Matteo Bianchi

The Variety Generated by all the Ordinal Sums of Perfect MV-Chains

Dedicated to my friend Erika, and to her invaluable talent in finding surprisingly deep connections among poetry, art, philosophy and logic.

Abstract. We present the logic $\mathrm{BL}_{\mathrm{Chang}}$, an axiomatic extension of BL (see [23]) whose corresponding algebras form the smallest variety containing all the ordinal sums of perfect MV-chains. We will analyze this logic and the corresponding algebraic semantics in the propositional and in the first-order case. As we will see, moreover, the variety of $\mathrm{BL}_{\mathrm{Chang}}$-algebras will be strictly connected to the one generated by Chang's MV-algebra (that is, the variety generated by all the perfect MV-algebras): we will also give some new results concerning these last structures and their logic.

Keywords: Many-valued logics, BL-algebras, Perfect MV-algebras, Łukasiewicz logic, Basic Logic, Wajsberg hoops.

1. Introduction

MV-algebras were introduced in [11] as the algebraic counterpart of Łukasiewicz (infinite-valued) logic. During the years these structures have been intensively studied (for a historical overview, see [12]): the book [13] is a reference monograph on this topic.

Perfect MV-algebras were firstly studied in [6] as a refinement of the notion of local MV-algebras: this analysis was expanded in [18], where it was also shown that the class of perfect MV-algebras $Perf(MV)$ does not form a variety, and the variety generated by $Perf(MV)$ is also generated by Chang's MV-algebra (see section 2.2 for the definition). Further studies, about this variety and the associated logic have been done in [4,5].

On the other side, Basic Logic BL and its correspondent variety, BL-algebras, were introduced in [23]: Łukasiewicz logic results to be one of the

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axiomatic extensions of BL and MV-algebras can also be defined as a subclass of BL-algebras. Moreover, the connection between MV-algebras and BL-algebras is even stronger: in fact, as shown in [2], every ordinal sum of MV-chains is a BL-chain.

For these reasons one can ask if there is a variety of BL-algebras whose chains are (isomorphic to) ordinal sums of perfect MV-chains: even if the answer to this question is negative, we will present the smallest variety (whose correspondent logic is called $\mathrm{BL}_{\mathrm{Chang}}$) containing this class of BL-chains.

As we have anticipated in the abstract, there is a connection between the variety of $\mathrm{BL}_{\mathrm{Chang}}$-algebras and the one generated by Chang's MV-algebra. In fact the first-one is axiomatized (over the variety of BL-algebras) with an equation that, over MV-algebras, is equivalent to the one that axiomatize the variety generated by Chang MV-algebras: however, the two equations are *not* equivalent, over BL.

The paper is structured as follows: in section 2 we introduce the necessary logical and algebraic background: moreover some basic results about perfect MV-algebras and other structures will be listed. In section 3 we introduce the main theme of the article: the variety of $\mathrm{BL}_{\mathrm{Chang}}$ and the associated logic. The analysis will be done in the propositional case: completeness results, algebraic and logical properties and also some results about the variety generated by Chang's MV-algebra. We conclude with section 4, where we will analyze the first-order versions of $\mathrm{BL}_{\mathrm{Chang}}$ and $\text{Ł}_{\mathrm{Chang}}$: for the first-one the completeness results will be much more negative.

To conclude, we list the main results.

- $\mathrm{BL}_{\mathrm{Chang}}$ enjoys the finite strong completeness (but not the strong one) w.r.t. $\omega\mathcal{V}$, where $\omega\mathcal{V}$ represents the ordinal sum of ω copies of the disconnected rotation of the standard cancellative hoop.
- $\text{Ł}_{\mathrm{Chang}}$ (the logic associated to the variety generated by Chang's MV-algebra) enjoys the finite strong completeness (but not the strong one) w.r.t. $\mathcal{V}$, $\mathcal{V}$ being the disconnected rotation of the standard cancellative hoop.
- There are two BL-chains $\mathcal{A}, \mathcal{B}$ that are strongly complete w.r.t., respectively $\text{Ł}_{\mathrm{Chang}}$ and $\mathrm{BL}_{\mathrm{Chang}}$.
- Every $\text{Ł}_{\mathrm{Chang}}$-chain that is strongly complete w.r.t. $\text{Ł}_{\mathrm{Chang}}$ is also strongly complete w.r.t $\text{Ł}_{\mathrm{Chang}}\forall$.
- There is no $\mathrm{BL}_{\mathrm{Chang}}$-chain to be complete w.r.t. $\mathrm{BL}_{\mathrm{Chang}}\forall$.

2. Preliminaries

2.1. Basic Concepts

Basic Logic BL was introduced by P. Hájek in [23]. It is based over the connectives $\{\&, \to, \bot\}$ and a denumerable set of variables VAR. The formulas are defined inductively, as usual (see [23] for details).

Other derived connectives are the following.

negation: $\neg\varphi := \varphi \to \bot$; *verum* or *top*: $\top := \neg\bot$; *meet*: $\varphi \wedge \psi := \varphi\&(\varphi \to \psi)$; *join*: $\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$.

BL is axiomatized as follows.

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) \tag{A1}$$

$$(\varphi\&\psi) \to \varphi \tag{A2}$$

$$(\varphi\&\psi) \to (\psi\&\varphi) \tag{A3}$$

$$(\varphi\&(\varphi \to \psi)) \to (\psi\&(\psi \to \varphi)) \tag{A4}$$

$$(\varphi \to (\psi \to \chi)) \to ((\varphi\&\psi) \to \chi) \tag{A5a}$$

$$((\varphi\&\psi) \to \chi) \to (\varphi \to (\psi \to \chi)) \tag{A5b}$$

$$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi) \tag{A6}$$

$$\bot \to \varphi \tag{A7}$$

Modus ponens is the only inference rule:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}. \tag{MP}$$

Among the extensions of BL (logics obtained from it by adding other axioms) there is the well known Łukasiewicz (infinitely-valued) logic Ł, that is, BL plus

$$\neg\neg\varphi \to \varphi. \tag{INV}$$

On Łukasiewicz logic we can also define a strong disjunction connective (in the following sections, we will introduce a strong disjunction connective, for BL, that will be proved to be equivalent to the following, over Ł)

$$\varphi \curlyvee \psi := \neg(\neg\varphi\&\neg\psi).$$

The notations φ^n and $n\varphi$ will indicate $\underbrace{\varphi\&\ldots\&\varphi}_{n \text{ times}}$ and $\underbrace{\varphi \curlyvee \cdots \curlyvee \varphi}_{n \text{ times}}$.

Given an axiomatic extension L of BL, a formula φ and a theory T (a set of formulas), the notation $T \vdash_L \varphi$ indicates that there is a proof of φ

from the axioms of Ł and the ones of T. The notion of proof is defined like in classical case (see [23]).

We now move to the semantics: for all the unexplained notions of universal algebra, we refer to [9, 22].

DEFINITION 2.1. A BL-algebra is an algebraic structure of the form $\mathcal{A} = \langle A, *, \Rightarrow, \sqcap, \sqcup, 0, 1\rangle$ such that

- $\langle A, \sqcap, \sqcup, 0, 1\rangle$ is a bounded lattice, where 0 is the bottom and 1 the top element.
- $\langle A, *, 1\rangle$ is a commutative monoid.
- $\langle *, \Rightarrow\rangle$ forms a residuated pair, i.e.

$$z * x \leq y \quad \text{iff} \quad z \leq x \Rightarrow y, \tag{res}$$

  it can be shown that the only operation that satisfies res is $x \Rightarrow y = \max\{z : \ z * x \leq y\}$.
- $\mathcal{A}$ satisfies the following equations

$$(x \Rightarrow y) \sqcup (y \Rightarrow x) = 1 \tag{pl}$$

$$x \sqcap y = x * (x \Rightarrow y). \tag{div}$$

Two important types of BL-algebras are the followings.

- A BL-chain is a totally ordered BL-algebra.
- A standard BL-algebra is a BL-algebra whose support is $[0, 1]$.

Notation: in the following, with $\sim x$ we will indicate $x \Rightarrow 0$.

DEFINITION 2.2. An MV-algebra is a BL-algebra satisfying

$$x = \sim\sim x. \tag{inv}$$

A well known example of MV-algebra is the standard MV-algebra $[0, 1]_{\text{Ł}} = \langle [0, 1], *, \Rightarrow, \min, \max, 0, 1\rangle$, where $x * y = \max(0, x + y - 1)$ and $x \Rightarrow y = \min(1, 1 - x + y)$.

In every MV-algebra we define the algebraic equivalent of $\curlyvee$, that is

$$x \oplus y := \sim (\sim x * \sim y).$$

The notations (where x is an element of some BL-algebra) x^n and nx will indicate $\underbrace{x * \cdots * x}_{n \text{ times}}$ and $\underbrace{x \oplus \cdots \oplus x}_{n \text{ times}}$.

Given a BL-algebra $\mathcal{A}$, the notion of $\mathcal{A}$-evaluation is defined in a truth-functional way (starting from a map $v : VAR \to A$, and extending it to formulas), for details see [23].

Consider a BL-algebra $\mathcal{A}$, a theory T and a formula φ. With $\mathcal{A} \models \varphi$ ($\mathcal{A}$ is a model of φ) we indicate that $v(\varphi) = 1$, for every $\mathcal{A}$-evaluation v; $\mathcal{A} \models T$ denotes that $\mathcal{A} \models \psi$, for every $\psi \in T$. Finally, the notation $T \models_{\mathcal{A}} \varphi$ means that if $\mathcal{A} \models T$, then $\mathcal{A} \models \varphi$.

A BL-algebra $\mathcal{A}$ is called L-algebra, where L is an axiomatic extension of BL, whenever $\mathcal{A}$ is a model for all the axioms of L.

Definition 2.3. Let L be an axiomatic extension of BL and K a class of L-algebras. We say that L is strongly complete (respectively: finitely strongly complete, complete) with respect to K if for every set T of formulas (respectively, for every finite set T of formulas, for $T = \emptyset$) and for every formula φ we have

$$T \vdash_L \varphi \quad \text{iff} \quad T \models_K \varphi.$$

2.2. Perfect MV-Algebras, Hoops and Disconnected Rotations

We recall that Chang's MV-algebra [11] is a BL-algebra of the form

$$C = \langle \{a_n : \ n \in \mathbb{N}\} \cup \{b_n : \ n \in \mathbb{N}\}, *, \Rightarrow, \sqcap, \sqcup, b_0, a_0 \rangle .$$

Where for each $n, m \in \mathbb{N}$, it holds that $b_n < a_m$, and, if $n < m$, then $a_m < a_n$, $b_n < b_m$; moreover $a_0 = 1$, $b_0 = 0$ (the top and the bottom element).

The operation $*$ is defined as follows, for each $n, m \in \mathbb{N}$:

$$b_n * b_m = b_0, \ b_n * a_m = b_{\max(0,n-m)}, \ a_n * a_m = a_{n+m}.$$

Definition 2.4. [6] Let $\mathcal{A}$ be an MV-algebra and let $x \in \mathcal{A}$: with $ord(x)$ we mean the least (positive) natural n such that $x^n = 0$. If there is no such n, then we set $ord(x) = \infty$.

- An MV-algebra is called *local*[1] if for every element x it holds that $ord(x) < \infty$ or $ord(\sim x) < \infty$.
- An MV-algebra is called *perfect* if for every element x it holds that $ord(x) < \infty$ iff $ord(\sim x) = \infty$.

[1]Usually, the local MV-algebras are defined as MV-algebras having a unique (proper) maximal ideal. In [6], however, it is shown that the two definitions are equivalent. We have preferred the other definition since it shows in a more transparent way that perfect MV-algebras are particular cases of local MV-algebras.

An easy consequence of this definition is that every perfect MV-algebra cannot have a negation fixpoint.

With $Perfect(MV)$ and $Local(MV)$ we will indicate the class of perfect and local MV-algebras. Moreover, given a BL-algebra $\mathcal{A}$, with $\mathbf{V}(\mathcal{A})$ we will denote the variety generated by $\mathcal{A}$.

THEOREM 2.1. [6] *Every MV-chain is local.*

Clearly there are local MV-algebras that are not perfect: $[0,1]_Ł$ is an example.

Now, in [18] it is shown that

THEOREM 2.2.

- $\mathbf{V}(C) = \mathbf{V}(Perfect(MV))$,
- $Perfect(MV) = Local(MV) \cap \mathbf{V}(C)$.

It follows that the class of chains in $\mathbf{V}(C)$ coincides with the one of perfect MV-chains. Moreover

THEOREM 2.3. [18] *An MV-algebra is in the variety $\mathbf{V}(C)$ iff it satisfies the equation $(2x)^2 = 2(x^2)$.*

As shown in [4], the logic correspondent to this variety is axiomatized as Ł plus $(2\varphi)^2 \leftrightarrow 2(\varphi^2)$: we will call it $Ł_{\text{Chang}}$.

We now recall some results about hoops

DEFINITION 2.5. [21,8] A *hoop* is a structure $\mathcal{A} = \langle A, *, \Rightarrow, 1\rangle$ such that $\langle A, *, 1\rangle$ is a commutative monoid, and $\Rightarrow$ is a binary operation such that

$$x \Rightarrow x = 1, \quad x \Rightarrow (y \Rightarrow z) = (x * y) \Rightarrow z \quad \text{and} \quad x * (x \Rightarrow y) = y * (y \Rightarrow x).$$

In any hoop, the operation $\Rightarrow$ induces a partial order $\leq$ defined by $x \leq y$ iff $x \Rightarrow y = 1$. Moreover, hoops are precisely the partially ordered commutative integral residuated monoids (pocrims) in which the meet operation $\sqcap$ is definable by $x \sqcap y = x * (x \Rightarrow y)$. Finally, hoops satisfy the following divisibility condition:

$$\text{If } x \leq y, \text{ then there is an element } z \text{ such that } z * y = x. \tag{div}$$

We recall a useful result.

DEFINITION 2.6. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras of the same language. Then we say that

- $\mathcal{A}$ is a partial subalgebra of $\mathcal{B}$ if $A \subseteq B$ and the operations of $\mathcal{A}$ are the ones of $\mathcal{B}$ restricted to A. Note that A could not be closed under these operations (in this case these last ones will be undefined over some elements of A): in this sense $\mathcal{A}$ is a partial subalgebra.

- $\mathcal{A}$ is partially embeddable into $\mathcal{B}$ when every finite partial subalgebra of $\mathcal{A}$ is embeddable into $\mathcal{B}$. Generalizing this notion to classes of algebras, we say that a class K of algebras is partially embeddable into a class M if every finite partial subalgebra of a member of K is embeddable into a member of M.

Definition 2.7. A bounded hoop is a hoop with a minimum element; conversely, an *unbounded* hoop is a hoop without minimum.

Let $\mathcal{A}$ be a bounded hoop with minimum a: with $\mathcal{A}^+$ we mean the (partial) subalgebra of $\mathcal{A}$ defined over the universe $A^+ = \{x \in A : x > x \Rightarrow a\}$.

A hoop is Wajsberg iff it satisfies the equation $(x \Rightarrow y) \Rightarrow y = (y \Rightarrow x) \Rightarrow x$.

A hoop is cancellative iff it satisfies the equation $x = y \Rightarrow (x * y)$.

Proposition 2.1. [21,8,1] *Every cancellative hoop is Wajsberg. Totally ordered cancellative hoops coincide with unbounded totally ordered Wajsberg hoops, whereas bounded Wajsberg hoops coincide with (the* 0*-free reducts of) MV-algebras.*

We now recall a construction introduced in [25] (and also used in [20,27]), called *disconnected rotation*.

Definition 2.8. Let $\mathcal{A}$ be a cancellative hoop. We define an algebra, $\mathcal{A}^*$, called the *disconnected rotation* of $\mathcal{A}$, as follows. Let $\mathcal{A} \times \{0\}$ be a disjoint copy of A. For every $a \in A$ we write a' instead of $\langle a, 0\rangle$. Consider $\langle A' = \{a' : a \in A\}, \leq\rangle$ with the inverse order and let $A^* := A \cup A'$. We extend these orderings to an order in A^* by putting $a' < b$ for every $a, b \in A$. Finally, we take the following operations in A^*: $1 := 1_{\mathcal{A}}$, $0 := 1'$, $\sqcap_{\mathcal{A}^*}, \sqcup_{\mathcal{A}^*}$ as the meet and the join with respect to the order over A^*. Moreover,

$$\sim_{\mathcal{A}^*} a \quad := \quad \begin{cases} a' & \textit{if } a \in A \\ b & \textit{if } a = b' \in A' \end{cases}$$

$$a *_{\mathcal{A}^*} b \quad := \quad \begin{cases} a *_{\mathcal{A}} b & \textit{if } a, b \in A \\ \sim_{\mathcal{A}^*} (a \Rightarrow_{\mathcal{A}} \sim_{\mathcal{A}^*} b) & \textit{if } a \in A, b \in A' \\ \sim_{\mathcal{A}^*} (b \Rightarrow_{\mathcal{A}} \sim_{\mathcal{A}^*} a) & \textit{if } a \in A', b \in A \\ 0 & \textit{if } a, b \in A' \end{cases}$$

$$a \Rightarrow_{\mathcal{A}^*} b \quad := \quad \begin{cases} a \Rightarrow_{\mathcal{A}} b & \textit{if } a, b \in A \\ \sim_{\mathcal{A}^*} (a *_{\mathcal{A}^*} \sim_{\mathcal{A}^*} b) & \textit{if } a \in A, b \in A' \\ 1 & \textit{if } a \in A', b \in A \\ (\sim_{\mathcal{A}^*} b) \Rightarrow_{\mathcal{A}} (\sim_{\mathcal{A}^*} a) & \textit{if } a, b \in A'. \end{cases}$$

THEOREM 2.4. [27, theorem 9] *Let $\mathcal{A}$ be an MV-algebra. The followings are equivalent:*

- *A is a perfect MV-algebra.*
- *A is isomorphic to the disconnected rotation of a cancellative hoop.*

To conclude the section, we present the definition of ordinal sum.

DEFINITION 2.9. [2] Let $\langle I, \leq\rangle$ be a totally ordered set with minimum 0. For all $i \in I$, let $\mathcal{A}_i$ be a hoop such that for $i \neq j$, $A_i \cap A_j = \{1\}$, and assume that $\mathcal{A}_0$ is bounded. Then $\bigoplus_{i\in I} \mathcal{A}_i$ (the *ordinal sum* of the family $(\mathcal{A}_i)_{i\in I}$) is the structure whose base set is $\bigcup_{i\in I} A_i$, whose bottom is the minimum of $\mathcal{A}_0$, whose top is 1, and whose operations are

$$x \Rightarrow y \quad = \quad \begin{cases} x \Rightarrow^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } \exists i > j (x \in A_i \text{ and } y \in A_j) \\ 1 & \text{if } \exists i < j (x \in A_i \setminus \{1\} \text{ and } y \in A_j) \end{cases}$$

$$x * y \quad = \quad \begin{cases} x *^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ x & \text{if } \exists i < j (x \in A_i \setminus \{1\},\ y \in A_j) \\ y & \text{if } \exists i < j (y \in A_i \setminus \{1\},\ x \in A_j) \end{cases}$$

When defining the ordinal sum $\bigoplus_{i\in I} \mathcal{A}_i$ we will tacitly assume that whenever the condition $A_i \cap A_j = \{1\}$ is not satisfied for all $i, j \in I$ with $i \neq j$, we will replace the $\mathcal{A}_i$ by isomorphic copies satisfying such condition. Moreover if all $\mathcal{A}_i$'s are isomorphic to some $\mathcal{A}$, then we will write $I\mathcal{A}$, instead of $\bigoplus_{i\in I} \mathcal{A}_i$. Finally, the ordinal sum of two hoops $\mathcal{A}$ and $\mathcal{B}$ will be denoted by $\mathcal{A} \oplus \mathcal{B}$.

Note that, since every bounded Wajsberg hoop is the 0-free reduct of an MV-algebra, then the previous definition also works with these structures.

THEOREM 2.5. [2, theorem 3.7] *Every BL-chain is isomorphic to an ordinal sum whose first component is an MV-chain and the others are totally ordered Wajsberg hoops.*

Note that in [10] it is presented an alternative and simpler proof of this result.

3. The Variety of $\mathbf{BL}_{\mathbf{Chang}}$-algebras

Consider the following connective

$$\varphi \veebar \psi := ((\varphi \to (\varphi \& \psi)) \to \psi) \wedge ((\psi \to (\varphi \& \psi)) \to \varphi)$$

Call $\uplus$ the algebraic operation, over a BL-algebra, corresponding to $\veebar$; we have that

LEMMA 3.1. *In every MV-algebra the following equation holds*

$$x \uplus y = x \oplus y.$$

PROOF. It is easy to check that $x \uplus y = x \oplus y$, over $[0,1]_{\text{Ł}}$, for every $x, y \in [0,1]$. ■

We now analyze this connective in the context of Wajsberg hoops.

PROPOSITION 3.1. *Let $\mathcal{A}$ be a linearly ordered Wajsberg hoop. Then*

- *If $\mathcal{A}$ is unbounded (i.e. a cancellative hoop), then $x \uplus y = 1$, for every $x, y \in \mathcal{A}$.*
- *If $\mathcal{A}$ is bounded, let a be its minimum. Then, by defining $\sim x := x \Rightarrow a$ and $x \oplus y = \sim(\sim x * \sim y)$ we have that $x \oplus y = x \uplus y$, for every $x, y \in \mathcal{A}$*

PROOF. An easy check. ■

Now, since the variety of cancellative hoops is generated by its linearly ordered members (see [20]), then we have that

COROLLARY 3.1. *The equation $x \uplus y = 1$ holds in every cancellative hoop.*

We now characterize the behavior of $\uplus$ for the case of BL-chains.

PROPOSITION 3.2. *Let $\mathcal{A} = \bigoplus_{i \in I} \mathcal{A}_i$ be a BL-chain. Then*

$$x \uplus y = \begin{cases} x \oplus y, & \textit{if } x, y \in \mathcal{A}_i \textit{ and } \mathcal{A}_i \textit{ is bounded} \\ 1, & \textit{if } x, y \in \mathcal{A}_i \textit{ and } \mathcal{A}_i \textit{ is unbounded} \\ \max(x, y), & \textit{otherwise.} \end{cases}$$

for every $x, y \in \mathcal{A}$.

PROOF. If x, y belong to the same component of $\mathcal{A}$, then the result follows from Lemma 3.1 and Proposition 3.1. For the case in which x and y belong to different components of $\mathcal{A}$, this is a direct computation. ■

Remark 3.1. From the previous proposition we can argue that $\uplus$ is a good approximation, for BL, of what that $\oplus$ represents for MV-algebras. Note that a similar operation was introduced in [3]: the main difference with respect to $\uplus$ is that, when x and y belong to different components of a BL-chain, then the operation introduced in [3] holds 1.

In the following, for every element x of a BL-algebra, with the notation $\overline{n}x$ we will denote $\underbrace{x \uplus \cdots \uplus x}_{n\ times}$; analogously $\overline{n}\varphi$ means $\underbrace{\varphi \veebar \cdots \veebar \varphi}_{n\ times}$.

Definition 3.1. We define $\mathrm{BL}_{\mathrm{Chang}}$ as the axiomatic extension of BL, obtained by adding

$$(\overline{2}\varphi)^2 \leftrightarrow \overline{2}(\varphi^2). \tag{cha}$$

That is, writing it in extended form

$$(\varphi^2 \to (\varphi^2 \& \varphi^2) \to \varphi^2) \leftrightarrow ((\varphi \to \varphi^2) \to \varphi)^2.$$

Clearly the variety corresponding to $\mathrm{BL}_{\mathrm{Chang}}$ is given by the class of BL-algebras satisfying the equation $(\overline{2}x)^2 = \overline{2}(x^2)$.

Moreover,

Definition 3.2. We will call pseudo-perfect Wajsberg hoops those Wajsberg hoops satisfying the equation $(\overline{2}x)^2 = \overline{2}(x^2)$.

Remark 3.2. Thanks to Lemma 3.1 we have that

$$\vdash_{\text{Ł}} ((\overline{2}\varphi)^2 \leftrightarrow \overline{2}(\varphi^2)) \leftrightarrow ((2\varphi)^2 \leftrightarrow 2(\varphi^2)),$$

that is, if we add $(\overline{2}\varphi)^2 \leftrightarrow \overline{2}(\varphi^2)$ or $(2\varphi)^2 \leftrightarrow 2(\varphi^2)$ to Ł, then we obtain the same logic $\text{Ł}_{\mathrm{Chang}}$.

These formulas, however are not equivalent over BL: see Remark 3.3 for details.

Theorem 3.1. *Every totally ordered pseudo-perfect Wajsberg hoop is a totally ordered cancellative hoop or (the 0-free reduct of) a perfect MV-chain.*

More in general, the variety of pseudo-perfect Wajsberg hoops coincides with the class of the 0-free subreducts of members of $\mathbf{V}(C)$.

Proof. In [20] it is shown that the variety of Wajsberg hoops coincides with the class of the 0-free subreducts of MV-algebras. The results easily follow from this fact and from Proposition 2.1, Theorem 2.3 and Definition 3.2. ∎

As a consequence, we have

Theorem 3.2. *Let* $\mathbb{WH}, \mathbb{CH}, ps\mathbb{WH}$ *be, respectively, the varieties of Wajsberg hoops, cancellative hoops, pseudo-perfect Wajsberg hoops. Then we have that*

$$\mathbb{CH} \subset ps\mathbb{WH} \subset \mathbb{WH}$$

Proof. An easy consequence of Theorem 3.1.

The first inclusion follows from the fact that $ps\mathbb{WH}$ contains all the totally ordered cancellative hoops and hence the variety generated by them. For the second inclusion note that, for example, the 0-free reduct of $[0, 1]_{\text{Ł}}$ belongs to $\mathbb{WH} \setminus ps\mathbb{WH}$. ■

We now describe the structure of BL_{Chang}-chains, with an analogous of the Theorem 2.5 for BL-chains.

Theorem 3.3. *Every* BL_{Chang}*-chain is isomorphic to an ordinal sum whose first component is a perfect MV-chain and the others are totally ordered pseudo-perfect Wajsberg hoops.*

It follows that every ordinal sum of perfect MV-chains is a BL_{Chang}*-chain.*

Proof. Thanks to Theorems 2.2, 2.3, Remark 3.2 and Definition 3.2, we have that every MV-chain (Wajsberg hoop) satisfying the equation $(\overline{2}x)^2 = \overline{2}(x^2)$ is perfect (pseudo-perfect): using these facts and Proposition 3.2 we have that a BL-chain satisfies the equation $(\overline{2}x)^2 = \overline{2}(x^2)$ iff it holds true in all the components of its ordinal sum. From these facts and Theorem 2.5 we get the result. ■

As a consequence, we obtain the following corollaries.

Corollary 3.2. *The variety of* BL_{Chang}*-algebras contains the ones of product-algebras and Gödel-algebras: however it does not contains the variety of MV-algebras.*

Proof. From the previous theorem it is easy to see that the variety of BL_{Chang}-algebras contains $[0, 1]_{\Pi}$ and $[0, 1]_G$, but not $[0, 1]_{\text{Ł}}$. ■

Corollary 3.3. *Every finite* BL_{Chang}*-chain is an ordinal sum of a finite number of copies of the two elements boolean algebra. Hence the class of finite* BL_{Chang}*-chains coincides with the one of finite Gödel chains.*

For this reason it is immediate to see that the finite model property does not hold for BL_{Chang}.

We conclude with the following remark.

Remark 3.3.

- One can ask if it is possible to axiomatize the class $\mathrm{BL}_{\mathrm{perf}}$ of BL-algebras, whose chains are the BL-algebras that are ordinal sum of perfect MV-chains: the answer, however, is negative. In fact, the class of bounded Wajsberg hoops does not form a variety: for example, it is easy to check that for every bounded pseudo-perfect Wajsberg hoop $\mathcal{A}$, its subalgebra $\mathcal{A}^+$ (see definition 2.7) forms a cancellative hoop. Hence $\mathrm{BL}_{\mathrm{perf}}$ cannot be a variety.

  However, as we will see in section 3.2, the variety of $\mathrm{BL}_{\mathrm{Chang}}$-algebras is the "best approximation" of $\mathrm{BL}_{\mathrm{perf}}$, in the sense that it is the smallest variety to contain $\mathrm{BL}_{\mathrm{perf}}$.

- In [19] (see also [14]) it is studied the variety, called P_0, generated by all the perfect BL-algebras (a BL-algebra $\mathcal{A}$ is perfect if, by calling $MV(\mathcal{A})$ the biggest subalgebra of $\mathcal{A}$ to be an MV-algebra, then $MV(\mathcal{A})$ is a perfect MV-algebra). P_0 is axiomatized with the equation

$$\sim((\sim(x^2))^2) = (\sim((\sim x)^2))^2. \tag{p_0}$$

  One can ask which is the relation between P_0 and the variety of $\mathrm{BL}_{\mathrm{Chang}}$-algebras. The answer is that the variety of $\mathrm{BL}_{\mathrm{Chang}}$-algebras is strictly contained in P_0. In fact, an easy check shows that a BL-chain is perfect if and only if the first component of its ordinal sum is a perfect MV-chain. Hence we have:

  – Every $\mathrm{BL}_{\mathrm{Chang}}$-chain is a perfect BL-chain.
  – There are perfect BL-chains that are not $\mathrm{BL}_{\mathrm{Chang}}$-chains: an example is given by $C \oplus [0,1]_{Ł}$.

  Now, since the variety of $\mathrm{BL}_{\mathrm{Chang}}$-algebras is generated by its chains (like any variety of BL-algebras, see [23]), then we get the result.

  Finally note that (p_0) is equivalent to $2(x^2) = (2x)^2$: hence, differently to what happens over Ł (see Remark 3.2), the equations $2(x^2) = (2x)^2$ and $\overline{2}(x^2) = (\overline{2}x)^2$ are not equivalent, over BL.

3.1. Subdirectly Irreducible and Simple Algebras

We begin with a general result about Wajsberg hoops.

Theorem 3.4. [21, Corollary 3.11] *Every subdirectly irreducible Wajsberg hoop is totally ordered.*

As a consequence, we have:

Corollary 3.4. *Every subdirectly irreducible pseudo-perfect Wajsberg hoop is totally ordered.*

We now move to simple algebras.

It is shown in [28, Theorem 1] that the simple BL-algebras coincide with the simple MV-algebras, that is, with the subalgebras of $[0,1]_{Ł}$ (see [13, Theorem 3.5.1]). Therefore we have:

Theorem 3.5. *The only simple BL_{Chang}-algebra is the two elements boolean algebra* $\mathbf{2}$*.*

An easy consequence of this fact is that the only simple $Ł_{\mathrm{Chang}}$-algebra is $\mathbf{2}$.

3.2. Completeness

We begin with a result about pseudo-perfect Wajsberg hoops.

Theorem 3.6. *The class pMV of 0-free reducts of perfect MV-chains generates $ps\mathbb{WH}$.*

Proof. From Theorems 2.4 and 3.1 it is easy to check that the variety generated by pMV contains all the totally ordered pseudo-perfect Wajsberg hoops.

From these facts and Corollary 3.4, we have that pMV must be generic for $ps\mathbb{WH}$. ■

Theorem 3.7. [15] *Let L be an axiomatic extension of BL, then L enjoys the finite strong completeness w.r.t a class K of L-algebras iff every countable L-chain is partially embeddable into K.*

As shown in [23] product logic enjoys the finite strong completeness w.r.t $[0,1]_{\Pi}$ and hence every countable product chain is partially embeddable into $[0,1]_{\Pi} \simeq \mathbf{2} \oplus (0,1]_C$, with $(0,1]_C$ being the standard cancellative hoop (i.e. the 0-free reduct of $[0,1]_{\Pi} \setminus \{0\}$). Since every totally ordered product chain is of the form $\mathbf{2} \oplus \mathcal{A}$, where $\mathcal{A}$ is a cancellative hoop (see [20]), it follows that:

Proposition 3.3. *Every countable totally ordered cancellative hoop partially embeds into* $(0,1]_C$*.*

Theorem 3.8. *Every countable perfect MV-chain partially embeds into $\mathcal{V} = (0,1]^*_C$ (i.e. the disconnected rotation of $(0,1]_C$).*

Proof. Immediate from Proposition 3.3 and Theorem 2.4. ■

Corollary 3.5. *The logic $Ł_{Chang}$ is finitely strongly complete w.r.t. $\mathcal{V}$.*

Theorem 3.9. *BL_{Chang} enjoys the finite strong completeness w.r.t. $\omega\mathcal{V}$. As a consequence, the variety of BL_{Chang}-algebras is generated by the class of all ordinal sums of perfect MV-chains and hence is the smallest variety to contain this class of algebras.*

Proof. Thanks to Theorem 3.7 it is enough to show that every countable $\mathrm{BL}_{\mathrm{Chang}}$-chain partially embeds into $\omega\mathcal{V}$ (i.e. the ordinal sum of "ω copies" of $\mathcal{V}$). This fact, however, follows immediately from Proposition 3.3 and Theorems 3.3, 3.8. ■

But we cannot obtain a stronger result: in fact

Theorem 3.10. *BL_{Chang} is not strongly complete w.r.t. $\omega\mathcal{V}$.*

Proof. Suppose not: from the results of [15, Theorem 3.5] this is equivalent to claim that every countable $\mathrm{BL}_{\mathrm{Chang}}$-chain embeds into $\omega\mathcal{V}$. But, this would imply that every countable totally ordered cancellative hoop embeds into $(0,1]_C$: this means that every countable product-chain embeds into $[0,1]_\Pi$, that is product logic is strongly complete w.r.t $[0,1]_\Pi$. As it is well known (see [23, Corollary 4.1.18]), this is false. ■

With an analogous proof we obtain.

Theorem 3.11. *$Ł_{Chang}$ is not strongly complete w.r.t. $\mathcal{V}$.*

However, thanks to [26, Theorem 3] we can claim

Theorem 3.12. *There exist a $Ł_{Chang}$-chain $\mathcal{A}$ and a BL_{Chang}-chain $\mathcal{B}$ such that $Ł_{Chang}$ is strongly complete w.r.t. $\mathcal{A}$ and BL_{Chang} is strongly complete w.r.t. $\mathcal{B}$.*

Problem 3.1. Which can be some concrete examples of such $\mathcal{A}$ and $\mathcal{B}$?

4. First-Order Logics

We assume that the reader is acquainted with the formalization of first-order logics, as developed in [23, 17].

Briefly, we work with (first-order) languages without equality, containing only predicate and constant symbols: as quantifiers we have $\forall$ and $\exists$. The notions of terms and formulas are defined inductively like in classical case.

As regards to semantics, given an axiomatic extension L of BL we restrict to L-chains: the first-order version of L is called L$\forall$ (see [23, 17] for an axiomatization). A first-order $\mathcal{A}$-interpretation ($\mathcal{A}$ being an L-chain) is a structure $\mathbf{M} = \langle M, \{r_P\}_{p\in\mathbf{P}}, \{m_c\}_{c\in\mathbf{C}}\rangle$, where M is a non-empty set, every r_P is a fuzzy *ariety*(P)-ary relation, over M, in which we interpretate the predicate P, and every m_c is an element of M, in which we map the constant c.

Given a map $v : VAR \to M$, the interpretation of $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v}$ in this semantics is defined in a Tarskian way: in particular the universally quantified formulas are defined as the infimum (over $\mathcal{A}$) of truth values, whereas those existentially quantified are evaluated as the supremum. Note that these inf and sup could not exist in $\mathcal{A}$: an $\mathcal{A}$-model $\mathbf{M}$ is called *safe* if $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v}$ is defined for every φ and v.

A model is called *witnessed* if the universally (existentially) quantified formulas are evaluated by taking the minimum (maximum) of truth values in place of the infimum (supremum): see [24,16,17] for details.

The notions of soundness and completeness are defined by restricting to safe models (even if in some cases it is possible to enlarge the class of models: see [7]): see [23,17,16] for details.

We begin with a positive result about $\text{Ł}_{\text{Chang}}\forall$.

Definition 4.1. Let L be an axiomatic extension of BL. With $\text{L}\forall^w$ we define the extension of $\text{L}\forall$ with the following axioms

$$(\exists y)(\varphi(y) \to (\forall x)\varphi(x)) \tag{C$\forall$}$$

$$(\exists y)((\exists x)\varphi(x) \to \varphi(y)). \tag{C$\exists$}$$

Theorem 4.1. [16, Proposition 6] *Ł$\forall$ coincides with Ł$\forall^w$, that is* *Ł$\forall$* $\vdash$(C$\forall$,C$\exists$).

An immediate consequence is:

Corollary 4.1. *Let L be an axiomatic extension of Ł. Then L$\forall$ coincides with L$\forall^w$.*

Theorem 4.2. [16, Theorem 8] *Let L be an axiomatic extension of BL. Then L$\forall^w$ enjoys the strong witnessed completeness with respect to the class K of L-chains, i.e.*

$$T \vdash_{L\forall^w} \varphi \quad \textit{iff} \quad \|\varphi\|^{\mathcal{A}}_{\mathbf{M}} = 1,$$

for every theory T, formula φ, algebra $\mathcal{A} \in K$ and witnessed $\mathcal{A}$-model $\mathbf{M}$ such that $\|\psi\|^{\mathcal{A}}_{\mathbf{M}} = 1$ for every $\psi \in T$.

Lemma 4.1. [26, Lemma 1] *Let L be an axiomatic extension of BL, let $\mathcal{A}$ be an L-chain, let $\mathcal{B}$ be an L-chain such that $A \subseteq B$ and let $\mathbf{M}$ be a witnessed $\mathcal{A}$-structure. Then for every formula φ and evaluation v, we have $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} = \|\varphi\|^{\mathcal{B}}_{\mathbf{M},v}$.*

Theorem 4.3. *There is a $Ł_{Chang}$-chain such that $Ł_{Chang}\forall$ is strongly complete w.r.t. it. More in general, every $Ł_{Chang}$-chain that is strongly complete w.r.t $Ł_{Chang}$ is also strongly complete w.r.t. $Ł_{Chang}\forall$.*

Proof. An adaptation of the proof for the analogous result, given in [26, Theorem 16], for Ł∀.

From Theorem 3.12 we know that there is a $Ł_{\mathrm{Chang}}$-chain $\mathcal{A}$ strongly complete w.r.t. $Ł_{\mathrm{Chang}}$: from [15, Theorem 3.5] this is equivalent to claim that every countable $Ł_{\mathrm{Chang}}$-chain embeds into $\mathcal{A}$. We show that $\mathcal{A}$ is also strongly complete w.r.t. $Ł_{\mathrm{Chang}}\forall$.

Suppose that $T \nvdash_{Ł_{\mathrm{Chang}}\forall} \varphi$. Thanks to Corollary 4.1 and Theorem 4.2 there is a countable $Ł_{\mathrm{Chang}}$-chain $\mathcal{C}$ and a witnessed $\mathcal{C}$-model $\mathbf{M}$ such that $\|\psi\|^{\mathcal{C}}_{\mathbf{M}} = 1$, for every $\psi \in T$, but $\|\varphi\|^{\mathcal{C}}_{\mathbf{M}} < 1$. Finally, from Lemma 4.1 we have that $\|\psi\|^{\mathcal{A}}_{\mathbf{M}} = 1$, for every $\psi \in T$ and $\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} = \|\varphi\|^{\mathcal{C}}_{\mathbf{M}} < 1$: this completes the proof. ■

For $\mathrm{BL}_{\mathrm{Chang}}\forall$, however, the situation is not so good.

Theorem 4.4. *$BL_{Chang}\forall$ cannot enjoy the completeness w.r.t. a single BL_{Chang}-chain.*

Proof. The proof is an adaptation of the analogous result given in [26, Theorem 17] for BL∀.

Let $\mathcal{A}$ be a $\mathrm{BL}_{\mathrm{Chang}}$-chain: call $\mathcal{A}_0$ its first component. We have three cases

- $\mathcal{A}_0$ is finite: from Theorem 3.3 we have that $\mathcal{A}_0 = \mathbf{2}$ and hence $\mathcal{A} \models (\neg\neg x) \to (\neg\neg x)^2$. However $\mathcal{V} \not\models (\neg\neg x) \to (\neg\neg x)^2$, where $\mathcal{V}$ is the chain introduced in section 3.2, and hence $\mathcal{A}$ cannot be complete w.r.t. $\mathrm{BL}_{\mathrm{Chang}}\forall$.
- $\mathcal{A}_0$ is infinite and dense. As shown in [26, Theorem 17] the formula $(\forall x)\neg\neg P(x) \to \neg\neg(\forall x)P(x)$ is a tautology in every BL-chain whose first component is infinite and densely ordered: hence we have that $\mathcal{A} \models (\forall x)\neg\neg P(x) \to \neg\neg(\forall x)P(x)$. However it is easy to check that this formula fails in $[0,1]_G$: take a $[0,1]_G$-model $\mathbf{M}$ with $M = (0,1]$ and such that $r_P(m) = m$. Hence, from Corollary 3.2, it follows that $\mathrm{BL}_{\mathrm{Chang}}\forall \nvdash (\forall x)\neg\neg P(x) \to \neg\neg(\forall x)P(x)$.

- $\mathcal{A}_0$ is infinite and not dense. As shown in [26, Theorem 17] the formula $(\forall x)\neg\neg P(x) \to \neg\neg(\forall x)P(x) \vee \neg(\forall x)P(x) \to ((\forall x)P(x))^2$ is a tautology in every BL-chain whose first component is infinite and not densely ordered: hence we have that $\mathcal{A} \models (\forall x)\neg\neg P(x) \to \neg\neg(\forall x)P(x) \vee \neg(\forall x)P(x) \to ((\forall x)P(x))^2$. Also in this case, however, this formula fails in $[0,1]_G$, using the same model $\mathbf{M}$ of the previous case. ■

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Matteo Bianchi
Dipartimento di Informatica e Comunicazione
Università degli Studi di Milano
via comelico 39/41, 20135 Milan, Italy
matteo.bianchi@unimi.it