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The Ontology of Justifications in the Logical Setting

Dedicated to the memory of Leo Esakia

Abstract. Justification Logic provides an axiomatic description of justifications and delegates the question of their nature to semantics. In this note, we address the conceptual issue of the logical type of justifications: we argue that justifications in the logical setting are naturally interpreted as *sets of formulas* which leads to a class of epistemic models that we call *modular models*. We show that Fitting models for Justification Logic naturally encode modular models and can be regarded as convenient pre-models of the former.

Keywords: Justification Logic, Kripke models, Fitting models.

1. Introduction

Since Plato, *justification* has been considered a principal element of epistemic analysis that was, until recently, conspicuously absent in formal logical models of knowledge and belief. Justification Logic augments epistemic logic by assertions $t{:}F$ that read

t is a justification for F,

hence incorporating the missing justification component.

Historically, the first system of Justification Logic was the Logic of Proofs LP (cf. [1, 2]), and the first formal semantics was the semantics of mathematical proofs. Extending to a general logical theory of justification required developing a generic epistemic semantics for justifications: Fitting models [8] and their modifications (cf. [4] for a comprehensive account) became the standard epistemic semantics for Justification Logic and played a pivotal role in its development, cf. [6, 7, 9, 10, 11, 12, 13, 15, 16, 17, 18].

However, from a conceptual perspective, Fitting models do not address the issue of the logical type of justifications, e.g., the truth value of a justification assertion $t{:}F$ is defined without introducing an interpretation for justification t.

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Why would one want an answer to the question of what a justification is? When asked "what is a real number?", we have an answer[1] ready: a Dedekind cut, i.e., essentially, a set of rational numbers with some conditions. We know a reasonable mathematical answer (within Kolmogorov's model) to the question "what is probability?": a function from σ-algebra of events to $[0, 1]$, again, with some natural conditions. Within an exact mathematical theory, there should be a similar kind of answer to the question "what is a justification?". In addition to its conceptual value, clarity in this issue could lead to cleaner mathematical models.

Of course, the logical type of justifications can be easily read from the format of Fitting models: at each possible world, justifications should be interpreted as *sets of formulas* with corresponding operations. But the story does not end there: it turns out that such an interpretation suggests refinement of Fitting models. Though rather minor on the mathematical scale, it produces a new class of *modular models* that could be viewed as a conceptually clean and potentially useful addition to the existing variety of models for knowledge, belief, and justification.

We retain a classical interpretation $*$ of the propositions (formulas Fm) in a model as subsets of the set W of possible worlds,

$$* : Fm \mapsto 2^W.$$

We will write F^* rather than $*(F)$ to denote the set of worlds that corresponds to formula F. The set F^* is usually understood as a *set of worlds at which F holds* and $u \Vdash F$ is a shorthand for $u \in F^*$.

In addition, we interpret justification terms Tm at each world as sets of formulas,

$$* : W \times Tm \mapsto 2^{Fm}.$$

We write t^*_u for the interpretation of term t at world u[2]. Each t^*_u is a *set of formulas for which t is a justification at u*. According to this reading,

$$u \Vdash t{:}F \qquad \text{iff} \qquad F \in t^*_u\ . \tag{1}$$

We call these models 'modular' because one can specify interpretations of justifications and atomic propositions and then build interpretations of all formulas from there in a uniform 'modular' way.

Note that whereas propositions in modular models are interpreted semantically, as sets of possible worlds, justifications are interpreted syntactically,

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[1] in fact, several answers

[2] though notation $*(u, t)$ would be formally more appropriate here

as sets of formulas. This is a principal feature: a modular model may treat distinct formulas F and G as equal, i.e. $F^* = G^*$, which yields

$$u \Vdash F \quad \text{iff} \quad u \Vdash G$$

for each possible world u, but still be able to distinguish justification assertions $t{:}F$ and $t{:}G$, e.g., when $F \in t^*_u$, but $G \not\in t^*_u$ yielding

$$u \Vdash t{:}F \quad \text{but} \quad u \not\Vdash t{:}G.$$

Modular models don't offer deep mathematical revelations but nevertheless provide a clear picture of what justifications are and how they relate to the world.

2. Basic Justification Logic

In Justification Logic, there is, in addition to the category of formulas, a category of *justifications* with a new sort of proposition $t : F$ stating t *is a justification of* F. In the basic setting, justifications are terms with operations *application* and *sum*.[3]

The *application* operation takes justifications s and t and produces a justification $s \cdot t$ such that if $s{:}(F \rightarrow G)$ and $t{:}F$, then $[s \cdot t]{:}G$. Symbolically,

$$s{:}(F \rightarrow G) \rightarrow (t{:}F \rightarrow [s \cdot t]{:}G).$$

This is a fundamental and widely assumed deductive property of justifications.

The second basic operation on justifications is *sum* '+.' If $s : F$, then whatever evidence t may be, the combined evidence $s + t$, as well as $t + s$, remains a justification for F. Operation '+,' given s and t, produces $s + t$, which is a justification for everything justified by s or by t

$$s{:}F \rightarrow [s + t]{:}F \quad \text{and} \quad s{:}F \rightarrow [t + s]{:}F.$$

As motivation, one might think of s and t as two volumes of a two-volume set, and $s + t$ as the set of those two volumes. Imagine that one of the volumes, say s, contains a sufficient justification for a proposition F, i.e., $s{:}F$ is the case. Then the larger set $s + t$ also contains a sufficient justification for F, $[s + t]{:}F$.

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[3] More elaborate justification logics allow additional operations on justifications.

In a more formal setting, justification terms, *Tm*, are built from justification variables and constants by means of the operations '$\cdot$' and '$+$.' Formulas, *Fm*, are built from propositional variables *Var* and truth constants by the usual Boolean connectives and the rule: if t is a term and F a formula, then $t{:}F$ is a formula.

Basic Logic of Justifications J_0:

Classical propositional axioms and the rule Modus Ponens;

$s{:}(F \rightarrow G) \rightarrow (t{:}F \rightarrow [s\cdot t]{:}G)$;

$s{:}F \rightarrow [s+t]{:}F, \quad s{:}F \rightarrow [t+s]{:}F$.

J_0 is the logic of general (not necessarily factive) justifications for a skeptical agent for whom no formula is justified *a priori*. Justification Logic offers a flexible additional mechanism of representing justified assumptions. When we want to assume that an axiom A is justified, we postulate $c_1{:}A$ for some justification constant c_1. Furthermore, if we want to assume that this new principle $c_1{:}A$ is also justified, we can postulate $c_2{:}(c_1{:}A)$ for a constant c_2, etc. The set of all assumptions of this kind for a given logic is called a *constant specification* (cf. [4] for formal definitions).

Let *CS* be a constant specification. J_{CS} is the logic

$$\mathsf{J}_0 + CS;$$

which axioms are those of J_0 with the members of *CS*, and the only rule of inference is *Modus Ponens*. J is defined as the logic with the union of all constant specifications.

For sample applications of Justification Logic in epistemology, cf. [4].

3. Basic modular models – Mkrtychev models

For sets of formulas X and Y, we define

$$X\cdot Y = \{F \mid G \rightarrow F \in X \text{ and } G \in Y \text{ for some } G\}.$$

Informally, $X\cdot Y$ is the result of applying *Modus Ponens* once to all members of X and of Y (in a given order).

Definition 1. A basic modular model is an evaluation $*$ which maps propositional variables Var to truth values $\{0,1\}$ and justification terms Tm to subsets of the set of formulas

$$* : Var \mapsto \{0,1\} \quad \text{and} \quad * : Tm \mapsto 2^{Fm}$$

such that

$$s^*\cdot t^* \subseteq (s\cdot t)^* \quad \text{and} \quad s^* \cup t^* \subseteq (s+t)^*. \tag{2}$$

As usual, we will write '$\Vdash F$' instead of '$F^* = 1$.' The truth value of formulas is defined inductively and respects Boolean logic, i.e.,

- $\Vdash F \wedge G$ iff $\Vdash F$ and $\Vdash G$;
- $\Vdash \neg F$ iff $\not\Vdash F$;
- $\Vdash t{:}F$ iff $F \in t^*$.

Mathematically, basic modular models are equivalent to the appropriate adaptation of Mkrtychev models[4] for J from [4]. Soundness and completeness of J with respect to basic modular models follow from [4], Theorem 5.2. (where basic modular models were referred to as Mkrtychev models), but we provide a direct proof of them here for the reader's convenience.

Let CS be a constant specification. *A model respects CS* if all formulas from CS hold in this model.

Theorem 1. *$\mathsf{J}_{CS} \vdash F$ iff F holds in any basic modular model respecting CS.*

Proof. Soundness is straightforward. Consider a basic modular model $*$ and run an induction on derivations in J_{CS}. Formulas from CS as well as Boolean axioms are obviously true. *Application*: suppose $\Vdash s{:}(F \to G)$ and $\Vdash t{:}F$. Then $F \to G \in s^*$ and $F \in t^*$. Therefore, $G \in s^*\cdot t^* \subseteq (s\cdot t)^*$, i.e., $\Vdash [s\cdot t]{:}G$. *Sum*: suppose $\Vdash s{:}F$. Then $F \in s^*$ and $F \in s^* \cup t^* \subseteq (s+t)^*$, hence $\Vdash [s+t]{:}F$. The rule of J_{CS} is *Modus Ponens*, respected by the semantics.

Completeness is established by a maximal consistent set construction. Once $\mathsf{J}_{CS} \not\vdash F$, the set $\{\neg F\}$ is consistent, and let Γ be its maximal consistent extension. Define interpretation $*$ such that for an atomic formula p and justification term t,

$$p^* = 1 \quad \text{iff} \quad p \in \Gamma; \qquad t^* = \{F \mid t{:}F \in \Gamma\}.$$

Conditions (2) follow immediately from the 'application' and 'sum' axioms.

By induction on formulas, we now establish the 'truth lemma': For each formula X,

$$\Vdash X \quad \text{iff} \quad X \in \Gamma.$$

The case of atomic X is covered by the definition of $*$; the Boolean cases are standard. Let X be $t{:}Y$ for some t and Y. By the definition of a basic modular model, $\Vdash t{:}Y$ iff $Y \in t^*$. Further, $Y \in t^*$ iff $t{:}Y \in \Gamma$ by the definition of this particular model. Hence $\Vdash t{:}Y$ iff $t{:}Y \in \Gamma$.

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[4] The original Mkrtychev models were introduced in [14] for the Logic of Proofs LP.

To complete the proof of Theorem 1, note that $F \notin \Gamma$, hence, by the truth lemma, $\not\Vdash F$. ■

It is unavoidable that evaluation sets for compound justifications are allowed to contain more formulas than required by the evaluation sets of the components, i.e., the inclusions '$\subseteq$' in (2) cannot be replaced by equalities. Consider the formula

$$F = [x+y]{:}P \rightarrow (x{:}P \vee y{:}P).$$

We observe that $\mathsf{J}_0 \nvdash F$. Indeed, a basic modular countermodel is provided by an evaluation $*$ such that $x^* = y^* = \emptyset$ and $t^* = Fm$ for all other justification terms t.[5] All necessary properties of $*$ obviously hold, so $*$ specifies a basic modular model. In this model, $\Vdash [x+y]{:}P$, but neither $\Vdash x{:}P$ nor $\Vdash y{:}P$, hence $\not\Vdash F$ and $\mathsf{J}_0 \nvdash F$. We show that F cannot be false in any modular model with $[x+y]^* = x^* \cup y^*$. Indeed, in such a model, $\Vdash [x+y]{:}P$, $\not\Vdash x{:}P$, and $\not\Vdash y{:}P$. By the definition of a basic modular model, $P \in [x+y]^*$, but $P \notin x^*$ and $P \notin y^*$. Therefore, $P \notin x^* \cup y^*$, hence $[x+y]^* \neq x^* \cup y^*$.

4. Introducing possible worlds

The main idea of introducing possible world semantics is, of course, to connect justification logic to mainstream epistemic logic which relies heavily on possible worlds models. The standard semantics of

F is believed at world u

is

F holds at all worlds considered possible at u.

How do justifications fit into this picture?

Take a Kripke frame (W, R), where W is a non-empty set of possible worlds, R is a binary 'accessibility' relation on W, and consider an interpretation $*$, which is a mapping of the format

$$* : \mathit{Var} \mapsto 2^W, \qquad * : W \times \mathit{Tm} \mapsto 2^{Fm}, \tag{3}$$

such that for each world u, it specifies a basic modular model $*_u$. To be precise,

$$p^*_u = 1 \ \text{ iff } \ u \in p^*, \qquad t^*_u = *(u,t),$$

and closure conditions (2) hold for each $*_u$.

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[5] Evaluations of propositional variables are irrelevant.

By Definition 1 and (3), such $*$ determines the truth value of each formula at each world. Technically,

$$\mathcal{M} = (W, R, *)$$

is already a possible worlds model for J_0: both soundness and completeness hold for these structures. However, such a model misses the goal of connecting justifications to the knowledge/belief semantics since evaluation $*$ may have nothing to do with the epistemic structure of the model represented by R. What we need here is a conceptually clean mathematical connection of $*$ and R reflecting the epistemic nature of justifications. A reasonable candidate for such a connection is a principle that

having a specific reason for F yields believing that F.

This principle has been the cornerstone of the Fitting semantics of justifications (cf. [8]); it has also been widely adopted in logical systems with explicit and implicit knowledge, cf. [3, 5].

Let us formulate a semantical condition that represents this principle in the modular model format. Given $\mathcal{M} = (W, R, *)$, let $\Box_u$ denote a set of formulas

$$\{F \mid v \Vdash F \textit{ for all } v \textit{ such that } uRv\}.$$

Conceptually, at a given world u, t^*_u reflects 'believing for a reason t,' whereas $\Box_u$ represents believing without providing a specific reason.

We say that *justification yields belief* in $\mathcal{M} = (W, R, *)$, if

$$t^*_u \subseteq \Box_u$$

for each justification term t and each $u \in W$. In other words, if t is a justification for F at u, then F is believed in u.

We now define modular models in the possible worlds setting.

Definition 2. A modular model is $\mathcal{M} = (W, R, *)$ in which

i) W is a non-empty set of worlds, and R is a binary relation of W;

ii) interpretation $*$ has the format

$$* : \mathit{Var} \mapsto 2^W; \quad * : W \times \mathit{Tm} \mapsto 2^{\mathit{Fm}}$$

and is a basic modular model at each world $u \in W$;

iii) justification yields belief, i.e., $t^*_u \subseteq \Box_u$ for each $t \in \mathit{Tm}$ and $u \in W$.

The truth values of formulas are determined by the basic modular model structure at each world, according to Definition 1.

The soundness and completeness theorem holds for modular models.[6]

Theorem 2. *$\mathsf{J}_{CS} \vdash F$ iff F holds in any modular model respecting CS.*

Proof. Soundness follows from Theorem 1 since each of the worlds is a basic modular model. Completeness is established by a maximal consistent set construction. Let W be the set of maximal consistent sets over J_{CS} and

$$\Gamma R \Delta \quad \text{iff} \quad \Gamma^\sharp \subseteq \Delta,$$

where $\Gamma^\sharp = \{F \mid t{:}F \in \Gamma \text{ for some } t\}$. Propositional variables and justifications are evaluated as usual for canonical models, namely,

$$\Gamma \in p^* \quad \text{iff} \quad p \in \Gamma, \qquad t^*_\Gamma = \{F \mid t{:}F \in \Gamma\},$$

which defines an interpretation $*$. Inclusions $s^*_\Gamma \cdot t^*_\Gamma \subseteq (s \cdot t)^*_\Gamma$ and $s^*_\Gamma \cup t^*_\Gamma \subseteq (s+t)^*_\Gamma$ are immediate. Therefore, each world Γ is a basic modular model from the proof of Theorem 1, hence $\Gamma \Vdash X$ iff $X \in \Gamma$ and there is no need to re-prove the truth lemma.

Let us check the 'justification yields belief' condition $t^*_\Gamma \subseteq \Box_\Gamma$. Suppose $F \in t^*_\Gamma$. By definitions, $t{:}F \in \Gamma$ and $F \in \Gamma^\sharp$. By the definition of R, if $\Gamma R \Delta$, then $F \in \Delta$. By the truth lemma, $\Delta \Vdash F$, hence $F \in \Box_\Gamma$.

To complete the proof of Theorem 2, consider F which is not derivable in J_{CS}. Then $\{\neg F\}$ is a consistent set, and let Γ be its maximal consistent extension, hence $\Gamma \in W$. Since Γ is consistent, $F \notin \Gamma$, hence $\Gamma \not\Vdash F$. ■

Note that basic modular models correspond to modular models with a single possible world and empty accessibility relation: $W = \{w\}$, $R = \emptyset$.

5. Modular models for justifications and beliefs

Modular semantics[7] allows us to model justification and beliefs simultaneously. Consider a logic of justifications and beliefs, KJ_0, in the joint language of J and K, containing

1. *modal logic* K *with principles*

   $\Box(F \to G) \to (\Box F \to \Box G)$,

   if $\vdash F$ *then* $\vdash \Box F$;

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[6] Technically, Theorem 2 follows easily from Theorem 3, Section 5, though the proofs of these theorems may produce different canonical models.

[7] as well as Fitting semantics

2. *axioms and rules for* J_0;
3. *the connection axiom*

$$t{:}F \rightarrow \Box F$$

(stating syntactically that justification yields belief).

As before, CS denotes a constant specification and $\mathsf{KJ}_{\mathit{CS}}$ a logic

$$\mathsf{KJ}_0 + \mathit{CS}.$$

Modular semantics seamlessly extends to this case. Formally, given a modular model $\mathcal{M} = (W, R, *)$, in addition to Definition 2, we assume the standard Kripkean clause:

$$u \Vdash \Box F \quad \textit{iff} \quad v \Vdash F \textit{ for all } v \textit{ such that } uRv.$$

The following soundness and completeness theorem with respect to modular models is now an easy exercise.

THEOREM 3. $\mathsf{KJ}_{\mathit{CS}} \vdash F$ *iff* F *holds in any modular model respecting* CS.

PROOF. Soundness of J-axioms follows from Theorem 2, soundness of K-axioms is straightforward, soundness of connection axiom 3. is secured by the 'justification yields belief' condition in modular models.

Completeness is established by a maximal consistent set construction. Let W be the set of maximal consistent sets over $\mathsf{KJ}_{\mathit{CS}}$ and

$$\Gamma R \Delta \quad \text{iff} \quad \Gamma^{\Box} \subseteq \Delta,$$

where $\Gamma^{\Box} = \{F \mid \Box F \in \Gamma\}$. Propositional variables and justifications are evaluated as usual:

$$\Gamma \in p^* \text{ iff } p \in \Gamma; \qquad t^*_{\Gamma} = \{F \mid t{:}F \in \Gamma\}.$$

Inclusions $s^*_{\Gamma} \cdot t^*_{\Gamma} \subseteq (s \cdot t)^*_{\Gamma}$ and $s^*_{\Gamma} \cup t^*_{\Gamma} \subseteq (s + t)^*_{\Gamma}$ are immediate.

The usual 'truth lemma' holds: $X \in \Gamma$ iff $\Gamma \Vdash X$. This is proved by induction on X. The propositional and Boolean cases are straightforward; the case $X = t{:}Y$ does not use R and is similar to that in the proof of Theorem 1. The case $X = \Box Y$ is similar to the standard modal proof since the accessibility relation R is defined in a canonical way.

To check the 'justification yields belief' condition $t^*_{\Gamma} \subseteq \Box_{\Gamma}$, suppose $F \in t^*_{\Gamma}$. By definition, $t{:}F \in \Gamma$. Since Γ is maximal consistent and contains connection axiom 3, $\Box F \in \Gamma$ as well. By the definition of R, for any Δ such that $\Gamma R \Delta$, $F \in \Delta$. By the truth lemma, $\Delta \Vdash F$, hence $F \in \Box_{\Gamma}$.

The claim of Theorem 3 now follows in the standard way. ■

6. Connections to Fitting models

A Fitting model for J_0 (cf. [4]) is a Kripke model $(W, R, \Vdash)$ for K enriched with an *admissible evidence function* $\mathcal{E}$: informally, $\mathcal{E}(t, F)$ specifies the set of possible worlds from W where t is considered an 'admissible,' but not necessarily actual, evidence for F. Formally, $\mathcal{E}(t, F) \subseteq W$ and $\mathcal{E}$ must satisfy the closure conditions with respect to operations '$\cdot$' and '$+$':

- $\mathcal{E}(s, F\to G) \cap \mathcal{E}(t, F) \subseteq \mathcal{E}(s\cdot t, G)$;
- $\mathcal{E}(s, F) \cup \mathcal{E}(t, F) \subseteq \mathcal{E}(s+t, F)$.

A justification assertion $t{:}F$ is true at u if and only if two conditions hold:

1. $v \Vdash F$ for all v such that uRv;
2. $u \in \mathcal{E}(t, F)$.

First, we note that each modular model $\mathcal{M} = (W, R, *)$ is a legitimate Fitting model in which there are no 'fake' justifications. Indeed, '$\Vdash$' can be defined as usual

$$u \Vdash p \quad \text{iff} \quad u \in p^*,$$

and $\mathcal{E}(t, F)$ as the set of worlds

$$\{u \in W \mid F \in t^*_u\}.$$

Obviously, closure conditions on $\mathcal{E}$ follow from (2). Truth condition 1. is vacuously subsumed by condition 2. since justification yields belief in $\mathcal{M}$.

So modular models can be identified with a subclass of Fitting models without 'fake' justifications: in such models, truth condition 1. becomes redundant.

However, the connection between Fitting models and modular models is more direct: each Fitting model encodes a modular model over the same frame and with the same truth evaluation of formulas at each node which we will now explain.

A Fitting model $\mathcal{M} = (W, R, \mathcal{E}, \Vdash)$ can be converted into a modular model $\mathcal{M}' = (W, R, *)$ by defining the evaluation of justifications as

$$t^*_u := \{F \mid u \in \mathcal{E}(t, F)\} \ \cap \ \Box_u$$

and leaving it as-is for propositional variables:

$$u \in p^* \quad \text{iff} \quad u \Vdash p.$$

Let '$\Vdash'$' be the forcing relation in $\mathcal{M}'$, i.e., $u \Vdash' p$ is $u \in p^*$, and $u \Vdash' t{:}F$ is $F \in t_u^*$.

We first note that '$\Vdash$' and '$\Vdash'$' coincide at each node:

$$u \Vdash' X \quad \text{iff} \quad u \Vdash X$$

for each node $u \in W$ and formula X. Induction on X. The base case holds since both $u \Vdash p$ and $u \Vdash' p$ are equivalent to $u \in p^*$. Let us check justification assertions: $X = t{:}Y$. By definitions, $u \Vdash t{:}Y$ iff '$u \in \mathcal{E}(t, Y)$ and $Y \in \Box_u$' iff $Y \in t_u^*$ iff $u \Vdash' t{:}Y$. Boolean steps are trivial.

We now check that $\mathcal{M}'$ is a legitimate modular model, namely *justification yields belief* in $\mathcal{M}'$, i.e., $t_u^* \subseteq \Box_u$. Let $F \in t_u^*$, then $u \Vdash' t{:}F$, hence $u \Vdash t{:}F$ as well. By 1., $v \Vdash F$ for all v such that uRv and hence $v \Vdash' F$ for all v such that uRv, hence $F \in \Box_u$ in $\mathcal{M}'$.

This observation shows that each Fitting model conceals an induced modular model that we call an '*induced modular model.*' A Fitting model's induced modular model has the same truth values of formulas at each node. In this respect, each Fitting model may be regarded as a pre-model for its induced modular model.

In addition to the basic categories of propositions and justifications, Fitting models rely on a conceptually new category – admissible justifications – and a two-stage truth definition 1.–2. that also requires explanation.

Modular models do not introduce auxiliary notions. They extend evaluation from the usual 'formulas are interpreted as sets of possible worlds' to include 'and justifications are interpreted as sets of formulas.' Once this is assumed, the semantics (1) for justification assertions suggests itself.

There is another rather subtle foundational reason for considering modular models: they treat justifications independently of beliefs. In Fitting models, for a final verdict of whether a justification assertion $t{:}F$ holds at a given world, one has to check that justification t obeys the given belief condition for F defined via the accessibility relation R. This makes the belief structure, i.e., relation R, appear to be the principal element of the model and justifications look like derivatives. This does not mesh well with Justification Logic's aim to provide a new, evidence-based semantics for beliefs (cf. [4]): in this light, justifications should precede beliefs, not *vice versa.* It appears modular models achieve this: one does not need a belief structure R to find truth values of justification assertions.

Fitting models have their advantages. For example, in a given structure, the 'justification yields belief' principle might not be easy to verify. Instead, it can be more practical to consider an appropriate Fitting model with truth conditions 1.–2. However, it may be good to know that this amounts to working in an induced modular model.

We conclude with two examples of Fitting models and their induced modular models.

EXAMPLE 1. Consider Fitting model $\mathcal{M}_1$ with

- $W = \{a, b\}$;
- $R = \{(a, b)\}$;
- $a, b \not\Vdash p$ and $a, b \Vdash q$ for all other q's;
- $\mathcal{E}(x, p) = \{a\}$ and $\mathcal{E}(t, F) = \emptyset$ for all other t, F.

Obviously, in $\mathcal{M}_1$ all justification assertions are false. In particular, x is a 'fake' justification for p at a since p is not believed at a. The corresponding induced modular model $\mathcal{M}'_1$ eliminates such justifications by maintaining $t^*_u = \emptyset$ for each t and $u \in W$.

In Example 1, moving from a Fitting model $\mathcal{M}_1$ to its induced modular model $\mathcal{M}'_1$ simplifies matters and eliminates some redundancies.

EXAMPLE 2. Consider Fitting model $\mathcal{M}_2$ with

- $W = \{a, b\}$;
- $R = \{(a, b)\}$;
- $a, b \not\Vdash p$ and $a, b \Vdash q$ for all other q's;
- $\mathcal{E}(t, F) = \{a, b\}$ for all t, F, i.e., each term is an admissible justification for each formula.

In the induced modular model $\mathcal{M}'_2$, for each justification term t, t^*_b is the set of all formulas, and t^*_a is the set of all formulas that are true at node b of model $\mathcal{M}_2$. In particular, $p \notin t^*_a$, but $q \in t^*_a$ for each q distinct from p.

In Example 2, the original Fitting model $\mathcal{M}_2$ appears simpler than its induced modular model $\mathcal{M}'_2$. Moreover, the easiest way to define the latter is to first invoke the former.

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