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Finite Basis Problem for Semigroups of Order Five or Less: Generalization and Revisitation

Abstract. A system of semigroup identities is *hereditarily finitely based* if it defines a variety all semigroups of which are finitely based. Two new types of hereditarily finitely based identity systems are presented. Two of these systems, together with eight existing systems, establish the hereditary finite basis property of every semigroup of order five or less with one possible exception.

Keywords: Semigroup, Monoid, Finitely based, Hereditarily finitely based.

1. Introduction

A semigroup is *finitely based* if the identities it satisfies are finitely axiomatizable. The well-known theorem of Oates and Powell, published in 1964, states that all finite groups are finitely based [25]. But any hope for the same result to also hold for all finite semigroups was quickly extinguished when in 1966, Perkins demonstrated that the Brandt monoid

$$B_2^1 = \left\langle\, a, b, 1 \,\middle|\, a^2 = b^2 = 0,\ aba = a,\ bab = b \,\right\rangle$$

of order six is non-finitely based [26]. The discovery of a non-finitely based semigroup with only six elements focused much attention upon the finite basis problem for semigroups of order five or less. This problem was explicitly raised by Tarski [36] in 1966 and attracted the interest of Bol'bot [2], Edmunds [6,7], Karnofsky [12], Tishchenko [37], and Trahtman [38]. A solution to this problem was eventually completed by Trahtman [39,40] in the early 1980s and published a few years later [41].

Theorem 1.1. *Every semigroup of order five or less is finitely based.*

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A more complete historical account of the proof of Theorem 1.1 can be found in the survey by Shevrin and Volkov [35].

A finitely based semigroup satisfies the stronger property of being *hereditarily finitely based* if it generates a variety all semigroups of which are finitely based. Examples of hereditarily finitely based semigroups include idempotent semigroups [1,9,10], commutative semigroups [27], and finite groups [25]. There exist finitely based semigroups that are not hereditarily finitely based [11,21,24], some of which have as few as six elements [8]. Recently, all semigroups of order four or less have been shown to be hereditarily finitely based [14]. In view of Theorem 1.1, it is natural to question whether or not all semigroups of order five are also hereditarily finitely based.

Two semigroups are *distinct* if they are neither isomorphic nor anti-isomorphic. There exist 1160 pairwise distinct semigroups of order five [29], among which 156 are monoids [4] and 1004 are non-unital. Edmunds *et al.* have shown that these 156 monoids, with the possible exception of

$$P_2^1 = \langle\, a, b, 1 \,|\, a^2 = ab = a,\ b^2 a = b^2 \,\rangle,$$

are hereditarily finitely based [8, Section 5]. They also announced that the 1004 non-unital semigroups are all hereditarily finitely based [8, Section 4]. This announced result will be confirmed in the present article.

An identity system is said to be *hereditarily finitely based* if it defines a variety all semigroups of which are finitely based. Pollák [30–32] pioneered the study of hereditarily finitely based identities in the 1970s and has immensely contributed to their classification; see the survey of his work [43]. A few other hereditarily finitely based identity systems have been found in the study of some specific classes of varieties [13,15–19,22,23]. The main aim of the present article is to establish two new types of hereditarily finitely based identity systems. Two of these systems, together with eight existing systems, are used to verify the aforementioned announced result of Edmunds *et al.* [8] and establish an almost complete generalization of Theorem 1.1.

THEOREM 1.2. *Every semigroup of order five or less that is distinct from* P_2^1 *is hereditarily finitely based.*

The main arguments of the proof of Theorem 1.2 are given in Section 3, while the finer details are deferred to Sections 4 and 5.

Theorem 1.2 can also be viewed as a revisitation of Theorem 1.1. Such a revisitation is useful since Theorem 1.1 involves numerous individual cases and its proof by Trahtman [41], published back in 1991, is neither well circulated nor translated into English.

REMARK 1.3. The identities

$$xh^m xt^n x \approx xh^m xt^n, \quad xh^m yt^n xy \approx xh^m yt^n yx, \quad m, n \in \{0, 1\},$$

constitute a basis for the semigroup P_2^1 [20, Corollary 6.6].

PROBLEM 1.4. Is the semigroup P_2^1 hereditarily finitely based?

A solution to Problem 1.4 completes the classification of hereditarily finitely based semigroups of order five or less.

2. Preliminaries

Denote by $\mathcal{X}^+$ and $\mathcal{X}^*$ respectively the free semigroup and free monoid over a countably infinite alphabet $\mathcal{X}$. Elements of $\mathcal{X}$ are called *letters* and elements of $\mathcal{X}^+$ and $\mathcal{X}^*$ are called *words*. The *content* of a word $\mathbf{w}$, denoted by $\mathsf{con}(\mathbf{w})$, is the set of letters occurring in $\mathbf{w}$.

Let x be any letter and $\mathbf{w}$ be any word. The number of times x occurs in $\mathbf{w}$ is denoted by $\mathsf{occ}(x, \mathbf{w})$. If $\mathsf{occ}(x, \mathbf{w}) = 1$, then x is said to be *simple* in $\mathbf{w}$. Denote by $\mathsf{sim}(\mathbf{w})$ the set of all simple letters occurring in $\mathbf{w}$. A word $\mathbf{w}$ is *simple* if all its letters are simple in it, that is, $\mathsf{sim}(\mathbf{w}) = \mathsf{con}(\mathbf{w})$. The *initial part* of $\mathbf{w}$, denoted by $\mathsf{ini}(\mathbf{w})$, is the simple word obtained by retaining the first occurrence of each letter in $\mathbf{w}$.

An identity is written as $\mathbf{u} \approx \mathbf{v}$ where $\mathbf{u}, \mathbf{v} \in \mathcal{X}^+$. An identity $\mathbf{u} \approx \mathbf{v}$ is *trivial* if the words $\mathbf{u}$ and $\mathbf{v}$ are equal. A semigroup S *satisfies* an identity $\mathbf{u} \approx \mathbf{v}$ if for any substitution φ of $\mathcal{X}$ into S, the elements $\mathbf{u}\varphi$ and $\mathbf{v}\varphi$ of S are equal. A variety of semigroups *satisfies* an identity if every semigroup in the variety satisfies the identity. The deducibility of an identity $\mathbf{u} \approx \mathbf{v}$ from a set Γ of identities is indicated by $\mathbf{u} \overset{\Gamma}{\approx} \mathbf{v}$; in this case, the set Γ is also said to *imply* the identity $\mathbf{u} \approx \mathbf{v}$.

Let L_2^1 and N_2^1 denote the monoids obtained by adjoining an identity element to the left-zero semigroup $L_2 = \langle\, a, b \mid ab = a,\, ba = b \,\rangle$ of order two and the null semigroup $N_2 = \langle\, a \mid a^2 = 0 \,\rangle$ of order two, respectively. Let $\mathbb{Z}_n$ denote the cyclic group of order n. The following characterizations of identities satisfied by the semigroups L_2^1, N_2^1, and $\mathbb{Z}_n$ are well known and easily verified.

LEMMA 2.1. *Let $\mathbf{u} \approx \mathbf{v}$ be any identity. Then*

(i) *L_2^1 satisfies $\mathbf{u} \approx \mathbf{v}$ if and only if* $\mathsf{ini}(\mathbf{u}) = \mathsf{ini}(\mathbf{v})$;

(ii) *N_2^1 satisfies $\mathbf{u} \approx \mathbf{v}$ if and only if* $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$ *and* $\mathsf{sim}(\mathbf{u}) = \mathsf{sim}(\mathbf{v})$;

(iii) *$\mathbb{Z}_n$ satisfies $\mathbf{u} \approx \mathbf{v}$ if and only if* $\mathsf{occ}(x, \mathbf{u}) \equiv \mathsf{occ}(x, \mathbf{v}) \pmod{n}$.

Refer to the monograph of Burris and Sankappanavar [3] for more information on universal algebra.

3. Proof of Theorem 1.2

Lemma 3.1. *The following identity systems are hereditarily finitely based*:

(0) $x^2 \approx x$;

(1) $xyx \approx xy^2$;

(2) $x^3y \approx xy, \ \ xyx \approx x^2y^3$;

(3) $xy^3 \approx xy, \ \ x^3y \approx xyx^2, \ \ xyxy \approx xy^2x$;

(4) $x^4 \approx x^2, \ \ x^3yx \approx xyx, \ \ x^2y \approx yx^2, \ \ xyxy \approx x^2y^2$;

(5) $xyzy \approx xy^2z$;

(6) $x^3 \approx x^2, \ \ xyxyx \approx xyx, \ \ xyxzx \approx xzxyx$;

(7) $x^3 \approx x^2, \ \ x^2yx \approx xyx, \ \ xyx^2 \approx xyx, \ \ xyxzx \approx xyzx$;

(8) $x^4 \approx x^2, \ x^3yx \approx xyx, \ x^2yx \approx xyx^2, \ x^2yz^2 \approx xyz^2x, \ xyxzx \approx xzxyx$;

(9) $x^4 \approx x^2, \quad xyxyxyx \approx xyx, \quad x^2y^3x^2 \approx x^2yx^2, \quad xyxzx \approx xzxyx$
$(x^2y^2)^3 \approx (x^2y^2)^2$.

The identity systems (2) and (8) are shown to be hereditarily finitely based in Sections 4 and 5, respectively. References for the other systems being hereditarily finitely based are as follows:

System	Reference
(0)	[1] or [9] or [10]
(1)	[33, Theorem 1]
(3)	[8, Theorem 6.1]
(4)	[8, Theorem 7.2]

System	Reference
(5)	[34, Proposition C]
(6)	[16, Theorem 1.3]
(7)	[19, Theorem 3.3]
(9)	[22, Proposition 3.2]

Using a computer, it is routinely shown that with the exception of the semigroup P_2^1, each of the other 1159 pairwise distinct semigroups of order five satisfies one of the identity systems (0)–(9) or its dual system and so is hereditarily finitely based by Lemma 3.1.

Remark 3.2. (i) The identity systems (0)–(9) are ordered first by the number of distinct letters involved, then followed by the total number of letters involved.

(ii) There is no redundancy with the identity systems (0)–(9) since they are distinguished by semigroups with the following multiplication tables:

S_0	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	4	5
3	3	3	3	3	3
4	1	2	5	4	5
5	5	5	5	5	5

S_1	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	3
3	3	3	3	3	3
4	1	2	3	4	1
5	5	5	5	5	5

S_2	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	2	3	4	5
4	1	2	4	3	5
5	5	5	5	5	5

S_3	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	1
3	1	1	3	4	5
4	4	4	4	4	4
5	4	4	5	1	3

S_4	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	2	2
3	1	1	1	3	3
4	1	2	3	4	5
5	1	3	2	5	4

S_5	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	2	1
3	1	1	1	3	1
4	1	1	1	4	1
5	1	1	2	3	2

S_6	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	2	3
3	1	2	3	2	3
4	1	1	1	4	5
5	1	4	5	4	5

S_7	1	2	3	4	5
1	1	1	1	1	1
2	1	2	4	4	2
3	1	1	3	1	3
4	1	1	4	1	4
5	1	2	3	4	5

S_8	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	2
3	1	1	1	1	3
4	1	1	2	1	1
5	1	2	1	4	5

S_9	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	2
3	1	2	3	4	2
4	1	2	4	3	2
5	1	1	1	1	5

Specifically, for any $m, n \in \{0, \dots, 9\}$, the semigroup S_m satisfies the identity system (n) if and only if $m = n$.

4. The Identity System (2)

For each $n \geq 1$, let $\mathbf{U}_n$ denote the variety defined by the identities

$$x^{n+1}y \approx xy, \quad xyx \approx x^2y^{n+1}. \tag{4.1}$$

THEOREM 4.1. *The variety $\mathbf{U}_n$ is hereditarily finitely based.*

COROLLARY 4.2. *The identity system* (2) *is hereditarily finitely based.*

REMARK 4.3. Corollary 4.2 is also deducible from a recent result of Luo and Zhang [23, Theorem 1.1 and Corollary 4.6].

In this section, a word of the form

$$x_1^{e_1} \cdots x_k^{e_k} y^e$$

is said to be in *canonical form* if the letters $x_1, \dots, x_k, y$ are distinct with $e_1, \dots, e_k \in \{1, \dots, n\}$, $e \in \{1, \dots, n+1\}$, and $k \geq 0$. (Note that the prefix $x_1^{e_1} \cdots x_k^{e_k}$ is empty when $k = 0$.) It is easily shown that the identities (4.1) can be used to convert any word into one in canonical form.

LEMMA 4.4. *Let S be any semigroup that satisfies the identities* (4.1) *but does not satisfy the identity*

$$x^n y^{n+1} \approx x^n y. \tag{4.2}$$

(i) *Suppose that the semigroups L_2^1 and $\mathbb{Z}_n$ are isomorphic to subsemigroups of S. Then the variety $\mathbf{U}_n$ is generated by S.*

(ii) *The variety* $\mathbf{U}_n$ *is generated by the group* $\mathbb{Z}_n$ *and the semigroup* U *with the following multiplication table*:

U	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	2	3	4
4	4	4	4	4

PROOF. (i) To show that the variety $\mathbf{U}_n$ is generated by the semigroup S, it suffices to show that any identity $\mathbf{u} \approx \mathbf{v}$ satisfied by S is implied by the identities (4.1). Since the semigroup S satisfies the identities (4.1), it follows from the observation preceding this lemma that the words $\mathbf{u}$ and $\mathbf{v}$ can be chosen to be in canonical form, say $\mathbf{u} = x_1^{e_1} \cdots x_k^{e_k} y^e$ and $\mathbf{v} = z_1^{f_1} \cdots z_\ell^{f_\ell} t^f$. Since the semigroups L_2^1 and $\mathbb{Z}_n$ are isomorphic to subsemigroups of S, it follows from Lemma 2.1 that $x_1^{e_1} \cdots x_k^{e_k} y = z_1^{f_1} \cdots z_\ell^{f_\ell} t$ and $e \equiv f \pmod{n}$. Suppose that $e \neq f$, that is, $\{e, f\} = \{1, n+1\}$. Then the identity $\mathbf{u} \approx \mathbf{v}$ is

$$x_1^{e_1} \cdots x_k^{e_k} y^{n+1} \approx x_1^{e_1} \cdots x_k^{e_k} y. \tag{4.3}$$

If the semigroup S satisfies the identity (4.3), then it also satisfies the identity (4.2), contradicting the assumption. Therefore $e = f$, whence the identity $\mathbf{u} \approx \mathbf{v}$ is trivial and is implied by the identities (4.1).

(ii) It is easily checked that the semigroup U satisfies the identities (4.1) so that by Lemma 2.1(iii), the semigroup $U \times \mathbb{Z}_n$ also satisfies (4.1). Now the semigroup U does not satisfy the identity (4.2) because $3^n \cdot 2^{n+1} \neq 3^n \cdot 2$ and the semigroup L_2^1 is isomorphic to the subsemigroup $\{1, 3, 4\}$ of U. Therefore by part (i), the variety $\mathbf{U}_n$ is generated by the semigroup $U \times \mathbb{Z}_n$. ■

LEMMA 4.5. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be any hereditarily finitely based varieties and let* $\mathbf{V} = \mathbf{A} \vee \mathbf{B}$ *be their varietal join. Suppose that the variety* $\mathbf{V}$ *is finitely based and that the lattice* $\mathcal{L}(\mathbf{V})$ *of subvarieties of* $\mathbf{V}$ *is modular. Then* $\mathbf{V}$ *is hereditarily finitely based.*

PROOF. A finitely based variety is hereditarily finitely based if and only if its lattice of subvarieties satisfies the descending chain condition. Therefore by assumption, the lattices $\mathcal{L}(\mathbf{A})$ and $\mathcal{L}(\mathbf{B})$ are modular and satisfy the descending chain condition. It follows that the lattice $\mathcal{L}(\mathbf{V}) = \mathcal{L}(\mathbf{A} \vee \mathbf{B})$ also satisfies the descending chain condition [28]. Consequently, the variety $\mathbf{V}$ is hereditarily finitely based. ■

PROOF OF THEOREM 4.1. Any semigroup that satisfies the identities

$$x^{n+1}y \approx xy, \quad (xy)^{n+1} \approx xy^{n+1}, \quad xyx^n zt \approx xyzt \tag{4.4}$$

generates a variety with modular lattice of subvarieties [42]. Let $\mathbf{U}$ and $\mathbf{Z}_n$ denote the varieties generated by the semigroups U and $\mathbb{Z}_n$, respectively. Since $\mathbf{U}_n = \mathbf{U} \vee \mathbf{Z}_n$ by Lemma 4.4(ii) and it is routinely verified that U and $\mathbb{Z}_n$ satisfy the identities (4.4), the lattice $\mathcal{L}(\mathbf{U}_n)$ is modular. The semigroup U satisfies the identity (1) and so is hereditarily finitely based by Lemma 3.1. As mentioned in Section 1, all finite groups are hereditarily finitely based [25] and so also is $\mathbb{Z}_n$. Since the join $\mathbf{U} \vee \mathbf{Z}_n = \mathbf{U}_n$ is finitely based, it is also hereditarily finitely based by Lemma 4.5. ■

5. The Identity System (8)

Let $\mathbf{Q}$ denote the variety defined by the identities

$$x^4 \approx x^2,\ x^3yx \approx xyx,\ x^2yx \approx xyx^2, \tag{5.1a}$$

$$x^2yz^2 \approx xyz^2x, \tag{5.1b}$$

$$xyxzx \approx xzxyx. \tag{5.1c}$$

THEOREM 5.1. *The variety* $\mathbf{Q}$ *is hereditarily finitely based. Equivalently, the identity system* (8) *is hereditarily finitely based.*

The proof of this theorem is given in Subsection 5.6.

LEMMA 5.2. (i) *The identities* (5.1a)–(5.1c) *imply the identity*

$$\mathbf{hxyt} \approx \mathbf{hyxt} \tag{5.1d}$$

for all $\mathbf{h}, \mathbf{x}, \mathbf{y} \in \mathcal{X}^+$ *and* $\mathbf{t} \in \mathcal{X}^*$ *such that* $\mathbf{h}$, $\mathbf{x}$, *and* $\mathbf{y}$ *end with non-simple letters of* $\mathbf{hxyt}$.

(ii) *The identities* (5.1a)–(5.1c) *imply the identity*

$$\mathbf{h}x\mathbf{k}x\mathbf{t} \approx \mathbf{h}x^2\mathbf{kt} \tag{5.1e}$$

for all $\mathbf{h}, \mathbf{t} \in \mathcal{X}^*$ *and* $\mathbf{k} \in \mathcal{X}^+$ *such that* $\mathbf{k}$ *ends with a non-simple letter of* $\mathbf{h}x\mathbf{k}x\mathbf{t}$.

PROOF.

(i) If the words $\mathbf{h}$, $\mathbf{x}$, and $\mathbf{y}$ end with the non-simple letters h, x, and y of $\mathbf{hxyt}$ respectively, then

$$\begin{aligned}\mathbf{hxyt} &\overset{(5.1a)}{\approx} \mathbf{h}h^4\mathbf{x}x^2\mathbf{y}y^2\mathbf{t} \overset{(5.1b)}{\approx} \mathbf{h}h^2(\mathbf{x}x^2)h(\mathbf{y}y^2)h\mathbf{t}\\ &\overset{(5.1c)}{\approx} \mathbf{h}h^2(\mathbf{y}y^2)h(\mathbf{x}x^2)h\mathbf{t} \overset{(5.1b)}{\approx} \mathbf{h}h^4\mathbf{y}y^2\mathbf{x}x^2\mathbf{t} \overset{(5.1a)}{\approx} \mathbf{hyxt}.\end{aligned}$$

(ii) If the word $\mathbf{k}$ ends with the non-simple letter y of $\mathbf{h}x\mathbf{k}x\mathbf{t}$, then

$$\mathbf{h}x\mathbf{k}x\mathbf{t} \overset{(5.1a)}{\approx} \mathbf{h}x\mathbf{k}y^2x\mathbf{t} \overset{(5.1b)}{\approx} \mathbf{h}x^2\mathbf{k}y^2\mathbf{t} \overset{(5.1a)}{\approx} \mathbf{h}x^2\mathbf{k}\mathbf{t}. \qquad \blacksquare$$

It is convenient to refer to the identities (5.1a)–(5.1e) collectively as (5.1). By Lemma 5.2, the identity system (5.1) also constitutes a basis for the variety $\mathbf{Q}$. For any set Γ of identities, let $\mathbf{Q}\Gamma$ denote the subvariety of $\mathbf{Q}$ defined by Γ.

5.1. Canonical Form

A word $\mathbf{w}$ with distinct non-simple letters $y_1, \ldots, y_m$ is said to be in *canonical form* if it can be written as

$$\mathbf{w} = \mathbf{w}_0 x_0 \cdot \mathbf{w}_1 x_1 \cdots \mathbf{w}_p x_p \cdot y_1^{e_1} \cdots y_m^{e_m} \cdot \mathbf{w}_*, \tag{5.2}$$

where $\mathbf{w}_0, \mathbf{w}_* \in \mathcal{X}^*$, $\mathbf{w}_1, \ldots, \mathbf{w}_p \in \mathcal{X}^+$, $x_0, \ldots, x_p \in \{y_1, \ldots, y_m\}$, and $e_1, \ldots, e_m \in \{0, 1, 2, 3\}$ are such that

(C1) the letters in $\mathbf{w}_0, \ldots, \mathbf{w}_p, \mathbf{w}_*$ are precisely all the simple letters of $\mathbf{w}$;

(C2) the letters $y_1, \ldots, y_m$ are in strict alphabetical order;

(C3) if $\mathsf{occ}(y_i, x_0 \cdots x_p) = 0$, then $e_i \in \{2, 3\}$;

(C4) if $\mathsf{occ}(y_i, x_0 \cdots x_p) = 1$, then $e_i \in \{1, 2\}$;

(C5) if $\mathsf{occ}(y_i, x_0 \cdots x_p) \geq 2$, then $e_i \in \{0, 1\}$.

Remark 5.3. (i) If the word $\mathbf{w}$ in (5.2) is simple, then it reduces to $\mathbf{w}_0$ and is vacuously in canonical form.

(ii) Note that the letters $x_0, \ldots, x_p$ need not be distinct, but by (C1), the words $\mathbf{w}_0 x_0, \mathbf{w}_1 x_1, \ldots, \mathbf{w}_p x_p$ are distinct.

Lemma 5.4. *The identities* (5.1) *can be used to convert any word into one in canonical form.*

Proof. Let $\mathbf{w}$ be any word. As observed in Remark 5.3(i), if the word $\mathbf{w}$ is simple, then it is already in canonical form. Therefore it suffices to assume that the word $\mathbf{w}$ is non-simple. Consider a factorization of $\mathbf{w}$ that displays all of its non-simple letters individually, that is,

$$\mathbf{w} = \mathbf{w}_0 x_0 \cdot \mathbf{w}_1 x_1 \cdots \mathbf{w}_r x_r \cdot \mathbf{w}_* \tag{5.3}$$

where the letters $x_0, \ldots, x_r$ are non-simple in $\mathbf{w}$ and the letters in the factors $\mathbf{w}_0, \ldots, \mathbf{w}_r, \mathbf{w}_* \in \mathcal{X}^*$ are simple in $\mathbf{w}$. Each letter in the list

$x_0, \ldots, x_r$ is non-simple in $\mathbf{w}$ and thus appears at least twice in the list, whence $r \geq 1$. Now the words $\mathbf{w}_0 x_0, \mathbf{w}_1 x_1, \ldots, \mathbf{w}_r x_r$ end with non-simple letters of $\mathbf{w}$ so that by applying the identities (5.1d), the factors $\mathbf{w}_1 x_1, \ldots, \mathbf{w}_r x_r$ of $\mathbf{w}$ can be permuted in any manner. In particular, the factors from $\mathbf{w}_1 x_1, \ldots, \mathbf{w}_r x_r$ with $\mathbf{w}_i \neq \emptyset$ can be gathered to the left, while the factors from $\mathbf{w}_1 x_1, \ldots, \mathbf{w}_r x_r$ with $\mathbf{w}_i = \emptyset$ can be gathered to the right in alphabetical order. The resulting word is of the form (5.2), with $e_1, \ldots, e_m \geq 0$, that satisfies (C1) and (C2). By applying the identities (5.1a), each exponent e_i can be reduced to a number in $\{0, 1, 2, 3\}$. If $\mathsf{occ}(y_i, x_0 \cdots x_p) = 0$, then since the letter y_i is non-simple in $\mathbf{w}$, it must occur at least twice in $\mathbf{w}$ so that $e_i \in \{2, 3\}$. Hence (C3) is satisfied.

Assume that $\mathsf{occ}(y_i, x_0 \cdots x_p) = 1$. Since the letter y_i is non-simple in $\mathbf{w}$, it must occur at least twice in $\mathbf{w}$ so that $e_i \in \{1, 2, 3\}$. If $e_i = 3$, then apply the identities (5.1a) to reduce e_i to 1. Hence (C4) is satisfied.

It remains to assume that $\mathsf{occ}(y_i, x_0 \cdots x_p) \geq 2$ and $e_i \in \{2, 3\}$. Then $y_i = x_j = x_k$ for some j and k with $j < k \leq m$, and $e_i = 2 + s$ for some $s \in \{0, 1\}$. Note that

$$\mathbf{w} = \cdots \mathbf{w}_j y_i \cdots \mathbf{w}_k y_i \cdot \underbrace{\mathbf{w}_{k+1} x_{k+1} \cdots \mathbf{w}_p x_p \cdot y_1^{e_1} \cdots y_{i-1}^{e_{i-1}}}_{\mathbf{u}} \cdot y_i^{2+s} \cdots$$

where the factor $\mathbf{u}$, if nonempty, ends with a non-simple letter of $\mathbf{w}$. Hence

$$\begin{aligned} \mathbf{w} &= \cdots \mathbf{w}_j y_i \cdots \mathbf{w}_k y_i \cdot \mathbf{u} \cdot y_i^2 \cdot y_i^s \cdots \\ &\overset{(5.1d)}{\approx} \cdots \mathbf{w}_j y_i \cdots \mathbf{w}_k y_i \cdot y_i^2 \cdot \mathbf{u} \cdot y_i^s \cdots \\ &\overset{(5.1a)}{\approx} \cdots \mathbf{w}_j y_i \cdots \mathbf{w}_k y_i \mathbf{u} y_i^s \cdots, \end{aligned}$$

that is, the exponent $e_i = 2 + s$ is reduced to $s \in \{0, 1\}$. Therefore (C5) is satisfied. ■

Remark 5.5. In the proof of Lemma 5.4, when the non-simple word $\mathbf{w}$ in (5.3) is converted into the word $\mathbf{w}$ in (5.2) in canonical form, the following remained unchanged: the prefix $\mathbf{w}_0$, the non-simple letter x_0, and the suffix $\mathbf{w}_*$. Therefore it is unambiguous to refer to $\mathbf{w}_0$ as the 0-*prefix* of $\mathbf{w}$, to x_0 as the *leading non-simple letter* of $\mathbf{w}$, and to $\mathbf{w}_*$ as the $*$-*suffix* of $\mathbf{w}$, regardless of whether or not $\mathbf{w}$ is in canonical form. It is convenient to write $0(\mathbf{w}) = \mathbf{w}_0$, $\ell(\mathbf{w}) = x_0$, and $*(\mathbf{w}) = \mathbf{w}_*$. In general, the identities (5.1) preserve the 0-prefix, the leading non-simple letter, and the $*$-suffix of any non-simple word.

5.2. Standard Identities

An identity $\mathbf{u} \approx \mathbf{v}$ is said to be a *standard identity* if $\mathbf{u}$ and $\mathbf{v}$ are words in canonical form such that $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$ and $\mathsf{sim}(\mathbf{u}) = \mathsf{sim}(\mathbf{v})$. Since the words $\mathbf{u}$ and $\mathbf{v}$ that constitute a standard identity $\mathbf{u} \approx \mathbf{v}$ are simultaneously simple, it is unambiguous to refer to the identity $\mathbf{u} \approx \mathbf{v}$ as a *simple* or *non-simple* identity depending on the simplicity of $\mathbf{u}$ and $\mathbf{v}$.

LEMMA 5.6. *Suppose that* $\mathbf{V}$ *is any subvariety of* $\mathbf{Q}$ *that satisfies an identity* $\mathbf{u} \approx \mathbf{v}$ *with* $\mathsf{con}(\mathbf{u}) \neq \mathsf{con}(\mathbf{v})$ *or* $\mathsf{sim}(\mathbf{u}) \neq \mathsf{sim}(\mathbf{v})$. *Then* $\mathbf{V}$ *is finitely based.*

PROOF. There are two cases to consider.
CASE 1. $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$ and $\mathsf{sim}(\mathbf{u}) \neq \mathsf{sim}(\mathbf{v})$, say with $y \in \mathsf{sim}(\mathbf{u}) \backslash \mathsf{sim}(\mathbf{v})$. Then $\mathsf{occ}(y, \mathbf{u}) = 1$ and $\mathsf{occ}(y, \mathbf{v}) = p + 1$ for some $p \geq 1$. Let φ denote the substitution

$$t \mapsto \begin{cases} x^2 & \text{if } t \in \mathcal{X} \backslash \{y\}, \\ x^2 y & \text{if } t = y. \end{cases}$$

Then $x^2(\mathbf{u}\varphi)x^2 \overset{(5.1a)}{\approx} x^2yx^2$ and $x^2(\mathbf{v}\varphi)x^2 \overset{(5.1a)}{\approx} (x^2y)^{p+1}x^2$ so that the variety $\mathbf{V}$ satisfies the identity $\alpha : x^2yx^2 \approx (x^2y)^{p+1}x^2$. The variety $\mathbf{V}$ then satisfies the identity $\beta : xy^3x \approx xyx$ because

$$xyx \overset{(5.1a)}{\approx} x^2yx^2 \overset{\alpha}{\approx} (x^2y)^{2p+1}x^2 \overset{(5.1a)}{\approx} (x^2y)^3x^2 \overset{(5.1e)}{\approx} x^6y^3x^2 \overset{(5.1a)}{\approx} xy^3x.$$

Now since

$$xyxyxyx \overset{(5.1e)}{\approx} x^3y^3x \overset{(5.1a)}{\approx} xy^3x \overset{\beta}{\approx} xyx,$$

the variety $\mathbf{V}$ satisfies the identity system (9) and so is finitely based by Lemma 3.1.
CASE 2. $\mathsf{con}(\mathbf{u}) \neq \mathsf{con}(\mathbf{v})$, say

$$\mathsf{con}(\mathbf{u}) \backslash \mathsf{con}(\mathbf{v}) = \{x_1, \ldots, x_r\} \quad \text{and} \quad \mathsf{con}(\mathbf{v}) \backslash \mathsf{con}(\mathbf{u}) = \{y_1, \ldots, y_s\}$$

for some $r, s \geq 0$ with $(r, s) \neq (0, 0)$. Then the variety $\mathbf{V}$ satisfies the identity $\mathbf{u}' \approx \mathbf{v}'$, where $\mathbf{u}' = \mathbf{u}x_1 \cdots x_r y_1 \cdots y_s$ and $\mathbf{v}' = \mathbf{v}x_1 \cdots x_r y_1 \cdots y_s$ are such that $\mathsf{con}(\mathbf{u}') = \mathsf{con}(\mathbf{v}')$ and $\mathsf{sim}(\mathbf{u}') \neq \mathsf{sim}(\mathbf{v}')$. Hence the variety $\mathbf{V}$ is finitely based by Case 1. ∎

5.3. $\mathsf{F_{SS}}$-consistent and $\mathsf{F_{SN}}$-consistent Identities

For any word $\mathbf{w}$, let $\mathsf{F_{SS}}(\mathbf{w})$ denote the set of factors of $\mathbf{w}$ of length two that consist of two simple letters, and let $\mathsf{F_{SN}}(\mathbf{w})$ denote the set of factors of $\mathbf{w}$ of length two that begin with a simple letter and end with a non-simple letter:

$$\mathsf{F_{SS}}(\mathbf{w}) = \{xy \mid \mathbf{w} \in \mathcal{X}^*xy\mathcal{X}^*,\ x, y \in \mathsf{sim}(\mathbf{w})\},$$
$$\mathsf{F_{SN}}(\mathbf{w}) = \{xy \mid \mathbf{w} \in \mathcal{X}^*xy\mathcal{X}^*,\ x \in \mathsf{sim}(\mathbf{w}),\ y \notin \mathsf{sim}(\mathbf{w})\}.$$

For any $\Diamond \in \{\mathsf{F_{SS}}, \mathsf{F_{SN}}\}$, a standard identity $\mathbf{u} \approx \mathbf{v}$ is said to be $\Diamond$-*consistent* if $\Diamond(\mathbf{u}) = \Diamond(\mathbf{v})$.

LEMMA 5.7. *Suppose that* $\mathbf{V}$ *is any subvariety of* $\mathbf{Q}$ *that satisfies some identity* $\mathbf{u} \approx \mathbf{v}$ *with* $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$, $\mathsf{sim}(\mathbf{u}) = \mathsf{sim}(\mathbf{v})$, *and* $\mathsf{F_{SS}}(\mathbf{u}) \neq \mathsf{F_{SS}}(\mathbf{v})$. *Then* $\mathbf{V}$ *satisfies the identity*

$$xyxzx \approx x^2yzx \tag{5.4}$$

and is finitely based.

PROOF. By symmetry, it suffices to assume that $yz \in \mathsf{F_{SS}}(\mathbf{u})\backslash\mathsf{F_{SS}}(\mathbf{v})$. Let φ denote the substitution

$$t \mapsto \begin{cases} x^2 & \text{if } t \in \mathcal{X}\backslash\{y, z\}, \\ x^2y & \text{if } t = y. \end{cases}$$

Then $x^2(\mathbf{u}\varphi)x \overset{(5.1a)}{\approx} x^2yzx$ and $x^2(\mathbf{v}\varphi)x \overset{(5.1a)}{\approx} \mathbf{w}$ with $\mathbf{w} \in \{xyxzx, xzxyx\}$. In view of the identity (5.1c), the variety $\mathbf{V}$ satisfies the identity (5.4).

Since any variety that satisfies the identities (5.1) and (5.4) is finitely based [23, Theorem 1.1 and Corollary 4.6], the variety $\mathbf{V}$ is finitely based. ■

REMARK 5.8. Let $\mathbf{u} \approx \mathbf{v}$ be any standard identity so that $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$ and $\mathsf{sim}(\mathbf{u}) = \mathsf{sim}(\mathbf{v})$.

(i) If the identity $\mathbf{u} \approx \mathbf{v}$ is simple, then it is $\mathsf{F_{SS}}$-consistent if and only if it is trivial.

(ii) If the identity $\mathbf{u} \approx \mathbf{v}$ is non-simple, then the words $\mathbf{u}$ and $\mathbf{v}$ share the same set of non-simple letters so that when written in canonical form,

$$\mathbf{u} = \mathbf{u}_0x_0 \cdot \mathbf{u}_1x_1 \cdots \mathbf{u}_px_p \cdot y_1^{e_1} \cdots y_m^{e_m} \cdot \mathbf{u}_*$$
$$\textit{and} \quad \mathbf{v} = \mathbf{v}_0z_0 \cdot \mathbf{v}_1z_1 \cdots \mathbf{v}_qz_q \cdot y_1^{f_1} \cdots y_m^{f_m} \cdot \mathbf{v}_*.$$

It is then easily seen that the identity $\mathbf{u} \approx \mathbf{v}$ is $\mathsf{F_{SS}}$-consistent if and only if $\{\mathbf{u}_0, \dots, \mathbf{u}_p, \mathbf{u}_*\} = \{\mathbf{v}_0, \dots, \mathbf{v}_q, \mathbf{v}_*\}$ with $p = q$.

LEMMA 5.9. *Suppose that* $\mathbf{V}$ *is any subvariety of* $\mathbf{Q}$ *that satisfies some identity* $\mathbf{u} \approx \mathbf{v}$ *with* $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$, $\mathsf{sim}(\mathbf{u}) = \mathsf{sim}(\mathbf{v})$, *and* $\mathsf{F}_{\mathsf{SN}}(\mathbf{u}) \neq \mathsf{F}_{\mathsf{SN}}(\mathbf{v})$. *Then* $\mathbf{V}$ *satisfies the identity*

$$x^2yz^2 \approx xyxz^2. \tag{5.5}$$

PROOF. By symmetry, it suffices to assume that $yz \in \mathsf{F}_{\mathsf{SN}}(\mathbf{u}) \backslash \mathsf{F}_{\mathsf{SN}}(\mathbf{v})$. Let φ denote the substitution

$$t \mapsto \begin{cases} x^2 & \text{if } t \in \mathcal{X} \backslash \{y, z\}, \\ x^2y & \text{if } t = y, \\ z^2x^2 & \text{if } t = z. \end{cases}$$

Since

$$\begin{aligned} x^2(\mathbf{u}\varphi) &\overset{(5.1a)}{\approx} x^2 \left(z^2x^2\right)^i x^2yz^2x^2 \left(z^2x^2\right)^j \quad \text{for some } i, j \geq 0 \text{ with } i + j \geq 1 \\ &\overset{(5.1c)}{\approx} x^2yz^2x^2 \left(z^2x^2\right)^i x^2 \left(z^2x^2\right)^j \overset{(5.1e)}{\approx} x^{6+2i+2j}yz^{2+2i+2j} \overset{(5.1a)}{\approx} x^2yz^2, \\ x^2 \left(\mathbf{v}\varphi\right) &\overset{(5.1a)}{\approx} x^2 \left(z^2x^2\right)^i x^2yx^2 \left(z^2x^2\right)^j \quad \text{for some } i, j \geq 0 \text{ with } i + j \geq 2 \\ &\overset{(5.1c)}{\approx} x^2yx^2(z^2x^2)^ix^2 \left(z^2x^2\right)^j \overset{(5.1e)}{\approx} x^{4+2i+2j}yx^2z^{2i+2j} \overset{(5.1a)}{\approx} xyxz^2, \end{aligned}$$

the variety $\mathbf{V}$ satisfies the identity (5.5). ■

COROLLARY 5.10. *Suppose that* $\mathbf{u} \approx \mathbf{v}$ *is any identity that is implied by the identity system* (5.1). *Then* $\mathsf{F}_{\mathsf{SS}}(\mathbf{u}) = \mathsf{F}_{\mathsf{SS}}(\mathbf{v})$ *and* $\mathsf{F}_{\mathsf{SN}}(\mathbf{u}) = \mathsf{F}_{\mathsf{SN}}(\mathbf{v})$.

PROOF. Let $\mathbf{V}$ be the variety generated by the semigroup V with the following multiplication table:

V	1	2	3	4	5
1	1	1	1	1	1
2	1	1	1	1	3
3	1	1	1	3	3
4	1	2	3	4	4
5	1	2	3	5	5

It is routinely verified that the semigroup V satisfies the identities (5.1) and so also the identity $\mathbf{u} \approx \mathbf{v}$. In particular, $\mathbf{V}$ is a subvariety of $\mathbf{Q}$. Since the subsemigroup $\{1, 3, 4\}$ of V is isomorphic to the semigroup N_2^1, it follows from Lemma 2.1(ii) that $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$ and $\mathsf{sim}(\mathbf{u}) = \mathsf{sim}(\mathbf{v})$.

Now the variety $\mathbf{V}$ does not satisfy the identities (5.4) and (5.5) because $4 \cdot 2 \cdot 4 \cdot 5 \cdot 4 \neq 4^2 \cdot 2 \cdot 5 \cdot 4$ and $4^2 \cdot 2 \cdot 5^2 \neq 4 \cdot 2 \cdot 4 \cdot 5^2$ in V. It then follows from Lemmas 5.7 and 5.9 that $\mathsf{F}_{\mathsf{SS}}(\mathbf{u}) = \mathsf{F}_{\mathsf{SS}}(\mathbf{v})$ and $\mathsf{F}_{\mathsf{SN}}(\mathbf{u}) = \mathsf{F}_{\mathsf{SN}}(\mathbf{v})$. ■

5.4. 0-Consistent, ℓ-Consistent, and $*$-Consistent Identities

For any $\Diamond \in \{0, \ell, *\}$, a non-simple, standard identity $\mathbf{u} \approx \mathbf{v}$ is said to be $\Diamond$-*consistent* if $\Diamond(\mathbf{u}) = \Diamond(\mathbf{v})$.

LEMMA 5.11. *Suppose that* $\mathbf{V}$ *is any subvariety of* $\mathbf{Q}$ *that satisfies some non-simple, standard identity* $\mathbf{u} \approx \mathbf{v}$ *that is* $\mathsf{F_{SS}}$-*consistent but either non-*0-*consistent or non-*$*$-*consistent. Then* $\mathbf{V}$ *is finitely based.*

PROOF. As observed in Remark 5.8(ii), when written in canonical form,

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_0 x_0 \cdot \mathbf{u}_1 x_1 \cdots \mathbf{u}_p x_p \cdot y_1^{e_1} \cdots y_m^{e_m} \cdot \mathbf{u}_* \\ \text{and} \quad \mathbf{v} &= \mathbf{v}_0 z_0 \cdot \mathbf{v}_1 z_1 \cdots \mathbf{v}_p z_p \cdot y_1^{f_1} \cdots y_m^{f_m} \cdot \mathbf{v}_* \end{aligned}$$

with $\{\mathbf{u}_0, \ldots, \mathbf{u}_p, \mathbf{u}_*\} = \{\mathbf{v}_0, \ldots, \mathbf{v}_p, \mathbf{v}_*\}$. Suppose that $\mathbf{u}_0 \neq \mathbf{v}_0$. By symmetry, it suffices to assume that $\mathbf{u}_0 \neq \emptyset$. Then $\mathbf{u}_0 = \mathbf{v}_i$ for some $i \geq 1$. Let φ denote the substitution that maps the first letter of $\mathbf{u}_0$ to z and any other letter to y^2. Since $(\mathbf{u}\varphi)y^2x \overset{(5.1a)}{\approx} zy^2x$ and $(\mathbf{v}\varphi)y^2x \overset{(5.1a)}{\approx} yzyx$, the variety $\mathbf{V}$ satisfies the identity $zy^2x \approx yzyx$, which is dual to (5), and so is finitely based by Lemma 3.1.

If $\mathbf{u}_* \neq \mathbf{v}_*$, then $\mathbf{V}$ is finitely based by a symmetrical argument. ■

5.5. Special Identities

A non-simple, standard identity is said to be *special* if it is $\Diamond$-consistent for all $\Diamond \in \{\mathsf{F_{SS}}, 0, \ell, *\}$. For each $k \geq 0$, define the identity

$$\theta_k : x_1 \cdots x_k y^2 z^2 \approx x_1 \cdots x_k z^2 y^2.$$

Note that the identity θ_0 is $y^2z^2 \approx z^2y^2$.

LEMMA 5.12. *Suppose that* $\mathbf{u} \approx \mathbf{v}$ *is any non-simple, standard identity that is* $\mathsf{F_{SS}}$-*consistent,* 0-*consistent,* $*$-*consistent, but non-*ℓ-*consistent. Then the equation* $\mathbf{Q}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{Q}\{(5.5), \theta_k, \sigma\}$ *holds for some* $k \geq 0$ *and some special identity* σ.

PROOF. By assumption and Remark 5.8(ii), when written in canonical form,

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_0 x_0 \cdot \mathbf{u}_1 x_1 \cdots \mathbf{u}_p x_p \cdot y_1^{e_1} \cdots y_m^{e_m} \cdot \mathbf{u}_* \\ \text{and} \quad \mathbf{v} &= \mathbf{u}_0 z_0 \cdot \mathbf{v}_1 z_1 \cdots \mathbf{v}_p z_p \cdot y_1^{f_1} \cdots y_m^{f_m} \cdot \mathbf{u}_* \end{aligned}$$

with $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and $x_0 \neq z_0$. If $\mathbf{u}_0 \neq \emptyset$, say with h being the last letter of $\mathbf{u}_0$, then $hx_0 \in \mathsf{F_{SN}}(\mathbf{u}) \backslash \mathsf{F_{SN}}(\mathbf{v})$ so that by Lemma 5.9,

(a) $\mathbf{Q}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{Q}\{\mathbf{u} \approx \mathbf{v}, (5.5)\}$.

If $\mathbf{u}_0 = \emptyset$, then picking any letter $t \notin \mathsf{con}(\mathbf{uv})$, the word tx_0 is a prefix of $t\mathbf{u}$ that is not a factor of $t\mathbf{v}$ so that $tx_0 \in \mathsf{F}_{\mathsf{SN}}(t\mathbf{u})\backslash\mathsf{F}_{\mathsf{SN}}(t\mathbf{v})$, whence (a) holds by Lemma 5.9. Thus (a) holds regardless of whether or not $\mathbf{u}_0$ is empty.

By assumption, the letters x_0 and z_0 are non-simple in both $\mathbf{u}$ and $\mathbf{v}$ with $x_0, z_0 \notin \mathsf{con}(\mathbf{u}_0)$. Let $k = |\mathbf{u}_0|$ and let φ denote the substitution

$$t \mapsto \begin{cases} z_0^2 & \text{if } t \in \mathcal{X}\backslash\mathsf{con}(\mathbf{u}_0 x_0), \\ x_0^2 & \text{if } t = x_0. \end{cases}$$

Then the deductions $\mathbf{u}\varphi \overset{(5.1)}{\approx} \mathbf{u}_0 x_0^2 z_0^2$ and $\mathbf{v}\varphi \overset{(5.1)}{\approx} \mathbf{u}_0 z_0^2 x_0^2$ hold so that

(b) $\mathbf{Q}\{\mathbf{u} \approx \mathbf{v}, (5.5)\} = \mathbf{Q}\{\mathbf{u} \approx \mathbf{v}, (5.5), \theta_k\}$.

Note that the word $\mathbf{u}$ can be written as $\mathbf{u} = \mathbf{u}_0 x_0 \mathbf{h} z_0 \mathbf{k} \mathbf{u}_*$ for some $\mathbf{h}, \mathbf{k} \in \mathcal{X}^*$ with $x_0, z_0 \in \mathsf{con}(\mathbf{hk})$. Define $\mathbf{w} = \mathbf{u}_0 z_0 \mathbf{h} x_0 \mathbf{k} \mathbf{u}_*$. By Lemma 5.4, there exists a word $\mathbf{w}'$ in canonical form such that the deduction $\mathbf{w} \overset{(5.1)}{\approx} \mathbf{w}'$ holds. Since $0(\mathbf{w}) = \mathbf{u}_0$, $\ell(\mathbf{w}) = z_0$, and $*(\mathbf{w}) = \mathbf{u}_*$, it follows from Remark 5.5 that $0(\mathbf{w}') = \mathbf{u}_0$, $\ell(\mathbf{w}') = z_0$, and $*(\mathbf{w}') = \mathbf{u}_*$. Hence the identity $\sigma : \mathbf{w}' \approx \mathbf{v}$ is 0-consistent, ℓ-consistent, and $*$-consistent. The equation $\mathsf{F}_{\mathsf{SS}}(\mathbf{u}) = \mathsf{F}_{\mathsf{SS}}(\mathbf{w})$ also holds since $\mathbf{w}$ is obtained from $\mathbf{u}$ by interchanging one occurrence of the non-simple letter x_0 with one occurrence of the non-simple letter z_0. Therefore $\mathsf{F}_{\mathsf{SS}}(\mathbf{v}) = \mathsf{F}_{\mathsf{SS}}(\mathbf{u}) = \mathsf{F}_{\mathsf{SS}}(\mathbf{w}) = \mathsf{F}_{\mathsf{SS}}(\mathbf{w}')$, where the first equation holds by assumption and the last equation holds by Corollary 5.10. Hence the identity σ is F_{SS}-consistent and thus also special. Since

$$\begin{aligned} \mathbf{u} &\overset{(5.1a)}{\approx} \mathbf{u}_0\big(x_0^3 \mathbf{h} z_0^3\big)\mathbf{k}\mathbf{u}_* \overset{(5.5)}{\approx} \mathbf{u}_0\big(x_0^2 \cdot \mathbf{h}x_0 \cdot z_0^3 \cdot \mathbf{k}\mathbf{u}_*\big) \overset{(5.1d)}{\approx} \big(\mathbf{u}_0 x_0^2 z_0^3\big)\mathbf{h}x_0\mathbf{k}\mathbf{u}_* \\ &\overset{\theta_k}{\approx} \mathbf{u}_0\big(z_0^2 \cdot x_0^2 \cdot z_0 \cdot \mathbf{h}x_0 \cdot \mathbf{k}\mathbf{u}_*\big) \overset{(5.1d)}{\approx} \mathbf{u}_0 z_0^3 \mathbf{h} x_0^3 \mathbf{k}\mathbf{u}_* \overset{(5.1a)}{\approx} \mathbf{w} \overset{(5.1)}{\approx} \mathbf{w}', \end{aligned}$$

the equation $\mathbf{Q}\{\mathbf{u} \approx \mathbf{v}, (5.5), \theta_k\} = \mathbf{Q}\{\sigma, (5.5), \theta_k\}$ holds. Hence by (a) and (b), the equation $\mathbf{Q}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{Q}\{(5.5), \theta_k, \sigma\}$ also holds. ■

LEMMA 5.13. *Suppose that $\mathbf{u} \approx \mathbf{v}$ is any special identity that is satisfied by the group $\mathbb{Z}_2$. Then either $\mathbf{Q}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{Q}$ or $\mathbf{Q}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{Q}\{(5.5)\}$.*

PROOF. By assumption and Remark 5.8(ii), when written in canonical form,

$$\begin{aligned} \mathbf{u} &= \mathbf{u}_0 x_0 \cdot \mathbf{u}_1 x_1 \cdots \mathbf{u}_p x_p \cdot y_1^{e_1} \cdots y_m^{e_m} \cdot \mathbf{u}_* \\ \text{and} \quad \mathbf{v} &= \mathbf{u}_0 x_0 \cdot \mathbf{v}_1 z_1 \cdots \mathbf{v}_p z_p \cdot y_1^{f_1} \cdots y_m^{f_m} \cdot \mathbf{u}_* \end{aligned}$$

with $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. There are three cases to consider.

CASE 1. $p \geq 1$ and $\{\mathbf{u}_1 x_1, \ldots, \mathbf{u}_p x_p\} = \{\mathbf{v}_1 z_1, \ldots, \mathbf{v}_p z_p\}$. Then there exists some permutation π on $\{1, \ldots, p\}$ such that $\mathbf{v}_{i\pi} z_{i\pi} = \mathbf{u}_i x_i$ for all i. Since the factors $\mathbf{v}_1 z_1, \ldots, \mathbf{v}_p z_p$ end with non-simple letters of $\mathbf{v}$, the identities

(5.1d) can be used to order them in any manner. Specifically,

$$\begin{aligned}\mathbf{v} &\overset{(5.1d)}{\approx} \mathbf{u}_0 x_0 \cdot \mathbf{v}_{1\pi} z_{1\pi} \cdots \mathbf{v}_{p\pi} z_{p\pi} \cdot y_1^{f_1} \cdots y_m^{f_m} \cdot \mathbf{u}_* \\ &= \underbrace{\mathbf{u}_0 x_0 \cdot \mathbf{u}_1 x_1 \cdots \mathbf{u}_p x_p \cdot y_1^{f_1} \cdots y_m^{f_m} \cdot \mathbf{u}_*}_{\mathbf{v}'}.\end{aligned}$$

Hence

(a) $\mathbf{Q}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{Q}\{\mathbf{u} \approx \mathbf{v}'\}$.

Since the group $\mathbb{Z}_2$ satisfies the identity $\mathbf{u} \approx \mathbf{v}$, it follows from Lemma 2.1(iii) that $e_i \equiv f_i \pmod 2$ for all i. Suppose that $e_i < f_i$ for some i. Then as $\mathbf{u}$ and $\mathbf{v}$ are in canonical form, $(e_i, f_i) \in \{(0,2),(1,3)\}$. Further, it follows from (C3)–(C5) that

$$\mathsf{occ}(y_i, x_0 \cdots x_p) \begin{cases} \geq 2 & \text{if } e_i = 0, \\ \geq 1 & \text{if } e_i = 1, \\ \leq 1 & \text{if } f_i = 2, \\ = 0 & \text{if } f_i = 3. \end{cases}$$

But this implies the contradiction $(e_i, f_i) \notin \{(0,2),(1,3)\}$. Therefore $e_i = f_i$ for all i, whence the identity $\mathbf{u} \approx \mathbf{v}'$ is trivial so that $\mathbf{Q}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{Q}$ by (a).

Case 2. $p = 0$. Then $\mathbf{u} = \mathbf{u}_0 x_0 \cdot y_1^{e_1} \cdots y_m^{e_m} \cdot \mathbf{u}_*$ and $\mathbf{v} = \mathbf{u}_0 x_0 \cdot y_1^{f_1} \cdots y_m^{f_m} \cdot \mathbf{u}_*$. By an argument that is similar to (and simpler than) Case 1, the equation $\mathbf{Q}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{Q}$ is obtained.

Case 3. $p \geq 1$ and $\{\mathbf{u}_1 x_1, \ldots, \mathbf{u}_p x_p\} \neq \{\mathbf{v}_1 z_1, \ldots, \mathbf{v}_p z_p\}$. The identity $\mathbf{u} \approx \mathbf{v}$ is F_{SN}-consistent so that $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\} = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Hence there exists some permutation π on $\{1, \ldots, p\}$ such that $\mathbf{u}_{i\pi} = \mathbf{v}_i$ for all i. By the assumption of this case, $\mathbf{u}_{j\pi} x_{j\pi} \neq \mathbf{v}_j z_j$ for some j; specifically, $\mathbf{u}_{j\pi} = \mathbf{v}_j$ and $x_{j\pi} \neq z_j$. Let t be the last letter of $\mathbf{u}_{j\pi}$ and $\mathbf{v}_j$, which is simple in both $\mathbf{u}$ and $\mathbf{v}$. Then $t x_{j\pi} \in \mathsf{F}_{\mathsf{SN}}(\mathbf{u}) \backslash \mathsf{F}_{\mathsf{SN}}(\mathbf{v})$ so that by Lemma 5.9,

(b) $\mathbf{Q}\{\mathbf{u} \approx \mathbf{v}\} = \mathbf{Q}\{\mathbf{u} \approx \mathbf{v}, (5.5)\}$.

Further,

$$\begin{aligned}\mathbf{u} &\overset{(5.1a)}{\approx} \mathbf{u}_0 x_0 \cdot x_0^{2p} \cdot \mathbf{u}_1 x_1^3 \cdot \mathbf{u}_2 x_2^3 \cdots \mathbf{u}_p x_p^3 \cdot y_1^{e_1} \cdots y_m^{e_m} \cdot \mathbf{u}_* \\ &\overset{(5.5)}{\approx} \mathbf{u}_0 x_0 \cdot x_0^p \cdot (\mathbf{u}_1 x_0) \cdot x_1^3 \cdot (\mathbf{u}_2 x_0) \cdot x_2^3 \cdots (\mathbf{u}_p x_0) \cdot x_p^3 \cdot y_1^{e_1} \cdots y_m^{e_m} \cdot \mathbf{u}_* \\ &\overset{(5.1d)}{\approx} \mathbf{u}_0 x_0 \cdot \mathbf{u}_{1\pi} x_0 \cdot \mathbf{u}_{2\pi} x_0 \cdots \mathbf{u}_{p\pi} x_0 \cdot x_0^p x_1^3 x_2^3 \cdots x_p^3 \cdot y_1^{e_1} \cdots y_m^{e_m} \cdot \mathbf{u}_*\end{aligned}$$

$$= \mathbf{u}_0 x_0 \cdot \mathbf{v}_1 x_0 \cdot \mathbf{v}_2 x_0 \cdots \mathbf{v}_p x_0 \cdot x_0^p x_1^3 x_2^3 \cdots x_p^3 \cdot y_1^{e_1} \cdots y_m^{e_m} \cdot \mathbf{u}_*$$

$$\overset{(5.1a)}{\approx} \underbrace{\mathbf{u}_0 x_0 \cdot \mathbf{v}_1 x_0 \cdot \mathbf{v}_2 x_0 \cdots \mathbf{v}_p x_0^p}_{\mathbf{h}} \cdot \underbrace{x_0 x_1 x_2 \cdots x_p \cdot y_1^{e_1} \cdots y_m^{e_m}}_{\mathbf{u}'} \cdot \mathbf{u}_* \quad (5.6)$$

and

$$\mathbf{v} \overset{(5.1a)}{\approx} \mathbf{u}_0 x_0 \cdot x_0^{2p} \cdot \mathbf{v}_1 z_1^3 \cdot \mathbf{v}_2 z_2^3 \cdots \mathbf{v}_p z_p^3 \cdot y_1^{f_1} \cdots y_m^{f_m} \cdot \mathbf{u}_*$$

$$\overset{(5.5)}{\approx} \mathbf{u}_0 x_0 \cdot x_0^p \cdot (\mathbf{v}_1 x_0) \cdot z_1^3 \cdot (\mathbf{v}_2 x_0) \cdot z_2^3 \cdots (\mathbf{v}_p x_0) \cdot z_p^3 \cdot y_1^{f_1} \cdots y_m^{f_m} \cdot \mathbf{u}_*$$

$$\overset{(5.1d)}{\approx} \mathbf{u}_0 x_0 \cdot \mathbf{v}_1 x_0 \cdot \mathbf{v}_2 x_0 \cdots \mathbf{v}_p x_0 \cdot x_0^p z_1^3 z_2^3 \cdots z_p^3 \cdot y_1^{f_1} \cdots y_m^{f_m} \cdot \mathbf{u}_*$$

$$\overset{(5.1a)}{\approx} \underbrace{\mathbf{u}_0 x_0 \cdot \mathbf{v}_1 x_0 \cdot \mathbf{v}_2 x_0 \cdots \mathbf{v}_p x_0^p}_{\mathbf{h}} \cdot \underbrace{x_0 z_1 z_2 \cdots z_p \cdot y_1^{f_1} \cdots y_m^{f_m}}_{\mathbf{v}'} \cdot \mathbf{u}_* \quad (5.7)$$

imply that

(c) $\mathbf{Q}\{\mathbf{u} \approx \mathbf{v}, (5.5)\} = \mathbf{Q}\{\widehat{\mathbf{u}} \approx \widehat{\mathbf{v}}, (5.5)\}$,

where $\widehat{\mathbf{u}} = \mathbf{h}\mathbf{u}'\mathbf{u}_*$ and $\widehat{\mathbf{v}} = \mathbf{h}\mathbf{v}'\mathbf{u}_*$ are the words in (5.6) and (5.7), respectively. The group $\mathbb{Z}_2$ satisfies the identities (5.1) and (5.5) by Lemma 2.1(iii), and it satisfies the identity $\mathbf{u} \approx \mathbf{v}$ by assumption. Therefore by (c), the group $\mathbb{Z}_2$ also satisfies the identity $\widehat{\mathbf{u}} \approx \widehat{\mathbf{v}}$, whence

(d) $\mathsf{occ}(x, \widehat{\mathbf{u}}) \equiv \mathsf{occ}(x, \widehat{\mathbf{v}}) \pmod 2$ for all $x \in \mathcal{X}$

by Lemma 2.1(iii). Observe also that

(e) the letters in $\mathbf{u}'$ are precisely all the non-simple letters of $\mathbf{u}$ counting multiplicity, while the letters in $\mathbf{v}'$ are precisely all the non-simple letters of $\mathbf{v}$ counting multiplicity.

Since the letters of $\mathbf{u}'$ are non-simple in $\widehat{\mathbf{u}}$, the identities (5.1d) can be used to order them within $\mathbf{u}'$ in any manner. Hence it follows from (e) that

$$\widehat{\mathbf{u}} = \mathbf{h}\mathbf{u}'\mathbf{u}_* \overset{(5.1d)}{\approx} \mathbf{h} y_1^{e_1'} \cdots y_m^{e_m'} \mathbf{u}_*$$

where $e_i' = \mathsf{occ}(y_i, \mathbf{u}) \geq 2$ for all i. Similarly,

$$\widehat{\mathbf{v}} = \mathbf{h}\mathbf{v}'\mathbf{u}_* \overset{(5.1d)}{\approx} \mathbf{h} y_1^{f_1'} \cdots y_m^{f_m'} \mathbf{u}_*$$

where $f_i' = \mathsf{occ}(y_i, \mathbf{v}) \geq 2$ for all i. It follows from (d) that $e_i' \equiv f_i' \pmod 2$ for all i. Therefore

$$\widehat{\mathbf{u}} \overset{(5.1d)}{\approx} \mathbf{h}y_1^{e_1'} \cdots y_m^{e_m'}\mathbf{u}_* \overset{(5.1a)}{\approx} \mathbf{h}y_1^{f_1'} \cdots y_m^{f_m'}\mathbf{u}_* \overset{(5.1d)}{\approx} \widehat{\mathbf{v}},$$

whence $\mathbf{Q}\{(5.5)\} = \mathbf{Q}\{\widehat{\mathbf{u}} \approx \widehat{\mathbf{v}}, (5.5)\} \overset{(c)}{=} \mathbf{Q}\{\mathbf{u} \approx \mathbf{v}, (5.5)\} \overset{(b)}{=} \mathbf{Q}\{\mathbf{u} \approx \mathbf{v}\}$. ■

5.6. Proof of Theorem 5.1

Theorem 5.1 is a consequence of the two results in this subsection.

PROPOSITION 5.14. *Any proper subvariety of* $\mathbf{Q}$ *that contains the group* $\mathbb{Z}_2$ *is finitely based.*

PROOF. Let $\mathbf{V}$ be any proper subvariety of $\mathbf{Q}$ with $\mathbb{Z}_2 \in \mathbf{V}$. Then it follows from Lemma 2.1(iii) that $\mathbf{V} = \mathbf{Q}\Gamma$ for some set Γ of nontrivial identities that are satisfied by the group $\mathbb{Z}_2$. By Lemma 5.4, the words that form the identities in Γ can be chosen to be in canonical form. Consider the following possibilities:

(a) every identity in Γ is standard;

(b) every identity in Γ is F_{SS}-consistent.

If (a) does not hold, then the variety $\mathbf{V}$ is finitely based by Lemma 5.6. If (a) holds and (b) does not hold, then the variety $\mathbf{V}$ is finitely based by Lemma 5.7. Hence assume that both (a) and (b) hold. In particular, since the identities in Γ are nontrivial, it follows from Remark 5.8(i) that they are all non-simple. Further, by Lemma 5.11, the variety $\mathbf{V}$ is finitely based if some identity in Γ is either non-0-consistent or non-$*$-consistent. Therefore assume that

(c) every identity in Γ is both 0-consistent and $*$-consistent.

Let $\Gamma = \Gamma_{\text{sp}} \cup \Gamma'_{\text{sp}}$ where Γ_{sp} consists of all special identities from Γ. By Lemma 5.13, the variety $\mathbf{Q}\Gamma_{\text{sp}}$ is either $\mathbf{Q}$ or $\mathbf{Q}\{(5.5)\}$ and so is finitely based. It follows from (a), (b), (c), and Lemma 5.12 that $\mathbf{Q}\Gamma'_{\text{sp}} = \mathbf{Q}(\{(5.5)\} \cup \Theta \cup \Sigma)$ for some $\Theta \subseteq \{\theta_0, \theta_1, \ldots\}$ and some set Σ of special identities. The variety $\mathbf{Q}\Theta$ is easily seen to be finitely based. By Lemma 5.13, the variety $\mathbf{Q}\Sigma$ is either $\mathbf{Q}$ or $\mathbf{Q}\{(5.5)\}$ and so is finitely based. Hence the variety $\mathbf{Q}\Gamma'_{\text{sp}}$ is finitely based. Consequently, the variety $\mathbf{V} = \mathbf{Q}\Gamma_{\text{sp}} \cap \mathbf{Q}\Gamma'_{\text{sp}}$ is also finitely based. ■

PROPOSITION 5.15. *Any proper subvariety of* $\mathbf{Q}$ *that does not contain the group* $\mathbb{Z}_2$ *is finitely based.*

PROOF. Let $\mathbf{V}$ be any proper subvariety of $\mathbf{Q}$ with $\mathbb{Z}_2 \notin \mathbf{V}$. Then $\mathbf{V} = \mathbf{Q}\Gamma$ for some set Γ of nontrivial identities such that

(a) some identity in Γ is not satisfied by the group $\mathbb{Z}_2$.

By Lemma 5.4, the words that form the identities in Γ can be chosen to be in canonical form. If some identity in Γ is nonstandard, then the variety $\mathbf{V}$ is finitely based by Lemma 5.6. Therefore assume that every identity in Γ is standard. Let $\Gamma = \Gamma_{\mathrm{g}} \cup \Gamma'_{\mathrm{g}}$ be a disjoint union, where Γ_{g} consists of all identities from Γ that are satisfied by the group $\mathbb{Z}_2$. Note that $\Gamma'_{\mathrm{g}} \neq \emptyset$ by (a). The variety $\mathbf{Q}\Gamma_{\mathrm{g}}$ is a subvariety of $\mathbf{Q}$ that contains the group $\mathbb{Z}_2$ and so is finitely based by Proposition 5.14.

Let $\gamma : \mathbf{u} \approx \mathbf{v}$ be any identity from Γ'_{g}. Since the group $\mathbb{Z}_2$ does not satisfy the identity γ, it follows from Lemma 2.1(iii) that $\mathsf{occ}(x, \mathbf{u}) \not\equiv \mathsf{occ}(x, \mathbf{v}) \pmod 2$ for some $x \in \mathcal{X}$. It is then routinely shown that the identities (5.1) and γ imply the identities

$$x^3 \approx x^2, \quad x^2yx \approx xyx^2 \approx xyx \tag{5.8}$$

so that

(b) $\mathbf{Q}\{\gamma\} = \mathbf{Q}\{(5.8), \gamma\}$.

Let $x_1, \ldots, x_m$ be all the letters such that $\mathsf{occ}(x_i, \mathbf{u}) \not\equiv \mathsf{occ}(x_i, \mathbf{v}) \pmod 2$. The identity γ is standard so that $\mathsf{sim}(\mathbf{u}) = \mathsf{sim}(\mathbf{v})$. Now if $\mathsf{occ}(x_i, \mathbf{u}) = 1$, then $x_i \in \mathsf{sim}(\mathbf{u}) = \mathsf{sim}(\mathbf{v})$ and $\mathsf{occ}(x_i, \mathbf{v}) = 1$, contradicting the choice of x_i. Therefore $\mathsf{occ}(x_i, \mathbf{u}), \mathsf{occ}(x_i, \mathbf{v}) \geq 2$ for all i. Let φ denote the substitution $x_i \mapsto x_i^2$ for all i. Then the deductions $\mathbf{u}\varphi \stackrel{(5.8)}{\approx} \mathbf{u}$ and $\mathbf{v}\varphi \stackrel{(5.8)}{\approx} \mathbf{v}$ hold so that $\mathbf{Q}\{(5.8), \gamma\} = \mathbf{Q}\{(5.8), \widehat{\gamma}\}$, where $\widehat{\gamma}$ is the identity $\mathbf{u}\varphi \approx \mathbf{v}\varphi$. Therefore $\mathbf{Q}\{\gamma\} = \mathbf{Q}\{(5.8), \widehat{\gamma}\}$ by (b). Now $\mathsf{occ}(x, \mathbf{u}\varphi) \equiv \mathsf{occ}(x, \mathbf{v}\varphi) \pmod 2$ for all $x \in \mathcal{X}$. Hence by Lemma 2.1(iii), the identity $\widehat{\gamma}$ is satisfied by the group $\mathbb{Z}_2$.

Since the identity γ was arbitrarily chosen from Γ'_{g}, the construction of $\widehat{\gamma}$ from γ in the preceding paragraph can be repeated on every identity in Γ'_{g} to obtain the set $\widehat{\Gamma}'_{\mathrm{g}} = \{\widehat{\gamma} \mid \gamma \in \Gamma'_{\mathrm{g}}\}$. Therefore $\mathbf{Q}\Gamma'_{\mathrm{g}} = \mathbf{Q}\{(5.8)\} \cap \mathbf{Q}\widehat{\Gamma}'_{\mathrm{g}}$, where the group $\mathbb{Z}_2$ satisfies the identities in $\widehat{\Gamma}'_{\mathrm{g}}$. Now the variety $\mathbf{Q}\widehat{\Gamma}'_{\mathrm{g}}$ contains $\mathbb{Z}_2$ and so is finitely based by Proposition 5.14. Consequently, the variety $\mathbf{V} = \mathbf{Q}\Gamma_{\mathrm{g}} \cap \mathbf{Q}\{(5.8)\} \cap \mathbf{Q}\widehat{\Gamma}'_{\mathrm{g}}$ is also finitely based. ■

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