = \top\}$ *is the unique maximal i-filter of $\boldsymbol{A}$*

PROOF. 1) Take $a \in A$, if $t^{\boldsymbol{A}}(a) \neq \top$, then, since $\top$ is join irreducible, by (2.22) and (2.23), we have $\neg t^{\boldsymbol{A}}(a) = t^{\boldsymbol{A}}(\neg a) = \top$, hence $t^{\boldsymbol{A}}(a) = \bot$.
2) and 3) are immediate consequences of 1) and (2.23).
To see 4), we denote by F the set $\{a \in A : t^{\boldsymbol{A}}(a) = \top\}$, then $\top \in F$. If $a \in F$ and $b \in A$ are such that $a \leq b$, then, by (2.25) $\top = t^{\boldsymbol{A}}(a) \leq t^{\boldsymbol{A}}(b)$ and so $b \in F$; in addition, (2.24) implies that F is closed under $*$. And so, since $\bot \notin F$, F is a proper i-filter. Moreover, for any $a \in A$ $a \notin F$ iff $\neg a \in F$, hence by Lemma 1.5 F is a maximal i-filter. Let M be a maximal i-filter of $\boldsymbol{A}$ and let $a \in F \setminus M$. By Lemma 1.5, there is $n > 0$ such that $\neg a^n \in M$. Thus, $t^{\boldsymbol{A}/M}(\neg(a^n)/M) = \top$. Now since $a \in F$, by (2.23) and (2.24), we have $t^{\boldsymbol{A}}(\neg(a^n)) = \neg t^{\boldsymbol{A}}(a)^n = \bot$, and so $\bot = t^{\boldsymbol{A}}(\neg(a^n))/M = t^{\boldsymbol{A}/M}(\neg(a^n)/M)$, contradiction. Therefore, $F \subseteq M$, and by maximality $F = M$. Thus F is the unique maximal i-filter and hence $Rad(\boldsymbol{A}) = F$. ■

Lemma 2.3. *If* (2.22), (2.23), (2.24) *and* (2.25) *hold in* $\mathbb{V}$, *then the following equations also hold in* $\mathbb{V}$*:*

$$t(x \to y) = t(x) \to t(y) \quad (2.26)$$
$$t(x \wedge y) = t(x) \wedge t(y) \quad (2.27)$$
$$t(x \vee y) = t(x) \vee t(y) \quad (2.28)$$

Proof. It is enough to check that (2.26) ,(2.27) and (2.28) hold in $\mathbb{V}_{si}$. Take then $\boldsymbol{A} \in \mathbb{V}_{si}$ and $a, b \in A$.
To see (2.26) we proceed by cases:

- if $t^{\boldsymbol{A}}(b) = \top$, since $b \leq a \to b$, then $\top = t^{\boldsymbol{A}}(a \to b) = t^{\boldsymbol{A}}(a) \to \top = t^{\boldsymbol{A}}(a) \to t^{\boldsymbol{A}}(b)$
- if $t^{\boldsymbol{A}}(a) = \bot$, then $a \notin Rad(\boldsymbol{A})$, by 3) of Lemma 2.2, $t^{A}(\neg a) = \top$ and since $\neg a \leq a \to b$, by (2.25) we have $\top = t^{\boldsymbol{A}}(\neg a) \leq t^{\boldsymbol{A}}(a \to b)$. Thus $\top = t^{\boldsymbol{A}}(a \to b) = \bot \to t^{\boldsymbol{A}}(b) = \top$
- if $t^{\boldsymbol{A}}(a) = \top$ and $t^{\boldsymbol{A}}(b) = \bot$, then $t^{\boldsymbol{A}}(a) \to t^{\boldsymbol{A}}(b) = \bot$. Since $a \in Rad(\boldsymbol{A})$ and $b \notin Rad(\boldsymbol{A})$, by **F4)**, $a \to b \notin Rad(\boldsymbol{A})$ and so $t^{\boldsymbol{A}}(a \to b) = \bot$.

Now by taking into account 3) of Lemma 2.2, (2.27) follows from the fact that $a \wedge b \in Rad(\boldsymbol{A})$ if and only if $a, b \in Rad(\boldsymbol{A})$, and (2.28) follows from the fact that $a \vee b \notin Rad(\boldsymbol{A})$ if and only if $a, b \notin Rad(\boldsymbol{A})$. ■

From Lemmas 2.1 and 2.3, we obtain the main result of this section:

Theorem 2.4. $\mathbb{V}$ *admits* t *as a Boolean retraction term if and only if the equations and quasi-equation* (2.22), (2.23), (2.24) *and* (2.25) *hold in* $\mathbb{V}$.

Observe that (2.25) can be replaced by (2.27), and hence the greatest variety $\mathbb{V}_{\boldsymbol{t}}$ having t as Boolean retraction term is given by the equations (2.22), (2.23), (2.24) and (2.27). Hence we have:

Corollary 2.5. *A subvariety* $\mathbb{V}$ *of* $\mathbb{BRL}$ *admits the unary term* t *as Boolean retraction term if and only if* $\mathbb{V} \subseteq \mathbb{V}_{\boldsymbol{t}}$.

Remark 2.6. *It follows from 4) in Lemma 2.2 that simple algebras in* $\mathbb{V}_{\boldsymbol{t}}$ *are isomorphic to the two element Boolean algebra* $\mathbf{2}$.

Theorem 2.7. *Let* t *be a Boolean retraction term for a variety* $\mathbb{V}$ *of bounded residuated lattices. For each* $\boldsymbol{A} \in \mathbb{V}$, $Rad(\boldsymbol{A}) = \{a \in A : t^{\boldsymbol{A}}(a) = \top\}$.

PROOF. The inclusion $Rad(\boldsymbol{A}) \subseteq \{a : t^{\boldsymbol{A}}(a) = \top\}$ follows from 4) of Lemma 2.2 and Remark 1.6. To see the other inclusion, take $a \in A$ such that $t^{\boldsymbol{A}}(a) = \top$. For each maximal i-filter M of $\boldsymbol{A}$, $t^{\boldsymbol{A}/M}(a/M) = \top$. Hence $a/M \in Rad(\boldsymbol{A}/M)$, and since $\boldsymbol{A}/M$ is simple, $Rad(\boldsymbol{A}/M) = \{\top\}$. Hence $a/M = \top/M$ and $a \in M$. ■

It follows from the above theorem that Boolean retraction terms are essentially unique on varieties of bounded residuated lattices:

COROLLARY 2.8. *If a variety* $\mathbb{V} \subseteq \mathbb{BRL}$ *admits* s, t *as Boolean retraction terms, then the equation* $s(x) = t(x)$ *holds in* $\mathbb{V}$.

PROOF. Let $\boldsymbol{A} \in \mathbb{V}_{si}$, and let $a \in A$. Then $t(a) = \top$ if and only if $a \in Rad(\boldsymbol{A})$ if and only if $s(a) = \top$, and since s, t take only the values $\top, \bot$, $t(a) = s(a)$ for all $a \in A$. This implies the result. ■

The next result proved to be useful to characterize varieties admitting a Boolean retraction term.

LEMMA 2.9. *Let* $\boldsymbol{A}$ *be a directly indecomposable bounded residuated lattice and let* t *be a unary term. If* $\boldsymbol{A}$ *satisfies the conditions*

(*i*) *The equations* (2.22) *and* (2.23) *hold in* $\boldsymbol{A}$, *and*
(*ii*) $Rad(\boldsymbol{A}) = \{a : t^{\boldsymbol{A}}(a) = \top\}$

then $\boldsymbol{A} \in \mathbb{V}_{\boldsymbol{t}}$.

PROOF. Since (2.22) and (2.23) holds in $\boldsymbol{A}$, then for all $a \in A$, $t^{\boldsymbol{A}}(a) \wedge \neg t^{\boldsymbol{A}}(a) \leq \neg\neg t^{\boldsymbol{A}}(a) \wedge \neg t^{\boldsymbol{A}}(a) = \neg(\neg t^{\boldsymbol{A}}(a) \vee t^{\boldsymbol{A}}(a)) = \bot$; hence $t^{\boldsymbol{A}}(a) \in \boldsymbol{B}(\boldsymbol{A}) = \{\top, \bot\}$. By (*ii*) $\boldsymbol{A} = Rad(\boldsymbol{A}) \cup \{a : \neg a \in Rad(\boldsymbol{A})\}$. Since $Rad(\boldsymbol{A})$ is upward directed and closed under $*$, then (2.24) and (2.25) hold in $\boldsymbol{A}$. ■

Let $\boldsymbol{A} \in \mathbb{V}_t$. Since for each $x \in A$, $t(\neg\neg x) = t(x)$, it follows that if $a \in A$ is *dense*, i.e., $\neg\neg a = \top$, then $a \in \mathrm{Rad}(\boldsymbol{A}) = \{x \in A : t(x) = \top\}$.

LEMMA 2.10. *Let* t *be a Boolean retraction term for* $\boldsymbol{A} \in \mathbb{BRL}$. *If all elements in* $\mathrm{Rad}(\boldsymbol{A})$ *are dense, that is, if for each* $a \in A$, $t(a) = \top$ *implies* $\neg\neg a = \top$, *then there is an automorphism* h *of the Boolean algebra* $\boldsymbol{B}(\boldsymbol{A})$ *such that* $t(x) = h(\neg\neg x)$ *for each* $x \in A$, *and* $\boldsymbol{A} \in \mathbb{SRL}$.

PROOF. Since $t(\neg\neg a \to a) = t(\neg\neg a) \to t(a) = \top$ for each $a \in A$, we have that $\boldsymbol{A}$ satisfies the Glivenko equation:

$$\neg\neg(\neg\neg x \to x) = \top. \tag{2.29}$$

Therefore by the results of [10], the double negation defines an endomorphism of $\boldsymbol{A}$. Since by hypothesis this endomorphism has the same kernel as t, there is an automorphism h of the Boolean algebra $\boldsymbol{B}(\boldsymbol{A})$ such that $t(x) = h(\neg\neg x)$ for each $x \in A$. By (2.22) and (2.28) we have that for each $a \in A$, $\top = t(\neg\neg a \vee \neg a) = h(\neg\neg(\neg\neg a \vee \neg a)) = h(\neg\neg a \vee \neg a)$, and since h is an automorphism, this implies $\neg\neg a \vee \neg a = \top$. Hence $\boldsymbol{A}$ satisfies the Stone equation (1.13). ∎

Corollary 2.11. *Let $\mathbb{V}$ be a subvariety of $\mathbb{BRL}$ admitting the Boolean retraction term t. If for each $\boldsymbol{A} \in \mathbb{V}_{si}$ and each $a \in A$, $t(a) = \top$ implies $\neg\neg a = \top$, then the equation $t(x) = \neg\neg x$ holds in all algebras in $\mathbb{V}$, and $\mathbb{V} \subseteq \mathbb{SRL}$.*

Proof. Since for each $\boldsymbol{A} \in \mathbb{V}_{si}$, $\boldsymbol{B}(\boldsymbol{A})$ is the two element Boolean algebra, it follows from Lemma 2.10 that $t(a) = \neg\neg a$ for each $a \in A$. Therefore the equation $t(x) = \neg\neg x$ holds in all algebras in $\mathbb{V}$, and again by Lemma 2.10, $\mathbb{V} \subseteq \mathbb{SRL}$. ∎

3. Radical classes

For each $\boldsymbol{A} \in \mathbb{BRL}$, $Rad(\boldsymbol{A})$ is an i-filter; hence, as noted after the definition of i-filters, $Rad(\boldsymbol{A})$ is the universe of a subalgebra $\boldsymbol{Rad}(\boldsymbol{A})$ of $\boldsymbol{A}^-$. Moreover, $Rad(\boldsymbol{A})$ is also closed under $\neg\neg$ and $\oplus$. Hence we can consider the algebra of type $(2, 2, 2, 2, 2, 1, 0)$

$$\boldsymbol{r}(\boldsymbol{A}) = \langle Rad(\boldsymbol{A}); *, \rightarrow, \vee, \wedge, \oplus, \neg\neg, \top\rangle.$$

We call it *the radical algebra* of $\boldsymbol{A}$. Observe that if $\boldsymbol{A}$ is pseudocomplemented, then $\neg\neg$ and $\oplus$ are the constant functions given by $\top$; and if $\boldsymbol{A}$ is involutive, then $\neg\neg$ is the identity and $x \oplus y = \neg x \rightarrow y$. Moreover, we have

Lemma 3.1. *For each $\boldsymbol{A} \in \mathbb{WL}_2$, $x \oplus y = \top$ for all $x, y \in Rad(\boldsymbol{A})$.*

Proof. Let $a, b \in Rad(\boldsymbol{A})$. Then $a * b \in Rad(\boldsymbol{A})$, and by Lemma 1.8, $\top = 2.(a*b) = \neg(a*b) \rightarrow \neg\neg(a*b)$. Hence $\neg a \leq \neg(a*b) \leq \neg\neg(a*b) \leq \neg\neg b$, and $a \oplus b = \top$. ∎

For a class $\mathbb{K}$ of bounded residuated lattices we define the *radical class* associated to $\mathbb{K}$ as

$$\mathbb{K}^{\boldsymbol{r}} = \mathbb{I}(\{\boldsymbol{r}(\boldsymbol{A}) : \boldsymbol{A} \in \mathbb{K}\}),$$

i.e., the class of isomorphic copies of algebras in $\{\boldsymbol{r}(\boldsymbol{A}) : \boldsymbol{A} \in \mathbb{K}\}$.

We shall show that for any subvariety $\mathbb{V}$ of $\mathbb{BRL}$, $\mathbb{V}^{\boldsymbol{r}} = (\mathbb{V}_{DI})^{\boldsymbol{r}}$.

For each $\boldsymbol{A} \in \mathbb{BRL}$, we define $coRad(\boldsymbol{A}) = \{a \in A : \neg a \in Rad(\boldsymbol{A})\}$.

LEMMA 3.2. *If $\boldsymbol{A}$ is a non trivial bounded residuated lattice and a, b are arbitrary elements of A, then we have*

(1) *$a \in Rad(\boldsymbol{A})$ if and only if $\neg\neg a \in Rad(\boldsymbol{A})$ if and only if $\neg a \in coRad(\boldsymbol{A})$; $a \in coRad(\boldsymbol{A})$ if and only if $\neg\neg a \in coRad(\boldsymbol{A})$.*

(2) *If $a \in coRad(\boldsymbol{A})$ and $b \leq a$, then $b \in coRad(\boldsymbol{A})$.*

(3) *If $a \in coRad(\boldsymbol{A})$, then $a * b \in coRad(\boldsymbol{A})$, and $a \to b \in Rad(\boldsymbol{A})$.*

(4) *If $a \in coRad(\boldsymbol{A})$ and $b \in Rad(\boldsymbol{A})$, then $b \to a \in coRad(\boldsymbol{A})$.*

PROOF. (1) follows from the (1.12) and the fact that for all $n, m \geq 0$, $n((\neg\neg a)^m) = n(a^m)$.
(2) If $b \leq a$ then $\neg a \leq \neg b$, and so, since $Rad(\boldsymbol{A})$ is an i-filter we have $b \in coRad(\boldsymbol{A})$.
(3) Follows from (2), because $a * b \leq a$ and $\neg a \leq a \to b$.
(4) $b \to a \leq b \to \neg\neg a = \neg(b * \neg a)$, then since $b, \neg a \in Rad(\boldsymbol{A})$, we have that $b * \neg a \in Rad(\boldsymbol{A})$ and so $b \to a \in coRad(\boldsymbol{A})$. ■

The next result generalizes [9, Theorem 3.5].

THEOREM 3.3. *If $\boldsymbol{A}$ is a bounded residuated lattice then*

$$\sigma(\boldsymbol{A}) = Rad(\boldsymbol{A}) \cup coRad(\boldsymbol{A})$$

is the universe of a subalgebra $\boldsymbol{\sigma}(\boldsymbol{A})$ of $\boldsymbol{A}$. Moreover, $\boldsymbol{\sigma}(\boldsymbol{A})$ is directly indecomposable.

PROOF. The fact that $\sigma(\boldsymbol{A})$ is the universe of a subalgebra of $\boldsymbol{A}$ follows at once from Lemma 3.2. To complete the proof, observe that from (1.21) and (1.12) we obtain that $\mathrm{Rad}(\boldsymbol{A}) \cap B(\boldsymbol{A}) = \{\top\}$, hence $B(\boldsymbol{\sigma}(\boldsymbol{A})) = \{\top, \bot\}$. ■

COROLLARY 3.4. *For each subvariety $\mathbb{V}$ of $\mathbb{BRL}$, $\mathbb{V}^{\boldsymbol{r}} = (\mathbb{V}_{DI})^{\boldsymbol{r}}$.*

PROOF. For each $\boldsymbol{A} \in \mathbb{V}$, $\boldsymbol{r}(\boldsymbol{A}) = \boldsymbol{r}(\boldsymbol{\sigma}(\boldsymbol{A}))$. ■

From the above theorem we have that for all $\boldsymbol{A} \in \mathbb{BRL}$, if $A = \sigma(\boldsymbol{A})$, then $\boldsymbol{A}$ is directly indecomposable. We are going to show that in the presence of a Boolean retraction term, the converse is also true.

In what follows, $\mathbb{V}_t$ will denote the variety corresponding to a Boolean retraction term t.

LEMMA 3.5. *Given $\boldsymbol{A} \in \mathbb{V}_{\boldsymbol{t}}$, $\boldsymbol{A}$ is directly indecomposable if and only if $A = \sigma(\boldsymbol{A})$.*

PROOF. We only need to prove the "only if" part. If $\boldsymbol{A}$ is directly indecomposable, then $B(\boldsymbol{A}) = \{\top, \bot\}$. Hence $A = t^{-1}(\{\top\}) \cup t^{-1}(\{\bot\})$, and the result follows from Theorem 2.7. ■

REMARK 3.6. *If $\boldsymbol{A}$ is directly indecomposable in $\mathbb{V}_t$, then $Rad(\boldsymbol{A})$ is the only maximal filter of $\boldsymbol{A}$. Indeed, $Rad(\boldsymbol{A})$ is a proper implicative filter, and by Theorem 2.7 it follows that for each $a \in A$, either $a \in Rad(\boldsymbol{A})$ or $\neg a \in Rad(\boldsymbol{A})$.*

LEMMA 3.7. *The following propositions hold true:*

(1) *If $\boldsymbol{A}$ is a directly indecomposable in $\mathbb{V}_{\boldsymbol{t}}$ and F is a proper implicative filter of $\boldsymbol{A}$, then $Rad(\boldsymbol{A}/F) = Rad(\boldsymbol{A})/F$, and $coRad(\boldsymbol{A}/F) = coRad(\boldsymbol{A})/F$*

(2) *If $(\boldsymbol{A}_i : i \in I)$ is a family of algebras admitting t as Boolean retraction term, then $\prod_{i\in I} Rad(\boldsymbol{A}_i) = Rad(\prod_{i\in I} \boldsymbol{A}_i)$.*

PROOF. 1) Observe that, by Remark 3.6, $F \subseteq Rad(\boldsymbol{A})$. By definition of radical, $Rad(\boldsymbol{A})/F \subseteq Rad(\boldsymbol{A}/F)$. Suppose that $a/F \in Rad(\boldsymbol{A}/F)$, then, by (1.12), for all $n \in \omega$, there is k_n, such that $k_n.(a^n)/F = \top/F$. In particular, there is $k > 0$ such that $k.a/F = \top/F$, hence $k.a \in F \subseteq Rad(\boldsymbol{A})$. If $a \notin Rad(\boldsymbol{A})$, then $k > 1$ and $k.a = \neg((k-1).a) \to \neg\neg a \in Rad(\boldsymbol{A})$, since $\neg\neg a \notin Rad(\boldsymbol{A})$, we have $(\neg a)^{k-1} = \neg((k-1).a) \notin Rad(\boldsymbol{A})$ and so $\neg a \notin Rad(\boldsymbol{A})$. Therefore $a \notin coRad(\boldsymbol{A})$ and $\boldsymbol{\sigma}(\boldsymbol{A}) \neq \boldsymbol{A}$, contradiction. Thus $a \in Rad(\boldsymbol{A})$. This shows that $Rad(\boldsymbol{A}/F) \subseteq Rad(\boldsymbol{A})/F$.
2) Let $a = (a_i)_{i\in I} \in \prod_{i\in I} A_i$, then since the operations in $\prod_{i\in I} \boldsymbol{A}_i$ are defined componentwise, $t^{\prod \boldsymbol{A}_i}(a) = \top$ iff for all $i \in I$ $t^{\boldsymbol{A}_i}(a_i) = \top_i$, hence, by Lemma 2.7, $a \in Rad(\prod_{i\in I} \boldsymbol{A}_i)$ iff $a_i \in Rad(\boldsymbol{A}_i)$, for all $i \in I$. Thus $\prod_{i\in I} Rad(\boldsymbol{A}_i) = Rad(\prod_{i\in I} \boldsymbol{A}_i)$. ■

Thus we have

THEOREM 3.8. *If $\mathbb{V}$ is a subvariety of $\mathbb{BRL}$ admitting t as Boolean retraction term, then $\mathbb{V}^{\boldsymbol{r}}$ is closed under homomorphic images and direct products.*

PROOF. Since $\mathbb{V}$ is a variety, then for all $\boldsymbol{A} \in \mathbb{V}$, $\boldsymbol{\sigma}(\boldsymbol{A}) \in \mathbb{V}_{DI}$. Without loss of generality, we can assume that $\mathbb{V}^{\boldsymbol{r}} = \{\boldsymbol{r}(\boldsymbol{A}) : \boldsymbol{A} \in \mathbb{V}_{DI}\}$. Let $\boldsymbol{A} \in \mathbb{V}_{DI}$, and let $\boldsymbol{S} = (S; *, \to, \vee, \wedge, \boxplus, \delta, \top^S)$ be an algebra of type $(2, 2, 2, 2, 2, 1, 0)$ such that the reduct $(S; *, \to, \vee, \wedge, \top^S, \top^S) \in \mathbb{RL}$. Suppose that h is a residuated lattice homomorphism from $\boldsymbol{r}(\boldsymbol{A})$ onto $\boldsymbol{S}$ such that $h(x \oplus y) =$

$h(x) \boxplus h(y)$ and $h(\neg\neg x) = \delta(x)$ for $x, y \in Rad(\boldsymbol{A})$. Clearly $F = h^{-1}(\top^S)$ is an i-filter of $\boldsymbol{Rad}(\boldsymbol{A})$ and so it is a proper i-filter of $\boldsymbol{A}$. Thus $\boldsymbol{A}/F \in \mathbb{V}$. By (1) of Lemma 3.7 it follows that $\boldsymbol{Rad}(\boldsymbol{A}/F) = \boldsymbol{Rad}(\boldsymbol{A})/F$. Since $(x, y) \in \theta(F)$ implies $(\neg\neg x, \neg\neg y) \in \theta(F)$, and $(x, y), (s, t) \in \theta(F)$ implies $(x \oplus s, y \oplus t) \in \theta(F)$ for all $x, y, s, t \in A$, $\theta(F)$ is a congruence of $\boldsymbol{r}(\boldsymbol{A})$, and the correspondence $a/\theta(F) \mapsto h(a)$ defines an isomorphism from $\boldsymbol{r}(\boldsymbol{\sigma}(\boldsymbol{A}/F))$ onto $\boldsymbol{S}$. Therefore $\mathbb{V}^{\boldsymbol{r}}$ is closed under homomorphic images.
Moreover, since in direct product operations are defined componentwise, it follows from (2) in Lemma 3.7 that $\mathbb{V}^{\boldsymbol{r}}$ is closed under direct products. ∎

The above results can be improved for varieties of involutive bounded residuated lattices.

THEOREM 3.9. *If $\mathbb{V}$ is a variety of involutive bounded residuated lattices admitting t as Boolean retraction term, then $\mathbb{V}^{\boldsymbol{r}}$ is a variety.*

PROOF. In the light of Theorem 3.8, it is enough to see that $\mathbb{V}^{\boldsymbol{r}}$ is closed under subalgebras; and, arguing as in Theorem 3.8, it suffices to take $\boldsymbol{A} \in \mathbb{V}_{DI}$. Let $\boldsymbol{B}$ be a subalgebra of $\boldsymbol{r}(\boldsymbol{A})$, i.e. $B \subseteq Rad(\boldsymbol{A})$ and it is closed under $*, \to, \wedge, \vee, \oplus, \neg\neg$ and contains $\top$. Clearly $\neg B = \{\neg a : a \in B\} \subseteq coRad(\boldsymbol{A})$. Taking into account Lemma 3.2 and that for any $a, b \in A$ $\neg a \to b = a \oplus b$, it follows easily that $s(B) = B \cup \neg B$ is universe of a subalgebra $\boldsymbol{s}(\boldsymbol{B})$ of $\boldsymbol{A}$ such that $B = Rad(\boldsymbol{s}(\boldsymbol{B}))$. Therefore $\boldsymbol{B} = \boldsymbol{r}(\boldsymbol{s}(\boldsymbol{B})) \in \mathbb{V}^{\boldsymbol{r}}$. ∎

4. Free algebras

Given a class of algebras $\mathbb{K}$, we represent by $\mathfrak{F}_{\mathbb{K}}(X)$ the $|X|$-free algebra over the class $\mathbb{K}$, if it exists. Any class closed under isomorphic images, subalgebras, and direct products has free algebras of any cardinality, in particular varieties have free algebras for each cardinal.

Given a subvariety $\mathbb{V}$ of $\mathbb{V}_{\boldsymbol{t}}$, since by Theorem 2.4 the map $x \mapsto t^{\mathfrak{F}_{\mathbb{V}}(X)}(x)$ is a retract from $\mathfrak{F}_{\mathbb{V}}(X)$ onto $\boldsymbol{B}(\mathfrak{F}_{\mathbb{V}}(X))$, taking into account that $\mathfrak{F}_{\mathbb{V}}(\emptyset) = \{\bot, \top\}$, we immediately obtain (cf.[9, Theorem 5.1]):

THEOREM 4.1. *For each subvariety $\mathbb{V}$ of $\mathbb{V}_{\boldsymbol{t}}$, $\boldsymbol{B}(\mathfrak{F}_{\mathbb{V}}(X))$ is the free Boolean algebra over the set $t(X) = \{t^{\mathfrak{F}_{\mathbb{V}}(X)}(x) : x \in X\}$, and the sets X and $t(X)$ have the same cardinal. That is $\boldsymbol{B}(\mathfrak{F}_{\mathbb{V}}(X))$ is isomorphic to the $|X|$-free Boolean algebra.*

We are going to give an explicit description of the representation of a bounded residuated lattice as Boolean product of directly indecomposable.

Firstly, note that the following lemma can be obtained in a standard way (see, for instance, [8]):

Lemma 4.2. *Let $\boldsymbol{A}$ be a bounded residuated lattice, and let F be a filter of the Boolean algebra $\boldsymbol{B}(\boldsymbol{A})$, then :*

$$\sim_F = \{(x,y) \in A^2 : x \wedge z = y \wedge z \text{ for some } z \in F\}$$

is a congruence relation on $\boldsymbol{A}$ that coincides with the congruence relation $\theta(\langle F\rangle)$ given by the i-filter $\langle F\rangle$ generated by F. Moreover, if F is a prime filter (i.e., an ultrafilter) of $\boldsymbol{B}(\boldsymbol{A})$, then $B(\boldsymbol{A}/{\sim_F}) = \{\bot/{\sim_F}, \top/{\sim_F}\}$.

We write $\boldsymbol{A}/\langle F\rangle$ in place of $\boldsymbol{A}/\sim_F$, and so $x/\langle F\rangle = x/\sim_F$ for the equivalence class of $x \in A$. We will represent by $Sp\,\boldsymbol{B}(\boldsymbol{A})$ the set of all prime filters (ultrafilters) of the Boolean algebra $\boldsymbol{B}(\boldsymbol{A})$. Then it is clear that for any $F \in Sp\,\boldsymbol{B}(\boldsymbol{A})$, $\boldsymbol{A}/\langle F\rangle$ is directly indecomposable. With these notations we have:

Theorem 4.3. *Each nontrivial bounded residuated lattice $\boldsymbol{A}$ is representable as the weak Boolean product of the family $(\boldsymbol{A}/\langle F\rangle : F \in Sp\,\boldsymbol{B}(\boldsymbol{A}))$ over the Boolean space given by the Stone topology on $Sp\,\boldsymbol{B}(\boldsymbol{A})$.*

Given a subvariety $\mathbb{V}$ of $\mathbb{V}_{\boldsymbol{t}}$, by Theorem 4.1, the mapping $x \mapsto t^{\mathfrak{F}_{\mathbb{V}}(X)}(x)$ is, up to isomorphism, a retract form $\mathfrak{F}_{\mathbb{V}}(X)$ onto $\mathfrak{F}_{\mathbb{B}}(X)$, where $\mathbb{B}$ denotes the variety of Boolean algebras. Since the Stone space of the free Boolean algebra over X is the Cantor space 2^X, the ultrafilters of $\mathfrak{F}_{\mathbb{B}}(X)$ are in one-one correspondence with the subsets of X. Hence, as in [9, Corollary 5.2], we have:

Corollary 4.4. *Let $\mathbb{V}$ be a subvariety of $\mathbb{V}_{\boldsymbol{t}}$. If $X \neq \emptyset$, then the correspondence:*

$$U \mapsto S_U = \{x \in X : t^{\mathfrak{F}_{\mathbb{V}}(X)}(x) \in U\}$$

is a bijection from the set of ultrafilters of $\boldsymbol{B}(\mathfrak{F}_{\mathbb{V}}(X))$ into the power set of X. The inverse mapping is given by

$$S \mapsto U_S = \text{ultrafilter generated by } t(S) \cup \neg t(S),$$

where $\neg t(S) = \{\neg t^{\mathfrak{F}_{\mathbb{V}}(X)}(x) : x \in X \setminus S\}$. □

Observe that by Lemma 4.2, for each $S \subseteq X$, $\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S\rangle$ is directly indecomposable.

From now on, we write $t(u)$ instead of $t^{\mathfrak{F}_{\mathbb{V}}(X)}(u)$. Setting $Y/\langle U_S\rangle = \{y/\langle U_S\rangle : y \in Y\}$, we have

Lemma 4.5. *Let $\mathbb{V}$ be variety of involutive residuated lattices admitting t as Boolean retraction term. Then for each $S \subseteq X$, the set $(S \cup \neg(X \smallsetminus S))/\langle U_S \rangle$ generates the algebra $\boldsymbol{r}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle)$.*

Proof. By the definition of U_S one has $X/\langle U_S \rangle \cap Rad\,(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle) = S/\langle U_S \rangle$. Taking into account that $t(\neg x) = \neg t(x) \in \langle U_S \rangle$ for each $x \in X \smallsetminus S$, one also has that $\neg(X/\langle U_S \rangle) \cap Rad\,(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle) = \neg(X \smallsetminus S)/\langle U_S \rangle$. Since $\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle$ is involutive and directly indecomposable, the set $G = (S \cup \neg(X \smallsetminus S))/\langle U_S \rangle$ also generates $\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle$. Indeed, let $\boldsymbol{H}$ be the subalgebra of $\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle$ generated by G. If $x/\langle U_S \rangle \in X/\langle U_S \rangle \cap Rad\,(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle)$, then $x/\langle U_S \rangle \in H$. If $x/\langle U_S \rangle \in coRad\,(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle)$, then $\neg x/\langle U_S \rangle \in H$, and $x/\langle U_S \rangle = \neg\neg x/\langle U_S \rangle \in H$. Hence, $X/\langle U_S \rangle \subseteq H$, and $\boldsymbol{H} = \mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle$. In particular G generates $\boldsymbol{r}\,(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle)$. ■

Each subvariety $\mathbb{V}$ of $\mathbb{V}_{\boldsymbol{t}}$ is generated by its subdirectly irreducible members. Hence if $\mathbb{V} \neq \mathbb{B}$, $\mathbb{V}_{si}$ has to contain an algebra with more than two elements. Any such algebra $\boldsymbol{C} \in \mathbb{V}_{si}$ will be called *a test algebra for* $\mathbb{V}$. It follows from item 3) in Theorem 2.2 that in every test algebra $\boldsymbol{C}$, we can find an element $a \in C$ such that $a \in Rad(\boldsymbol{C}) \smallsetminus \{\top\}$, and hence

$$\bot < a < t^{C}(a) = \top \text{ and } \bot = t^{C}(\neg a) = \neg t^{C}(a) \leq \neg a < \top.$$

Such a will be called *a test element*(cf. [11, p.72]).

With the notations of Corollary 4.4, for each subvariety $\mathbb{V}$ of $\mathbb{V}_{\boldsymbol{t}}$ we have:

Lemma 4.6. *For each $S \subseteq X$ one has:*

(i) *For each $y \in X$, $y/\langle U_S \rangle \neq \top/\langle U_S \rangle$.*

(ii) *For each $y \in S$, $y/\langle U_S \rangle \neq \bot/\langle U_S \rangle$.*

(iii) *If y, z are in X and $y \neq z$, then $y/\langle U_S \rangle = z/\langle U_S \rangle$ implies that y, z are in $X \smallsetminus S$.*

Proof. Observe that for each α in $\mathfrak{F}_{\mathbb{V}}(X)$, $\alpha \in \langle U_S \rangle$ if and only if there are finite sets $V \subseteq S$ and $W \subseteq X \smallsetminus S$ such that $V \cup W \neq \emptyset$ and

$$\bigwedge_{t \in V} t(u) \wedge \bigwedge_{w \in W} \neg t(w) \leq \alpha. \tag{4.30}$$

Let $\boldsymbol{C}$ be a test algebra in $\mathbb{V}$, and let $a \in C$ be a test element. To prove (i) suppose that $y \in \langle U_S \rangle$ (absurdum hypothesis). Let V, W as in (4.30), with $\alpha = y$, and let $f : X \to C$ be the function defined as follows:

$$f(x) = \begin{cases} a & \text{if } x \in X \smallsetminus W, \\ \bot & \text{if } x \in W. \end{cases} \tag{4.31}$$

If there were a homomorphism $\widehat{f}\colon \mathfrak{F}_{\mathbb{V}}(X) \to \boldsymbol{C}$ extending f, then $\widehat{f}$ would assign the value $\top$ to the left member of (4.30), while $\widehat{f}(y) \in \{a, \bot\}$. Since this contradicts the inequality (4.30), f cannot be extended to a homomorphism, in contradiction with the definition of free algebra. Hence we conclude that $y \notin \langle U_S \rangle$, and (i) holds. To prove (ii), suppose that $y \in S$ and $\neg y \in \langle U_S \rangle$. Let V, W as in (4.30) with $\alpha = \neg y$. Since $y \notin W \subseteq X \smallsetminus S$, we can show that the same function f defined by (4.31) cannot be extended to a homomorphism $\widehat{f}\colon \mathfrak{F}_{\mathbb{V}}(X) \to \boldsymbol{C}$, and this proves (ii). To prove (iii), suppose that $y \to z \in \langle U_S \rangle$. Let V, W as in (4.30), with $\alpha = y \to z$. If $y \notin W$, then the function $g\colon X \to C$ defined as follows:

$$g(x) = \begin{cases} a & \text{if } x \in X \smallsetminus (\{y\} \cup W); \\ \top & \text{if } x = y, \\ \bot & \text{if } x \in W. \end{cases}$$

cannot be extended to a homomorphism $\widehat{g}\colon \mathfrak{F}_{\mathbb{V}}(X) \to \mathbf{C}$. Hence if $(y \to z)$ and $(z \to y)$ are in $\langle U_S \rangle$, then y and z are in $X \smallsetminus S$. ∎

If $\mathbb{V}$ is a non-pseudocomplemented subvariety of $\mathbb{V}_{\boldsymbol{t}}$ and $\boldsymbol{C}$ is a non-pseudocomplemented test algebra for $\mathbb{V}$ and a is a test element, then, since $\boldsymbol{C}$ is directly indecomposable, we have $\bot < \neg(a \vee \neg a)$ and $a \vee \neg a < \top$. Hence, without loss of generality, we can take a test element b satisfying $\bot = t^{\boldsymbol{C}}(\neg b) < \neg b$ and $b < t^{\boldsymbol{C}}(b) = \top$.

Lemma 4.7. *Let $\mathbb{V}$ be a subvariety of $\mathbb{V}_{\boldsymbol{t}}$, which is not pseudocomplemented. Then for each $x \in X \smallsetminus S$, $x/\langle U_S \rangle \neq \bot/\langle U_S \rangle$, and X and $(S \cup \neg(X \smallsetminus S))/\langle U_S \rangle$ have the same cardinal.*

Proof. Suppose that $y \in X \smallsetminus S$ and that $y/\langle U_S \rangle = \bot/\langle U_S \rangle$. Then there is a test algebra $\boldsymbol{C} \in \mathbb{V}$, and a test element $a \in C$ such that $\bot = t^{\boldsymbol{C}}(\neg a) < \neg a$ and $a < t^{\boldsymbol{C}}(a) = \top$. Let V, W be as in (4.30) of the proof of Lemma 4.6 with $\alpha = \neg y$, and define $f\colon X \to C$ by the prescription

$$f(x) = \begin{cases} a & \text{if } x \in X \smallsetminus W, \\ \neg a & \text{if } x \in W. \end{cases}$$

If there were a homomorphism $\widehat{f}\colon \mathfrak{F}_{\mathbb{V}}(X) \to \boldsymbol{C}$ extending f, then $\widehat{f}$ would assign the value $\top$ to the left member of (4.30), while $\widehat{f}(\neg y) \in \{a, \neg a\}$, contradicting the inequality (4.30). Hence f cannot be extended to a homomorphism, in contradiction with the definition of free algebra. Therefore $y/\langle U_S \rangle \neq \bot/\langle U_S \rangle$. In the proof of (iii) in Lemma 4.6 we have shown that for y, z in X, $y \neq z$, if $y \to z \in \langle U_S \rangle$, then $y \in X \smallsetminus S$. Let us see now that

$z \in S$. Indeed, let V, W be as in (4.30), with $\alpha = y \to z$, and let a be the same test element as before. If $z \notin V$, then the function $g\colon X \to C$ defined as follows

$$g(x) = \begin{cases} a & \text{if } x \in V, \\ \bot & \text{if } x = z, \\ \neg a & \text{if } x \in X \setminus (V \cup \{z\}). \end{cases}$$

cannot be extended to a homomorphism $\widehat{g}\colon \mathfrak{F}_{\mathbb{V}}(X) \to \mathbf{C}$. Hence $y \to z \in \langle U_S \rangle$ implies that $z \in V \subseteq S$. Consequently, $y/\langle U_S \rangle = z/\langle U_S \rangle$ would imply that y, z are simultaneously in S and $X \setminus S$, absurdum. Similar arguments show that y, z in S and $y \neq z$ imply that $\neg y/\langle U_S \rangle \neq \neg z/\langle U_S \rangle$ and $y/\langle U_S \rangle \neq \neg z/\langle U_S \rangle$. Hence $|X| = |(S \cup \neg \widetilde{S})/\langle U_S \rangle|$. ■

Theorem 4.8. *If $\mathbb{V}$ is a variety of involutive residuated lattices admitting t as Boolean retraction term, then for any $S \subseteq X$, $\boldsymbol{r}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle)$ is, up isomorphism, the $|X|$-free algebra in $\mathbb{V}^{\boldsymbol{r}}$.*

Proof. Let $\boldsymbol{B} \in \mathbb{V}^{\boldsymbol{r}}$. Without loss of generality, we can assume that $\boldsymbol{B} = \boldsymbol{r}(\boldsymbol{A})$ for some $\boldsymbol{A} \in \mathbb{V}_{DI}$. Given a map $f\colon (S \cup \neg(X \setminus S))/\langle U_S \rangle \to Rad(\boldsymbol{A})$, we can define a function $\bar{f}\colon X \to A$ by the prescription:

$$\bar{f}(x) = \begin{cases} f(x/\langle U_S \rangle) & \text{if } x \in S, \\ \neg f(\neg x/\langle U_S \rangle) & \text{if } x \in X \setminus S. \end{cases}$$

Then there is a unique homomorphism $\bar{g}\colon \mathfrak{F}_{\mathbb{V}}(X) \to \boldsymbol{A}$ which extends $\bar{f}$. Since $\{t(x) : x \in S\} \cup \{\neg t(x) : x \in X \setminus S\} \subseteq \ker \bar{g} = \bar{g}^{-1}(\{\top\})$, it follows that $\langle U_S \rangle \subseteq \ker(\bar{g})$. Therefore the correspondence $\alpha/\langle U_S \rangle \mapsto \bar{g}(\alpha)$ gives a homomorphism $h\colon \mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle \to \boldsymbol{A}$, and the restriction of h to $Rad(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle)$ gives a homomorphism from $\boldsymbol{r}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle)$ into $\boldsymbol{r}(\boldsymbol{A}) = \boldsymbol{B}$ that extends f. Since by Lemma 4.5 the set $(S \cup \neg(X \setminus S))/\langle U_S \rangle$ generates $\boldsymbol{r}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle)$, and, by Lemma 4.7, it has the same cardinal than $|X|$; we have that $\boldsymbol{r}(\mathfrak{F}_{\mathbb{V}}(X)/\langle U_S \rangle)$ is $|X|$-free algebra in $\mathbb{V}^{\boldsymbol{r}}$. ■

5. A sequence of Boolean retraction terms

In this section we exhibit examples of varieties with a Boolean retraction term. For each positive integer $0 < n$, we consider the term

$$\nabla_n(x) := n.x^n$$

Let $\mathbb{V}^n$ the subvariety of $\mathbb{BRL}$ given by the equation

$$\nabla_n(x) \vee \nabla_n(\neg x) = \top \qquad (2.22)_n$$

Our next aim is to show that ∇_n is a Boolean retraction term for each $\mathbb{V}^n$, that $\mathbb{V}^n = \mathbb{V}_{\nabla_n}$, and that for any $n > 0$, $\mathbb{V}_{\nabla_n} \subsetneq \mathbb{V}_{\nabla_{n+1}}$.

THEOREM 5.1. *Each* $\boldsymbol{A} \in \mathbb{V}^n_{si}$ *satisfies:*

1) $\neg\nabla_n(x) = \nabla_n(\neg x)$, $(2.23)_n$
2) $\nabla_n(x) \in \{\top, \bot\}$,
3) $\nabla_n(x) = (n.x)^n$,
4) $Rad(\boldsymbol{A}) = \{x : \nabla_n(x) = \top\}$.

PROOF. Through this proof, a, b will denote arbitrary elements of A. To see 1), suppose that $\nabla_n(a) = \top$, since $\nabla_n(a) = n.a^n \leq n.a$, we have $n.a = \top$, hence $(n.a)^n = \top$, and so, by 8 of Lemma 1.3, $\nabla_n(\neg a) = n.(\neg a)^n = \neg(n.a)^n = \bot$. If $\nabla_n(a) \neq \top$, then $\nabla_n(\neg a) = \top$, hence $\nabla_n(\neg\neg a) = \bot$, and by 4) of Lemma 1.3, we have $\nabla_n(a) = \bot$. Hence $\nabla_n(\neg a) = \neg\nabla_n(a)$. This proves (2.23).
To prove 2), note that by $(2.22)_n$ and 1), $\top = \nabla_n(a) \vee \neg\nabla_n(a)$. Hence $\nabla_n(a) \in B(\boldsymbol{A}) = \{\bot, \top\}$ for each $a \in A$.
3) follows from 1), 2) and item 8) in Lemma 1.3.
To prove 4), take first $c \in Rad(\boldsymbol{A})$. By (1.12), there exists k_n such that $k_n.c^n = \top$. If $k_n \leq n$, then $\nabla_n(c) = \top$. If $k_n > n$, then there are $k > 0$ and $0 < r < n$ such that $k_n = kn + r$, and by 5) and 6) of Lemma 1.3, $k_n.c^n = (k.(n.c^n)) \oplus (r.c^n) = \top$. Now, if $\nabla_n(c) = n.c^n = \bot$, then, since $r < n$, by 8) of Lemma 1.3, $(r.c^n) = \bot$; and so $k_n.c^n = \bot$, a contradiction. Therefore $\nabla_n(c) = \top$. Conversely, suppose that $\nabla_n(a) = n.a^n = \top$. If $m = \min\{r > n : n.a^r = \bot\}$, then, $\top = (n.a^{m-1})^n$, and by 3), $\top = n.a^{n(m-1)} \leq n.a^m$ a contradiction. Hence such m does not exists and so $n.a^m = \top$. Therefore $a \in Rad(\boldsymbol{A})$. ∎

COROLLARY 5.2. *For each positive integer* n, ∇_n *is a Boolean retraction term for* $\mathbb{V}^n$, *and* $\mathbb{V}_{\nabla_n} = \mathbb{V}^n$.

PROOF. By $(2.22)_n$, and items 1) and 4) of Theorem 5.1, each $\boldsymbol{A} \in \mathbb{V}^n_{si}$ satisfies conditions (i) and (ii) of Proposition 2.9, and since subdirectly irreducible algebras are directly indecomposable, we have that $\mathbb{V}^n_{si} \subseteq \mathbb{V}_{\nabla_n}$. To complete the proof, observe that (2.22) and (2.23) imply $(2.22)_n$. Hence we also have $\mathbb{V}_{\nabla_n} \subseteq \mathbb{V}^n$. ∎

Moreover we have:

LEMMA 5.3. *For all positive integer* n, *and any* $0 < r \leq n$, $\mathbb{V}_{\nabla_r} \subseteq \mathbb{V}_{\nabla_n}$.

Proof. By 5) in Theorem 5.1, for each $\boldsymbol{A} \in \mathbb{V}_{\nabla_r}$ and every $a \in A$, $r.(a^n)^r = (r.a^n)^r \leq r.a^n \leq n.a^n$. The assertion follows from these inequalities. ■

In the next section we shall show that the inclusions in the above lemma are proper.

Remark 5.4. *For any* $0 < r \leq n$ *the variety* $\mathbb{V}_{\nabla_n}$ *satisfies the equation* $n.x^r \vee n.(\neg x)^r = \top$.

Since $\nabla_1(x) = \neg\neg x$, we have that $(2.22)_1$ coincides with (1.13). Consequently, $\mathbb{V}_{\nabla_1} = \mathbb{SRL}$. Thus (cf [11, 6]):

Corollary 5.5. *For each* $\boldsymbol{A} \in \mathbb{SRL}$, *the double negation* $\neg\neg$ *defines a retract from* $\boldsymbol{A}$ *onto* $\boldsymbol{B}(\boldsymbol{A})$. □

We say that a variety $\mathbb{V}$ of bounded residuated lattices has the *Boolean retraction property* provided that for each $\boldsymbol{A} \in \mathbb{V}_{si}$ there is a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}(\boldsymbol{A})$.

We are going to show that $\mathbb{V}_{\nabla_k}$ is the greatest subvariety of $\mathbb{WL}_k$ having the Boolean retraction property.

Theorem 5.6. *For each* $k > 0$, $\mathbb{V}_{\nabla_k}$ *is the subvariety of* $\mathbb{WL}_k$ *given by the equation*

$(\boldsymbol{d}_k)$ $k.x^k = (k.x)^k$.

Proof. Since by Remark 5.4 $\mathbb{V}_{\nabla_k} \subseteq \mathbb{WL}_k$, and by 3) of Theorem 5.1, $\mathbb{V}_{\nabla_k}$ satisfies $(\boldsymbol{d}_k)$, then it suffices to see that whenever the equation $k.x^k = (k.x)^k$ holds in $\boldsymbol{A} \in \mathbb{WL}_{k\,si}$, then $\boldsymbol{A} \in \mathbb{V}_{\nabla_k}$. Take then $\boldsymbol{A} \in \mathbb{WL}_{k\,si}$ satisfying $(\boldsymbol{d}_k)$. Since $\top$ is join irreducible in $\boldsymbol{A}$, for each $a \in A$, $k.a = \top$ or $k(\neg a) = \top$. If $k.a = \top$, then $k.a^k = (k.a)^k = \top$. If $k.(\neg a) = \top$, then $k.(\neg a)^k = (k.(\neg a))^k = \top$. Therefore $\boldsymbol{A} \in \mathbb{V}_{\nabla_k}$. ■

Theorem 5.7. *The following are equivalent conditions for each subvariety* $\mathbb{V}$ *of* $\mathbb{WL}_k$*:*

(i) $\mathbb{V}$ *has the Boolean retraction property.*

(ii) $\mathbb{V}$ *satisfies* $(2.22)_k$.

(iii) ∇_k *is a Boolean retraction term for* $\mathbb{V}$.

Proof. Let $\boldsymbol{A} \in \mathbb{V}_{si}$ and let $h\colon \boldsymbol{A} \to \boldsymbol{B}(\boldsymbol{A}) = \{\bot, \top\}$ be a homomorphism. By Corollary 1.9, $Rad(\boldsymbol{A}) = h^{-1}(\top)$. Hence by Lemma 1.8, if $h(a) = \top$, then $k.a^n = \top$ for all n, in particular $k.(a^k) = \top$. If $h(a) = \bot$, then $h(\neg a) = \top$, and so $\neg a \in Rad(\boldsymbol{A})$, hence, as above, $k.(\neg a)^k = \top$. Since $\mathbb{V}$ is generated by its subdirectly irreducibles members, $(2.22)_k$ holds in all algebras in $\mathbb{V}$. This shows that (i) implies (ii). That (ii) implies (iii) follows from Corollary 5.2, and (iii) trivially implies (i). ■

Now from Theorems 5.6 and 5.7 and Corollary 2.8 we have:

Corollary 5.8. *A subvariety $\mathbb{V}$ of $\mathbb{WL}_k$ has the Boolean retraction property if and only if ($\boldsymbol{d}_k$) holds in $\mathbb{V}$. In this case ∇_k is a Boolean retraction term for $\mathbb{V}$, and a unary term t is a Boolean retraction term for $\mathbb{V}$ if and only if the equation $t(x) = \nabla_n(x)$ holds in $\mathbb{V}$.*

Since $\mathbb{MTL} \subseteq \mathbb{WL}_2$, we have:

Corollary 5.9. *A subvariety $\mathbb{V}$ of $\mathbb{MTL}$ has the Boolean retraction property if and only if ($\boldsymbol{d}_2$) holds in $\mathbb{V}$. In this case ∇_2 is a Boolean retraction term for $\mathbb{V}$, and a unary term t is a Boolean retraction term for $\mathbb{V}$ if and only if the equation $t(x) = 2.x^2$ holds in $\mathbb{V}$.*

Note that the above corollary improves [11, Theorem 3.2], because the Glivenko equation (2.29) is not required and the essential uniqueness of the Boolean retraction term ∇_2 is established.

5.1. An example of an involutive variety contained in $\mathbb{V}_{\nabla_n}$.

For $n > 0$, let $\boldsymbol{L_{n+1}} = \langle L_{n+1} = \{0, 1, 2, \ldots, n\}; *, \to, \vee, \wedge, 0, n\rangle$ be the residuated lattice characterized by the following properties:

- Its lattice order is given by $0 < 1 < 2 < \cdots < n-1 < n$,
- $r * s = \max\{0, r + s - n\}$,
- $r \to s = \min\{n, n - r + s\}$.

in others words $\boldsymbol{L_{n+1}}$ is a copy of the totally ordered Wajsberg hoop with $n+1$ elements (see [2]).

Consider $\widehat{\boldsymbol{L_{n+1}}} = \langle \widehat{L_{n+1}} = \{0,1\} \times L_{n+1}; \odot, \div, \vee, \wedge, (1, n), (0, n)\rangle$ where

- $\langle \widehat{L_{n+1}}; \wedge, \vee, (1, n), (0, n)\rangle$ is the (distributive[1]) bounded lattice given by the diagram depicted in Figure 1.

Moreover, if $0 \leq x, y \leq n$,

[1]Distributive because it neither contains the pentagon nor the diamond

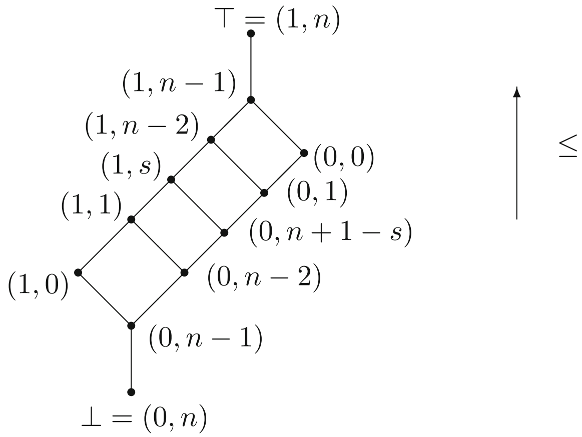

Figure 1. The lattice reduct of $\widehat{\boldsymbol{L_{n+1}}}$

- $(i,x) \odot (j,y) = (i,y) \odot (j,x)$ is given by:
  - $(1,x) \odot (1,y) = (1, x * y)$,
  - $(0,x) \odot (0,y) = (0, \min\{x+y+1, n\})$,
  - $(1,x) \odot (0,y) = (0,y) \odot (1,x) = (0, x \to y)$.
- It is straightforward to show that $\langle \widehat{L_{n+1}}; \odot, \top = (1,n)\rangle$ is a commutative monoid, and that the following distributive law holds:

$$x \odot (y \vee z) = (x \odot y) \vee (x \odot z).$$

Therefore, since it is finite, it is a bounded integral commutative residuated lattice, with residual:

$$(i,x) \div (j,y) = \max\{(k,z) : (i,x) \odot (k,z) \leq (j,y)\}.$$

Now if $\sim (i,x) = (i,x) \div \bot = (i,x) \div (0,n)$, then we have:

$$\sim (1,x) = (0,x), \text{ and } \sim (0,x) = (1,x).$$

Hence it is involutive, and then:

$$(i,x) \div (j,y) = \sim ((i,x) \odot \sim (j,y)).$$

Remark 5.10. *$\widehat{\boldsymbol{L_2}}$ coincides with the minimum nilpotent algebra $\boldsymbol{A_4}$ considered in [14] (see also [11]).*

THEOREM 5.11. *For each $n > 0$, $\widehat{\boldsymbol{L_{n+1}}} \in \mathbb{V}_{\nabla_{n+1}}$.*

PROOF. Observe first that if $0 \leq x \leq n-1$, then

$$(1,x)^{n+1} = (1,x)^n = (1,0),$$

and moreover,

$$(0,0)^n = (0,n-1) \neq \bot, \text{ and } (0,0)^{n+1} = (0,n) = \bot.$$

To see that every $\alpha \in 2 \times L_{n+1}$ satisfies $(2.22)_{n+1}$, we are going to consider all possible cases.

- if $\alpha = (1,x)$ with $0 \leq x \leq n-1$, then we have that

$$\begin{aligned} (n+1).(\alpha^{n+1}) &= (n+1).(1,0) =\sim ((\sim (1,0))^{n+1}) \\ &= \sim ((0,0)^{n+1}) =\sim (0,n) = (1,n) = \top \end{aligned}$$

  Since $(n+1).\top^{n+1} = \top$, we have that $(n+1).(\alpha^{n+1}) = \top$ for $\alpha \geq (1,0)$.
- Trivially $(n+1).(\sim \bot)^{n+1} = \top$.
  If $\bot < \alpha \leq (0,0)$, then $\alpha = (0,x)$ for some $0 \leq x \leq n-1$. Thus $\sim \alpha = (1,x)$, and then $(n+1).(\sim \alpha)^{n+1} = \top$.

We have verified that all $\alpha \in 2 \times L_{n+1}$ satisfies the equation $(2.22)_{n+1}$. ■

COROLLARY 5.12. *For each $n \geq 1$ the variety generated by $\widehat{\boldsymbol{L_{n+1}}}$ is an $n+1$-potent involutive subvariety of $\mathbb{V}_{\nabla_{n+1}}$.*

THEOREM 5.13. *For each $0 < r < n+1$, $\widehat{\boldsymbol{L_{n+1}}} \notin \mathbb{V}_{\nabla_r}$.*

PROOF. Observe that in $\widehat{\boldsymbol{L_{n+1}}}$,

$$\begin{aligned} n.((1,1)^n) &= n.(1,0) =\sim ((\sim (1,0))^n) =\sim ((0,0)^n) \\ &= \sim (0,n-1) = (1,n-1) \neq \top. \\ n.(\sim (1,1))^n &= n.(0,1)^n = n.(0,n) = n.\bot = \bot \end{aligned}$$

Hence $(n.(1,1)^n) \vee (n.(\sim (1,1)^n)) \neq \top$. This shows that $\widehat{\boldsymbol{L_{n+1}}} \notin \mathbb{V}_{\nabla_n}$. In the light of Lemma 5.3, the proof is completed. ■

Since $Rad(\widehat{\boldsymbol{L_{n+1}}}) = \{(1,x) : 0 \leq x \leq n\}$, it is easy to see that $\boldsymbol{r}(\widehat{\boldsymbol{L_{n+1}}})$ is isomorphic to $\boldsymbol{L_{n+1}}$, enriched with the operation:

- $x \oplus y = \min\{x + y + 1, n\}$.

Remark 5.14. *For each $n > 1$, $\{0, n\}$ is the universe of a subalgebra of $\boldsymbol{L_{n+1}}$ which is not closed under $\oplus$. This shows that $\oplus$ does not belong to the clone of operations of $\boldsymbol{L_{n+1}}$. On the other hand, $\boldsymbol{L_2}$ satisfies the equation $x \oplus y = \top$ (cf. Lemma 3.1).*

Since $\boldsymbol{L_{n+1}}$ is a simple Wajsberg hoop (see [2, Example 2.4]), it follows that $\boldsymbol{r}(\widehat{\boldsymbol{L_{n+1}}})$ is also simple.

Trough this subsection for $n > 1$, $\mathbb{V}_{n+1}$ will denote the subvariety of $\mathbb{V}_{\nabla_{n+1}}$ generated by $\widehat{\boldsymbol{L_{n+1}}}$, and $\mathbb{WH}^{\oplus}_{n+1}$ will denote the variety of enriched Wajsberg hoops generated by $\boldsymbol{r}(\widehat{\boldsymbol{L_{n+1}}})$. Notice that by Remark 5.14, $\mathbb{WH}^{\oplus}_{2}$ coincides with the variety of Wajsberg hoops generated by $\mathbf{L_2}$.

Lemma 5.15. *For each $n > 0$, $(\mathbb{V}_{n+1})^{\boldsymbol{r}} = \mathbb{WH}^{\oplus}_{n+1}$.*

Proof. The inclusion $\mathbb{WH}^{\oplus}_{n+1} \subseteq (\mathbb{V}_{n+1})^{\boldsymbol{r}}$ follows from $\boldsymbol{r}(\widehat{\boldsymbol{L_{n+1}}}) \in (\mathbb{V}_{n+1})^{\boldsymbol{r}}$. We shall see the other inclusion. Since $\widehat{\boldsymbol{L_{n+1}}}$ is a finite subdirectly irreducible algebra and $\mathbb{V}_{n+1}$ is a congruence distributive variety, by a well known result of Jónsson in [16, Corollary 3.4], see also [3, Corollary IV-6.10], the subdirectly irreducibles in $\mathbb{V}_{n+1}$ are the homomorphic images of subalgebras of $\widehat{\boldsymbol{L_{n+1}}}$. Clearly, each subalgebra $\boldsymbol{S}$ of $\widehat{\boldsymbol{L_{n+1}}}$ is directly indecomposable, and by Lemma 3.5, $S = Rad(\boldsymbol{S}) \cup coRad(\boldsymbol{S})$. By Remark 3.6 the i-filters of $\boldsymbol{S}$ are all contained in $Rad(\boldsymbol{S}) = S \cap Rad(\widehat{\boldsymbol{L_{n+1}}})$. Then it follows that $\boldsymbol{r}(\boldsymbol{S})$ is isomorphic to a subalgebra of $\boldsymbol{r}(\widehat{\boldsymbol{L_{n+1}}})$. This implies that the only i-filter properly contained in $Rad(\boldsymbol{S})$ is $\{(1, n)\}$, and consequently the only non trivial homomorphic image of $\boldsymbol{S}$ has $\{(0, n), (1, n)\}$ as a universe, and so it is a subalgebra of $\boldsymbol{S}$. Hence the subdirectly irreducibles in $\mathbb{V}_{n+1}$ are the subalgebras of $\widehat{\boldsymbol{L_{n+1}}}$. Therefore for each $\boldsymbol{A} \in \mathbb{V}_{n+1}$ there is a family $\{\boldsymbol{S}_i : i \in I\}$ of subalgebras of $\widehat{\boldsymbol{L_{n+1}}}$ and an embedding $h\colon \boldsymbol{A} \to \prod_{i \in I} \boldsymbol{S}_i$. Since $\oplus$ is in the clone of operations of algebras in $\mathbb{V}_{n+1}$, it follows from (2) in Lemma 3.7 that the restriction of h to $Rad(\boldsymbol{A})$ is an embedding into $\prod_{i \in I} Rad(\boldsymbol{S}_i)$ that preserves $\oplus$. Therefore $\boldsymbol{r}(\boldsymbol{A})$ is isomorphic to a subalgebra of a direct product of algebras in $\mathbb{WH}^{\oplus}_{n+1}$. ∎

With the notations of Section 4, from above lemma and Theorem 4.8 we deduce:

Theorem 5.16. *For any integer $n > 0$, each set $X \neq \emptyset$ and each $S \subseteq X$, $\boldsymbol{r}(\mathfrak{F}_{\mathbb{V}_{n+1}}(X)/\langle U_S \rangle)$ is, up to isomorphism, the $|X|$-free algebra in $\mathbb{WH}^{\oplus}_{n+1}$.* □

A description of free algebras in the variety generated by $\widehat{\boldsymbol{L_2}}$, considered as a minimum nilpotent algebra (see Remark 5.10), was given in [11].

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