Hiroakira Ono

Crawley Completions of Residuated Lattices and Algebraic Completeness of Substructural Predicate Logics

In memoriam Leo Esakia

Abstract. This paper discusses Crawley completions of residuated lattices. While MacNeille completions have been studied recently in relation to logic, *Crawley completions* (i.e. complete ideal completions), which are another kind of regular completions, have not been discussed much in this relation while many important algebraic works on Crawley completions had been done until the end of the 70's.

In this paper, basic algebraic properties of ideal completions and Crawley completions of residuated lattices are studied first in their conncetion with the join infinite distributivity and Heyting implication. Then some results on algebraic completeness and conservativity of Heyting implication in substructural predicate logics are obtained as their consequences.

Keywords: regular completions, Crawley completions, infinite distributivities, residuated lattices, algebraic completeness, substructural predicate logics.

1. Preliminaries

Algebraic methods have been applied successfully to nonclassical propositional logics via universal algebra and algebraic logic. Algebraization is one of the most important key concepts which ensure what algebraic methods can do in the study of logics. Many interesting connections between algebra and logic have been discovered recently. As a continuation of our previous works [20] and [22] of the study of substructural predicate logics, we will discuss in this paper to what extent algebraic methods can work well for substructural predicate logics and where the liminations of these methods will be, in particular by focusing on Crawley completions of residuated lattices.

To begin with, we will consider superintuitionistic logics. A *superintuitionistic propositional logic* (SIL) (*superintuitionistic predicate logic* (QSIL)) is an axiomatic extension of intuitionistic propositional logic (intuitionistic predicate logic, respectively). Algebraic semantics for superintuitionistic

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propositional logics is given by classes of Heyting algebras. A *SIL* $\mathbf{L}$ is *complete* with respect to a class $\mathcal{C}$ of Heyting algebras, when for each formula φ, $\varphi \in \mathbf{L}$ iff $\mathbf{A} \models f(\varphi) = 1$ for every $\mathbf{A} \in \mathcal{C}$ and every valuation f on $\mathbf{A}$. By a standard argument using Lindenbaum algebras, it can be shown that every *SIL* is complete with respect to a class of Heyting algebras. In fact, it is shown that the class of Heyting algebras in which all formulas in a given SIL $\mathbf{L}$ forms a subvariety of the variety $\mathcal{H}$ of all Heyting algebras, and moreover these exists an inverse lattice isomorphism between the lattice of all SILs and the lattice of all subvarieties of the variety $\mathcal{H}$.

To define an algebraic semantics for superintuitionistic predicate logics, there will be some alternatives on how to interpret quantifiers. Following ideas proposed by A. Mostowski, H. Rasiowa and R. Sikorski, we consider here an algebraic semantics in which universal and existential quantifiers are interpreted by infinite meets and infinite joins, respectively. (We assume here that we will consider the pure first-order language, i.e. our language contains neither individual constant symbols nor function symbols.)

Tentatively we call here a structure $\langle \mathbf{A}, V\rangle$, an algebraic frame for a QSIL, if $\mathbf{A}$ is a Heyting algebra and V a non-empty set, called the *individual domain*. (A formal definition of algebraic frames is given in the next section.) Each valuation f is extended to first-order formulas by requiring the following. (Here, we identify each element in V with its name, just for the simplicity's sake.)

1. $f(\forall x\varphi(x)) = \bigwedge\{f(\varphi(w)) : w \in V\}$
2. $f(\exists x\varphi(x)) = \bigvee\{f(\varphi(w)) : w \in V\}$

But, this definition may be ambiguous, as it is not always the case that infinite meets and infinite joins exist in a given Heyting algebra. In such a case, we cannot define values of a given evaluation.

A possible way out of this is to take an arbitrary Heyting algebra but to consider only *safe* valuations. Here, a valuation f is safe for a given set Φ of closed formulas (or, sentences)[1], if for any $\alpha \in \Phi$, all infinite joins and infinite meets appearing in the calculation of $f(\alpha)$ exist always. If we choose this way, then by using the Lindenbaum algebra, we can show that every QSIL is complete with respect to a class of algebraic frames restricted only to safe valuations. But Lindenbaum algebras will not give us much information. From an algebraic point of view, Lindembaum algebras for a propositional

[1]More precisely, here by closed formulas we mean closed formulas in the language augmented by constant symbols corresponding to elements of V.

logic are *free* algebras of the corresponding variety, and thus they have the *universal mapping property.* On the other hand, Lindembaum algebras for a predicate logics do not have such a nice property.

Taking a different way, here we will restrict our attention only to *complete* Heyting algebras. As every valuation on a complete Heyting algebra becomes always safe in this case, we can take arbitrary valuations. Now let $\mathcal{W}$ be a class of algebraic frames composed of complete Heyting algebras. Then the algebraic completeness of a given QSIL $\mathbf{L}$ with respect to $\mathcal{W}$ means that for each first-order closed formula φ, φ is provable in $\mathbf{L}$ iff $f(\varphi)$ takes the value 1 for every algebraic frame in $\mathcal{W}$ and every valuation f on it. On the other hand, since Lindenbaum algebras are not complete algebras, we cannot use them directly to show the algebraic completeness. So it becomes necessary to find complete Heyting algebras which can act in the place of Lindenbaum algebras. For this purpose, it is necessary to consider ways of *completing* algebras without losing necessary information which original algebras have, in particular information on infinite joins and meets.

In this paper we assume familiarity with basic notions and results of residuated lattices and substructural propositional logics (see e.g. [11]). After giving a basic framework of the algebraic framework of algebraic completeness of substructural predicate logics in Section 2, regular completions of residuated lattices are introduced in Section 3. Ideal and Crawley completions and Heyting implication are discussed in the succeeding sections. Their applications to algebraic completeness and conservativity of Heyting implication in substructural predicate logics are shown in Section 7.

2. Algebraic frames of substructural predicate logics

Hereafter, we will focus mainly on algebraic completeness of *substructural predicate logics* and completions of algebras related to these logics. In the following, we will discuss only substructural logics with exchange rule just for the simplicity's sake, though in most cases it is not hard to extend arguments to the noncommutative case. As for general information on substructural propositional logics and residuated lattices, see [11]. Let $\mathbf{QFL_e}$ be the sequent system for predicate language, obtained from the sequent system for the substructural propositional logic $\mathbf{FL_e}$ by adding the following rules for quantifiers $\forall$ and $\exists$.

$$\frac{\alpha(t), \Gamma \Rightarrow \varphi}{\forall x\alpha, \Gamma \Rightarrow \varphi}\ (\forall \Rightarrow) \qquad\qquad \frac{\Gamma \Rightarrow \alpha(y)}{\Gamma \Rightarrow \forall x\alpha}\ (\Rightarrow \forall)$$

$$\frac{\alpha(y), \Gamma \Rightarrow \varphi}{\exists x\alpha, \Gamma \Rightarrow \varphi}\ (\exists \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha(t)}{\Gamma \Rightarrow \exists x\alpha}\ (\Rightarrow \exists)$$

Here, $\alpha(y)$ and $\alpha(t)$ denote formulas obtained from the formula α by substituting the variable y and the term t, respectively, for every free occurrence of the variable x in α . In applying each of these rules, t is an arbitrary term while y is a variable which does not appear freely in the lower sequent.

As usual, we identify a formal system with the logic determined by it. The logic $\mathbf{QFL_e}$ is sometimes called intuitionistic linear predicate logic without exponentials. A *substructural predicate logic* (QSL) $\mathbf{L}$ over $\mathbf{QFL_e}$ (or, a commutative QSL) is any axiomatic extension of $\mathbf{QFL_e}$. As usual, sometimes a logic $\mathbf{L}$ is identified with a set of formulas $F_{\mathbf{L}}$ satisfying the following conditions:

1. $F_{\mathbf{L}}$ contains all formulas which are provable in $\mathbf{QFL_e}$,
2. $F_{\mathbf{L}}$ is closed under *modus ponens* and *rule of adjunction*, i.e. $\alpha, \beta \in \mathbf{L}$ implies $\alpha \wedge \beta \in \mathbf{L}$,
3. $F_{\mathbf{L}}$ is closed under *rule of generalization*,
4. $F_{\mathbf{L}}$ is closed under substitution.

It is easy to see that for each substructural propositional logic $\mathbf{K}$ there exists the minimum QSL $\mathbf{L}$ among QSLs that are extensions of $\mathbf{K}$. In fact, this $\mathbf{L}$ is obtained from $\mathbf{QFL_e}$ by adding all formulas provable in $\mathbf{K}$ as axioms. We call this $\mathbf{L}$ the *minimum predicate extension* of $\mathbf{K}$. It is clear that $\mathbf{QFL_e}$ is the minimum predicate extension of $\mathbf{FL_e}$. Similarly, we can introduce $\mathbf{QFL_{ew}}$ and $\mathbf{QFL_{ec}}$ etc. as minimum predicate extensions of $\mathbf{FL_{ew}}$ and $\mathbf{FL_{ec}}$, respectively.

An algebra $\mathbf{A} = \langle A, \vee, \wedge, \cdot, \rightarrow, 1\rangle$ is a *commutative residuated lattice* (CRL), iff it satisfies the following:

1. $\langle A, \vee, \wedge\rangle$ is a lattice,
2. $\langle A, \cdot, 1\rangle$ is a commutative monoid,
3. for all $x, y, z \in A$, $x \cdot y \leq z$ iff $y \leq x \rightarrow z$ (*law of residuation*).

Note that the unit 1 in a residuated lattice is not always the greatest element of A. An $\mathbf{FL_e}$-*algebra* is a CRL with a fixed element 0. Using 0, we can define the *negation* $-$ by $-x = x \rightarrow 0$. *Complete* CRLs (*complete* $\mathbf{FL_e}$-algebras) are CRLs ($\mathbf{FL_e}$-algebras, respectively), in which the join $\bigwedge X$ and the meet $\bigvee X$ exist for any subset X. By using the law of residuation, we can show that in any complete CRL, the equality $\bigvee_i a_i \cdot b = \bigvee_i (a_i \cdot b)$ holds always. An $\mathbf{FL_e}$-algebra is an $\mathbf{FL_{ew}}$-*algebra* if and only if the unit 1 is equal to the greatest element, or equivalently $x \cdot y \leq x$ for all x and y (integrality), and moreover the element 0 is equal to the least element.

It is known that the class $\mathcal{FL}_e$ of all $\mathbf{FL_e}$-algebras forms a variety and every subvariety $\mathcal{W}$ of $\mathcal{FL}_e$ determines uniquely a substructural propositional logic $\mathbf{L}_{\mathcal{W}}$ over $\mathbf{FL_e}$. Conversely, every substructural propositional logic $\mathbf{K}$ over $\mathbf{FL_e}$ determines uniquely a subvariety $\mathcal{V}_{\mathbf{K}}$ of $\mathcal{FL}_e$. Moreover, these two correspondences are mutually inverse, dual lattice isomorphisms. For details, see [11].

We now give a formal definition of algebraic frames. A structure $\langle \mathbf{A}, V \rangle$ is a *pre-algebraic frame* (for QSLs) when $\mathbf{A}$ is an arbitrary $\mathbf{FL_e}$-algebra and V is a non-empty set. A (first-order) formula φ is *true* in a pre-algebraic frame $\langle \mathbf{A}, V \rangle$ under a (safe) valuation f iff $f(\varphi^*) \geq 1$, where φ^* is the universal closure of φ. A pre-algebraic frame $\langle \mathbf{A}, V \rangle$ is an *algebraic frame* when $\mathbf{A}$ is a complete $\mathbf{FL_e}$-algebra.

A formula φ is *valid* in an algebraic frame $\langle \mathbf{A}, V \rangle$ iff φ is true under every valuation f on $\langle \mathbf{A}, V \rangle$. A formula φ is *valid* in a complete $\mathbf{FL_e}$-algebra $\mathbf{A}$, if it is valid in any algebraic frame of the form $\langle \mathbf{A}, V \rangle$ with some V. Note that the validity of two algebraic frames $\langle \mathbf{A}, V \rangle$ and $\langle \mathbf{A}, W \rangle$ is the same as long as $|V| = |W|$ holds, where $|X|$ denotes the cardinality of a set X. The following proposition, a Löwenheim-type theorem, shown firstly for QSILs in [18], can be easily extended to QSLs. Though we assume that our language is countable, this proposition can be easily generalized for the case where the language is of an arbitrary infinite cardinality.

PROPOSITION 2.1. Suppose that $\langle \mathbf{A}, V \rangle$ is an algebraic frame such that $|A| < |V|$. Then there exists a set W with $|W| \leq \max\{|A|, \aleph_0\}$ such that for each formula φ, φ is valid in $\langle \mathbf{A}, V \rangle$ iff φ is valid in $\langle \mathbf{A}, W \rangle$.

COROLLARY 2.2. For each complete $\mathbf{FL_e}$-algebra $\mathbf{A}$ and for each set V with the cardinality $\max\{|A|, \aleph_0\}$, a formula φ is valid in $\mathbf{A}$ iff φ is valid in the algebraic frame $\langle \mathbf{A}, V \rangle$.

PROOF. Note that the only-if part holds always by the definition. Let us take an arbitrary set V whose cardinality is equal to $\max\{|A|, \aleph_0\}$. Suppose that φ is not valid in $\mathbf{A}$. Then there is an algebraic frame of the form $\langle \mathbf{A}, U \rangle$ in which φ is not valid, where the individual domain U may be taken depending on a given φ. By Proposition 2.1, we may suppose that $|U| \leq |V|$. Then, by taking an arbitrary surjective map from V to U, we can show that φ is not valid in $\langle \mathbf{A}, V \rangle$ either. Note that the set V is chosen independently of the formula φ. ■

A predicate logic $\mathbf{L}$ is *algebraically complete* with respect to a class $\mathcal{K}$ of complete $\mathbf{FL_e}$-algebras, when for each formula φ, φ is provable in $\mathbf{L}$ iff φ is valid in every algebra in $\mathcal{K}$. Our main goal of the present paper is to see how far algebraic methods can work well for the problem of algebraic completeness of substructural predicate logics. As a matter of fact, a substantial limitation in this completeness problem is already known even in the case of superintuitionisitic predicate logics, as the following result in [18] shows.

Proposition 2.3. There exist uncountably many algebraically incomplete superintuitionistic predicate logics.

Some attempts have been made from the middle of the 1980s to introduce various kinds of stronger semantics. For details, see [10]. Though some are general enough to get the completeness of a wide class of predicate logics, these semantics are often too complicated to see clearly what are mathematical consequences of completeness of a given logic, and also to apply them to a given concrete problem.

3. Completions and nuclei

A *completion* of a given CRL ($\mathbf{FL_e}$-algebra) $\mathbf{A}$ is a pair $(\mathbf{C}, h)$ of a complete CRL ($\mathbf{FL_e}$-algebra, respectively) $\mathbf{C}$ and an embedding h from $\mathbf{A}$ to $\mathbf{C}$. Often we omit h and say simply that $\mathbf{C}$ is a completion of an algebra $\mathbf{A}$, whenever the mapping h is clear from the context. An embedding of an algebra $\mathbf{A}$ into another algebra $\mathbf{C}$ is *regular* when all existing *infinite joins and meets* in $\mathbf{A}$ are preserved. A completion $(\mathbf{C}, h)$ of $\mathbf{A}$ is called *regular* when the embedding h is regular. Though MacNeille completions which are discussed below are always regular, canonical extensions which are also well-known completions are never regular.

Regular completions will be useful in showing the algebraic completeness of a given predicate logic $\mathbf{L}$. Let $\mathcal{K}_{\mathbf{L}}$ be the class of all complete $\mathbf{FL_e}$-algebras in which every formula in $\mathbf{L}$ is valid. Also, let $\mathbf{A_L}$ be the Lindenbaum algebra and f be the canonical mapping. The mapping f can be considered as a safe valuation on a prealgebraic frame $\langle \mathbf{A_L}, V\rangle$ for a countable set V. Thus, for each formula ψ not in $\mathbf{L}$, $f(\psi) \not\geq 1$ in $\langle \mathbf{A_L}, V\rangle$. Now suppose that $(\mathbf{C}, h)$ is a regular completion of $\mathbf{A_L}$. Then, the composition $h \circ f$ is a valuation on an algebraic frame $\langle \mathbf{C}, V\rangle$ since h is regular, and moreover $(h \circ f)(\psi) \not\geq 1$ holds in $\langle \mathbf{C}, V\rangle$ for each formula ψ not in $\mathbf{L}$. Thus, every formula ψ not in $\mathbf{L}$ is refuted in a complete algebra $\mathbf{C}$ in $\mathcal{K}_{\mathbf{L}}$. Then the algebraic completeness of the logic $\mathbf{L}$ with respect to the class $\mathcal{K}_{\mathbf{L}}$ follows if $\mathbf{C}$ belongs also to $\mathcal{K}_{\mathbf{L}}$. In particular, we have the following.

PROPOSITION 3.1. Let $\mathbf{K}$ be an axiomatic extension of the substructural propositional logic $\mathbf{FL_e}$, and $\mathcal{V}_\mathbf{K}$ be the corresponding subvariety of the variety $\mathcal{FL}_e$ of all $\mathbf{FL_e}$-algebras. If $\mathcal{V}_\mathbf{K}$ is closed under regular completion, then the minimum predicate extension $\mathbf{K}_*$ of $\mathbf{K}$ is algebraically complete.

A standard way of constructing complete algebras is to use *nuclei* on a given monoid $\mathbf{M}$. Here we give a brief sketch of nuclei (see [11] for the details). Let $\mathbf{M}$ be a monoid. A mapping E on $\wp(\mathbf{M})$ is a *closure operator* on $\wp(\mathbf{M})$ iff for all $X, Y \in \wp(\mathbf{M})$,

- $X \subseteq E(X)$,
- $E(E(X)) \subseteq E(X)$,
- $X \subseteq Y$ implies $E(X) \subseteq E(Y)$.

We define two binary operations $\cdot$ and $\Rightarrow$ on $\wp(\mathbf{M})$ by $U \cdot V = \{u \cdot v : u \in U$ and $v \in V\}$ and $U \Rightarrow V = \{z : z \cdot u \in V$ for all $u \in U\}$ for all subsets U, V of M. Clearly, the following relation holds for all subsets U, V, W of M:

$U \cdot V \subseteq W$ if and only if $U \subseteq V \Rightarrow W$.

A closure operator E on $\wp(\mathbf{M})$ is a *nucleus* if it satisfies

- $E(X) \cdot E(Y) \subseteq E(X \cdot Y)$.

For a given nucleus E on $\wp(\mathbf{M})$ let $\wp(M)_E$ be the set of all E-*closed* subsets X of M, i.e. subsets X for which $E(X) = X$ holds. Define an algebra $\wp(\mathbf{M})_E = \langle \wp(M)_E, \vee_E, \wedge_E, \circ_E, \Rightarrow_E, E(\{1\})\rangle$, where operations $\vee_E, \wedge_E, \circ_E$ and $\Rightarrow_E$ are defined as follows; for all $X, Y \in \wp(M)_E$

1. $X \vee_E Y = E(X \cup Y)$,
2. $X \wedge_E Y = X \cap Y$,
3. $X \circ_E Y = E(X \cdot Y)$,
4. $X \Rightarrow_E Y = X \Rightarrow Y$.

It can be shown that $X \Rightarrow Y$ is E-closed when Y is E-closed. The algebra $\wp(\mathbf{M})_E$ is a complete residuated lattice (and a complete $\mathbf{FL_e}$-algebra if we add E($\{0\}$) to it), which is called the *E-retraction* of $\wp(\mathbf{M})$. In particular, the law of residuation $X \circ_E Y \subseteq Z \Leftrightarrow X \subseteq Y \Rightarrow_E Z$ holds for all E-closed subsets X, Y and Z. We omit the subscript E of these operations when it is clear from the context.

Sometimes, we take for $\mathbf{M}$ a partially ordered (p.o.) monoid, a monoid with a partial order such that the monoid operaton $\cdot$ is monotone. For instance, we take the p.o.monoid-reduct $\mathbf{A}^\dagger$ of a given $\mathbf{FL_e}$-algebra $\mathbf{A}$. In such a case, we express the E-retraction of $\wp(\mathbf{A}^\dagger)$ simply as $\mathbf{A}^E$. Though $\mathbf{A}^E$ is complete, we cannot expect in general that $\mathbf{A}$ can be embedded into $\mathbf{A}^E$.

A typical example of regular completions is MacNeille completion, which is defined as follows. Let $\mathbf{M}$ be a p.o. monoid. For each subset X of M, $U(X)$ ($L(X)$) denotes the set of all upper bounds (lower bounds, respectively) of X. Define $M(X) = L(U(X))$. Then M is shown to be a nucleus, if M is moreover residuated. Now let us consider the complete $\mathbf{FL_e}$-algebra $\mathbf{A}^M$ of a given $\mathbf{FL_e}$-algebra $\mathbf{A}$. Define the *canonical mapping* h from $\mathbf{A}$ to $\mathbf{A}^M$ by $h(a) = (a] = \{x : x \leq a\}$. Then h is shown to be a regular embedding. The pair $(\mathbf{A}^M, h)$ is called the *MacNeille completion* (or the *Dedekind-MacNeille completion*) of an algebra $\mathbf{A}$. For general information on MacNeille completions, see e.g. [30]. We say that a class $\mathcal{K}$ of $\mathbf{FL_e}$-algebras is closed under MacNeille completion if the MacNeille completion $\mathbf{A}^M$ of $\mathbf{A}$ belongs to $\mathcal{K}$ whenever $\mathbf{A} \in \mathcal{K}$ Using MacNeille completions of residuated lattices with regular embeddings, we can show the following algebraic completeness of substructural predicate logics.

In [24] Rasiowa proved the algebraic completeness of intuitionistic predicate logic $\mathbf{QIL}$ by using MacNeille completion of Boolean algebras with S4-modality (see also [25]). Algebraic completeness of substructural predicate logics $\mathbf{QFL_{ew}}$, $\mathbf{QFL_{ec}}$ and $\mathbf{QFL_e}$ are shown by using MacNeille completion (see e.g. [19]).

4. Infinite distributivity

In this section we will discuss algebras satisfying the distributive law and its infinite forms, in particular the *join infinite distributivity* (JID) (or, $(\wedge, \bigvee)$-Dis) and the *meet infinite distributivity* (MID) (or, $(\vee, \bigwedge)$-Dis).

(JID) : $\bigvee_i a_i \wedge b = \bigvee_i (a_i \wedge b)$,

(MID) : $\bigwedge_i a_i \vee b = \bigwedge_i (a_i \vee b)$.

Precisely speaking, the (JID) and the (MID) respectively mean as follows, as a given algebra may not be complete.

- (JID) : when the join $\bigvee_i a_i$ exists then the join $\bigvee_i (a_i \wedge b)$ exists and $\bigvee_i a_i \wedge b = \bigvee_i (a_i \wedge b)$ holds,

- (MID) : when the meet $\bigwedge_i a_i$ exists then the meet $\bigwedge_i(a_i \vee b)$ exists and $\bigwedge_i a_i \vee b = \bigwedge_i(a_i \vee b)$.

From a logical point of view, the (JID) and the (MID) correspond to the following axiom schemes:

$(\wedge, \exists)$: $(\exists x\alpha(x) \wedge \beta) \to \exists x(\alpha(x) \wedge \beta)$,
$(\vee, \forall)$: $\forall x(\alpha(x) \vee \beta) \to (\forall x\alpha(x) \vee \beta)$,

respectively. Here the variable x does not have any free occurrences in β. The axiom scheme $(\vee, \forall)$ is known as the axiom scheme of constant domain (CD). In fact, we can show that the (JID) (the (MID)) holds in a given $\mathbf{FL_e}$-algebra if and only if $(\wedge, \exists)$ ($(\vee, \forall)$, respectively) is valid in it.

It is well-known that while usual distributivity is self-dual, each of the (JID) and the (MID) does not imply always the other. For instance, the (JID) holds in any Heyting algebra but there exists a complete Heyting algebra in which the (MID) does not hold. We note that $\bigvee_i a_i \cdot b = \bigvee_i(a_i \cdot b)$ holds in every $\mathbf{FL_e}$-algebra and moreover that $a \cdot b = a \wedge b$ in every Heyting algebra. (See also an example in p.104 of [7].)

Sufficient conditions for $\mathbf{FL_e}$-algebras to satisfy the (JID) are given in the following two lemmas. The following equality is known to be the *divisibility* (div):

(div) : $a \wedge b = a \cdot (a \to b)$.

It is obvious that the (div) holds always in every Heyting algebra since the fusion $\cdot$ is equal to the meet $\wedge$ and $b \leq a \to b$ holds in it. We note that the following weak divisibility (wdiv) follows immediately from the integrality and in fact they are equivalent (in $\mathbf{FL_e}$-algebras). For, taking 1 for a in (wdiv), we have $1 \geq 1 \wedge b \geq 1 \cdot (1 \to b) = b$, which means the integrality.

(wdiv) : $a \wedge b \geq a \cdot (a \to b)$.

Lemma 4.1. The (JID) holds in every $\mathbf{FL_e}$-algebra which validates the (div).

Proof. Suppose that $\bigvee_{i \in I} a_i$ exists. We show that the supremum of the set $\{a_i \wedge b : i \in I\}$ exists and is equal to $\bigvee_{i \in I} a_i \wedge b$. It is clear that $\bigvee_{i \in I} a_i \wedge b$ is an upperbound of $\{a_i \wedge b : i \in I\}$. Let c be any upperbound of $\{a_i \wedge b : i \in I\}$. Since $a_k \leq \bigvee_i a_i$, $\bigvee_{i \in I} a_i \to b \leq a_k \to b$. Hence, $a_k \cdot (\bigvee_{i \in I} a_i \to b) \leq a_k \cdot (a_k \to b) = a_k \wedge b \leq c$ for each k, by using the assumptions that c is an upperbound of $\{a_i \wedge b : i \in I\}$ and that the (div) holds. Thus by the law of residuation, $a_k \leq (\bigvee_{i \in I} a_i \to b) \to c$ for each k. Since $\bigvee_{i \in I} a_i$ exists, $\bigvee_{i \in I} a_i \leq (\bigvee_{i \in I} a_i \to b) \to c$. Hence $\bigvee_{i \in I} a_i \wedge b = \bigvee_{i \in I} a_i \cdot (\bigvee_{i \in I} a_i \to b) \leq c$. Hence, $\bigvee_{i \in I} a_i \wedge b$ is the least upperbound of $\{a_i \wedge b : i \in I\}$. ∎

The (JID) holds in another important class of $\mathbf{FL_e}$-algebras. The following result was suggested by P. Cintula and then was improved by C. Tsinakis. Here the (prelin) means the following equality, which is known to be the *prelinearity*:

$$\text{(prelin)}: \quad ((a \to b) \wedge 1) \vee ((b \to a) \wedge 1) \geq 1.$$

LEMMA 4.2. The (JID) holds in every $\mathbf{FL_e}$-algebra if it satisfies both the (prelin) and the (MID).

PROOF. Again, it is enough to show that $\bigvee_{i\in I} a_i \wedge b$ is the least upperbound of $\{a_i \wedge b : i \in I\}$, assuming the existence of $\bigvee_{i\in I} a_i$. It is shown in Theorem 3.4 of [15] that for all a, b, c, $(a \wedge b) \to c = (a \to c) \vee (b \to c)$ holds in any $\mathbf{FL_e}$-algebra satisfying the (prelin). Also, it can be easily verified that for all c, $\bigwedge_{i\in I}(a_i \to c)$ exists and is equal to $\bigvee_{i\in I} a_i \to c$, by using the commutativity and the law of residuation. Now, let d be any upperbound of $\{a_i \wedge b : i \in I\}$. Then by using the (MID)

$$(\textstyle\bigvee_{i\in I} a_i \wedge b) \to d = (\bigvee_{i\in I} a_i \to d) \vee (b \to d) = \bigwedge_{i\in I}(a_i \to d) \vee (b \to d)$$
$$= \textstyle\bigwedge_{i\in I}((a_i \to d) \vee (b \to d)) = \bigwedge_{i\in I}((a_i \wedge b) \to d) \geq 1.$$

Thus, $\bigvee_{i\in I} a_i \wedge b \leq d$. Therefore, $\bigvee_{i\in I} a_i \wedge b$ is the least upperbound of $\{a_i \wedge b : i \in I\}$. ■

Replacing algebraic arguments in proofs of the above two lemmas on algebras by syntactic ones, we can get the following theorem. We denote here $(\gamma \to \delta) \wedge (\delta \to \gamma)$ as $\gamma \leftrightarrow \delta$. We define axiom schemes of the divisibility (Div) and the prelinearity (Prelin), respectively, as follows.

$$\text{(Div)}: \quad (\alpha \wedge \beta) \leftrightarrow (\alpha \cdot (\alpha \to \beta)).$$
$$\text{(Prelin)}: \quad ((\alpha \to \beta) \wedge 1) \vee ((\beta \to \alpha) \wedge 1).$$

Uninorm logic **UL** is obtained from $\mathbf{FL_e}$ by adding the axiom scheme (Prelin), and the predicate logic **UL**∀ is obtained from the minimum predicate extension of **UL** by adding the axiom scheme $(\vee, \forall)$. The logic obtained from **UL**∀ by adding weakening rules is known to be **MTL**∀, whose algebraic completeness is shown in [17].

THEOREM 4.3. *1. Every instance of both* $(\wedge, \exists)$ *and the distributive law is provable in the substructural predicate logic* $\mathbf{QFL_e}$ *with the additional axiom scheme (Div).*
2. Every instance of $(\wedge, \exists)$ *and the distributive law is provable in the substructural predicate logic* **UL**∀.

5. Ideal completions and Crawley completions

In this section, we will discuss another type of completions called *ideal completions* and *Crawley completions*. The distributive law is preserved under the former and even the infinite distributive law (JID) is preserved under the latter. This is a special feature of these two completions, while MacNeille completions lack this. From algebraic point of view, many works have been already done not only on ideal completions and Crawley completions but also join-completions of *lattices* e.g. in [1, 8, 6, 5, 27] and [28]. Here we concentrate our attention mainly on algebraic properties of these two completions of residuated lattices which will be relevant to their applications to substructural logics. Since they can be obtained similarly to the case of lattices, we omit proofs of them. Note that join-completions of residuated lattices are discussed in [32].

In the following, we will introduce the notion of ideals of a given $\mathbf{FL_e}$-algebra $\mathbf{A}$. A mapping J on $\wp(\mathbf{A})$ is defined so that $J(X)$ denotes the smallest ideal containing X for each subset X of A. Then it is shown that the mapping J is a nucleus on $\wp(\mathbf{A})$ and that $\mathbf{A}$ is embedded into the complete algebra $\mathbf{A}^J$ by the canonical mapping h which is defined by $h(a) = (a]$ for each $a \in A$.

Here we take notice of the following. Suppose that E is an arbitrary nucleus on a p.o. monoid $\mathbf{M}$, and moreover that $(a]$ is E-closed for all a. When $\mathbf{M}$ has the smallest element $\perp$, it is desirable that h is moreover $\perp$-*preserving*, i.e. $h(\perp) = (\perp] = E(\emptyset)$. Note here that $E(\emptyset)$ is the smallest E-closed subset of M. Hence, $\perp \in U$ for every E-closed set U. On the other hand, we can show that if $\mathbf{M}$ does not have the smallest element then $E(\emptyset)$ must be empty. This means that the empty set is E-closed. To see this, suppose that an element c belongs to $E(\emptyset)$. Then $c \leq x$ since $E(\emptyset) \subseteq (x]$ for each x. Thus c is the smallest element in M. But this is a contradiction.

These considerations lead us to the following definition of ideals. Let $\mathbf{M}$ be a join semilattice. A subset X is an *ideal* of $\mathbf{M}$ if and only if X is downward closed and is closed under join, i.e. $a, b \in X$ implies $a \vee b \in X$ for all a, b. Different from the standard definition of ideals, we do not assume that X is nonempty as long as $\mathbf{M}$ does not have the smallest element. On the other hand, when it has the smallest $\perp$, we assume moreover that $\perp \in X$, or equivalently that X is nonempty. In the rest of the paper, we always follow this convention of the definition of ideals. See also the footnote 6 in [27] for some discussions on the definition of ideals.

Now let $\mathbf{A}$ be an $\mathbf{FL_e}$-algebra and $\mathbf{A}^\ddagger$ be its $\{\vee, \cdot, 1\}$-reduct. Note that $\mathbf{A}^\ddagger$ forms a join semilattice-ordered monoid. We consider the set of all ideals of $\mathbf{A}^{\ddagger}$. For each subset X of $A^{\ddagger}$, $J(X)$ denotes the smallest ideal containing X, which exists always and is called the ideal generated by X. In fact, $J(X)$ is expressed as $\{c \in M : c \leq (b_1 \vee \ldots \vee b_m)$ for some m and some $b_1, \ldots, b_m \in X\}$. We can show that J is a closure operator, and moreover a nucleus on $\wp(\mathbf{A}^{\ddagger})$, by using the distributivity of the join $\vee$ over the monoid operation $\cdot$ on $\mathbf{A}$. We express the complete $\mathbf{FL_e}$-algebra $\wp(\mathbf{A}^{\ddagger})_J$ as $\mathbf{A}^J$, and call it the *ideal completion* of the algebra $\mathbf{A}$. Clearly, $J(\{a\}) = (a]$ for each element a.

THEOREM 5.1. *The canonical mapping h defined by $h(a) = (a]$ for each $a \in A$ is an embedding (of residuated lattices) from $\mathbf{A}$ to $\mathbf{A}^J$ which preserves moreover the smallest element and all infinite meets if they exist.*

For more information, see e.g. [21]. Note that the mapping h is not always regular. The distributive law is always preserved by ideal completions (see e.g. [3] p.114 and [26] Theorem 9.32). We can show in fact the following.

THEOREM 5.2. *If an $\mathbf{FL_e}$-algebra $\mathbf{A}$ is distributive then its ideal completion $\mathbf{A}^J$ is join infinite distributive.*

It is easy to see that for the operator M of MacNeille completions, every M-closed subset is an ideal. Therefore, M-closed subsets are sometimes called *normal ideals*. Next, we will introduce another type of ideals. A subset X of an $\mathbf{FL_e}$-algebra $\mathbf{A}$ is a *complete ideal*[2] iff

1. I is downward closed,
2. If $a_j \in I$ for each $j \in S$ and moreover $\bigvee_{j \in S} a_j$ exists, then $\bigvee_{j \in S} a_j \in I$.

Obviously, every complete ideal is an ideal and every normal ideal is a complete ideal. The set A itself is a complete ideal and that the set of all complete ideals of $\mathbf{A}$ is closed under arbitrary intersection, i.e., if I_k is a complete ideal for all k then $\bigcap_k I_k$ is also a complete ideal. Now define a mapping K on $\wp(A)$ by the condition that $K(X)$ is the smallest complete ideal containing X for each subset X of A. Then K is a closure operator. The ideal $K(X)$ is called the complete ideal generated by X. (For more information on complete ideals, see [8, 6] and [5].)

THEOREM 5.3. *The closure operator K is a nucleus. Thus, $\mathbf{A}^K$, i.e. $\wp(\mathbf{A}^{\ddagger})_K$, forms a complete $\mathbf{FL_e}$-algebra. Moreover, the canonical mapping h defined by $h(a) = (a]$ for each $a \in A$ is a regular embedding from $\mathbf{A}$ to $\mathbf{A}^K$.*

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[2] Again, we assume moreover that $\bot \in X$ in the definition when $\mathbf{A}$ has the smallest $\bot$.

PROOF. To show that $K(X) \cdot K(Y) \subseteq K(X \cdot Y)$ for subsets X and Y of A, it is enough to prove that $X \Rightarrow Y$ is K-closed whenever Y is K-closed (see e.g. Lemma 3.33 in [11]). It is easy to verify that $X \Rightarrow Y$ is downward closed when Y is so. Suppose that $\bigvee_i z_i$ exists where $z_i \in X \Rightarrow Y$ for each $i \in S$. Take an arbitrary element $x \in X$. Then $z_i \cdot x \in Y$. The least upper bound $\bigvee_i (z_i \cdot x)$ of $\{z_i \cdot x : i \in S\}$ is given by $(\bigvee_i z_i) \cdot x$, and belongs to Y since Y is K-closed. It means that $(\bigvee_i z_i) \cdot x \in Y$ for every $x \in X$. Thus, $\bigvee_i z_i \in X \Rightarrow Y$. Hence, $X \Rightarrow Y$ is K-closed. The remaining statements can be verified easily. Note that the above argument tells us that K on $\wp(A)$ is a nucleus whenever $\mathbf{A}$ is a p.o. monoid satisfying $\bigvee_i (y \cdot z_i \cdot x) = y \cdot (\bigvee_i z_i) \cdot x$ for every y, x and every existing $\bigvee_i z_i$. ■

See also §2 of [21]. Clearly, $\mathbf{A}^K$ is isomorphic to $\mathbf{A}$ when $\mathbf{A}$ is already complete. The complete $\mathbf{FL_e}$-algebra $\mathbf{A}^K$ is said to be the *Crawley completion* (or *complete ideal completion*) of $\mathbf{A}$. Crawley completions of lattices are discussed in e.g. [6] and [5]. When $\mathbf{A}$ satisfies the (JID), we can give a simple explicit representation of the complete ideal $K(X)$ for a given non-empty subset X of A, which is essentially due to W.H. Cornish (see [5] Lemma 2.1). Let

$$K_\circ(X) = \{y : y = \textstyle\bigvee_i x_i \text{ for existing } \bigvee_i x_i \text{ such that } x_i \in (X] \text{ for each } i\}.$$

where $(X]$ is the downward closure of a set X.

THEOREM 5.4. *If an $\mathbf{FL_e}$-algebra $\mathbf{A}$ is join infinite distributive then $K(X) = K_\circ(X)$ for each subset X of A.*

A corresponding result is shown in [5] under a weaker assumption called the *conditional upper continuity* which is the (JID): $\bigvee_i a_i \wedge b = \bigvee_i (a_i \wedge b)$, but only for *directed* subsets $\{a_i\}$ such that $\bigvee_i a_i$ exists. The following theorem is closely related to results in [6], [16] Lemma 3.2 and [5] Theorem 2.2.

THEOREM 5.5. *For an $\mathbf{FL_e}$-algebra $\mathbf{A}$, the following are equivalent.*
1. $\mathbf{A}$ is join infinite distributive.
2. For all subsets U and V of A, $K(U) \cap K(V) = K(U \cap V)$.
3. The Crawley completion $\mathbf{A}^K$ is join infinite distributive.
4. There exists a nucleus E on $\wp(\mathbf{A})$ such that the algebra $\mathbf{A}^E$ is join infinite distributive, which satisfies (i) $E(\{a\}) = (a]$ for any element a and (ii) every E-closed subset X of M is a complete ideal.

Since every normal ideal is a complete ideal, the following result is an immediate consequence of Theorem 5.5

COROLLARY 5.6. If the MacNeille completion $\mathbf{A}^M$ of an $\mathbf{FL_e}$-algebra $\mathbf{A}$ is join infinite distributive then $\mathbf{A}$ is join infinite distributive.

6. Heyting implication

It is well-known that the (JID) holds in any Heyting algebra. An operation $\rightsquigarrow$ is called a *Heyting implication* of a meet-semilattice **A**, if the following law of residuation holds between the meet $\wedge$ and $\rightsquigarrow$:

$$a \wedge c \leq b \text{ iff } c \leq a \rightsquigarrow b.$$

If a Heyting implication exists, it is uniquely determined. It is easy to see that the (JID) holds in any algebra in which Heyting implication exists. Conversely, if the (JID) holds in a complete algebra, then we can introduce an operation $\rightsquigarrow$ by defining

$$a \rightsquigarrow b = \bigvee \{c : a \wedge c \leq b\},$$

which is shown to be the Heyting implication (see e.g. [4] p.69). A lattice with a Heyting implication is sometimes called *Brouwerian.*

We consider now in particular the Heyting implication in the E-retraction $\mathbf{A}^E$ of an algebra **A** where E is a nucleus. We define binary operations $\rhd$ and $\triangle$ by

$$X \rhd Y = \{w \in A : x \wedge w \in Y \text{ for every } x \in X\}$$
$$X \triangle Y = \{x \wedge y : x \in X \text{ and } y \in Y\}$$

for all subsets X and Y of A. Clearly, $X \triangle Z \subseteq Y$ if and only if $Z \subseteq X \rhd Y$ for all subsets X, Y and Z of A. By using the standard argument (see e.g. the proof of Lemma 3.33 of [11]), we can show the following.

LEMMA 6.1. Let E be a closure operator on $\wp(\mathbf{A})$. Then $E(X) \triangle E(Y) \subseteq E(X \triangle Y)$ holds for all X and Y if and only if $X \rhd Y$ is E-closed whenever Y is E-closed.

When both X and Y are downward closed sets, $X \triangle Y = X \cap Y$. Therefore, for all downward closed subsets X, Y and Z the following holds between $\cap$ and $\rhd$: $X \cap Z \subseteq Y$ if and only if $Z \subseteq X \rhd Y$. In particular, when E is a *downward nucleus*, i.e. when every E-closed subset is downward closed, this relation holds for all E-closed subsets X, Y and Z though $X \rhd Y$ is not necessarily E-closed. The following theorem can be regarded as a natural generalization of Lemma 2.3 in [2] on MacNeille completions of Heyting algebras.

THEOREM 6.2. *Suppose that* **A** *is an* $\mathbf{FL_e}$*-algebra and* E *is a downward nucleus on* $\wp(\mathbf{A})$*. Then, the following three conditions are mutually equivalent:*

1. $X \rhd Y$ is E-closed for all E-closed X and Y,
2. $\mathbf{A}^E$ satisfies the join infinite distributivity,
3. $X \rhd Y$ is the Heyting implication in $\mathbf{A}^E$, i.e. $X \rhd Y = X \rightsquigarrow_E Y$ where $X \rightsquigarrow_E Y = \bigvee_E \{Z : Z \text{ is } E\text{-closed and } X \cap Z \subseteq Y\}$.

Obviously, every normal ideal of a given $\mathbf{FL_e}$-algebra $\mathbf{A}$ is a complete ideal. We show that the converse holds when the MacNeille completion $\mathbf{A}^M$ of $\mathbf{A}$ is join infinite distributive. That is, whenever $\mathbf{A}^M$ satisfies the (JID), $\mathbf{A}^K$ is equal to $\mathbf{A}^M$. The corresponding result on lattices is already shown in e.g. [5]. Here we give a direct proof of it.

THEOREM 6.3. *The following are equivalent for each $\mathbf{FL_e}$-algebra $\mathbf{A}$:*
1. The MacNeille completion $\mathbf{A}^M$ is join infinite distributive.
2. The algebra $\mathbf{A}$ is join infinite distributive, and moreover every complete ideal of $\mathbf{A}$ is normal (and hence $\mathbf{A}^M = \mathbf{A}^K$).

PROOF. The statement (1) follows from (2) immediately by using Theorem 5.5. So let us show the converse direction. Suppose that the MacNeille completion $\mathbf{A}^M$ satisfies the (JID). Then, $\mathbf{A}$ satisfies the (JID) by Corollary 5.6. We show that each complete ideal X of $\mathbf{A}$ is normal. Let us suppose that $a \in M(X) = L(U(X))$ for an arbitrary element a. Our goal is to show that $a \in X$. Define $X_a = X \cap (a]$. Obviously, X_a is a complete ideal, and a is an upper bound of it. Take any upper bound b of X_a. Then $X \cap (a] = X_a \subseteq (b]$ and hence $X \subseteq (a] \rhd (b]$. By taking M for E in Theorem 6.2 and using the assumption that $\mathbf{A}^M$ is join infinite distributive, $(a] \rhd (b]$ is M-closed as both $(a]$ and $(b]$ are M-closed. Hence, $M(X) \subseteq (a] \rhd (b]$. Therefore $a \wedge x \in (b]$ for any $x \in M(X)$. As $a \in M(X)$, by taking a for x in particular we have $a = a \wedge a \leq b$. Thus, a is the least upper bound of X_a. In other words, $\bigvee X_a$ exists and is equal to a. Since X is a complete ideal and $X_a \subseteq X$, $a = \bigvee X_a \in X$. Hence $M(X) \subseteq X$. Therefore X is a normal ideal for each complete ideal X. ■

MacNeille completions of algebras with Heyting implication are discussed e.g. in [24, 29, 28] and [20]. The first claim of the following proposition is shown in them. A proof of the second consequence is given in [5] for lattices and also in [2] for Heyting algebras.

PROPOSITION 6.4. Suppose that Heyting implication exists in an algebra $\mathbf{A}$. Then Heyting implication exists also in the MacNeille completion $\mathbf{A}^M$ which is equal to $\rhd$. Therefore, $\mathbf{A}^M = \mathbf{A}^K$.

7. Discussions on logical consequences

We discuss logical consequences of completions of algebras developed in the previous sections for algebraic completeness of substructural predicate logics. From an algebraic point of view, Crawley completions have some unique features which are distinct from MacNeille completions. On the other hand, our results on logical consequences of Crawley completions remain partial at this moment.

(1) Algebraic completeness using Crawley completions From Proposition 3.1 algebraic completeness of some substructural predicate logics follows. For instance, using Theorem 5.3 we can give an alternative proof of algebraic completeness of $\mathbf{QFL_e}$. The argument can be modified easily to some other cases, including noncommutative residuated lattices. Thus, using Crawley completions, we can show algebraic completeness of **QFL**, $\mathbf{QFL_e}$, $\mathbf{QFL_{ew}}$, $\mathbf{QFL_{ec}}$ and **QIL**. (A proof of the algebraic completeness of **QIL** using Crawley completion is show in Chapter 13 of [31].)

(2) Substructural logics with $(\wedge, \exists)$ The original goal of our present study was to show algebraic completeness of substructural predicate logics satisfying both the axiom scheme of distributivity (Dis) : $\alpha \wedge (\beta \vee \gamma) \to (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$ and the axiom scheme $(\wedge, \exists)$. Let **L** be a substructural predicate logic satisfying them, and **A** be the Lindenbaum algebra of **L**. Though the distributivity of the $\mathbf{FL_e}$-algebra **A** follows from the axiom scheme (Dis), it is uncertain for us whether the (JID) follows from $(\wedge, \exists)$.[3] So an interesting question is:

> does the Lindenbaum algebra of a QSL **L** satisfy the (JID) whenever $(\wedge, \exists)$ is provable in **L**?

If the answer is positive, the (JID) holds also in the Crawley completion $\mathbf{A}^{\mathbf{K}}$ of **A** and thus $(\wedge, \exists)$ is valid in it. In this way, we will be able to show algebraic completeness of **L**, as long as other axioms of **L** are preserved under Crawley completions. Note that when the axiom scheme (Div) is provable in **L**, (Div) is valid in the Lindenbaum algebra **A** of **L** and hence the (JID) holds in it by Lemma 4.1.

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[3] For a given formula ϕ let $[\phi]$ be the equivalence class of formulas to which the formula ϕ belongs (and thus an element of Lindenbaum algebra **A**). For a given existing infinite disjunction $\bigvee[\varphi_i]$ in **A**, $\bigvee_{i\in I}[\varphi_i]$ is not necessarily of the form $[\exists x\eta(x)]$ for some formula $\exists x\eta(x)$. For example, this happens when $[\varphi_k]$ is the greatest among $\{[\varphi_i] : i \in I\}$ and is not an existential formula. But in this case $\bigvee_{i\in I}[\varphi_i] \wedge [\psi] = [\varphi_k] \wedge [\psi] = \bigvee_{i\in I}([\varphi_i] \wedge [\psi])$ holds.

(3) Conservativity of Heyting implication Suppose that a QSL $\mathbf{L}$ is given. We expand our language by adding a new binary connective $\rightsquigarrow$ and consider the following two rules for Heyting implication $\rightsquigarrow$, which are called $\wedge$-residuation.

$(\rightsquigarrow 1)$: From $(\alpha \wedge \beta) \rightarrow \gamma$ infer $\alpha \rightarrow (\beta \rightsquigarrow \gamma)$.
$(\rightsquigarrow 2)$: From $\alpha \rightarrow (\beta \rightsquigarrow \gamma)$ infer $(\alpha \wedge \beta) \rightarrow \gamma$.

Define a logic $\mathbf{L}^H$ of the expanded language to be a QSL obtained from $\mathbf{L}$ by adding these rules for $\rightsquigarrow$. We say that $\mathbf{L}^H$ is *conservative over* $\mathbf{L}$ if and only if for every formula φ of the original language, if φ is provable in $\mathbf{L}^H$ then it is provable in $\mathbf{L}$. In this case, we say also that Heyting implication is *conservative* over $\mathbf{L}$. See [13].

Consider again any QSL in which both (Dis) and $(\wedge, \exists)$ are provable, and suppose that Heyting implication is conservative over such a logic $\mathbf{L}$. Let $\mathbf{A}^*$ be the Lindembaum algebra of the logic $\mathbf{L}^H$ with Heyting implication. Then the (JID) holds in $\mathbf{A}^*$, which is preserved by Crawley completion, and hence $(\wedge, \exists)$ is valid in the Crawley completion $(\mathbf{A}^*)^{\mathbf{K}}$. In this way, we will be able to show the algebraic completeness of $\mathbf{L}$. In particular, we have the following, where $\mathbf{K}_*\exists$ denotes the logic $\mathbf{K}_*$ with $(\wedge, \exists)$.

LEMMA 7.1. Let $\mathbf{K}$ be an axiomatic extension of the substructural propositional logic $\mathbf{FL_e}$ with the axiom scheme (Dis), and $\mathcal{V}_{\mathbf{K}}$ is the corresponding variety of $\mathbf{FL_e}$-algebras. If Heyting implication is conservative over $\mathbf{K}_*\exists$ and $\mathcal{V}_{\mathbf{K}}$ is closed under Crawley completion, then $\mathbf{K}_*\exists$ is algebraically complete.

To get algebraic completeness results using the above lemma, it is still necessary to prove the conservativity of Heyting implication in some way. For instance, this is proved for a QSL $\mathbf{QDFL}_e\exists$, which is obtained from $\mathbf{QFL_e}$ by adding both axiom scheme of distributivity (Dis) and the axiom scheme $(\wedge, \exists)$, by using cut elimination of a sequent system for $(\mathbf{QDFL}_e\exists)^H$ (see [20]). Thus we have the following.

PROPOSITION 7.2. The substructural logic $\mathbf{QDFL}_e\exists$ is algebraically complete.

(4) Ideal completions and $\exists$-free fragments of logics Though ideal completions are not necessarily regular, the canonical mappings preserve at least all existing infinite meets, as mentioned in Theorem 5.1. Moreover, as shown in Theorem 5.2, the ideal completion $\mathbf{A}^J$ is join infinite distributive when a given $\mathbf{FL_e}$-algebra $\mathbf{A}$ is distributive. These facts suggest us that ideal completions will be useful as long as we restrict our attention to $\exists$-free fragments of substructural predicate logics. We owe the idea also to §7 of the paper [13] by Goldblatt. In the following, $\mathbf{L}^{\ominus\exists}$ denotes the $\exists$-free fragment of a substructural predicate logic $\mathbf{L}$. We have the following.

THEOREM 7.3. *Let* $\mathbf{K}$ *be an axiomatic extension of the substructural propositional logic* $\mathbf{FL_e}$*, and* $\mathcal{V}_{\mathbf{K}}$ *is the corresponding variety of* $\mathbf{FL_e}$*-algebras. If* $\mathcal{V}_{\mathbf{K}}$ *is closed under ideal completion, then* ${\mathbf{K}_*}^{\ominus\exists}$ *is algebraically complete, where* $\mathbf{K}_*$ *is the minimum predicate extension of* $\mathbf{K}$*. Moreover, Heyting implication is conservative over* ${\mathbf{K}_*}^{\ominus\exists}$ *when the axiom of distributivity (Dis) is provable in* $\mathbf{K}$*.*

It will be interesting to know which propositional formulas can be preserved under ideal completions of residuated lattices. The following result is shown in [12] by using the monotonicity of connectives $\vee, \wedge$ and $\cdot$ (see also [14]).

PROPOSITION 7.4. Every inequality $s \leq t$ with terms s, t that contain only connectives and constants in $\{\vee, \wedge, \cdot, 0, 1\}$ is preserved under ideal completions.

COROLLARY 7.5. Let α and β be propositional formulas that contain only connectives and constants in $\{\vee, \wedge, \cdot, 0, 1\}$. Then the formula $\alpha \to \beta$ is preserved under ideal completions. That is, for every $\mathbf{FL_e}$-algebra $\mathbf{A}$, if the formula $\alpha \to \beta$ is valid in $\mathbf{A}$ then it is also valid in its ideal completion $\mathbf{A}^J$.

At first sight, the above proposition and the corollary look rather limited. But these results in fact can cover much wider results than what one might expect. To give an informative example, we consider noncommutative case. Let us recall that the *left division* (i.e. left implication) $\backslash$ satisfies that $x \cdot y \leq z$ if and only if $y \leq x \backslash z$. Now consider the following inequality

(a) $z \cdot (y \cdot x) \leq (z \cdot y) \cdot (z \cdot x)$.

Clearly, this inequality can be preserved under ideal completions by Proposition 7.4. Using this fact, we can prove that the formula

(b) $(\alpha\backslash(\beta\backslash\gamma))\backslash((\alpha\backslash\beta)\backslash(\alpha\backslash\gamma))$

is preserved under ideal completions. This is obtained by showing that for every $\mathbf{FL_e}$-algebra $\mathbf{A}$ the inequality (a) holds in $\mathbf{A}$ if and only if the following inequality (c) holds in $\mathbf{A}$, which is the algebraic counterpart of (b):

(c) $a\backslash(b\backslash c) \leq (a\backslash b)\backslash(a\backslash c)$

To show that the inequality (a) implies (c), it is enough to substitute x, y and z in (a) for $a\backslash(b\backslash c)$, $a\backslash b$ and a, respectively. For the converse, we substitute a, b and c in (c) for z, $z \cdot y$ and $(z \cdot y) \cdot (z \cdot x)$, respectively. Then, from the fact that both inequalities $x \leq z\backslash((z \cdot y)\backslash((z \cdot y) \cdot (z \cdot x)))$ and $y \leq z\backslash(z \cdot y)$ hold, we can derive (a). Thus, (b) is preserved under ideal completions.

This example looks ad hoc. But the above example shows that the preservation of a given formula φ is obtained as long as we can find a term equivalent to φ which fits for applying Proposition 7.4. Such connections were studied already, but implicitly, as *correspondence results* on Kripke completeness (in the sense of [23]) of substructural propositional logics. For, Kripke completeness is essentially the same as completeness with respect to a class of ideal completions, as pointed out in [21]. Though we can not go into the details here, correspondences between inequalities and the validity of axiom schemes given in [9] (in particular, the table in p. 59) tell us how the above argument works in general.

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Hiroakira Ono
Research Center for Integrated Science
Japan Advanced Institute of Science and Technology
Nomi, Ishikawa, Japan
ono@jaist.ac.jp