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Constraint Satisfaction, Irredundant Axiomatisability and Continuous Colouring

Abstract. We observe a number of connections between recent developments in the study of constraint satisfaction problems, irredundant axiomatisation and the study of topological quasivarieties. Several restricted forms of a conjecture of Clark, Davey, Jackson and Pitkethly are solved: for example we show that if, for a finite relational structure **M**, the class of **M**-colourable structures has no finite axiomatisation in first order logic, then there is no set (even infinite) of first order sentences characterising the continuously **M**-colourable structures amongst compact totally disconnected relational structures. We also refute a rather old conjecture of Gorbunov by presenting a finite structure with an infinite irredundant quasi-identity basis.

Keywords: Quasivariety, Antivariety, Irredundant axiomatisability, Standard topological quasivarieties, Graph dualities, Constraint satisfaction problems.

1. Introduction

In this article we provide some links between the axiomatisability of certain kinds of universal Horn classes, recent developments in the logical study of constraint satisfaction problems and finite model theory, and the axiomatisability of classes of topological structures. The main theme is applications to the last of these topics, including a positive solution to a restricted version of a problem in Clark, Davey, Jackson and Pitkethly [8, §10, Problem 3]. This problem asks whether the property of nonfinite axiomatisability of the universal Horn sentences of a finite structure **M** forces the absence of a universal-Horn-theoretic axiomatisation of the topological quasivariety of **M** amongst compact totally disconnected structures. We answer this in the positive for antivarieties generated by relational structures; in fact we establish a stronger version of the implication, showing that there is no first order characterisation of the topological quasivariety (in [8] this stronger version was shown *not* to hold for finite algebras). The reverse implication is shown

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to hold when $\mathbf{M}$ is a directed bipartite graph (again, this implication is known not to hold in general). We also establish a link between this property and the problem of finding a finite structure with an infinite irredundant axiomatisation for its quasi-identities. In the 1970s, Gorbunov conjectured that no finite structure with this latter property exists. We make some simple observations explaining the prevalence of examples in support of this widely reiterated conjecture, but also present a relatively basic counterexample.

Section 2 contains three subsections detailing background information on antivarieties and universal Horn classes, on constraint satisfaction problems, and finally on topological prevarieties. We make no attempt to present exhaustive surveys on these topics, rather just select suitable background for the main results and their proofs. The main results themselves are listed in precise detail in Section 3. Section 4 contains some refinements to the relationships between universal Horn classes and antivarieties, while Section 5 develops basic observations about homomorphisms between relational structures and irredundant axiomatisability by anti-identities. The negative solutions to Gorbunov's conjectures are contained in this section. Sections 6–8 contain the proofs of Theorems A–C, respectively. The article concludes with some open problems.

2. Preliminaries

2.1. Antivarieties and Universal Horn Classes

(For more information and background on the basic definitions of this subsection, consult a book such as Burris and Sankappanavar [4, Chapter V] or Gorbunov [12]. The articles [8] and Stronkowski [31] also review similar material.) A *universal Horn class* is a class of similar structures closed under taking induced substructures S, nonempty direct products P and ultraproducts P_u. We often use the terminology $\forall_\mathrm{H}$-*class* to refer to universal Horn classes. The smallest $\forall_\mathrm{H}$-class containing a class of structures $\mathcal{K}$ is known to be equal to $\mathsf{SPP}_\mathrm{u}(\mathcal{K})$, and if $\mathcal{K}$ consists of finitely many finite structures, then $\mathsf{SPP}_\mathrm{u}(\mathcal{K}) = \mathsf{SP}(\mathcal{K})$. A structure $\mathbf{A}$ in $\mathsf{SP}(\mathcal{K})$ can be residually separated into members of $\mathcal{K}$ using projections: if $a \neq b$ then there is a member $\mathbf{B} \in \mathcal{K}$ and a homomorphism $\psi : \mathbf{A} \to \mathbf{B}$ with $\psi(a) \neq \psi(b)$, and if R is a fundamental relation, and $(a_1, \dots, a_n) \notin R^\mathbf{A}$ then there is $\mathbf{B} \in \mathcal{K}$ and a homomorphism $\psi : \mathbf{A} \to \mathbf{B}$ with $(\psi(a_1), \dots, \psi(a_n)) \notin R^\mathbf{B}$. In fact it is not hard to see that a structure residually separable into $\mathcal{K}$ is contained in $\mathsf{SP}(\mathcal{K})$ provided there is at least one homomorphism into one member of $\mathcal{K}$. This equivalence holds even if $\mathcal{K}$ is not closed under taking ultraproducts.

An *antivariety* of relational structures is a first order (elementary) class of similar structures closed under the operator H^{-1}, which closes a class $\mathcal{K}$ by including all structures admitting a homomorphism into a member of $\mathcal{K}$. Equivalently, an antivariety is a class of similar structures equal to the models of some set of anti-identities: universally quantified first order sentences taking the form of disjunctions of negated atomic formulas. Universal Horn classes also admit a well known syntactic characterisation. Anti-identities are allowed, but also quasi-identities: universally quantified implications whose premise is a conjunction of atomic sentences, and whose conclusion is a single atomic formula. We often use the notation $\forall_{\mathrm{H}}$-*sentences* and $\forall_{\mathrm{H}}$-*axiomatisable* to denote universal Horn sentences and axiomatisability by universal Horn sentences, respectively.

It is easy to see that an antivariety is in fact a $\forall_{\mathrm{H}}$-class with the property that whenever a structure $\mathbf{A}$ admits a homomorphism into a member of $\mathcal{A}$, then $\mathbf{A}$ is itself in $\mathcal{A}$. It is also routine to show that for any finite structure $\mathbf{M}$, the class $\mathsf{H}^{-1}(\mathbf{M})$ is the smallest antivariety containing $\mathbf{M}$; the *antivariety generated by* $\mathbf{M}$. A set of anti-identities Σ is a *basis* for (the anti-identities of) a finite structure $\mathbf{M}$ if the class of models of Σ (in symbols, $\mathrm{Mod}(\Sigma)$) coincides with the antivariety $\mathsf{H}^{-1}(\mathbf{M})$ generated by $\mathbf{M}$. For details on antivarieties, consult the book by Gorbunov [12] or an article such as Gorbunov and Kravchenko [13]. Classes of the form $\mathsf{H}^{-1}(\mathbf{M})$ are also widely encountered in the the graph theory literature: see the book by Hell and Nešetřil [16].

An *obstruction* for an antivariety $\mathcal{A}$ is any structure not in $\mathcal{A}$. If $\mathbf{A}$ is an obstruction for $\mathcal{A}$ and there is a homomorphism from $\mathbf{A}$ into some structure $\mathbf{B}$ (that is, $\mathbf{A}$ *obstructs* $\mathbf{B}$), then $\mathbf{B}$ cannot be in $\mathcal{A}$ either. So obstructions are closed under taking homomorphisms.

The following lemma is at least folklore and has an easy proof.

Lemma 1. *The following are equivalent for a* $\forall_{\mathrm{H}}$*-class* $\mathcal{K}$ *of finite relational type*:

(1) $\mathcal{K}$ *is the class of models of some first order sentence*;

(2) $\mathcal{K}$ *is the class of models of some finite set of* $\forall_{\mathrm{H}}$*-sentences*;

(3) *there is a number* n *such that a structure* $\mathbf{S}$ *is contained in* $\mathcal{K}$ *if and only if all* n*-element induced substructures are contained in* $\mathcal{K}$;

(4) *there is a finite set* $\mathcal{S}$ *of finite structures such that a structure* $\mathbf{S}$ *is contained in* $\mathcal{K}$ *if and only if no member of* $\mathcal{S}$ *is an induced substructure of* $\mathbf{S}$;

(5) *there is a finite set* $\mathcal{S}$ *of finite structures such that a finite structure* $\mathbf{S}$ *is contained in* $\mathcal{K}$ *if and only if no member of* $\mathcal{S}$ *is an induced substructure of* $\mathbf{S}$;

Moreover, if $\mathcal{K}$ *is an antivariety these are equivalent to*:

(6) *there is a finite set* $\mathcal{T}$ *of finite structures such that a finite structure* $\mathbf{S}$ *is contained in* $\mathcal{K}$ *if and only if no member of* $\mathcal{T}$ *admits a homomorphism into* $\mathbf{S}$.

Proof. We give a sketch for completeness. (1)⇔(2) is very well known consequence of the compactness theorem. (2)⇔(3) is an easy consequence of the fact that a $\forall_{\mathrm{H}}$-sentence in at most n variables that fails on a relational structure $\mathbf{S}$ fails on an induced substructure of $\mathbf{S}$ having at most n elements. The equivalence (3)⇔(4) follows from the fact that there are only finitely many structures of given finite cardinality in a finite relational type. (4)⇒(5) is trivial, while for the converse, assume (5) holds and that $\mathbf{T}$ is a structure not containing any member of $\mathcal{S}$ as an induced substructure. By (5), every finite induced substructure of $\mathbf{T}$ is in $\mathcal{K}$. However $\mathbf{T}$ embeds into an ultraproduct of its finitely generated substructures, which are all finite since the type is relational. Hence $\mathbf{T} \in \mathsf{SP_u}(\mathcal{K}) = \mathcal{K}$. So (4) holds.

Now say that $\mathcal{K}$ is an antivariety. If (5) holds, we can assume without loss of generality that the finite set $\mathcal{S}$ contains all finite structures of some bounded cardinality other than those in $\mathcal{K}$. In particular, choosing $\mathcal{T} := \mathcal{S}$ we have that $\mathcal{T}$ is closed under taking homomorphic images and under adding new hyperedges to its members. Hence a structure $\mathbf{S}$ (finite or otherwise) has an induced substructure from $\mathcal{T}$ if and only if some member of $\mathcal{T}$ admits a homomorphism into $\mathbf{S}$. So (6) holds. Conversely, if (6) holds then we may let $\mathcal{S}$ be the class of all structures for which there is a surjective homomorphism from a member of $\mathcal{T}$. Now, $\mathcal{S}$ is finite and a structure embeds a member of $\mathcal{S}$ if and only if there is a homomorphism into it from a member of $\mathcal{T}$. Hence (5) holds. ∎

A class of obstructions $\mathcal{T}$ for an antivariety $\mathcal{K}$ is said to be *complete* if $\mathcal{K}$ is the class of all structures avoiding a homomorphism from a member of $\mathcal{T}$ (cf. part 6 of Lemma 1).

Lemma 1 requires a relational signature. If $\mathcal{K}$ were a $\forall_{\mathrm{H}}$-class of algebras then, even if $\mathcal{K}$ is locally finite, in (3)–(6) one cannot in general restrict to finite algebras, only finitely generated algebras. This possible discrepancy is closely related to the rather old problem of Eilenberg and Schützenberger [11] which asks whether or not there is a finite algebra $\mathbf{A}$ without a finite basis for its identities, but for which there is a finite system of identities for testing membership of finite algebras in the variety of **A** (see Jackson [19, §7.4] for the $\forall_{\mathrm{H}}$-class version of the problem and its relationship with the Eilenberg/Schützenberger problem and Jackson [17] and McNulty, Szekely and Willard [25] for further discussion and other related problems).

In a relational signature we may restrict to anti-identities in which there are no instances of the equality symbol $\approx$ (except in the case of the empty antivariety, which can be axiomatised by the single anti-identity $\neg x \approx x$). Indeed, let $\Phi := \bigvee_{1\leq i\leq n} \neg\alpha_i$ be an anti-identity having at least one model. Then we may assume that at least one of the α_i is not an equality; say α_1 is not an equality. If α_n is an equality—say, $x \approx y$—then consider the anti-identity $\Phi' := \bigvee_{1\leq i\leq n-1} \neg\alpha_i'$ obtained by dropping $\neg\alpha_n$ and replacing every occurrence of y in each α_i by x to produce α_i'. It is routine to verify that (in a relational signature) a structure satisfies Φ if and only if it satisfies Φ'. Hence Φ and Φ' are logically equivalent. Continuing, we may find a logically equivalent anti-identity with no occurrences of equality. Such an anti-identity will be said to be *equality free*. Note in particular that this shows that when $\Sigma = \{\Phi_i \mid i \in I\}$ is a complete system of anti-identities for a non-empty antivariety, then we may assume without loss of generality that the anti-identities Φ_i are equality free.

The relationship between equality-free anti-identities of an antivariety and finite obstructions for the antivariety can be made even more transparent. With every equality-free anti-identity $\Phi := \bigvee_{1\leq i\leq n} \neg\alpha_i$ one may associate a structure $\mathbf{M}(\Phi)$ on the variables of Φ by including the hyperedge α_i for each $i = 1, \dots, n$. Similarly, with a finite structure $\mathbf{C}$, we may associate an equality-free anti-identity $\Phi_{\mathbf{C}}$ whose variables are the elements of $\mathbf{C}$ and whose negated atomic expressions are the negations of the hyperedges in $\mathbf{C}$. It is easy to see that $\mathbf{M}(\Phi_{\mathbf{C}})$ is identical to $\mathbf{C}$. The following lemma is routine and well known; its proof is omitted.

LEMMA 2. *The following are equivalent for an equality free anti-identity* $\Phi := \bigvee_{1\leq i\leq n} \neg\alpha_i$ *and a structure* $\mathbf{A}$:

- $\mathbf{A} \models \Phi$;
- *there is no homomorphism from* $\mathbf{M}(\Phi)$ *into* $\mathbf{A}$.

In particular, if Σ *is an equality-free anti-identity basis for an antivariety* $\mathcal{A}$ *and* $\mathcal{O}$ *is a complete set of finite obstructions for* $\mathcal{A}$ *then*:

- $\{\mathbf{M}(\Phi) \mid \Phi \in \Sigma\}$ *is a complete system of finite obstructions for* $\mathcal{A}$; *and*
- $\{\Phi_{\mathbf{C}} \mid \mathbf{C} \in \mathcal{O}\}$ *is an equality-free anti-identity basis for* $\mathcal{A}$.

In the study of graph homomorphisms (of relational structures of some fixed finite type), a *duality pair* $(\mathcal{F}, \mathbf{M})$ is a pair in which $\mathcal{F}$ is a set of structures, $\mathbf{M}$ is a structure and for any structure $\mathbf{A}$, there is a homomorphism from some member of $\mathcal{F}$ into $\mathbf{A}$ if and only if there is no homomorphism from $\mathbf{A}$ into $\mathbf{M}$. In other words, $\mathcal{F}$ is a complete set of obstructions for the antivariety of $\mathbf{M}$, or by Lemma 2 (and when $\mathbf{M}$ and the members of $\mathcal{F}$ are finite), $\mathcal{F}$ corresponds to a complete system of anti-identities for the antivariety generated by $\mathbf{M}$.

If no proper subset $\Sigma' \subsetneq \Sigma$ has $\mathrm{Mod}(\Sigma') = \mathsf{H}^{-1}(\mathbf{M})$, then Σ is said to be an *irredundant* (equivalently, *independent*, or *irreducible*) basis for the anti-identities of $\mathbf{M}$. One can equivalently refer to a system $\mathcal{O}$ of finite structures as being irredundant provided that for every $\mathbf{O} \in \mathcal{O}$ there is no homomorphism from some member of $\mathcal{O}\backslash\{\mathbf{O}\}$ into $\mathbf{O}$: in other words, the system $\mathcal{O}$ forms an antichain in the class of all finite structures, as preordered by homomorphism (precise details of this are considered in Section 5).

The compactness theorem ensures that if $\mathbf{M}$ has a finite anti-identity basis (equivalently, a finite set of obstructions), then every basis of anti-identities for $\mathbf{M}$ contains a finite subset that is also a basis and is irredundant. However $\mathbf{M}$ need not have a finite anti-identity basis and to date all resolved examples without a finite anti-identity basis also have no irredundant basis (at least, when $\mathbf{M}$ is finite). A conjecture due to Gorbunov is that all finite structures without a finite basis for their anti-identities also have no irredundant basis for their anti-identities. (Expressed in the language of homomorphism dualities, the conjecture states that it is not possible to have a duality pair $(\mathcal{F}, \mathbf{M})$, where the family $\mathcal{F}$ is an infinite antichain of finite structures in the homomorphism order and $\mathbf{M}$ is finite.) A more widely stated conjecture, also due to Gorbunov, is the same conjecture but phrased in terms of quasi-identities and quasivarieties, rather than anti-identities and antivarieties respectively; see for example, Adams, Adaricheva, Dziobiak and Kravchenko [1, Conjecture 24], Sapir [30, Problem 1], Gorbunov [12], and Gorbunov and Smirnov [14, Problem 1] (though in [14], the problem is phrased only for algebras). We provide a counterexample to both these conjectures in the present article. (It is interesting to note that for algebras, the identity/variety analogue of these problems was also moderately recently resolved in the negative: see Jackson [18]. The algebra versions of Gorbunov's quasi-identity and antivariety problems are not solved in the present article, though the evidence in favour of the conjecture is, we think, substantially weakened by the example of the present article.)

2.2. Constraint Satisfaction Problems

The finite membership problem for $\mathsf{H}^{-1}(\mathbf{M})$ is usually called the *constraint satisfaction problem relative to* $\mathbf{M}$ (at least when the signature is finite; the infinite type formulation requires some subtleties in definition that we do not discuss), and the finite models in $\mathsf{H}^{-1}(\mathbf{M})$ are often denoted by CSP($\mathbf{M}$). Understanding the possible computational complexities of constraint satisfaction problems is currently a very active research area in theoretical computer science.

The constraint satisfaction problem over $\mathbf{M}$ is said to have *finite duality* if there is a finite complete set of obstructions for CSP($\mathbf{M}$) (this definition comes from the notion of homomorphism dualities described in the previous subsection). So the finite duality property of CSP($\mathbf{M}$) is the same as the finite axiomatisability of the antivariety of $\mathbf{M}$ by (2)⇔(6) of Lemma 1. (Atserias [2] showed that this is also equivalent to the finite axiomatisability of CSP($\mathbf{M}$) *amongst finite structures*: this does not follow from Lemma 1 as item 1 of that lemma refers to $\mathsf{H}^{-1}(\mathbf{M})$ rather than to CSP($\mathbf{M}$).)

A remarkable result of Larose, Loton and Tardif [23] is that the problem of deciding the finite duality property for a finite relational structure is decidable (NP-complete even), which in view of the above discussion can be restated as follows.

Theorem 3. (*Larose, Loton and Tardif* [23].) *The problem of deciding the finite basis property for the antivariety of an arbitrary finite relational structure is* NP*-complete*; *in particular it is decidable.*

Aspects of the classification of first order CSPs in [23] will play a central role in the present article.

We mention that in contrast to Theorem 3, the undecidability of recognising finite axiomatisability for the *variety* generated by a finite algebra was a landmark result of McKenzie [24]. The possible decidability or undecidability for the quasivariety version of the problem (amongst finite algebras or other finite structures) is one of the tantalising open problems in universal algebra.

2.3. Boolean Topological Prevarieties

Recall that a *Boolean space* (sometimes, a *Stone space*) is a compact, Hausdorff, totally disconnected topological space (they are the duals of Boolean algebras under Stone duality). A structure $\mathbf{X}$ on a Boolean space $(X, \mathcal{T})$ is a *Boolean topological structure* if all fundamental operations $f : X^n \to X$ of $\mathbf{X}$ are continuous and every fundamental relation $R^{\mathbf{X}} \subseteq X^n$ is closed in the product topology. A finite space is a Boolean space if and only if it is discrete and we tacitly assume that all finite structures have this underlying topology. For any class $\mathcal{K}$, let $\mathcal{K}_{\mathcal{T}}$ denote the class of all Boolean topological structures that lie in $\mathcal{K}$ if the topology is dropped. We also let $\mathrm{Mod}_{\mathcal{T}}(\Sigma)$ abbreviate $[\mathrm{Mod}(\Sigma)]_{\mathcal{T}}$, the set of all Boolean topological models of the set of first order sentences Σ.

A (Boolean) *topological prevariety* is a class of similar structures with compatible Boolean topologies and closed under taking continuous isomorphic copies of closed substructures and products (with the product topology). Topological prevarieties arise frequently in the study of finite objects: for any pseudovariety $\mathcal{V}$ of semigroups, the pro-$\mathcal{V}$ semigroups (the class of semigroups continuously isomorphic to an inverse limit of finite semigroups from $\mathcal{V}$) are a topological prevariety. Indeed it is precisely the class $\mathsf{S_c P}(\mathcal{V})$, where $\mathsf{S_c}$ denotes taking closed substructures (semigroups in this case) and P is the class operator of (nonempty) direct products; here infinite products of finite structures are given the product topology.

Topological prevarieties arise in the study of natural dualities (see Clark and Davey [5] for example, where they are called topological quasivarieties) and more generally in other category-theoretic dualities based on finite structures (see Johnstone [21] for example). If $\mathcal{K} = \mathrm{Mod}_{\mathcal{T}}(\Sigma)$ for some set of first order sentences, then $\mathcal{K}$ is said to be *first order axiomatisable* (amongst Boolean topological structures). If Σ consists of universal sentences, then $\mathcal{K}$ is said to be *universal axiomatisable*. In analogy with our notation for universal Horn sentences, we also use the notation $\forall$-*sentence* for universal sentence and $\forall$-*axiomatisable* for universal axiomatisable.

For example, if Σ consists of the associativity law in the language of a single binary operation, then the class of profinite semigroups coincides with $\mathrm{Mod}_{\mathcal{T}}(\Sigma)$ (Numakura [28]). However, if Σ consists of the usual axioms for lattices, then $\mathrm{Mod}_{\mathcal{T}}(\Sigma)$ does not consist of the class of profinite lattices: there is a Boolean topological lattice that is not an inverse limit of finite lattices (see Clark, Davey, Freese and Jackson [7] and Clinkenbeard [9]).

Simple graphs (antireflexive, symmetric digraphs) provide further interesting examples, experiencing a "reversal of fortune" when it comes to axiomatisability and topology. The following examples come from [8]. The class of all finitely colourable graphs (that is, admitting a homomorphism into a finite graph) is obviously not axiomatisable in first order logic, however the class of k-colourable graphs is an antivariety, albeit one without a finite axiomatisation in first order logic. But in the realm of Boolean topological structures, the class of continuously finitely colourable graphs is nothing other than the class of all Boolean topological simple graphs (so is finitely axiomatisable), while the class of continuously k-colourable graphs cannot be axiomatised in first order logic (amongst Boolean topological structures).

Of particular interest for a finite structure $\mathbf{M}$ is how $\mathsf{S_cP}(\mathbf{M})$ relates to the $\forall_{\mathrm{H}}$-class $\mathsf{SP}(\mathbf{M})$. Certainly, $\mathsf{S_cP}(\mathbf{M}) \subseteq [\mathsf{SP}(\mathbf{M})]_{\mathcal{T}}$, because Boolean topological structures are closed under taking non-empty direct products and closed substructures. When equality holds, it is said that $\mathsf{S_cP}(\mathbf{M})$ is *standard* [6] (because the standard axiomatisation of $\mathsf{SP}(\mathbf{M})$ by $\forall_{\mathrm{H}}$-sentences continues to hold). It is not too difficult to verify that (when $\mathbf{M}$ is a finite structure) $\mathsf{S_cP}(\mathbf{M})$ is standard if and only if it is $\forall$-axiomatisable if and only if it is $\forall_{\mathrm{H}}$-axiomatisable [8, Proposition 2.17]. In general however, $\mathsf{S_cP}(\mathbf{M})$ need not be standard, nor even first order axiomatisable. The "standardness" property has been characterised for finite cyclic semigroups, quasi-orders and three element unary algebras in [8], for reflexive antisymmetric digraphs, simple graphs, and antireflexive antisymmetric digraphs whose symmetric closure is bipartite in Trotta [32–34], respectively. "Inherent" non-standardness is characterised for groups and completely simple semigroups in Jackson [20]. First order axiomatisability of topological prevarieties is known to coincide with standardness for finite cyclic semigroups, quasi-orders, three-element unary algebras, reflexive antisymmetric graphs, and antireflexive antisymmetric digraphs whose symmetric closure is bipartite, but is open in the remaining cases cited here. In general the notions are distinct: if $\mathbf{M}$ is a finite lattice, then $\mathsf{S_cP}(\mathbf{M})$ is always first order axiomatisable, however need not be $\forall$-axiomatisable [8, Theorem 4.2 and Example 4.3].

3. Main Results

In this section we state the details of the main results of the article.

A very large number of examples examined so far led the authors of [8] to pose the following problem.

Problem 4. ([8, p. 1651]) If a finite structure generates a standard topological prevariety, must it have a finite basis for its universal Horn sentences?

This problem has an obvious antivariety analogue and a number of the motivating examples for Problem 4 in [8] are in fact antivarieties. Moreover, we prove in Lemmas 6 and 8 below that every finitely generated antivariety is a finitely generated $\forall_{\mathrm{H}}$-class, so that the *the antivariety variant of this* *problem is just a restriction of Problem 4.* With Theorem B we solve this antivariety problem in the positive for relational structures.

We now state the first result.

Theorem A. *Let $\mathcal{A}$ be any antivariety of relational structures of finite type. If $\mathcal{A}$ fails to have an irredundant basis for its anti-identities, then $[\mathcal{A}]_{\mathcal{T}}$ contains a member for which there is no continuous map into a finite member of $\mathcal{A}$. Furthermore, the class $\mathsf{S_c P}(\mathcal{A}_{\mathrm{fin}})$ is axiomatisable by neither $\forall$-sentences nor any single first order sentence.*

The proof of this Theorem is given in Section 6.

The property of having no irredundant basis by anti-identities seems to be very widely held by finitely generated antivarieties. Moreover, in the introduction we explained that it has been conjectured that every nonfinitely based but finitely generated antivariety is without an irredundant anti-identity basis (Gorbunov's conjecture): Theorem A would then resolve the antivariety version of Problem 4. However, we observe in the present article that this conjecture is in fact false: the counterexample is quite routine and is described in Example 16 below. Example 17 contains the corresponding counterexample in the quasivariety setting.

The second main result shows that the antivariety version of Problem 4 does, nevertheless, have a positive solution (in relational signatures). Compared to Theorem A, the result requires the antivariety to be generated by a single finite structure (in common with Problem 4), but has the pay-off that one only needs the absence of a finite anti-identity basis, rather than the absence of an irredundant one, and the non-axiomatisability statement is stronger.

Theorem B. *Let $\mathbf{M}$ be any finite relational structure of finite type and $\mathcal{A}$ be the antivariety generated by $\mathbf{M}$. If $\mathbf{M}$ has no finite basis for its anti-identities, then $[\mathcal{A}]_{\mathcal{T}}$ contains a member that is not continuously $\mathbf{M}$-colourable. Moreover, the class of continuously $\mathbf{M}$-colourable structures is not first order axiomatisable.*

The proof of this theorem is given in Section 7; it depends heavily on the proof of Theorem 3. In fact our proof will show something that is possibly slightly stronger: $[\mathcal{A}]_{\mathcal{T}}$ cannot be axiomatised amongst Boolean topological structures by any logical language that does not refer to topology itself. This is because we identify a single structure $\mathbf{X} \in \mathcal{A}$ admitting two different Boolean topologies: one placing it outside of $\mathsf{S_c P}(\mathbf{M})$ and one placing it inside of $\mathsf{S_c P}(\mathbf{M})$.

In general, the reverse implication of Problem 4 does not hold (though there do not seem to be any known antivariety counterexamples at this stage). The final main result shows that the reverse implication does hold for antivarieties generated by a fairly natural class of digraphs. We will say that a directed graph $\mathbf{G} = \langle G, \sim \rangle$ is *bipartite* if G is the disjoint union of two independent subsets A and B. Every directed tree is bipartite. We say that $\mathbf{G}$ is *strictly bipartite* if $x \sim y$ in G implies $x \in A$ and $y \in B$.

Theorem C. *The following are equivalent for a finite bipartite digraph* $\mathbf{B}$:

(1) $\mathbf{B}$ *has a finite basis for its anti-identities*;

(2) $\mathbf{B}$ *is strictly bipartite*;

(3) *every Boolean topological digraph that is* $\mathbf{B}$*-colourable is topologically* $\mathbf{B}$*-colourable*;

(4) *there is a set* Σ *of first order sentences in the language of digraphs such that a Boolean topological digraph* $\mathbf{X}$ *is topologically* $\mathbf{B}$*-colourable if and only if* $\mathbf{X} \models \Sigma$.

The proof of this Theorem is given in Section 8.

4. Antivarieties Versus Universal Horn Classes

In this section we observe a convenient tightening of the connection between antivarieties and $\forall_{\mathrm{H}}$-classes: with any finite structure $\mathbf{M}$ of finite signature $\mathcal{R}$, we effectively construct a new structure $\mathbf{M}^\sharp$ with the property that the antivariety of $\mathbf{M}$ is equal to the $\forall_{\mathrm{H}}$-class of $\mathbf{M}^\sharp$. Similar facts are already observed in the literature: for example [13, Theorem 2.4] demonstrates this within the class of simple graphs (that is, antireflexive symmetric digraphs) and other related instances can be found in Nešetřil and Pultr [26].

We first describe the construction of $\mathbf{M}^\sharp$ from $\mathbf{M}$. Let k be the maximal variety of any relation in the signature $\mathcal{R}$. We construct a finite set $\mathcal{M}$ of structures with the following properties:

(1) each structure in $\mathcal{M}$ is obtained from $\mathbf{M}$ by adding at most k new elements and some new hyperedges;

(2) each structure in $\mathcal{M}$ retracts onto $\mathbf{M}$.

The structure $\mathbf{M}^\sharp$ will be constructed by amalgamating the structures in $\mathcal{M}$ on the common substructure $\mathbf{M}$. We will show below that $\mathbf{M}^\sharp$ has the desired properties by showing the following properties hold:

(3) if $\mathbf{A}$ is in the antivariety of $\mathbf{M}$, then $\mathbf{A}$ is in the $\forall_{\mathrm{H}}$-class of $\mathcal{M}$,

(4) the $\forall_{\mathrm{H}}$-class of $\mathcal{M}$ equals that of $\mathbf{M}^\sharp$.

We will also prove a topological analogue of these statements.

Let N be a set and $f : N \to M$ be any function. We define the *inflation of M relative to f* to be the structure on N such that, for all relations R in the type of $\mathbf{M}$, we have $(n_1, \dots, n_r) \in R^{\mathbf{N}}$ if and only if $(f(n_1), \dots, f(n_r)) \in R^{\mathbf{M}}$. The members of $\mathcal{M}$ will be constructed based on inflations of $\mathbf{M}$ according to two recipes (which are only minor variants of each other). It will be clear that the desired properties (1) and (2) hold for both kinds of construction.

Consider a hyperedge h; say $(a_1, \dots, a_r) \in R^{\mathbf{M}}$. The a_i might not necessarily be pairwise distinct. Let $\{a_{1,h}, \dots, a_{r,h}\}$ be an ordered set of new elements not in M and let θ be any equivalence relation on $\{a_{1,h}, \dots, a_{r,h}\}$ with the property that $a_{i,h}\ \theta\ a_{j,h}$ implies $a_i = a_j$. In the first recipe, we construct a structure $\mathbf{N}$ in $\mathcal{M}$ based on this selection of hyperedge h and equivalence relation θ. The universe of $\mathbf{N}$ will be $M \cup \{a_{1,h}/\theta, \dots, a_{r,h}/\theta\}$. Let $\nu : N \to M$ be the map fixing M and sending $a_{i,h}/\theta$ to a_i. Let $\mathbf{N}^+$ be the inflation of $\mathbf{M}$ by ν. Let $\mathbf{N}$ be the structure obtained from $\mathbf{N}^+$ by removing the tuple $(a_{1,h}/\theta, \dots, a_{r,h}/\theta)$ from the definition of the relation R on $\mathbf{N}^+$.

Note that if e denotes the number of equivalence relations on a k-element set, then the total number of structures to be included in this way is at most the number of hyperedges times e, which is polynomial in $\mathbf{M}$, provided that the signature $\mathcal{R}$ is fixed. (And each such structure can be constructed from $\mathbf{M}$—considered, say, as a set with a family of subsets of powers—in polynomial time.) We include these structures in $\mathcal{M}$.

The remaining structures in $\mathcal{M}$ are formed in a similar way but involving the relation of equality. In this second recipe, for each $x \in M$, we add a new point x' and include the inflation of $\mathbf{M}$ by the map from $M \cup \{x'\}$ that fixes M and has $x' \mapsto x$. There are only $|M|$ such structures of this kind. Thus $\mathcal{M}$ can be constructed in polynomial time, and so may the amalgam $\mathbf{M}^\sharp$.

Example 5. Let $\mathbf{E}$ be the two element structure $\langle \{0,1\}; \sim, B, W \rangle$, where $\sim$ denotes the usual $\leq$ order relation, B ("black") is the unary relation $\{0\}$ and W ("white") is the unary relation $\{1\}$. Then $\mathbf{E}^\sharp$ has 14 elements.

Proof. There are 5 hyperedges $(0,0) \in \sim$, $(1,1) \in \sim$, $(0,1) \in \sim$, $(0) \in B$ and $(1) \in W$. The first recipe applied to the first two of these hyperedges creates two new structures in both cases, with one and two new elements respectively. Applied to the third hyperedge we get just one further structure, with two new elements, and applied to the fourth and fifth hyperedge, we get one new structure each, both with a single new vertex. The second recipe creates two new structures, each with one new element. After amalgamation we find that $\mathbf{E}^\sharp$ has the original two elements plus a further $(1+2)+(1+2)+(2)+(1)+(1)+(1)+(1) = 12$ new elements. ■

It is easy to see that if property (3) is true for $\mathcal{M}$, then property (4) holds for $\mathbf{M}^\sharp$ and $\mathcal{M}$. Indeed, each $\mathbf{N} \in \mathcal{M}$ is an induced substructure of $\mathbf{M}^\sharp$, hence the $\forall_{\mathrm{H}}$-class of $\mathcal{M}$ is contained within that of $\mathbf{M}^\sharp$. However $\mathbf{M}^\sharp$ is in the antivariety of $\mathbf{M}$, which by (3) implies it lies in the $\forall_{\mathrm{H}}$-class of $\mathcal{M}$.

We establish property (3) in the proof of the following lemma (which subsumes property (3) and (4)).

LEMMA 6. *Let* $\mathbf{M}$ *be a finite relational structure of finite signature* $\mathcal{R}$. *A structure* $\mathbf{A}$ *is in the antivariety of* $\mathbf{M}$ *if and only if it is in the* $\forall_{\mathrm{H}}$-*class of* $\mathbf{M}^\sharp$.

PROOF. If $\mathbf{A}$ lies in the $\forall_{\mathrm{H}}$-class $\mathsf{SP}(\mathbf{M}^\sharp)$ of $\mathbf{M}^\sharp$, then there exists at least one homomorphism from $\mathbf{A}$ to $\mathbf{M}^\sharp$. But as $\mathbf{M}^\sharp$ retracts onto $\mathbf{M}$ we have $\mathbf{A} \in \mathsf{H}^{-1}(\mathbf{M})$. So $\mathsf{SP}(\mathbf{M}^\sharp) \subseteq \mathsf{H}^{-1}(\mathbf{M})$.

Now we turn to the reverse inclusion, which is just property (3) for the class $\mathcal{M}$. Let $\phi : \mathbf{A} \to \mathbf{M}$ be a homomorphism. We must show that $\mathbf{A} \in \mathsf{SP}(\mathcal{M})$. Recall from the start of Subsection 2.1, that it suffices to show that $\mathbf{A}$ is residually separable into $\mathcal{M}$. We build the separating homomorphisms from ϕ.

First assume that $(b_1, \dots, b_r) \notin R^{\mathbf{A}}$. We construct a homomorphism $\phi' : \mathbf{A} \to \mathbf{N} \in \mathcal{M}$ with $(\phi'(b_1), \dots, \phi'(b_r)) \notin R^{\mathbf{N}}$. If $(\phi(b_1), \dots, \phi(b_r)) \notin R^{\mathbf{M}}$ then this follows because $\mathbf{M}$ is an induced substructure of every $\mathbf{N} \in \mathcal{M}$. Now assume that $(\phi(b_1), \dots, \phi(b_r)) \in R^{\mathbf{M}}$, and for notational convenience, let $a_1, \dots, a_r$ denote the points $\phi(b_1), \dots, \phi(b_r)$. Consider the structure $\mathbf{N}$ constructed according to the first recipe from the hyperedge $h = (a_1, \dots, a_r) \in R^{\mathbf{M}}$, with the equivalence relation θ defined by $a_{i,h}\ \theta\ a_{j,h}$ if $b_i = b_j$. Then the map $\phi' : \mathbf{A} \to \mathbf{N}$ defined by

$$\phi'(b) := \begin{cases} \phi(b) & \text{if}\, b \notin \{b_1, \dots, b_r\} \\ a_{i,h}/\theta & \text{if}\, b = b_i \end{cases}$$

is a homomorphism with $(\phi'(b_1), \dots, \phi'(b_r)) \notin R^{\mathbf{N}}$. To see why this is true, first note that the homomorphism ϕ is obtained by following the function ϕ' by the natural retract ν from $\mathbf{N}$ to $\mathbf{M}$. Thus the image of a hyperedge under ϕ' is always a hyperedge in the inflation $\mathbf{N}^+$ of $\mathbf{M}$ by ν. But $\mathbf{N}$ differs from $\mathbf{N}^+$ only by the single hyperedge $(\phi'(b_1), \ldots, \phi'(b_r)) \in R^{\mathbf{N}^+} \backslash R^{\mathbf{N}}$, and the choice of $\mathbf{N}$ was made precisely so that the only tuple mapping onto $(\phi'(b_1), \ldots, \phi'(b_r))$ was $(b_1, \ldots, b_r)$.

A very similar argument shows that if $a \neq b$ in $\mathbf{A}$, then we may adjust the map ϕ to make a homomorphism from $\mathbf{A}$ into a structure $\mathbf{N} \in \mathcal{M}$ constructed according to the second recipe. ∎

The following remark is of interest as we later show that the structure in Example 5 is a counterexample to the Gorbunov conjecture.

Remark 7. There is a 10-element generator $\mathbf{F}$ for the $\forall_{\mathrm{H}}$-class described in Example 5.

Proof. When applying the first construction to $(0,0) \in \sim$ using the equivalence relation of equality, there are two new points added, say a and b (the order will not matter for the following argument). Similarly, let c and d denote the two points added by applying the first recipe to the tuple $(1,1) \in \sim$ using the equivalence relation of equality. Now select one element from $\{a, b\}$, say a, and one from $\{c, d\}$, say c. Then the induced subgraph of $\mathbf{M}^\sharp$ on $M \cup \{a, c\}$ is isomorphic to the structure $\mathbf{N}$ obtained by applying the first recipe to the hyperedge $(0,1) \in \sim$. Thus applying the first recipe to the hyperedge $(0,1) \in \sim$ is redundant. Similarly there is no need to apply the second recipe to either of the points 0 or 1, because the resulting structures are isomorphic to the induced substructure of $\mathbf{M}^\sharp$ on $M \cup \{a\}$ and on $M \cup \{c\}$ respectively. Thus only 10 elements are required. We denote this structure by $\mathbf{F}$; it is depicted in Figure 1. ∎

The next lemma is essentially an extension of [8, Proposition 7.2] (which concerns colourings of topological simple graphs into complete graphs).

Lemma 8. *A Boolean topological structure* $\mathbf{A}$ *has a continuous homomorphism into* $\mathbf{M}$ *if and only if it is in the topological prevariety of* $\mathbf{M}^\sharp$*.*

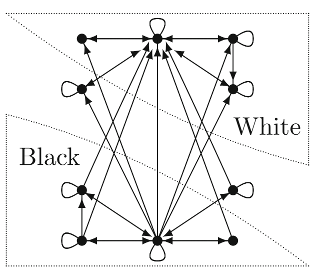

Figure 1. The structure $\mathbf{F}$: a 10-element generator for the universal Horn class of the structure $\mathbf{E}^\sharp$

Proof. If $\mathbf{A}$ is in the topological prevariety of $\mathbf{M}^\sharp$, then there is at least one continuous homomorphism from $\mathbf{A}$ into $\mathbf{M}^\sharp$, hence (as $\mathbf{M}$ is a retract of $\mathbf{M}^\sharp$), there is a continuous homomorphism from $\mathbf{A}$ into $\mathbf{M}$.

For the converse, we simply follow the proof of Lemma 6, demonstrating separating *continuous* homomorphisms from $\mathbf{A}$ into members of $\mathcal{M}$. Fix some continuous homomorphism $\phi : \mathbf{A} \to \mathbf{M}$ and consider some $(b_1, \ldots, b_r) \notin R^{\mathbf{A}}$. If $(\phi(b_1), \ldots, \phi(b_r)) \notin R^{\mathbf{M}}$ then there is nothing to do, so we may assume that $(\phi(b_1), \ldots, \phi(b_r)) \in R^{\mathbf{M}}$.

Let $D_1, \ldots, D_m$ be a list of the blocks of the kernel of ϕ, which by continuity must be clopen sets. For each $i \leq r$, let E_i be the member of this list that contains the element b_i (the b_i need not be distinct). As $(b_1, \ldots, b_r) \notin R^{\mathbf{A}}$ and $R^{\mathbf{A}}$ is closed in the topology on A^r, there are clopen sets $C_1, \ldots, C_r$ in A such that $(b_1, \ldots, b_r) \in (C_1 \times C_2 \times \cdots \times C_r) \subseteq A^r \backslash R^{\mathbf{A}}$. Since we can replace C_i by $C_i \cap E_i$, it does no harm to assume that $C_i \subseteq E_i$. Let $C := C_1 \cup \cdots \cup C_r$. Now (as in the proof of Lemma 6), let $a_1, \ldots, a_r$ denote $\phi(b_1), \ldots, \phi(b_r)$, select the structure $\mathbf{N} \in \mathcal{M}$ formed from the hyperedge $h = (\phi(b_1), \ldots, \phi(b_r)) \in R^{\mathbf{M}}$ and the equivalence relation θ formed on $\{a_{1,h}, \ldots, a_{r,h}\}$ by $a_{i,h} \ \theta \ a_{j,h}$ if $b_i = b_j$. Now define a map ϕ' from $\mathbf{A}$ to $\mathbf{N}$ by sending elements of $D_i \backslash C$ to $\phi(D_i)$, and sending $\phi(C_i)$ to $a_{i,h}/\theta$. This is continuous, as its kernel has clopen blocks. Also, $(\phi'(b_1), \ldots, \phi'(b_r)) \notin R^{\mathbf{N}}$. So it remains to show that ϕ' is a homomorphism, and here the argument is essentially the same as in Lemma 6 (the preimage of the tuple $(a_{1,h}/\theta, \ldots, a_{r,h}/\theta)$ is precisely $C_1 \times C_2 \times \cdots \times C_r$, which is disjoint from $R^{\mathbf{A}}$). ∎

5. The Homomorphism Order and Irredundant Bases

In this section we collect together some basic observations regarding the class of finite obstructions for an antivariety and provide an example of a finite structure with an infinite irredundant anti-identity basis, and a further structure with an infinite irredundant quasi-identity basis.

It is well known that an antivariety can be characterised by its finite obstructions: this is essentially Lemma 1. An obstruction $\mathbf{B}$ for an antivariety $\mathcal{A}$ is *critical* if every (not necessarily induced) substructure of $\mathbf{B}$ belongs to $\mathcal{A}$. This means that removing points and/or hyperedges from $\mathbf{B}$ produces a structure in $\mathcal{A}$. Note that any homomorphism between critical obstructions must be a surjection (moreover, all hyperedges in the codomain structure must be the image of a hyperedge in the domain structure). As the following lemma shows, critical obstructions are always finite.

Lemma 9. *Every antivariety $\mathcal{A}$ is characterised by its finite critical obstructions $\mathcal{C}$ in the sense that $\mathbf{A} \in \mathcal{A}$ if and only if there is no homomorphism from a member of $\mathcal{C}$ into $\mathbf{A}$.*

Proof. Certainly if a member of $\mathcal{C}$ admits a homomorphism into $\mathbf{A}$ then $\mathbf{A} \notin \mathcal{A}$. Conversely, if $\mathbf{A} \notin \mathcal{A}$ then it fails some anti-identity of $\mathcal{A}$. A failure of an n-variable anti-identity happens on some substructure of $\mathbf{B}$ of $\mathbf{A}$ with at most n elements; evidently, $\mathbf{B} \notin \mathcal{A}$ and any critical substructure of $\mathbf{B}$ is in $\mathcal{C}$ and obstructs $\mathbf{A}$. ■

Let $\mathcal{A}$ be an antivariety, and $\mathcal{A}_o$ denote a set formed by taking some representative of each isomorphism class from the class of finite obstructions for $\mathcal{A}$. The set $\mathcal{A}_o$ admits a well studied preorder, where $\mathbf{A} \leq \mathbf{B}$ means that there is a homomorphism from $\mathbf{A}$ into $\mathbf{B}$ (see for example, Chapter 3 of Hell and Nešetřil [16]). If $\mathbf{A} \leq \mathbf{B}$ and $\mathbf{B} \leq \mathbf{A}$ then $\mathbf{A}$ and $\mathbf{B}$ are *homomorphism equivalent*. Let $\mathcal{A}_c$ denote the subset of $\mathcal{A}_o$ consisting of the critical obstructions for $\mathcal{A}$. The homomorphism preorder restricted to $\mathcal{A}_c$ is an order. The following lemma is trivial.

Lemma 10. *A subset $\mathcal{O}$ of $\mathcal{A}_o$ is a complete set of obstructions for $\mathcal{A}$ if and only if the upset of $\mathcal{O}$ is equal to $\mathcal{A}_o$; equivalently, if and only if $\mathcal{A}_c \subseteq {\uparrow}\mathcal{O}$. In particular, every complete set of obstructions for $\mathcal{A}$ contains* (*up to homomorphism equivalence*) *all obstructions that are minimal in the homomorphism preorder on $\mathcal{A}_o$.*

We will say that an obstruction $\mathbf{O} \in \mathcal{A}_o$ for an antivariety $\mathcal{A}$ is *minimal* if it is minimal in the homomorphism preorder on $\mathcal{A}_o$: if $\mathbf{A} \notin \mathcal{A}$ and there is a homomorphism $\mathbf{A} \to \mathbf{O}$ then there is a homomorphism in the other direction $\mathbf{O} \to \mathbf{A}$. Note that as every obstruction embeds a critical obstruction, we can assume that minimal obstructions are critical.

It is well known that an ordered set satisfies the descending chain condition (DCC) if and only if every nonempty subset has a minimal element (see for example, [10, Lemma 2.39]). Thus a corollary of Lemma 10 is the following.

Lemma 11. *If the set of minimal obstructions* (*with respect to the homomorphism order*) *for an antivariety $\mathcal{A}$ is not a complete set of obstructions for $\mathcal{A}$ then there is an infinite strictly decreasing chain of obstructions for $\mathcal{A}$.*

Proof. By Lemma 10 there is an obstruction $\mathbf{O}$ not above any minimal obstruction for $\mathcal{A}$. Then the principal downset of $\mathbf{O}$ has no minimal element, hence the obstructions for $\mathcal{A}$ fail the DCC. ■

The following easy fact isn't strictly necessary, but demonstrates that a restriction to critical obstructions is a rather natural one from the perspective of complete sets of obstructions.

PROPOSITION 12. *Let $\mathcal{O}$ be a complete set of obstructions for an antivariety $\mathcal{A}$. If $\mathbf{O} \in \mathcal{O}$ is not homomorphism-equivalent to a critical obstruction for $\mathcal{A}$, then $\mathcal{O} \backslash \{\mathbf{O}\}$ is also a complete set of obstructions for $\mathcal{A}$.*

PROOF. Let $\mathbf{C}$ be a critical obstruction mapping into $\mathbf{O}$. By completeness there is $\mathbf{P} \in \mathcal{O}$ with $\mathbf{P} \to \mathbf{C}$. As $\mathbf{O}$ is not homomorphism equivalent to $\mathbf{C}$ we have $\mathbf{P} \neq \mathbf{O}$ and $\mathbf{P} \to \mathbf{O}$. Hence $\mathcal{O} \backslash \{\mathbf{O}\}$ is complete. ■

We will say that a class $\mathcal{C}$ of obstructions for an antivariety is *redundant* if there is a homomorphism between two distinct members of $\mathcal{C}$. Otherwise, $\mathcal{C}$ is irredundant.

LEMMA 13. *The following are equivalent for an antivariety $\mathcal{A}$*:

(1) *$\mathcal{A}$ has a complete and irredundant set of obstructions that are critical*;

(2) *$\mathcal{A}$ has a complete and irredundant set of obstructions*;

(3) *the minimal obstructions for $\mathcal{A}$ form a complete set of obstructions*;

(4) *the minimal obstructions for $\mathcal{A}$ form a complete set of obstructions and every complete irredundant set of obstructions is equivalent up to homomorphism equivalence to the set of minimal obstructions for $\mathcal{A}$*;

(5) *every complete set of obstructions contains a subset that is complete and irredundant.*

PROOF. (3)⇒(4) and (4)⇒(5) both follow from Lemma 10 while (5)⇒(1) follows because the critical obstructions for $\mathcal{A}$ are complete. (1)⇒(2) is trivial. We prove (2)⇒(3) in the contrapositive.

Assume that there is some obstruction $\mathbf{C}$ with the property that no minimal obstruction admits a homomorphism into $\mathbf{C}$. Consider any complete system $\mathcal{B}$ of finite obstructions. As $\mathbf{C}$ can be replaced by any critical obstruction $\mathbf{C}'$ admitting a homomorphism into $\mathbf{C}$, we may assume that $\mathbf{C}$ is critical. As $\mathcal{B}$ is complete, there is $\mathbf{B} \in \mathcal{B}$ with $\mathbf{B} \leq \mathbf{C}$. By assumption, $\mathbf{B}$ cannot be a minimal obstruction, so there is some critical obstruction $\mathbf{D}$ strictly lower than $\mathbf{B}$ in the homomorphism order. As $\mathcal{B}$ is complete there is $\mathbf{E} \in \mathcal{B}$ admitting a homomorphism into $\mathbf{D}$, whence also into $\mathbf{B}$. Then $\mathbf{E} \leq \mathbf{D} < \mathbf{B}$, so that $\mathcal{B} \backslash \{\mathbf{B}\}$ is also complete. Thus $\mathcal{B}$ is redundant. ■

The following is just a restatement of (2) ⇔ (4) ⇔ (5) in Lemma 13, interpreted in terms of anti-identities using Lemma 2.

Corollary 14. *Let $\mathcal{A}$ be an antivariety of relational structures. If $\mathcal{A}$ has an irredundant basis of anti-identities, then it is essentially unique, and every complete anti-identity basis for $\mathcal{A}$ contains a complete irredundant subset.*

In comparison to the first statement of Corollary 14, we mention that [18, Proposition 4.11] shows that a finite algebra can have uncountably many different irredundant equational bases. In comparison to the second statement, the following problem in equational logic appears to be open [18, Problem (4), p. 421]: is there a finitely generated variety of algebras with two identity bases, one irredundant and the other containing no equivalent irredundant subsystem?

Given a relational structure $\mathbf{T}$ construct the *incidence graph* of $\mathbf{T}$ on the set $T \cup H$, where H denotes the set of hyperedges; that is, expressions of the form $(a_1, \dots, a_n) \in R^{\mathbf{T}}$ for fundamental relations R. Construct the (undirected) edges of the graph on $T \cup H$ by connecting each element $a \in T$ to every hyperedge $(a_1, \dots, a_n) \in R^{\mathbf{T}}$ in which it appears (as some a_i). The structure $\mathbf{T}$ will be said to be a *generalised tree* if this incidence graph is a tree in the usual graph theoretic sense. A *generalised forest* is a disjoint union of generalised trees.

A structure $\mathbf{A}$ is said to have *tree duality* if it has a complete system of obstructions consisting of generalised trees (recall the definition of a homomorphism duality from Subsection 2.1). The following proposition is essentially a corollary of a famous result due to Erdős, as generalised by Feder and Vardi [15, Theorem 5]. Moreover it is just the general signature version of a statement in [13] (who prove this for antivarieties relativised to simple graphs). Amongst other things, it shows that the hypothesis of Theorem A holds quite frequently.

Proposition 15. *Let $\mathcal{A}$ be the antivariety generated by a finite relational structure $\mathbf{A}$. If $\mathbf{A}$ fails to have tree duality then $\mathcal{A}$ fails to have a complete irredundant set of obstructions; equivalently, it has no irredundant anti-identity basis.*

Proof. By assumption there is a critical obstruction $\mathbf{O}$ for $\mathbf{A}$ such that every generalised tree $\mathbf{T}$ with $\mathbf{T} \to \mathbf{O}$ has $\mathbf{T} \to \mathbf{A}$. By (2) $\Leftrightarrow$ (3) of Lemma 13, it suffices to prove that no minimal element of $\mathcal{A}_o$ obstructs $\mathbf{O}$.

Let $\mathbf{R} \notin \mathcal{A}$ admit a homomorphism into $\mathbf{O}$. By the result of Feder and Vardi [15, Theorem 5], for any n we can find an obstruction $\mathbf{P}$ satisfying all of the following:

(1) $\mathbf{P} \to \mathbf{R}$;

(2) $\mathbf{P} \in \mathcal{A}_o$;

(3) every n-element substructure of $\mathbf{P}$ is a generalised forest.

Let n be greater than $|R|$. Without loss of generality, we may assume that $\mathbf{P}$ is critical. If $\mathbf{S}$ is an $|R|$-element substructure of $\mathbf{P}$, then $\mathbf{S} \to \mathbf{R} \to \mathbf{O}$, and therefore $\mathbf{S} \to \mathbf{A}$ since $\mathbf{S}$ is a union of generalised trees. Thus no $|R|$-element substructure of $\mathbf{P}$ is an obstruction for $\mathbf{A}$. Therefore $\mathbf{R}$ does not obstruct $\mathbf{P}$, whence $\mathbf{P}$ is strictly lower than $\mathbf{R}$ in the homomorphism order on $\mathcal{A}_c$ and $\mathbf{R}$ is not minimal. ■

The tree duality property was characterised by Feder and Vardi [15, Theorem 21] and is known to be a rather strong restriction. All finite structures with finite duality (equivalently, a finite anti-identity basis) have tree duality, but with a finite complete set of tree obstructions. So any counterexample to Gorbunov's conjecture would have to have tree duality but not bounded tree duality. This indicates that, informally at least, counterexamples to Gorbunov's antivariety conjecture should be reasonably "uncommon". On the other hand, Nešetřil and Tardif [27] have shown that every finite set of (generalised) trees $\mathcal{F}$ is in a homomorphism duality with some finite structure $\mathbf{M}$ (see also [16, Theorem 3.37] where this is presented in the directed graph setting). However, the size of $\mathbf{M}$ grows with the number of elements of $\mathcal{F}$, thus one cannot start with an infinite antichain of trees and obtain a finite $\mathbf{M}$. We now present a counterexample to the antivariety version of Gorbunov's conjecture.

Example 16. The two-element structure $\mathbf{E} = \langle \{0,1\}; \sim, B, W \rangle$ from Example 5 has an infinite irredundant anti-identity basis.

Proof. Recall that $B = \{0\}$ and $W = \{1\}$. For each $n = 0, 1, 2, \ldots$ let $\mathbf{O}_n$ denote the structure on $\{0, 1, \ldots, n\}$, where n is black and 0 is white, and where $i \sim i + 1$ for $i = 0, \ldots, n - 1$. We leave it to the reader to verify that $\{\mathbf{O}_n \mid n = 0, 1, \ldots\}$ is a complete set of obstructions for $\mathbf{E}$ and that $\mathbf{O}_n \to \mathbf{O}_m$ if and only if $n = m$. ■

The next example provides a counterexample to the quasi-identity version of Gorbunov's conjecture.

Example 17. The structure $\mathbf{E}^\sharp$ constructed in Example 5 is a 14-element structure generating a quasivariety with an infinite irredundant quasi-identity

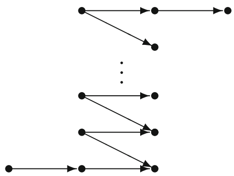

Figure 2. The directed graph $\mathbf{Z}_n$

basis. An equivalent generator for this quasivariety is the 10 element structure $\mathbf{F}$ constructed in Remark 7 and depicted in Figure 1.

PROOF. Consider an anti-identity $\bigvee_{0\leq i\leq n}\neg\phi_i$, and let x, y be new variables. It is well known and easy to see that a structure $\mathbf{M}$ with more than one element satisfies $\bigvee_{0\leq i\leq n}\neg\phi_i$ if and only if it satisfies $(\&_{0\leq i\leq n}\phi_i) \rightarrow x \approx y$. Thus the quasivariety generated by $\mathbf{E}^\sharp$ is a subclass of that defined by the following set of quasi-identities

$$\{(x_0\in W \ \& \ x_n\in B \ \& \ x_0\sim x_1\sim \cdots \sim x_n) \rightarrow x\approx y \mid n=0,1,2,\dots \quad \} \quad (*)$$

and differs only by some subset of the set of eight one-element structures in the signature $\{\sim, B, W\}$ (all of which trivially satisfy (*) because the conclusion of the implications cannot be falsified if there is only one element). Also, the only one-element structures that do not homomorphically map into $\mathbf{E}$ are the one-element total structure $\langle\{0\};\{(0,0)\},\{0\},\{0\}\rangle$ and the structure $\langle\{0\};\{\},\{0\},\{0\}\rangle$ (as no point of $\mathbf{E}$ is both black and white). By Lemma 6, the antivariety $\mathsf{H}^{-1}(\mathbf{E})$ coincides with the $\forall_{\mathrm{H}}$-class of $\mathbf{E}^\sharp$, whence the quasivariety generated by $\mathbf{E}^\sharp$ differs from the quasivariety defined by (*) only by the single structure $\langle\{0\};\{\},\{0\},\{0\}\rangle$. To eliminate this structure, we add one new quasi-identity to those in (*): the law $(x\in B \ \& \ x\in W)\rightarrow x\sim x$. The resulting system is irredundant and axiomatises the quasivariety of $\mathbf{E}^\sharp$.

Note that as $\mathbf{E}^\sharp$ and $\mathbf{F}$ generate the same $\forall_{\mathrm{H}}$-class, they also generate the same quasivariety. ■

We mention in passing that not every finite relational structure with tree duality also has an irredundant axiomatisation for its anti-identities. For example the three vertex directed path has the structures $\mathbf{Z}_n$ shown in Figure 2 as a complete set of critical obstructions. This system has no irredundant subsystem, so by Lemma 13, no irredundant anti-identity basis exists.

6. Proof of Theorem A

In this section we prove Theorem A, which we now reiterate for easy reference.

THEOREM A. *Let $\mathcal{A}$ be any antivariety of relational structures of finite type. If $\mathcal{A}$ fails to have an irredundant basis for its anti-identities, then $[\mathcal{A}]_{\mathcal{T}}$ contains a member for which there is no continuous map into a finite member of $\mathcal{A}$. Furthermore, the class $\mathsf{S_c P}(\mathcal{A}_{\text{fin}})$ is axiomatisable by neither $\forall$-sentences nor any single first order sentence.*

We first connect the homomorphism order on critical obstructions to continuous colourings.

LEMMA 18. *Let $\mathcal{A}$ be an antivariety for which the set of finite critical obstructions contains an infinite strictly descending chain under the homomorphism order. Then there is a structure $\mathbf{X} \in \mathcal{A}$ admitting a compatible Boolean topology under which $\mathbf{X}$ cannot be continuously coloured onto a finite member of $\mathcal{A}$. Specifically, $\mathbf{X}$ can be taken to be the inverse limit over the given strictly descending chain.*

PROOF. This is a straightforward application of one of the main techniques in [8]. Let $\mathbf{C}_1 > \mathbf{C}_2 > \mathbf{C}_3 > \cdots$ be a strictly descending chain of critical obstructions. As $|C_{i+1}| > |C_i|$ (by criticality) we may assume without loss of generality that $|C_n| > n$ for each $n \in \mathbb{N}$. Hence for each n, the n-element substructures of $\mathbf{C}_n$ are in $\mathcal{A}$ but no homomorphisms exist from $\mathbf{C}_n$ into any member of $\mathcal{A}$. The Second Inverse Limit Technique of Clark et al. [8, SILT 3.9] shows that the inverse limit $\mathbf{X}$ over $\mathbf{C}_1 \leftarrow \mathbf{C}_2 \leftarrow \cdots$ is a Boolean topological structure whose underlying nontopological structure is in $\mathcal{A}$. However, every continuous homomorphism from $\mathbf{X}$ into a finite discrete structure $\mathbf{S}$ factors through a homomorphism from some $\mathbf{C}_i$ to $\mathbf{S}$. (In [8] this part of the technique is listed as Lemma 3.2.) As the $\mathbf{C}_i$ are obstructions for $\mathcal{A}$, it follows that no continuous homomorphism exists from $\mathbf{X}$ into any finite member of $\mathcal{A}$. ■

Now we prove the first part of Theorem A and the non-$\forall$-axiomatisability claim; we withhold the proof of the statement about non-first-order axiomatisability until Proposition 19.

PROOF. Let $\mathcal{A}$ be an antivariety of relational structures of finite type but without an irredundant anti-identity basis. By Lemma 13, the set of minimal critical obstructions for $\mathcal{A}$ is not complete. By Lemma 11 there is a strictly decreasing chain of critical obstructions for $\mathcal{A}$ and then by Lemma 18 there is a structure $\mathbf{X} \in \mathcal{A}$ admitting a compatible Boolean topology under which $\mathbf{X}$ cannot be continuously coloured onto a finite member of $\mathcal{A}$, as required. The fact that $\mathsf{S_cP}(\mathcal{A}_{\text{fin}})$ is not $\forall$-axiomatisable is one of the conclusions of [8, SILT 3.9] (which we used to construct $\mathbf{X}$). ∎

In [8, Theorem 4.2] it is shown that every finite lattice $\mathbf{L}$ has the property that there is a single first order sentence Φ such that $\mathrm{Mod}_{\mathcal{T}}(\Phi) = \mathsf{S_cP}(\mathbf{L})$, even though there may not be any set Σ of $\forall$-sentences with $\mathrm{Mod}_{\mathcal{T}}(\Sigma) = \mathsf{S_cP}(\mathbf{L})$. The following proposition establishes the last statement in Theorem A, which shows that such behaviour cannot be exhibited by an antivariety without an irredundant anti-identity basis.

Proposition 19. *Let $\mathcal{A}$ be an antivariety with no finite basis for its anti-identities. There is no first order sentence Φ such that $\mathsf{S_cP}(\mathcal{A}_{\text{fin}}) = \mathrm{Mod}_{\mathcal{T}}(\Phi)$.*

Proof. Say that Φ is a sentence such that $\mathrm{Mod}_{\mathcal{T}}(\Phi) = \mathsf{S_cP}(\mathcal{A}_{\text{fin}})$. So the class of finite models of Φ is $\mathcal{A}_{\text{fin}}$, and the class of finite models of $\neg\Phi$ is closed under taking homomorphic images. The remainder of the argument is alluded to by Atserias [2, §5]. By the recently proved Homomorphism Preservation Theorem of finite model theory (Rossman [29]), we have that $\neg\Phi$ is equivalent, for finite structures, to an existential positive sentence Ψ. Hence $\neg\Psi$ is logically equivalent, for finite structures, to a single $\forall$-sentence (in fact to a finite conjunction of anti-identities). Hence $\mathcal{A}_{\text{fin}}$ has only finitely many critical obstructions, showing that $\mathcal{A}$ has a finite axiomatisation by Lemma 1. ∎

This does not show that $\mathsf{S_cP}(\mathcal{A}_{\text{fin}})$ is not first order axiomatisable (amongst Boolean topological structures) because there may be an infinite set of sentences characterising $\mathsf{S_cP}(\mathcal{A}_{\text{fin}})$. However, Theorem B will rule out even this, provided that $\mathcal{A}$ is generated by a single finite relational structure.

7. Proof of Theorem B

Theorem B. *Let $\mathbf{M}$ be any finite relational structure of finite type and $\mathcal{A}$ be the antivariety generated by $\mathbf{M}$. If $\mathbf{M}$ has no finite basis for its anti-identities, then $[\mathcal{A}]_{\mathcal{T}}$ contains a member that is not continuously $\mathbf{M}$-colourable. Moreover, the class of continuously $\mathbf{M}$-colourable structures is not first order axiomatisable.*

The *n-link* in the type $\mathcal{R}$ is the structure $\mathbf{L}_n$ on the set $\{0, 1, \dots, n\}$ with each relation $R \in \mathcal{R}$ defined by $R^{\mathbf{L}_n} = \cup_{j=0}^{n-1}\{j, j+1\}^r$. We also consider the ω-link on $\{0, 1, \ldots, \omega\}$ with $R^{\mathbf{L}_\omega} = \{\omega\}^r \cup \bigcup_{j=0}^{\omega} \{j, j+1\}^r$. Now, for $n \in \mathbb{N} \cup \{\omega\}$, let us define a structure $\mathbf{P}_n$ as follows. First consider the direct product structure $\mathbf{L}_n \times \mathbf{M} \times \mathbf{M}$. Define an equivalence relation $\sim_n$ by

$$(i, a, b) \sim (i', a', b') \text{ if } \begin{cases} (i, a, b) = (i', a', b'), & \text{or} \\ i = i' = 0 \text{ and } a = a', & \text{or} \\ i = i' = n \text{ and } b = b'. \end{cases}$$

The structure $\mathbf{P}_n(\mathbf{M})$ is given by $(\mathbf{L}_n \times \mathbf{M} \times \mathbf{M})/{\sim_n}$. When no confusion arises, we abbreviate $\mathbf{P}_n(\mathbf{M})$ to $\mathbf{P}_n$ and refer to it as the *n-pinch* over $\mathbf{M}$. Larose, Loten and Tardif [22, 23] show that the finite duality property for $\mathrm{CSP}(\mathbf{M})$ (hence, equivalently, the finite axiomatisability of the antivariety of $\mathbf{M}$) is equivalent to testing membership of $\mathbf{P}_n$ in $\mathrm{CSP}(\mathbf{M})$ for some $n \in \mathbb{N}$ less than or equal to $|M|^{|M|^2}$. On the other hand, $\mathbf{P}_\omega$ always admits a homomorphism into $\mathbf{M}$: map $(i, a, b) \mapsto a$ if $i \in \omega$ and $(\omega, a, b) \mapsto b$.

The following facts are proved in [23]; the *diameter* of a relational structure is defined to be half the diameter of its incidence graph.

Theorem 20. (1) ([23, Proposition 4.1]) *For any n, if $\mathbf{C}$ is a critical obstruction for* $\mathrm{CSP}(\mathbf{M})$ *and has diameter at least n, then there is a homomorphism from $\mathbf{C}$ into $\mathbf{P}_n$.*

(2) ([23, Lemma 4.6][1]) *For any n, the n-element induced substructures of $\mathbf{P}_n$ are contained in* $\mathrm{CSP}(\mathbf{M})$.

(3) ([23, Theorem 4.7]) *The following are equivalent*:

(a) $\mathrm{CSP}(\mathbf{M})$ *has finite duality*;

(b) *there exists $n \in \mathbb{N}$ such that $\mathbf{P}_n$ is contained in* $\mathrm{CSP}(\mathbf{M})$;

(c) *$\mathbf{P}_k$ is contained in* $\mathrm{CSP}(\mathbf{M})$ *for some $k \leq |M|^{|M|^2}$.*

For each $n \in \mathbb{N}$ define a map $\phi_n : \mathbf{P}_{n+1} \to \mathbf{P}_n$ by

$$(i, a, b)/{\sim_{n+1}} \mapsto \begin{cases} (i, a, b)/{\sim_n} & \text{if } i \leq n \\ (n, a, b)/{\sim_n} & \text{if } i = n+1. \end{cases}$$

This is trivially seen to be well defined. The following lemma is also routine.

Lemma 21. *The map $\phi_n : \mathbf{P}_{n+1}(\mathbf{M}) \to \mathbf{P}_n(\mathbf{M})$ is a surjective homomorphism.*

[1] Strictly speaking, Lemma 4.6 of [23] only shows that certain substructures of $\mathbf{P}_n$ admit homomorphisms into $\mathbf{M}$. However it is easy to see that any substructure of $\mathbf{P}_n$ on at most n elements is a substructure of the disjoint union of the two key substructures described in [23, Lemma 4.6].

Proof. Say that $(p_1, \dots, p_r) \in R^{\mathbf{P}_{n+1}}$, where $p_j = (i_j, a_j, b_j)/{\sim_{n+1}}$. If there is no $j \leq r$ such that $i_j = n+1$, then it is trivial that $(\phi_n(p_1), \dots, \phi_n(p_r)) \in R^{\mathbf{P}_n}$. Otherwise, $\{n+1\} \subseteq \{i_1, \dots, i_r\} \subseteq \{n, n+1\}$ and $(b_1, \dots, b_r) \in R^{\mathbf{M}}$. Then $\phi(p_j) = (n, a_j, b_j)/{\sim_n} = (n, b_j, b_j)/{\sim_n}$ and because $(b_1, \dots, b_r) \in R^{\mathbf{M}}$, we have

$$(\phi_n(p_1), \dots, \phi_n(p_r)) = ((n, b_1, b_1)/{\sim_n}, \dots, (n, b_r, b_r)/{\sim_n}) \in R^{\mathbf{P}_n}$$

as required. The surjectivity of ϕ_n is trivial. ∎

The homomorphisms ϕ_n and their composites provide an inverse system on $\{\mathbf{P}_n(\mathbf{M}) \mid n \in \mathbb{N}\}$. The inverse limit of this system is rather obviously equal to $\mathbf{P}_\omega(\mathbf{M})$ as a nontopological structure. The inherited topology is that of a disjoint union of $|M|$ distinct one-point compactifications of infinite discrete spaces with an $|M|$-element discrete space. A more precise consideration of the topology is given below, however a rough description is that all points except those in $\{(\omega, a, b)/{\sim_\omega} \mid a, b \in M\}$ are isolated (form clopen singletons) and that each point $(\omega, a, b)/{\sim_\omega}$ is a limit point of the set $\{(i, a', b)/{\sim_\omega} \mid a' \in M, i \in \mathbb{N}\}$.

Proposition 22. *There is a continuous homomorphism from $\mathbf{P}_\omega$ to $\mathbf{M}$ if and only if there is $n \in \mathbb{N}$ such that there is a homomorphism from $\mathbf{P}_n$ into $\mathbf{M}$. In particular,* CSP($\mathbf{M}$) *has finite duality if and only if $\mathbf{P}_\omega$ is continuously $\mathbf{M}$-colourable.*

Proof. If there exists a homomorphism from $\mathbf{P}_n$ to $\mathbf{M}$, then as the natural map from $\mathbf{P}_\omega$ onto $\mathbf{P}_n$ is continuous, there is a continuous map from $\mathbf{P}_\omega$ into $\mathbf{M}$. Conversely, assume no $\mathbf{P}_n$ admits a homomorphism into $\mathbf{M}$. We now follow the proof of Lemma 18 (the lemma does not directly apply as the $\mathbf{P}_i$ are not critical). By Theorem 20(2), every n-element substructure of $\mathbf{P}_n$ admits a homomorphism into $\mathbf{M}$. As $\mathbf{P}_i \geq \mathbf{P}_{i+1}$ by Lemma 21, we have that $\mathbf{P}_1 \geq \mathbf{P}_2 \geq \dots$ is an unbounded descending chain of obstructions in the homomorphism order. Now [8, Lemma 3.2] shows that the the inverse limit $\mathbf{P}_\omega$ has no continuous homomorphism into $\mathbf{M}$, as in the proof of Lemma 18. ∎

Now [8, SILT 3.9] would also show that $\mathbf{P}_\omega$ is in the antivariety of $\mathbf{M}$ provided topology is ignored; however we've already made direct observation of this. Then Proposition 22 shows that $\mathbf{M}$ is nonstandard in the sense of [8]. Theorem B, however, refers to the non first order axiomatisability of the continuously $\mathbf{M}$-colourable Boolean topological structures, which requires separate proof. For this we could construct an application of the Second Ultraproduct Technique of [8, SUPT 5.3]. However we instead provide an alternative approach, which takes about the same amount of work and might have some independent interest.

The following concept is used in [8] and is useful here also.

Definition 23. Let (S, λ) be a set S with a unary operation $\lambda : S \to S$ satisfying $\lambda \circ \lambda = \lambda$ and with λ fixing only finitely many points $\{s_1, \dots, s_n\}$. The *λ-topology on S*, or the *topology induced by* λ, is the Boolean topology on S formed by taking the topological sum of the one-point compactification spaces on $\lambda^{-1}(s_i)$, with s_i the compactification point for the (possibly empty) discrete space $\lambda^{-1}(s_i)\backslash\{s_i\}$.

To elucidate further: if $\lambda^{-1}(s_i)$ is finite, then the topology induced by λ is discrete, while if $\lambda^{-1}(s_i)$ is infinite then its clopen sets consist of the finite subsets of $\lambda^{-1}(s_i)\backslash\{s_i\}$ and the cofinite subsets of $\lambda^{-1}(s_i)$ containing s_i.

Let $\mathbf{X} = \langle X; \mathcal{R}\rangle$ be a relational structure. For $R \in \mathcal{R}$ and $x \in X$, we define the set $R(x) = \{(a_1, \dots, a_n) \in R^{\mathbf{X}} \mid a_i = x \text{ for some } i\}$. We define the *degree* of x to be the maximum of the cardinalities of the sets $R(x)$ for $R \in \mathcal{R}$. We say $\mathbf{X}$ has *finite degree* if every vertex has finite degree.

Theorem 24. *Let $\mathcal{A}$ be the antivariety generated by a finite relational structure $\mathbf{M} = \langle M; \mathcal{R}\rangle$ and suppose $\mathsf{S_c P}(\mathcal{A}_{\text{fin}})$ is not $\forall_{\text{H}}$-axiomatisable; so there exists a Boolean topological structure $\mathbf{X}$ such that $\mathbf{X} \in \mathcal{A}$ but there is no continuous homomorphism from $\mathbf{X}$ to $\mathbf{M}$. If $\mathbf{X}$ has finite degree, then $\mathsf{S_c P}(\mathcal{A}_{\text{fin}})$ is not first-order axiomatisable.*

Proof. Since $\mathbf{X} \in \mathcal{A}$, there is a homomorphism $\phi : \mathbf{X} \to \mathbf{M}$. If $\mathbf{M}$ has the discrete topology, then clearly the disjoint union $\mathbf{X} \mathbin{\dot\cup} \mathbf{M}$ is not in $\mathsf{S_c P}(\mathcal{A}_{\text{fin}})$. We now describe another Boolean topology on the structure $\mathbf{X} \mathbin{\dot\cup} \mathbf{M}$ for which there exists a continuous homomorphism from $\mathbf{X} \mathbin{\dot\cup} \mathbf{M}$ to $\mathbf{M} \in \mathcal{A}_{\text{fin}}$. We use the concept in Definition 23. Write $\mathbf{Y} := \mathbf{X} \mathbin{\dot\cup} \mathbf{M}$, and let $\lambda : Y \to Y$ be the map such that $\lambda(x) = \phi(x)$ for $x \in X$ and $\lambda(x) = x$ for $x \in M$. Note that λ is a homomorphism. We show that the topology induced by λ is compatible with the relations of R.

Let $R \in \mathcal{R}$ and suppose $(a_1, \dots, a_n) \notin R^{\mathbf{X}}$. We find an open set $U \ni (a_1, \dots, a_n)$ such that U is disjoint from $R^{\mathbf{X}}$. For each $i \le n$, define U_i in the following way. If $a_i \notin f(Y)$, let $U_i = \{a_i\}$. Otherwise, let

$$U_i := \lambda^{-1}(a_i) \setminus \{b \mid (a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) \in R^{\mathbf{X}}\}.$$

Since $\mathbf{X}$ has finite degree, each U_i is open. Thus $U = U_1 \times \cdots \times U_n$ is open. Also, U contains $(a_1, \dots, a_n)$. If $a_1, \dots, a_n \in \lambda(Y)$ then, since λ is a homomorphism, U is disjoint from R as required. Otherwise, $a_i \notin \lambda(Y)$ for some i. Since a_i has finite degree, U contains only finitely many elements of R.

So $V := U \backslash R$ is open, contains $(a_1, \ldots, a_n)$, and is disjoint from R. Hence R is closed, as required.

We have shown that $\mathbf{Y}$ with the compactification induced by λ is a Boolean topological structure. Since λ is a continuous map from $\mathbf{Y}$ to $\mathbf{M}$, the structure $\mathbf{Y}$ with the compactification topology is in $\mathsf{S_c P}(\mathcal{A}_{\text{fin}})$. ∎

The final statement of Theorem B now follows immediately from Theorem 24, as $\mathbf{P}_\omega$ has finite degree.

The non-first order axiomatisability of the $\mathsf{S_c P}$-class of the complete graph K_n (for $n > 1$) is given already in [8, Theorem 7.6]. The nonstandardness of (loopless) graphs has been completely classified by Trotta [33], although first order axiomatisability of the corresponding topological prevarieties remains open for all but a handful of cases and those covered by Theorem B.

8. Proof of Theorem C

Theorem C. *The following are equivalent for a finite bipartite digraph* $\mathbf{B}$:

(1) $\mathbf{B}$ *has a finite basis for its anti-identities*;

(2) $\mathbf{B}$ *is strictly bipartite*;

(3) *every Boolean topological digraph that is* $\mathbf{B}$*-colourable is topologically* $\mathbf{B}$*-colourable*;

(4) *there is a set* Σ *of first order sentences in the language of digraphs such that a Boolean topological digraph* $\mathbf{X}$ *is topologically* $\mathbf{B}$*-colourable if and only if* $\mathbf{X} \models \Sigma$.

If $\mathbf{A}$ is a digraph, define the digraphs ${}^1\mathbf{A}^n$ for $n \in \mathbb{N}$ as follows. The underlying set of ${}^1\mathbf{A}^n$ is A^n, and the edge relation consists of those pairs (a, b) such that there is at most one $i \leq n$ with $a(i) \not\sim b(i)$. Note that ${}^1\mathbf{A}^1$ is not isomorphic to $\mathbf{A}$ except in degenerate cases. In [23] the construction ${}^1\mathbf{A}^n$ is called the *1-tolerant* n^{th} *power* of $\mathbf{A}$ (the construction is defined for arbitrary structures, but we only use it for digraphs here).

The following fact will be used in the proof of Theorem C.

Lemma 25. [23, Corollary 4.3] *Let* $\mathbf{A}$ *be a finite digraph. Then* $\mathbf{A}$ *has finite duality if and only if there is a homomorphism mapping* ${}^1\mathbf{A}^n$ *to* $\mathbf{A}$ *for some* $n \in \mathbb{N}$.

We first prove the equivalence of (1) and (2) in Theorem C. If $\mathbf{B}$ is strictly bipartite, then either it has no edges (which can be axiomatised by $\forall x \forall y\ x \not\sim y$), or it retracts onto a single edge (which can be axiomatised

$$x^n \to y^n \to yz^{n-1} \leftarrow x^2y^{n-2} \to y^3z^{n-3} \leftarrow x^4y^{n-2} \cdots \leftarrow x^{n-2}y^2 \to y^{n-1}z \leftarrow x^n$$

Figure 3. The structure **S**

by the property $\forall x \forall y \forall z\ x \not\sim y \vee y \not\sim z$). So (2) implies (1). Now say that (2) fails. In this case there are vertices x, y, z with $x \sim y \sim z$ (possibly x equals z; this does not matter). We show that ${}^1\mathbf{B}^n$ does not map into **B** (for every n) and apply Lemma 25. As ${}^1\mathbf{B}^{n+1}$ maps homomorphically onto ${}^1\mathbf{B}^n$ it suffices to show that ${}^1\mathbf{B}^n$ does not map into **B** in the case that n is even. Consider the substructure, **S**, of ${}^1\mathbf{B}^n$ (where n is even) shown in Figure 3. We use the notations $a \to b$ and $a \leftarrow b$ to denote the presence of the directed edges $a \sim b$ and $b \sim a$ (respectively), and for any positive integers k, j with $k + j = n$, we write $a^k b^j$ to denote the tuple c of ${}^1\mathbf{B}^n$ whose first k coordinates are a and whose remaining j coordinates are b.

Now, since the symmetric closure of S is an odd cycle, ${}^1\mathbf{B}^n$ is not bipartite, and hence it admits no homomorphism to **B**. Thus by Lemma 25, it follows that (1) fails (that is, the antivariety of **B** has no finite basis for its anti-identities). Thus (1) and (2) are equivalent.

The implications (3) implies (1) and (4) implies (1) are just Theorem B for bipartite digraphs. Also, (3) implies (4) is trivial (let Σ be the $\forall_{\mathrm{H}}$-theory of **B**). So it remains to show (2) implies (3).

If **B** is strictly bipartite, then either it has no edges at all, or it retracts onto a single edge. We consider these two cases separately. If **B** has no edges, then (3) holds trivially. If **B** retracts onto a single edge, then we may assume without loss of generality that **B** is the single directed edge graph $\langle \{0, 1\}; \{(0, 1)\} \rangle$. We must show that Boolean topological digraphs in $[\mathsf{H}^{-1}(B)]_{\mathcal{T}}$ can be mapped by a continuous homomorphism into **B**. This can be extracted from Trotta [34, Theorem 2.12], but the direct argument is just as short. If **G** is a Boolean topological digraph in $[\mathsf{H}^{-1}(B)]_{\mathcal{T}}$, then let X be the set of points with an outwards edge, and Y be the set of points with an inwards edge. Now X and Y are closed and disjoint, so there is a clopen U containing X and disjoint from Y. The map sending U to 0 and $G \backslash U$ to 1 is a continuous homomorphism from **G** to **B**. Thus (3) holds for **B**, completing the proof that (2) implies (3) (and then (4)), which in turn completes the proof of Theorem C.

9. Some Problems

We finish the article with two problems arising from the article. The first relates to irredundant axiomatisations. The phrase "axiomatisation" is frequently used to refer to any set of laws defining a class of interest (see for example Burris and Sankappanavar [4, Definition V.2.15]). A stricter definition of axiomatisation however, requires it to be a recursively enumerable set of defining laws (see [3, pp. 191] for example). The distinction is uninteresting in many cases of finite structures, since for many natural kinds of finitely generated classes, recursively enumerable axiomatisations always exist. For example, the set of all anti-identities (over some fixed countably infinite set of variables) satisfied by any finite structure is a recursive set, whence an axiomatisation, albeit one with many redundancies. In the infinite case, however, the situation is different—witness Gödel's Incompleteness Theorem for example. With the restriction of irredundancy of axioms, the situation seems unclear even for finite structures.

PROBLEM 26. Is it true that every finite structure of finite type either has no irredundant basis for its anti-identities, or has a recursively enumerable irredundant basis for its anti-identities?

The identity basis analogue of this problem (for algebras) is [18, Problem 3].

The following question might possibly have a positive answer, in view of the positive solution in the antivariety setting (Theorem 3 above, due to Larose, Loton and Tardiff [23]).

PROBLEM 27. Is the following problem decidable: given a finite relational structure of finite type, does the quasivariety generated by $\mathbf{M}$ have a finite quasi-identity basis?

Finally, we recall again that Gorbunov's conjecture is still unresolved in the case of algebraic structures, though the evidence in its favour seems rather weakened by Example 16 above. Similarly, in this article we have given a partial positive solution to Problem 4, but the full version remains unresolved.

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References

[1] ADAMS, M. E., K. V. ADARICHEVA, W. DZIOBIAK, and A. V. KRAVCHENKO, Open questions related to the Problem of Birkhoff and Maltsev, *Studia Logica* 78:357–378, 2004.

[2] Atserias, A., On digraph coloring problems and treewidth duality, *European Journal of Combination* 29:796–820, 2008 [Preliminary version in LICS 2005, pp. 106–115, 2005.].

[3] Boolos, G. S., J. P. Burgess, and R. C. Jeffrey, *Computability and Logic*, 4th ed., Cambridge University Press, 2002.

[4] Burris, S., and H. P. Sankappanavar, *A Course in Universal Algebra*, Graduate Texts in Mathematics 78, Springer Verlag, 1980.

[5] Clark, D. M., and D. A. Davey, *Natural Dualities for the Working Algebraist*, Cambridge University Press, Cambridge, 1998.

[6] Clark, D. M., B. A. Davey, M. Haviar, J. G. Pitkethly, and M. R. Talukder, Standard topological quasi-varieties, *Houston Journal of Mathematics* 29:859–887, 2003.

[7] Clark, D. M., B. A. Davey, R. S. Freese, and M. Jackson, Standard topological algebras: Syntactic and principal congruences and profiniteness, *Algebra Universalis* 52:343–376, 2004.

[8] Clark, D. M., B. A. Davey, M. G. Jackson, and J. G. Pitkethly, The axiomatizability of topological prevarieties, *Advances in Mathematics* 218:1604–1653, 2008.

[9] Clinkenbeard, D. J., Simple compact topological lattices, *Algebra Universalis* 9:322–328, 1979.

[10] Davey, B. A., and H. A. Priestley, *An Introduction to Lattices and Order*, 2nd ed., Cambridge University Press, 2002.

[11] Eilenberg, S., and M. P. Schützenberger, On pseudovarieties, *Advances in Mathematics* 19:413–418, 1976.

[12] Gorbunov, V. A., *Algebraic Theory of Quasivarieties*, Consultants Bureau, New York, 1998.

[13] Gorbunov, V., and A. Kravchenko, Antivarieties and colour-families of graphs, *Algebra Universalis* 46:43–67, 2001.

[14] Gorbunov, V. A., and D. M. Smirnov, Finite algebras and the general theory of quasivarieties, in *Finite Algebra and Multiple-Valued Logic*, Szeged (Hungry 1979), pp. 325–332, *Colloquim of Mathematical Society János Bolyai*, 28. North-Holland, Amsterdam-New York, 1981.

[15] Feder, T., and M. Y. Vardi, The Computational Structure of Monotone Monadic SNP and Constraint Satisfaction: A Study through Datalog and Group Theory, *SIAM Journal on Computing* 28:57–104, 1998.

[16] Hell, P., and J. Nešetřil, *Graphs and Homomorphisms*, Oxford Lecture Series in Mathematics and its Applications 28, Oxford University Press, 2004.

[17] Jackson, M., Finiteness properties and the restriction to finite semigroups, *Semigroup Forum* 70:159–187, 2005.

[18] Jackson, M., Finite semigroups with infinite irredundant identity bases, *International Journal of Algebra and Computation* 15:405–422, 2005.

[19] Jackson, M., Flat algebras and the translation of universal Horn logic to equational logic, *Journal of Symbolic Logic*, 73:90–128, 2008.

[20] Jackson, M., Residual bounds for compact totally disconnected algebras, *Houston Journal of Mathematics*, 34:33–67, 2008.

[21] Johnstone, P. T., *Stone Spaces*, Cambridge University Press, Cambridge, 1982.

[22] Larose, B., C. Loten, and C. Tardif, A characterisation of first-order constraint satisfaction problems, in *Proceedings of the 21st IEEE Symposium on Logic in Computer Science (LICS'06),* IEEE, 2006, pp. 201–210.

[23] Larose, B., C. Loten, and C. Tardif, A characterisation of first-order constraint satisfaction problems, *Logical Methods in Computer Science* 3(4–6):1–22, 2007.

[24] McKenzie, R., Tarski's finite basis problem is undecidable, *International Journal of Algebra and Computation* 6:49–104, 1996.

[25] McNulty, G. F., Z. Szekely, and R. Willard, Equational complexity of the finite algebra membership problem, *International Journal of Algebra and Computation* 18:1283–1319, 2008.

[26] Nešetřil, J., and A. Pultr, On classes of relations and graphs determined by subobjects and factorobjects, *Discrete Mathematics* 22:287–300, 1978.

[27] Nešetřil, J., and C. Tardif, Duality theorems for finite structures (characterising gaps and good characterizations), *Journal of Combinatorial Theory B* 80:80–97, 2000.

[28] Numakura, K., Theorems on compact totally disconnected semigroups and lattices, *Proceedings of American Mathematical Society* 8:623–626, 1957.

[29] Rossman, B., Homomorphism Preservation Theorems, *Journal of the ACM* 55(3):53, Art. 15, 2008.

[30] Sapir, M., On the quasivarieties generated by finite semigroups, *Semigroup Forum* 20:73–88, 1980.

[31] Stronkowski, M. M., Quasi-equational bases for graphs of semigroups, monoids and groups. *Semigroup Forum* 82:296–306, 2011.

[32] Trotta, B., Residual properties of reflexive anti-symmetric digraphs, *Houston Journal of Mathematics* 37:27–46, 2011.

[33] Trotta, B., Residual properties of simple graphs, *Bulletin of Australian Mathematical Society* 82:488–504, 2010.

[34] Trotta, B., Residual properties of pre-bipartite digraphs, *Algebra Universalis*, 64:161–186, 2010.

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