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Finite Cardinals in Quasi-set Theory

Abstract. Quasi-set theory is a ZFU-like axiomatic set theory, which deals with two kinds of ur-elements: M-atoms, objects like the atoms of ZFU, and m-atoms, items for which the usual identity relation is not defined. One of the motivations to advance such a theory is to deal properly with collections of items like particles in non-relativistic quantum mechanics when these are understood as being *non-individuals* in the sense that they may be indistinguishable although identity does not apply to them. According to some authors, this is the best way to understand quantum objects. The fact that identity is not defined for m-atoms raises a technical difficulty: it seems impossible to follow the usual procedures to define the cardinal of collections involving these items. In this paper we propose a definition of finite cardinals in quasi-set theory which works for collections involving m-atoms.

Keywords: Finite cardinals, Quasi-set theory, Identity.

1. Introduction

One of the many philosophical puzzles advanced by quantum mechanics, namely, the problem of the identity and individuality of indiscernible particles, still raises as many controversies and difficulties as it did in the beginnings of the theory. It has been since then a matter of philosophical debate whether the underlying metaphysics of non-relativistic quantum mechanics is some kind of "classical" metaphysics, dealing only with individuals or, alternatively, by a kind of a non-classical metaphysics, in the sense that at least some of the items with which it deals are not individuals. The last point of view was privileged by some of the founding fathers of the theory, like Schrödinger, and it is still widely held nowadays, but that understanding of the theory came to be challenged in recent years. Also, it is assumed by most thinkers that the famous Principle of Identity of Indiscernibles fails for these items, so if the metaphysics of individuals is chosen, it will hardly be one in which this principle is taken into account, that is, items are allowed that are indiscernible but not numerically identical (for a detailed study wholly dedicated to this problem, see [5]).

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The view that quantum particles are not individuals can be best understood if we take that claim to consist in the fact that these items do not figure in the identity relation. Such a restriction brings together its own metaphysical novelties and limitations. Related with this view that holds that it is senseless to speak about the identity of these items there is the problem that it seems we cannot apply the usual notion of "counting" for them. In the usual sense, to count a collection of things, intuitively speaking, means that we stipulate a one-to-one correspondence between the collection of things to be counted and an ordinal number. The heart of the matter is that, being indiscernible and not having identity conditions, it is not possible to decide if more than one item has not been counted for one, or, if one item has not been counted twice, and also, since identity does not make sense for them, it is not possible even to say that in this process different items should be assigned different numbers. The identity relation results are crucial for the **usual** counting process.

In this paper, we shall provide a formalization of a different way of counting finite collections of items that applies also for non-individuals as characterized above. Our underlying framework is quasi-set theory, a formal theory whose motivations are based on the suggestion that a mathematical framework differing from usual set-theories should be provided when it comes to treat adequately with items for which identity and difference do not make sense. We sketch the theory in general lines in the next section. In the usual approaches to quasi-set theory, since, as we said, cardinality cannot be defined by the usual procedures, this notion is introduced as a primitive, and called *quasi-cardinal*.

The idea that for finite collections quasi-cardinality can be defined rather than taken as primitive was already explored in [2]. The motivation behind this work lies in the fact that we can define a procedure of elimination of the elements of a collection of objects without having to appeal to their identification. Being able to count the number of steps used to make a collection empty, and making sure that we are taking one element at each time, we can define the number of elements of that collection as the number of steps taken to empty the collection. Despite the simple motivation, and the alleged gain in simplicity when we dispense with the primitive notion of quasi-cardinal, their own approach to the formalization of the idea, conducted also in quasi-set theory, was not as simple as one would desire, being sometimes even counter-intuitive. Also, there is the drawback that these authors needed the addition of two extra axioms to make their proposal work. Our definition labors on the motivation provided by them, but retains the simplicity of the intuitive motivation and needs no extra axioms. We shall leave the discussion of the relation of our definition to the one provided by Domenech and Holik in [2] for future work.

2. Quasi-set theory

Quasi-set theory is a first-order theory based on a logic whose postulates are those of classical first order predicate calculus without the symbol of identity (see [1] for further discussions on the underlying logic of the theory). As we mentioned before, the main motivation behind the theory is the need to give a mathematical treatment of objects for which identity and difference are not defined. Besides this, these objects can also be indistinguishable. This is achieved by the proposal of a ZFU-like theory with two kinds of atoms, m-atoms and M-atoms. For the first kind, identity is not defined, but the indistinguishability relation holds. For the second kind of atoms, identity is defined and coincides with indistinguishability. The collections formed in this theory are called q-sets, and they also can be distinguished as belonging to two kinds: those whose transitive closure contains m-atoms, and those for which it does not happen, which are called "classical sets", and satisfy the predicate Z of the language, as we shall see below. Objects satisfying the predicates M or Z are called "classical objects" of the theory, for, when restricted to them, we can construct inside quasi-set theory a kind of copy of ZFU.

Now, we pass to the language of the theory. Besides logical constants and a denumerable collection of variables indexed by the natural numbers, the language of the theory is composed of the following proper symbols:

1. Three unary predicate symbols, m, M and Z, where $m(x)$, $M(x)$ and $Z(x)$ are read as "x is an m-atom", "x is an M-atom" and "x is a classical set", respectively;
2. The binary relation symbol $\in$, for the usual membership relation, and the binary relation symbol $\equiv$ for indistinguishability, so that $x \equiv y$ means that x is indistinguishable from y.

One should note that the identity symbol is not a primitive of the language of the theory. This is important, for we shall restrict the scope of identity precisely in its definition. We will not present the theory in its full details, restricting ourselves to what is relevant for our purposes and what is needed to keep the work self-contained. For more details on this theory, the reader can check the exposition in [5, chap. 7] and also [6].

Definition 2.1. x is a q-set if $(\neg M(x) \wedge \neg m(x))$.

That is, a q-set is something that is not an atom. Now, we define the identity symbol:

Definition 2.2. $x = y =_{def} [(Q(x) \wedge Q(y) \wedge \forall z(z \in x \leftrightarrow z \in y)) \vee (M(x) \wedge M(y) \wedge \forall z(x \in z \leftrightarrow y \in z))]$

We put as an axiom the substitution law for the defined symbol of identity, submitted to the usual restrictions, and it is easy to show that the reflexive property follows as a theorem from the definition when we deal with q-sets or M-atoms. Identity relates to the indistinguishability through another postulate, which states that when both x and y satisfy Z or M, that is, when they are classical objects, then identity is equivalent to indistinguishability. Furthermore, we postulate that indistinguishability is an equivalence relation not compatible with membership in the case of m-atoms, that is, if x and y are indistinguishable m-atoms, then for any q-set z it does not follow necessarily that if $x \in z$ then $y \in z$. This will ensure us that identity and indistinguishability do not collapse into the same relation for m-atoms.

Another relevant axiom accounts for the desired fact that if anything belonging to some q-set is a classical object then this q-set is a set in the classical sense (it satisfies Z), and conversely. In the formal language of the theory, this can be stated as $\forall x \forall y(y \in x \rightarrow (M(y) \vee Z(y)) \leftrightarrow Z(x))$. This postulate guarantees that those objects satisfying Z will not have m-atoms as its members, and also that its member's members too will obey this condition, and so on. Having classical objects in the theory allows us to develop all the classical mathematics that can be developed in ZFU, including the theory of cardinal numbers for classical sets.

The other postulates are existence postulates. They are the usual postulates conveniently adapted to the language of our quasi-set theory. A small difference occurs for the unordered pairs postulate. To obtain the collection corresponding to the usual unordered pairs, we first assume, as usual, that for objects x and y there always is some q-set z that contains them as elements. Then, with the help of the separation axiom we obtain a q-set whose elements are the ones indistinguishable from x and from y that belong to z. Pairs are always relative to some q-set from which they were obtained, and, in the case of our example, they are denoted by $[x, y]_z$, which is the q-set whose elements are the elements of z indistinguishable from x and from y. For classical objects, this represents no real limitation, but for m-atoms, there may be some pairs that are composed of more than two objects, for there may be some z with more than one object indistinguishable from x or y in it, and since identity is not defined for them, we cannot require that only x and y be in this new set; we must restrict ourselves to indistinguishability.

The other set-theoretical operations, like power set, union, intersection and difference are introduced as usual. The main difference occurs for ordered pairs when there are m-atoms involved. If x and y are indistinguishable m-atoms, then, it follows from the definition of ordered pairs, given in terms of unordered pairs in the Kuratowski style that one cannot distinguish between $\langle x, y\rangle$ and $\langle y, x\rangle$. As a consequence, for the cases involving m-atoms order relations cannot be defined, for they cannot order the elements. Also the concept of function, which relies on the notion of ordered pairs and identity, has to be weakened to what we call *q-functions*; in quasi-set theory, q-functions are q-sets of "ordered pairs" that obey only the weak condition that indistinguishable objects in the domain must be correlated with indistinguishable objects in the image q-set. That is, if f is a q-function, and x and y are indistinguishable m-atoms in the domain, then $f(x)$ must be indistinguishable from $f(y)$, and also, if x and y are in the counter domain and are indistinguishable m-atoms, then the object z mapped by f to x must be indistinguishable from the object w mapped to y. If there are no m-atoms involved, the concept assumes its classical form. When there is no danger of confusion, we may call a q-function simply a function.

These adaptations in the definition of functions stems from the fact that identity is not defined for m-atoms. In these cases, the concept of bijection cannot be defined also, for it presupposes identity. This raises difficulties for the definition of all concepts that rely on the notion of bijection, like ordinal number of collections of m-atoms, cardinal numbers for these collections and also equipollence for these kinds of sets. So, the problem of how to define these notions without the use of identity is a pressing one. The problem is faced in the usual presentations of the theory through the introduction in the language of the theory of a function symbol qc that denotes the cardinal of any collection. This symbol was not taken as a primitive in our presentation of the theory, for our goal, in the next section, is to define this notion for finite q-sets only.

3. Finite cardinals

The motivation for the following definition, as we said before, comes from the idea that we can intuitively count the number of elements of a finite collection by a simple procedure: we eliminate the members of the collection one at each time, and count the number of times this procedure is repeated until the collection is empty. This number is the number of elements originally in the collection and, so, we can say that it is its cardinal. In principle, the identification of the elements being taken from the collection need not play any role in this process, and, as we shall show, it can be done in quasi-set theory for collections of m-atoms, that is, objects for which identity and difference are not defined. Let us see how this can be done.

First of all, we shall employ the following form of the axiom of choice: (AC) If A is a q-set whose elements are non-empty q-sets, then there is a q-function f such that for every $B \in A$, $f(B) \in B$

It follows from the definition of q-function that for some $B \in A$ whose elements are indiscernible m-atoms the choice function will not specify a unique well defined element of B.

Now, since bijections as usually defined are not available to us, the usual definition of finite sets as those that can be put in one-to-one correspondence with some natural number is also not available. Luckily we have an alternative fit to our purposes, given by Tarski. We need a previous definition though.

Definition 3.1. Given a q-set B whose elements are q-sets, we call an element A of B $\subset$-minimal if $\forall C(C \in B \rightarrow \neg(C \subset A))$.

This definition applies to q-sets whose elements are q-sets only; no m-atoms are involved here.

Definition 3.2. A q-set A is finite in the sense of Tarski if every non-empty collection of subsets of A has a $\subset$-minimal element.

It is important to notice that we can talk about finite q-sets, even q-sets whose elements are m-atoms, without having to use the usual definition in which we employ a one-to-one function between natural numbers and the elements of the set. As we have presented it above, the usual definition of counting and the usual definition of finiteness are strictly linked, and both make essential use of the notion of identity. Tarski-finiteness helps us to avoid the use of identity when stipulating the finite q-sets to be counted. In the following, when we talk about finite q-sets, we mean Tarski-finite q-sets, unless otherwise stated.

The next point is the definition of the strong singleton of any item A, which we will denote by $\langle A \rangle$.

Definition 3.3.
1. Given any object A, we call S_A the q-set $[S \in \mathcal{P}([A]) : A \in S]$ (here, $\mathcal{P}$ denotes the power set operation);
2. $\langle A \rangle =_{def} \bigcap_{t \in S_A} t$.

In the usual presentations of quasi-set theory the notion of strong singleton is frequently associated with the concept of quasi-cardinal, being defined in terms of it, that is, the strong singleton of A is a q-set with only one element indistinguishable from A (if A is m-atom, we cannot prove, obviously, that the element in question is A itself). Since we are not taking a primitive symbol for quasi-cardinal, we define this notion as above without appeal to quasi-cardinals, and, as Domenech and Holik have argued (see [2]), this notion encapsulates what we would intuitively take to be a q-set with only one element. So, without using the concept of cardinal, the strong singleton formalizes the intuitive idea of having only one element.

The next step is the definition of a function which picks out elements of a q-set and makes intuitive the taking of "one at each time". Let us call it the *subtraction function*. Given a finite q-set A, by AC there is a choice function g for $\mathcal{P}(A\backslash\{\emptyset\})$, where $\backslash$ denotes the difference operation between sets. Now, the subtraction function h from $\mathcal{P}(A)$ to $\mathcal{P}(A)$ can be defined as follows:

Definition 3.4. 1. If B is not empty, then $h(B) = B\backslash\langle g(B)\rangle$;

2. If B is empty, then $h(B) = \emptyset$.

Intuitively speaking, h picks from each subset of A an element, which results in a subset of A with one element less than the original one in which we were operating. In the cases in which we apply the function to the empty set, since there is nothing to be picked out, the function results in the empty set again. It is good to remember that we are speaking only intuitively that we pick "only one element" at a time, for the proof of it would need the concept of cardinal. A partial proof can be given for the classical elements of quasi-set theory, since for them we have the classical definition of cardinal and through the application of it we have the desired result. Later, with our definition of cardinal we will also be able to show this result.

We must also remember that $\mathcal{P}(A)$ is a q-set whose elements are also q-sets, and being so the concept of identity is defined for it (for it is defined for q-sets), and so the recursion theorem can be applied. We define by recursion a q-function f from ω to $\mathcal{P}(A)$

Definition 3.5. 1. $f(0) = A$;

2. $f(n + 1) = h(f(n))$.

We can always be sure that the sequence so defined will always arrive at the empty set for some $n \in \omega$, for on the contrary we would have a collection of subsets of A without a $\subset$-minimal element, contradicting the fact that A is Tarski-finite. So, intuitively speaking, any finite q-set will have only a finite number of stages in the process of "eliminating" its elements and coming to be empty. Given this situation, we find it plausible to define the cardinal of a finite q-set as the least number of steps in which we have "emptied" the set.

DEFINITION 3.6. The cardinal of A, denoted $qc(A)$, is the least natural number n such that $f(n) = \emptyset$.

Now, we must note that the q-function f was defined with the help of the notion of the subtraction function h, and this one was defined with a choice function g. Now we need to show that it makes sense to talk about a unique cardinal of A, that is, given choice functions g_1 and g_2 for $\mathcal{P}(A\backslash\{\emptyset\})$, we will not have different results $qc_1(A)$ and $qc_2(A)$ defined respectively with these functions. That is what we do now.

Let g_1 and g_2 be choice functions for $\mathcal{P}(A\backslash\{\emptyset\})$, and let h_1 and h_2 be their respective subtraction functions. By the recursion theorem we can define as above functions f_1 and f_2 from ω to $\mathcal{P}(A)$. We have to show that if $qc_1(A) = n$ and $qc_2(A) = m$, then $n = m$, that is,the result is independent of the order in which we take the elements of A out of it. For the empty set the result can be established directly.

LEMMA 3.1. *If* $A = \emptyset$*, then* $qc_1(A) = \emptyset = qc_2(A)$*.*

PROOF. If A is empty, by definition $f_1(0) = A = \emptyset$ and $f_2(0) = A = \emptyset$, from which we have that $qc_1(A) = qc_2(A)$. ■

Now, suppose that A is non-empty. By the separation axiom we can obtain from $\mathcal{P}(A)$ the following q-sets:

1. $C =_{def} [\langle g_1(f_1(t))\rangle : 0 \leq t \leq n]$;
2. $D =_{def} [\langle g_2(f_2(t))\rangle : 0 \leq t \leq m]$.

One must remember that identity here is always *extensional identity*, as it was defined above.

LEMMA 3.2. $\bigcup C = A$*.*

PROOF. $x \in \bigcup C \rightarrow x \in A$ is immediate. Now, to show the other direction suppose that $x \in A$ but that it is not the case that $x \in \bigcup C$. Note that $x \in f_1(0) = A$ and it does not happen that $x \in f_1(n) = \emptyset$. Take the least k such that x is not in $f_1(k)$. Notice that $k \leq n$, and so $x \in f_1(k-1)$. But $f_1(k) = h_1(f_1(k-1)) = f_1(k-1)\backslash\langle g_1(f_1(k-1))\rangle$. Suppose that it is not the case that $x \in \langle g_1(f_1(k-1))\rangle$, then $x \in f_1(k)$, which is absurd. So, $x \in \bigcup C$. This gives us the desired result. ■

LEMMA 3.3. $\bigcup D = A$.

PROOF. Easy adaptation of the previous result. ■

LEMMA 3.4. *For $t \neq k$, with $t < k \leq n$, we have that $\langle g_1(f_1(t))\rangle \cap \langle g_1(f_1(k))\rangle = \emptyset$.*

PROOF. Note that $f_1(k) \subseteq f_1(t+1)$ and that $g_1(f_1(t)) \cap f_1(t+1) = \emptyset$, but $\langle g_1(f_1(k))\rangle \subseteq f_1(k) \subseteq f_1(t+1)$, and so the mentioned q-sets cannot have common elements. ■

LEMMA 3.5. *For $t \neq k$, with $t < k \leq m$, we have that $\langle g_2(f_2(t))\rangle \cap \langle g_2(f_2(k))\rangle = \emptyset$.*

PROOF. Adaptation of the previous result. ■

LEMMA 3.6. $[x \in C : \forall y \in D(x \cap y) = \emptyset] = \emptyset$.

PROOF. Suppose that some x belongs to this q-set. Then, for some $k \leq n$ we have that $x = \langle g_1(f_1(k))\rangle$. Since $\langle g_1(f_1(k))\rangle \subseteq A = \bigcup D$ (by lemma 3.3) there is at least one $y \in D$ such that $x \cap y$ is not empty, contradicting the hypothesis. ■

LEMMA 3.7. $[x \in D : \forall y \in C(x \cap y) = \emptyset] = \emptyset$.

PROOF. Adaptation of the previous result. ■

LEMMA 3.8. $\forall x \in C, \exists y \in D(x \cap y) \neq \emptyset$.

PROOF. It follows from lemma 3.6. ■

LEMMA 3.9. $\forall x \in D, \exists y \in C(x \cap y) \neq \emptyset$.

PROOF. It follows from lemma 3.7. ■

We also use the following lemma, taken from [2], where it appears as proposition 4.4.

LEMMA 3.10. *If $y \in \mathcal{P}(\langle x\rangle)$, then $y = \emptyset$ or $y = \langle x\rangle$.*

PROOF. Let X be a q-set such that $x \in X$, and consider $y \in \mathcal{P}(\langle x\rangle)$, from which we have in particular that y is a q-set. From the underlying logic we know that $(x \in y \vee \neg(x \in y))$. Consider the first case. Since $y \subseteq \langle x\rangle$, then $y \subseteq X$, and since by hypothesis $x \in y$, the $y \in S_x$, and by the definition of $\langle x\rangle$ we have that $\langle x\rangle \subseteq y$, so, $\langle x\rangle = y$. In the second case, since $\neg(x \in y)$, the $x \in \langle x\rangle\backslash y \subseteq \langle x\rangle \subseteq X$ from which we have that $\langle x\rangle\backslash y \in S_x$, from which it follows that $\langle x\rangle \subseteq \langle x\rangle\backslash y$, and so $\langle x\rangle = \langle x\rangle\backslash y$. If y were not empty, we would have that there is at least one z such that $z \in y$. By hypothesis,$y \subseteq \langle x\rangle$, from which it follows that $z \in \langle x\rangle$, but, as we saw, $\langle x\rangle = \langle x\rangle\backslash y$, and so $z \in \langle x\rangle\backslash y$, that is, $\neg(z \in y)$, which is absurd. So, $y = \emptyset$. ■

LEMMA 3.11. *For $x \in C$, if $u \leq m$, $k \leq m$ and $u \neq k$, then it is not the case that $x \cap \langle g_2(f_2(u))\rangle \neq \emptyset$ and $x \cap \langle g_2(f_2(k))\rangle \neq \emptyset$.*

PROOF. If $x \cap \langle g_2(f_2(u))\rangle = \emptyset$ and $x \cap \langle g_2(f_2(k))\rangle = \emptyset$ there is nothing to prove. Remember that $x = \langle g_1(f_1(t))\rangle \neq \emptyset$, for some $t \leq n$. Suppose that $x \cap \langle g_2(f_2(u))\rangle \neq \emptyset$. By lemma 3.5, $\langle g_2(f_2(u))\rangle \cap \langle g_2(f_2(k))\rangle = \emptyset$ when $k \neq u$. Since $z =_{def} x \cap \langle g_2(f_2(u))\rangle \neq \emptyset$ we have in particular that $z \subseteq x$ and by the previous lemma $z = x$. If $w =_{def} x \cap \langle g_2(f_2(k))\rangle \neq \emptyset$ it also follows from previous lemma that $w = x$. Now note that $w \cap z = (x \cap \langle g_2(f_2(u))\rangle) \cap (x \cap \langle g_2(f_2(k))\rangle) = x \cap x = x = \emptyset$ which is absurd. ■

LEMMA 3.12. *For $x \in D$, if $u \leq n$, $k \leq n$ and $u \neq k$, then it is not the case that $x \cap \langle g_1(f_1(u))\rangle \neq \emptyset$ and $x \cap \langle g_1(f_1(k))\rangle \neq \emptyset$.*

PROOF. Adaptation from previous proof. ■

THEOREM 3.7. *Given $qc_1(A)$ and $qc_2(A)$ defined as above, we have that $qc_1(A) = qc_2(A)$, that is, $n = m$.*

PROOF. By lemmas 3.8 and 3.11 there is for each $x \in C$ only one $y \in D$ such that $x \cap y \neq \emptyset$, that is, $n \leq m$, and by lemmas 3.9 and 3.12 we have for each $y \in D$ only one $x \in C$ such that $x \cap y \neq \emptyset$, that is, $m \leq n$, from which it follows that $m = n$. ■

Having proved this theorem, we guarantee that the cardinal of a finite q-set is independent of the particular choice function used in the definition of the subtraction function. We call *counting function* the q-function f given in the definition of cardinal above when it is restricted to the least $n \in \omega$ such that $f(n) = 0$. With this in mind, we can establish the cardinal of a finite q-set by providing it with a counting function. In the case of q-sets satisfying the predicate Z, in many cases we can stipulate an adequate choice function for its subsets, making easier the proof of some results, as we can see in the following result. As in a previous definition, for a classical q-set A, let us denote by $card(A)$ the cardinal of A as defined in the classical part of quasi-set theory, that is, defined in von Neumann style as the least ordinal equipollent to it and not equipollent to no smaller ordinal.

THEOREM 3.8. *Given $n \in \omega$, $qc(n) = n = card(n)$.*

PROOF. Take as choice function $u(B) =$ *the greatest number belonging to* B (B is finite). The corresponding counting function is given by $h(B) = B \backslash \langle u(B) \rangle$. In particular, if B is a natural number,it follows from these definitions that $h(B) = B - 1$, if $B \neq 0$. Then, by definition: $f(0) = n$, and $f(k+1) = h(f(k))$. From the previous observation, we have that in particular $f(n) = n - n = 0 = \emptyset$. If n were not the least number for which it happens, then there would be some $k \in \omega$ such that $k < n$ and $f(k) = 0$, that is, $n - k = 0$, which is absurd. ■

We can use the idea behind the proof of the last theorem to show that for any finite q-set A satisfying Z, the cardinal of A defined above coincides with its cardinal as defined *à la* von Neumann. So the counting process given here gives the same results as the usual one when applied to classical objects.

THEOREM 3.9. *If A is a Tarski-finite classical set, then $card(A) = n$ iff $qc(A) = n$.*

PROOF. If $qc(A) = n$, then, by definition there is a q-function f that counts the elements o A such that $f(n) = \emptyset$. We can stipulate a bijection t in the classical part of Q between A and n as follows: $t(k) = g(f(k-1))$, for $1 \leq k \leq n$, and $t(0) = A \backslash \{g(f(k-1)) : 1 \leq k \leq n\}$. It is simple to verify that t is a bijection, and so that $card(A) = n$.
Now, if $card(A) = n$, there is a bijection t between A and n in the classical part of Q. We can put as choice function for the definition of counting function the following function g: for each subset B of A, $g(B) = t(k)$, with k the least natural number such that $t(k) \in B$. Then, f is defined as follows: $f(0) = A$, $f(1) = A \backslash \langle g(f(0)) \rangle$ and in general, $f(k) = f(k-1) \backslash \langle g(f(k-1)) \rangle$, that is, the choice function takes from each subset of A the element that is the image of the least number by the function t. We have only to show that $f(n) = \emptyset$. Note that $f(n) = f(n-1) \backslash \langle g(f(n-1)) \rangle$. If this set were not empty, there would be an element in A that is not an image of the function t, contradicting the hypothesis. So, $qc(A) = n$. ■

Now, we also show that for any x, $qc(\langle x \rangle) = 1$. This gives a formal counterpart for the previous informal explanation that the cardinal of $\langle x \rangle$ is 1. With lemma 3.10 it is easy to show that $\langle x \rangle$ is finite, and then the definition can be applied.

THEOREM 3.10. *For any x, $qc(\langle x \rangle) = 1$.*

PROOF. We have to show that for any counting function f, $f(1) = \emptyset$. By definition $f(0) = \langle x \rangle$ and $f(1) = h(f(0)) = f(0) \backslash \langle g(\langle x \rangle) \rangle$. Suppose this last q-set is not empty. Since $\langle x\rangle\backslash\langle g(\langle x\rangle)\rangle \subseteq \langle x\rangle$, then $\langle x\rangle\backslash\langle g(\langle x\rangle)\rangle \in \mathcal{P}(\langle x\rangle)$, and by lemma 3.10 we have that $\langle x\rangle\backslash\langle g(\langle x\rangle)\rangle = \langle x\rangle$, from which it follows that $\langle g(\langle x\rangle)\rangle = \emptyset$, which is absurd. So, $f(1) = \emptyset$, and this is the least number for which it happens, and so $qc(\langle x\rangle) = 1$. ■

With the help of this theorem then, we guarantee that we are always taking one element from the set being counted when we apply our counting procedure. Some more results can be shown which help us grasp the idea that the strong singleton has only one element in it.

THEOREM 3.11. *If $qc(A) = n$ and $\neg(x \in A)$, then $qc(A \cup \langle x\rangle) = n + 1 = qc(A) + qc(\langle x\rangle)$.*

PROOF. Let g be the choice function for $\mathcal{P}(\langle x\rangle)$ (such that $g(\langle x\rangle) \in \langle x\rangle$) which is used in the counting with which we obtain that $qc(\langle x\rangle) = 1$ by theorem 3.4 and let h be the choice function used in the counting used to obtain $qc(A) = n$. We can use the following choice function j for $\mathcal{P}(A \cup \langle x\rangle)$: if B is a non-empty subset of A, then $j(B \cup \langle x\rangle) = h(B)$ and if $B = \emptyset$ then $j(B \cup \langle x\rangle = g(\langle x\rangle))$. Now, there is a counting function f for $A \cup \langle x\rangle$, with a subtraction function s. By definition of counting we have that $f(n + 1) = s(f(n)) = f(n)\backslash\langle j(f(n))\rangle$. If $j(f(n)) = h(f(n))$ then there are still elements of A in the $n - th$ step, contradicting the fact that $qc(A) = n$ and our construction of the counting function, which should make A empty first. So, $j(f(n)) = g(f(n))$ and $g(f(n)) \in \langle x\rangle$ and so we must have that $f(n) = \langle x\rangle$ and $f(n + 1) = \emptyset$, for otherwise we would still have an element of A to be counted, contradicting again our definition of j. ■

THEOREM 3.12. *If $A \neq \emptyset$ is a finite q-set such that $qc(A) = n$ and $x \in A$ then $qc(A\backslash\langle x\rangle) = n - 1$.*

PROOF. Note that $n = qc(A) = qc(A\backslash\langle x\rangle \cup \langle x\rangle)$. By the previous theorem, we have that $qc(A\backslash\langle x\rangle \cup \langle x\rangle) = qc(A\backslash\langle x\rangle) + qc(\langle x\rangle)$, from which we have $qc(A\backslash\langle x\rangle) + 1 = n$, that is, $qc(A\backslash\langle x\rangle) = n - 1$. ■

4. Further discussions on counting and cardinality

As we said before, one of the reasons that prohibits us from defining cardinality for q-sets in general is that even for some finite q-set X we cannot in particular define a one-to-one correspondence between $\langle n, \in\rangle$ and $\langle X, R\rangle$, where R well-orders X. This process is a simple way to formalize in usual set theories the usual counting process mentioned before. With the help of previous definitions and results, though, we can make something similar even for q-sets whose elements are m-atoms.

First of all, we must notice that when $qc(X) = n$, the relation $\subset$ defined on the q-set $[f(k) : 0 \leq k \leq n]$ is irreflexive, transitive and connected. Secondly, since it is impossible to define order relations over X when some of its elements are m-atoms, we can apply the following *manoeuvre*: we pretend we are fixing the elements of X as they are eliminated in the counting process defined above. This does not assume that we are identifying the element being eliminated, but only that we are sure that one element is taken out in each step. In the first step, $f(0)$, nothing is being taken out, so we can keep with $[f(k) : 1 \leq k \leq n]$. We are counting the steps used to empty the set X, as explained before, and the $\subset$ relation, when defined over it, has some similarity with the properties of membership for transitive sets.

If we denote the set defined before with the more perspicuous notation $f(k)_{1 \leq k \leq n}$, we can define for X a certain kind of structure, which we will call the $\subset$-image of X, to make clear the intuitive similarity with the usual $\in$-images of classical set theory.

Definition 4.1. The $\subset$-image of a finite q-set X is the pair $\langle f(k)_{1 \leq k \leq n}, \subset \rangle$, where $f(k)_{1 \leq k \leq n}$ is defined as above.

This notion may help us associate an ordinal with X in an indirect way, without having to identify the elements of X. We only need to verify that there is an order preserving isomorphism between $\langle f(k)_{1 \leq k \leq n}, \subset \rangle$ and $\langle n, \in \rangle$. The function j from n to $f(k)_{1 \leq k \leq n}$ defined by $j(k) = f(k+1)$ is such an isomorphism. It is easy to notice that for each element of n there is only one element of $f(k)_{1 \leq k \leq n}$ associated and conversely. Also, if $k \in m$, with $k < m < n$, since $j(k) = f(k+1)$ and $j(m) = f(m+1)$, we have that $j(k) \subset j(m)$.

Proceeding this way we associate an ordinal with any finite q-set A in an indirect way. The $\subset$-images allow this strategy to work for any finite q-set, in particular for those whose elements are m-atoms. This gives a precise sense in saying that we are somehow counting the elements of a q-set when we apply the process defined in the previous section. Since we cannot count the elements, we count the number of steps employed to make the q-set empty. In the case of classic q-sets, the ordinal associated in our process with the help of $\subset$-images and the ordinal associated in the usual process through $\in$-images are the same, as it is easy to verify.

One more point where our approach is conservative with the usual approach is that the ordinal associated with a finite set X through $\subset$-images and its cardinal as defined above are just the same natural number. To be sure, this happens in the classical approach when the von Neumann definitions are used. One point to be noted is that although a $\subset$-image has a last stage, associated with the least element of its $\in$-image, the set X itself may not have a least element, for it does not follow from our discussion that it is possible to define an order relation over it in case its elements are m-atoms.

Following this discussion one can see more clearly the relations between counting and identity. The usual counting process is obviously tied to this notion but, as we are proposing in this paper, this is not the only reasonable notion of counting one can adopt. Applying alternative definitions one can circumvent the difficulties that appear when trying to count items for which identity does not make sense.

As a last consequence of our definitions, we can show the following result:

THEOREM 4.2. *Given finite q-sets X and Y, $qc(X) = qc(Y)$ if and only if there is a bijection between $\langle f(k)_{1\leq k\leq n}, \subset\rangle$ and $\langle g(k)_{1\leq k\leq m}, \subset\rangle$, where these are the $\subset$-images of X and Y respectively.*

PROOF. Given these $\subset$-images, if there is a one-one correspondence between them, that is, if for each $f(k)$ there is only one $g(k)$ and conversely, then both of these $\subset$-images are isomorphic to the same $\in$-image, and so $m = n$ and the cardinal of these q-sets is the same. On the other way, suppose that $qc(X) = n = qc(Y)$. As we saw, the $\in$-image is isomorphic to the $\subset$-images $\langle f(k)_{1\leq k\leq n}, \subset\rangle$ and $\langle g(k)_{1\leq k\leq n}, \subset\rangle$ of X and Y, from which a composition of functions gives us the desired correspondence between the counting steps. ■

This is a generalization for finite q-sets of what is known as *Hume's principle*. The standard statement of this principle uses the notion of equipollence, which is not available for us in general, for it uses identity. Our generalization seems intuitive and plausible and, since Hume's principle is, according to some authors (see [4, p.96]), the main law that a purported definition of cardinality must be shown to obey, we feel that our goal to show ours is a reasonable definition of finite cardinality is now achieved.

5. Concluding remarks

The concepts of identity and counting are very important notions for philosophical discussions on individuality. In general, they come together and under these conditions work very well for usual items of our experience. When it comes to deal with objects whose status on the matter of individuality is so problematical as it is the case for quantum particles, the simplicity and sometimes even the desirability of the link between these notions may be questioned. Our goal in this work was to show that when looked from a formal point of view, these notions need not to be so closely tied, they can be treated independently, and, as we proposed, if an alternative account of counting is advanced, then we need not presuppose identity of the items of the collection being counted.

Obviously, controversy on the topics of individuality may not be decided by a formalization of some set theory, standard or non-standard, but it may help us to escape from conceptual fuzziness and imprecision. The philosophical position according to which it is at least plausible to maintain that some of the objects dealt with by non-relativistic quantum mechanics may be non-individuals in the sense presented above has already been gifted by the proposal of a quasi-set theory, allowing it even to formulate rigorously a version of quantum mechanics in which their main claim that identity is not defined for these particles is taken into account in the very construction of the theory (see [3]). Now, since we can accommodate in this formal framework discussion about cardinality and some non-standard counting process, it seems that this position can be made even more plausible and less objectionable on the grounds that it does not account for intuitive features of physicists' daily practice, like attribution of cardinality to collections of particles. Furthermore, the stage is open for this kind of research in quantum field theory, and some clues are given in [5, chap. 9].

Also, besides this philosophical gain, there is the formal aspect which can be profitably investigated. The whole relation of identity and cardinality in the framework under consideration can be best understood when this kind of investigation is conduced. We can weigh the benefits of the proposed definition restricted to finite q-sets against the generality of the usual approach, in which the notion is introduced through axioms, and decide what is more convenient for the applications we have in mind. If one is looking for simplicity in the formulation of quasi-set theory and economy of primitive notions, one has another good motivation for the search for alternative definitions of cardinality. The extension of the definition for infinite q-sets is still to be done and has formal interest, but the lack of such definition represents no real limitation in the applications of the theory to philosophical problems, since in these cases we deal typically with only a finite number of objects.

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