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Interpolation and Definability over the Logic Gl

To Professor Ryszard Wójcicki on his 80th Birthday

Abstract. In a previous paper [21] all extensions of Johansson's minimal logic J with the weak interpolation property WIP were described. It was proved that WIP is decidable over J. It turned out that the weak interpolation problem in extensions of J is reducible to the same problem over a logic Gl, which arises from J by adding tertium non datur.

In this paper we consider extensions of the logic Gl. We prove that only finitely many logics over Gl have the Craig interpolation property CIP, the restricted interpolation property IPR or the projective Beth property PBP. The full list of Gl-logics with the mentioned properties is found, and their description is given. We note that IPR and PBP are equivalent over Gl. It is proved that CIP, IPR and PBP are decidable over the logic Gl.

Keywords: Minimal logic, interpolation, definability, amalgamation.

Introduction

Consequence relation is one of central notions in the theory of logical systems. There is a numerous literature devoted to various aspects of this notion, in particular, [32, 33]. In the present paper we consider interpolation problem for consequence relations associated with Johansson's minimal logic [5].

Interpolation theorem proved by W. Craig [2] in 1957 for the classical first order logic was a source of a lot of investigations devoted to interpolation problem in classical and non-classical logical theories [1, 4]. Now interpolation is considered as a standard property of logics and calculi like consistency, completeness and so on. For the intuitionistic predicate logic and for the predicate version of Johansson's minimal logic, the interpolation theorem was proved by K. Schütte [29].

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In this paper we consider some variants of the interpolation property in extensions of the minimal logic. The minimal logic introduced by I. Johansson [5] has the same positive fragment as the intuitionistic logic but has no special axioms for negation. Unlike the classical and intuitionistic logics, the minimal logic admits non-trivial theories containing some proposition together with its negation.

The original definition of interpolation admits different analogs which are equivalent in the classical logic but are not equivalent in other logics. It is known that in classical theories the interpolation property is equivalent to the joint consistency RCP, which arises from the joint consistency theorem proved by A. Robinson [28] for the classical predicate logic. It was proved by D. Gabbay [3] that in the intuitionistic predicate logic the full version of RCP does not hold. But some weaker version of RCP is valid, and this weaker version is equivalent to CIP in all superintuitionistic predicate logics.

A weak version WIP of the interpolation property was introduced in [14]. In [20] it was proved that WIP is equivalent to some weak version WRP of Robinson consistency property in all extensions of the minimal logic. In [14] we noted that all propositional superintuitionistic logics have WIP, although it does not hold for superintuitionistic predicate logics. Since only finitely many propositional superintuitionistic logics possess CIP [8], WIP and WRP are not equivalent to CIP and RCP over the intuitionistic logic. It follows that WIP is not equivalent to CIP in propositional logics over J. In addition, WIP is non-trivial in propositional extensions of the minimal logic. There are continua of propositional J-logics with WIP and of J-logics without WIP [20].

In [19] we defined a logic Gl, which is axiomatized over J by tertium non datur, and proved that the problem of weak interpolation in J-logics is reducible to the same problem over Gl. A description of J-logics with WIP was found in [21]; it turned out that the set of J-logics with WIP is divided into eight pairwise disjoint intervals. It was proved in that paper that the weak interpolation property WIP is decidable over the logics Gl and J. Note that there is a continuum of Gl-logics with WIP and a continuum of Gl-logics without WIP.

In this paper we concentrate on extensions of the logic Gl. In addition to CIP and WIP, we consider also the restricted interpolation property IPR and the projective Beth property PBP. All positively axiomatizable J-logics with the properties CIP and PBP were described in [11]. We use a description of J-logics with WIP [21] in order to prove that there are only finitely many Gl-logics with CIP, IPR or PBP and to describe all of these logics. In particular, IPR and PBP turn out to be equivalent in all Gl-logics. We show that all the considered properties are decidable over the logic Gl.

1. Interpolation and definability

If $\mathbf{p}$ is a list of non-logical symbols, let $A(\mathbf{p})$ denote a formula whose all non-logical symbols are in $\mathbf{p}$, and $\mathcal{F}(\mathbf{p})$ the set of all such formulas.

Let L be a logic, $\vdash_L$ deducibility relation in L. Suppose that $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ are disjoint lists of non-logical symbols, and $A(\mathbf{p},\mathbf{q})$, $B(\mathbf{p},\mathbf{r})$ are formulas. *The Craig interpolation property* CIP and *the deductive interpolation property* IPD are defined as follows:

CIP. If $L \vdash A(\mathbf{p},\mathbf{q}) \to B(\mathbf{p},\mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $L \vdash A(\mathbf{p},\mathbf{q}) \to C(\mathbf{p})$ and $L \vdash C(\mathbf{p}) \to B(\mathbf{p},\mathbf{r})$.

IPD. If $A(\mathbf{p},\mathbf{q}) \vdash_L B(\mathbf{p},\mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L C(\mathbf{p})$ and $C(\mathbf{p}) \vdash_L B(\mathbf{p},\mathbf{r})$.

The restricted interpolation property was introduced in [12]:

IPR. If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$.

In [14] *the weak interpolation property* was introduced:

WIP. If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L \bot$.

It is clear that WIP is a special case of IPR.

Suppose that $\mathbf{p}$, $\mathbf{q}$, $\mathbf{q}'$ are disjoint lists of variables that do not contain x and y, $\mathbf{q}$ and $\mathbf{q}'$ are of the same length, and $A(\mathbf{p},\mathbf{q},x)$ is a formula. We define *the projective Beth property*:

PBP. If $A(\mathbf{p},\mathbf{q},x), A(\mathbf{p},\mathbf{q}',y) \vdash_L x \leftrightarrow y$, then $A(\mathbf{p},\mathbf{q},x) \vdash_L x \leftrightarrow B(\mathbf{p})$ for some $B(\mathbf{p})$.

A weaker *Beth property* BP arises from PBP by omitting $\mathbf{q}$ and $\mathbf{q}'$.

It was proved by G. Kreisel [7] that all superintuitionistic logics possess the Beth property BP. The same proof is valid for all J-logics. In addition, PBP follows from CIP, and IPR follows from PBP in every J-logic [13]. There is a continuum of J-logics without WIP [20] although all superintuitionistic logics have WIP [14]. Thus in all extensions of the minimal logic J we have

$$\text{CIP} \iff \text{IPD} \Rightarrow \text{PBP} \Rightarrow \text{IPR} \Rightarrow \text{WIP}.$$

Moreover, PBP does not imply IPD, and WIP does not imply IPR even on the class of superintuitionistic logics [9]. The problem of equivalence of IPR and PBP in J-logics still remains open; it is proved in [18, 15] that these properties are equivalent in superintuitionistic, positive and negative logics.

In [8] a description of all propositional superintuitionistic logics with the Craig interpolation property was obtained. In [9] all superintuitionistic logics with the projective Beth properties were found. There are only finitely many superintuitionistic logics with these properties. All positive logics with CIP and PBP were described in [11], where a study of these properties was initiated for extensions of Johansson's minimal logic, too.

The language of the logic J contains $\&, \vee, \rightarrow, \bot, \top$ as primitive; negation is defined by $\neg A = A \rightarrow \bot$; $(A \leftrightarrow B) = (A \rightarrow B)\&(B \rightarrow A)$. A formula is said to be *positive* if contains no occurrences of $\bot$. The logic J can be given by the calculus, which has the same axiom schemes as the positive intuitionistic calculus Int^+, and the only rule of inference is modus ponens: $A, A \rightarrow B \,/\, B$. More exactly, J has the following axiom schemes:

1. $A \rightarrow (B \rightarrow A)$
2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
3. $A\&B \rightarrow A$
4. $A\&B \rightarrow B$
5. $A \rightarrow (B \rightarrow A\&B)$
6. $A \rightarrow A \vee B$
7. $B \rightarrow A \vee B$
8. $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$

By a J-*logic* we mean an arbitrary set of formulas containing all the axioms of J and closed under modus ponens and substitution rules. We denote

$$\mathrm{Int} = \mathrm{J} + (\bot \rightarrow A),\ \mathrm{Cl} = \mathrm{Int} + (A \vee \neg A),\ \mathrm{Neg} = \mathrm{J} + \bot,\ \mathrm{Gl} = \mathrm{J} + (A \vee \neg A).$$

A logic is *non-trivial* if it differs from the set of all formulas. A J-logic is *superintuitionistic* if it contains the intuitionistic logic Int, and *negative* if contains the logic Neg; L is *paraconsistent* if contains neither Int nor Neg. One can prove that a J-logic is negative if and only if it is not contained in Cl. For any J-logic L we denote by $E(L)$ the family of all J-logics containing L.

The problem of weak interpolation in J-logics is reducible to the same problem in extensions of the logic Gl.

THEOREM 1.1. [20] *For any* J-*logic* L*, the logic* L *has* WIP *if and only if* $L + (A \vee \neg A)$ *has* WIP.

The denotation Gl is caused by well known Glivenko's theorem, which says that a formula $\neg A$ is valid in the intuitionistic logic if and only if it is valid in the classical logic $\mathrm{Cl} = \mathrm{Int} + (A \vee \neg A)$. The following generalisation of Glivenko's theorem holds.

PROPOSITION 1.2. [26] *For any* J*-logic* L *and any formula* A*:*

$$L \vdash \neg A \iff L + \mathrm{Gl} \vdash \neg A.$$

2. Algebraic semantics

Relational semantics for the logic J and some of its extensions is presented in [30, 31, 16]. Algebraic semantics for extensions of the minimal logic is built with the help of so-called J-*algebras*, i.e. the algebras $\mathbf{A} = < A;\ \&, \vee, \to, \bot, \top >$ satisfying the conditions:

$< A; \&, \vee, \to, \top >$ is an implicative lattice, i.e. a lattice w.r.t. & and $\vee$, with the greatest element $\top$, where for any $x, y.z \in A$,

$z \le x \to y \iff z\&x \le y$,

and $\bot$ is an arbitrary element of A.

A J-algebra is said to be a *Heyting algebra*, or a *pseudoboolean algebra* [27], if $\bot$ is the least element of the set A, and a *negative algebra* if $\bot$ is the greatest element of A. The one-element J-algebra $\mathbf{E}$ is said to be *degenerate*; it is a unique J-algebra, which is a negative algebra and a Heyting algebra at the same time.

A J-algebra $\mathbf{A}$ is *non-degenerate* if it contains at least two elements; $\mathbf{A}$ is *well-connected*, or *strongly compact*, if it satisfies the condition $x \vee y = \top \Leftrightarrow (x = \top$ or $y = \top)$ for all $x, y \in \mathbf{A}$. An element Ω of an algebra $\mathbf{A}$ is said to be an *opremum* of $\mathbf{A}$ if it is the greatest among the elements of $\mathbf{A}$ different from $\top$. By B_0 we denote the two-element boolean algebra.

Recall that a non-degenerate algebra $\mathbf{A}$ is *subdirectly irreducible* if it can not be represented as a subdirect product of factors different from $\mathbf{A}$. An algebra is *finitely indecomposable* if it can not be represented as a subdirect product of finitely many factors different from it.

The following lemma known for Heyting algebras (see, for example, [8]) can easily be extended to J-algebras.

LEMMA 2.1. *For any* J*-algebra* $\mathbf{A}$*:*

a) $\mathbf{A}$ *is finitely indecomposable if and only if the one-element filter* $\nabla = \{\top\}$ *is prime, i.e.* $\mathbf{A}$ *is well-connected;*

b) $\mathbf{A}$ *is subdirectly irreducible if and only if* $\mathbf{A}$ *has an opremum.*

The proof of the following lemma is analogous to the proof of a similar lemma for Heyting algebras given in [9].

LEMMA 2.2. *For any* J*-algebra* $\mathbf{A}$*, if* $a \not\le b$ *in* $\mathbf{A}$*, then there exist a subdirectly irreducible* $\mathbf{B}$ *with an opremum* Ω *and a homomorphism* $f : \mathbf{A} \to \mathbf{B}$ *such that* $f(a) = \top$ *and* $f(b) = \Omega$*.*

For a negative algebra $\mathbf{A}$ we denote by $\mathbf{A}^\Lambda$ a new J-algebra arising from $\mathbf{A}$ by adding a new greatest element $\top = \top_{\mathbf{A}^\Lambda}$. Thus $\bot_{\mathbf{A}^\Lambda} = \bot_{\mathbf{A}} = \top_{\mathbf{A}}$ becomes an opremum of the algebra $\mathbf{A}^\Lambda$, and the algebra $\mathbf{A}^\Lambda$ itself is subdirectly irreducible.

It is clear that $\mathbf{A}$ is a sublattice of $\mathbf{A}^\Lambda$. Moreover, for all $x, y \in \mathbf{A}^\Lambda$:

$$x \to_{\mathbf{A}^\Lambda} y = \begin{cases} \top_{\mathbf{A}^\Lambda}, & \text{if} \quad x \leq y, \\ x \to_{\mathbf{A}} y, & \text{if} \quad x, y \in \mathbf{A}, x \not\leq y, \\ y, & \text{if} \quad x = \top_{\mathbf{A}^\Lambda}, y \in \mathbf{A}. \end{cases}$$

It follows easily from the definition

LEMMA 2.3. *Any algebra* $\boldsymbol{A}$ *is a homomorphic image of* $\boldsymbol{A}^\Lambda$ *under a homomorphism*

$$f(z) = z \& \bot.$$

The following lemma immediately follows from [13, Proposition 2.5].

LEMMA 2.4. *Let* $\boldsymbol{A}$ *and* $\mathbf{B}$ *be negative algebras,* $\mathbf{C}$ *a* J-*algebra.*

(1) *A mapping* $\alpha : \boldsymbol{A}^\Lambda \to \mathbf{B}^\Lambda$ *is a monomorphism if and only if its restriction* α^l *onto* $\boldsymbol{A}$ *is a monomorphism of* $\boldsymbol{A}$ *to* $\mathbf{B}$.

(2) *For any homomorphism* $h : \boldsymbol{A}^\Lambda \to \mathbf{C}$ *exactly one of the following conditions is satisfied:*

(a) $h(\bot_{\boldsymbol{A}^\Lambda}) = \top_{\mathbf{C}}$ *and the restriction* h^l *of* h *onto* $\boldsymbol{A}$ *is a homomorphism of* $\boldsymbol{A}$ *to* $\mathbf{C}$*;*

(b) $h(\bot_{\boldsymbol{A}^\Lambda}) \neq \top_{\mathbf{C}}$ *and* h *is a monomorphism of* $\boldsymbol{A}^\Lambda$ *into* $\mathbf{C}$.

(3) *Let* $h_1 : \boldsymbol{A} \to \mathbf{C}$ *be a homomorphism. Then* $\mathbf{C}$ *is a negative algebra and the following mapping is a homomorphism of* $\boldsymbol{A}^\Lambda$ *to* $\mathbf{C}$:

$$h(x) = \begin{cases} \top_{\mathbf{C}}, & \textit{if} \quad x = \top_{\boldsymbol{A}^\Lambda}, \\ h_1(x), & \textit{if} \quad x \in \mathbf{A}. \end{cases}$$

It is well known that the family of all J-algebras forms a variety and there exists a one-to-one correspondence between J-logics and varieties of J-algebras. If A is a formula and $\mathbf{B}$ is an algebra, we say that A is *valid* in $\mathbf{B}$ and write $\mathbf{B} \models A$ if the identity $A = \top$ is satisfied in $\mathbf{B}$. We write $\mathbf{B} \models L$ instead of $(\forall A \in L)(\mathbf{B} \models A)$.

To any logic $L \in E(\mathrm{J})$ there corresponds a variety

$$V(L) = \{\mathbf{A} | \mathbf{A} \models L\}.$$

Every logic L is characterized by the variety $V(L)$. We say that a logic L is generated by some class of algebras if the variety $V(L)$ is generated by this class. If $V(L)$ is generated by an algebra $\mathbf{A}$, we sometimes write $L = L\mathbf{A}$.

If $L \in E(\mathrm{Int})$, then $V(L)$ is a variety of Heyting algebras, and if $L \in E(\mathrm{Neg})$, then $V(L)$ is a variety of negative algebras.

It is clear that any intersection of two J-logics is also a J-logic. An axiomatization of the intersection can easily be found from the axiomatization of the initial logics. For formulas A and B, let us denote by $A \vee' B$ a disjunction $A \vee B'$, where B' is obtained from B by replacing all variables by new variables not contained in A.

LEMMA 2.5. *Let L be an intersection of two* J*-logics L_1 and L_2. Then*

1. *L is axiomatizable by all formulas $A \vee' B$, where A is an axiom of L_1 and B is an axiom of L_2;*
2. *a finitely indecomposable algebra $\boldsymbol{A}$ belongs to $V(L)$ if and only if $\boldsymbol{A} \in (V(L_1) \cup V(L_2))$.*

PROOF. (1) By analogy with Miura's theorem [24].

(2) Follows from (1) and Lemma 2.1. ■

For $L_1 \in E(\mathrm{Neg})$, we denote by $L_1 \uparrow \mathrm{Cl}$ a logic characterized by all algebras of the form $\mathbf{A}^{\Lambda}$, where $\mathbf{A} \models L_1$. By $L_1 \Uparrow \mathrm{Cl}$ we denote a logic characterized by the class of algebras of the form $\mathbf{A}^{\Lambda}$, where $\mathbf{A}$ is a finitely indecomposable algebra in $V(L_1)$. In particular, if L_1 is the trivial logic For, then $L_1 \uparrow \mathrm{Cl}$ and $L_1 \Uparrow \mathrm{Cl}$ are equal to Cl.

Note that the definition of $L_1 \uparrow \mathrm{Cl}$ and $L_1 \Uparrow \mathrm{Cl}$ is slightly different from the definition introduced in [13]. One can easily prove that the definitions are equivalent.

As an example, we consider the logic $\mathrm{Gl} = \mathrm{J} + (p \vee \neg p)$.

PROPOSITION 2.6. [21] *The logic* Gl *coincides with* Neg $\uparrow$ Cl *and is generated by the class $\{\boldsymbol{A}^{\Lambda} |\ \boldsymbol{A}$ is a negative algebra$\}$.*

In [13] an axiomatization was found for logics of the form $L_1 \uparrow \mathrm{Cl}$ and $L_1 \Uparrow \mathrm{Cl}$, where L_1 is a negative logic.

PROPOSITION 2.7. [13] *For any negative logic L_1:*

$$L_1 \uparrow \mathrm{Cl} = \mathrm{Gl} + \{\bot \to A |\ A \in L_1\},$$

$$L_1 \Uparrow \mathrm{Cl} = (L_1 \uparrow \mathrm{Cl}) + ((\bot \to A \vee B) \to (\bot \to A) \vee (\bot \to B)).$$

If $L_1 = \mathrm{Neg} + Ax$, then $L_1 \uparrow \mathrm{Cl} = \mathrm{Gl} + \{\bot \to A |\ A \in Ax\}$.

PROOF. The first and the second identities are proved in [13, Corollary 3.5(2) and Theorem 3.4(2)]. The last line follows from the results by S.Odintsov [25]. ■

3. Interpolation, projective Beth property, and amalgamation

Recall [11] that a J-logic has the Craig interpolation property if and only if the variety $V(L)$ has the amalgamation property AP. In the case of J-algebras AP is equivalent to the super-amalgamation property SAP. We recall necessary definitions.

Let V be a class of algebras invariant under isomorphisms. The class V is said to be *amalgamable* if it satisfies the following condition AP for any algebras $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$ in V:

AP. If $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$, then there exist $\mathbf{D}$ in V and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$.

The triple $(\mathbf{D}, \delta, \varepsilon)$ is called an *amalgam* for $\mathbf{A}, \mathbf{B}, \mathbf{C}$.

Say that a class V has a property *SAP* (the *super-amalgamation property*) if for any algebras $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$ in V the condition AP is satisfied and, moreover, in $\mathbf{D}$ the following holds :

$$\delta(x) \leq \varepsilon(y) \iff (\exists z \in \mathbf{A})(x \leq z \text{ and } z \leq y),$$

$$\delta(x) \geq \varepsilon(y) \iff (\exists z \in \mathbf{A})(x \geq z \text{ and } z \geq y).$$

A class V has the *restricted amalgamation property RAP* [12] if the following is satisfied:

RAP. For any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ such that $\mathbf{A}$ is a common subalgebra of algebras $\mathbf{B}$ and $\mathbf{C}$, there exist $\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$ and the restriction δ' of δ onto $\mathbf{A}$ is a monomorphism.

In another way the notion of restricted amalgamation was defined in [9, 11]: we say that a class V has the property *RAP** if *AP* is satisfied for any subdirectly irreducible algebras $\mathbf{A}, \mathbf{B}, \mathbf{C}$ having the same opremum.

A class V of algebras possesses *strong epimorphisms surjectivity SES* if the following condition is satisfied:

SES. For any $\mathbf{A}, \mathbf{B}$ in V such that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$, and for any $b \in \mathbf{B} - \mathbf{A}$ there exist $\mathbf{C} \in V$ and homomorphisms $g : \mathbf{B} \to \mathbf{C}$, $h : \mathbf{B} \to \mathbf{C}$ such that $g(x) = h(x)$ for all $x \in \mathbf{A}$ and $g(b) \neq h(b)$.

An algebraic equivalent of the weak interpolation property is found in [20]. We define the *weak amalgamation property* for a class V of J-algebras.

WAPJ. For any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V$ and monomorphisms $\beta : \mathbf{A} \to \mathbf{B}$, $\gamma : \mathbf{A} \to \mathbf{C}$ there exist an algebra $\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$

such that $\delta\beta(x) = \varepsilon\gamma(x)$ for all $x \in \mathbf{A}$, where $\bot \neq \top$ in $\mathbf{D}$ whenever $\bot \neq \top$ in $\mathbf{A}$.

A variety of J-algebras is said to be *weakly amalgamable* if it has the property WAPJ.

Note that the definition introduced above differs from the definition of a weak amalgamation property WAP considered in [17]. The property WAP is a particular case of WAPJ.

Note that if a class V is closed under isomorphisms then WAPJ is equivalent to the following condition:

For any $\mathbf{B}, \mathbf{C} \in V$ with a common subalgebra $\mathbf{A}$, there exist an algebra $\mathbf{D}$ in V and homomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathbf{A}$, where $\bot \neq \top$ in $\mathbf{D}$ whenever $\bot \neq \top$ in $\mathbf{A}$.

THEOREM 3.1. [20] *Let L be a* J*-logic. Then L has* WIP *if and only if $V(L)$ has WAPJ.*

The following theorems were proved in [11].

THEOREM 3.2. *For any logic L in $E(J)$ the following are equivalent:*

1. *L possesses the Craig interpolation property;*
2. *$V(L)$ is amalgamable;*
3. *$V(L)$ has SAP;*
4. *the condition AP is satisfied for all finitely indecomposable* $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$ *in $V(L)$.*

THEOREM 3.3. *For any logic L in $E(J)$ the following are equivalent:*

1. *L has the projective Beth property;*
2. *$V(L)$ has SES;*
3. *$V(L)$ has RAP* and the class $FI(V(L))$ of finitely indecomposable algebras of $V(L)$ has SES.*

We add that from the description of all superintuitionistic and negative logics with the interpolation property found in [8] and in [11], it follows

THEOREM 3.4. *For any logic L in $E(\mathrm{Int})$ or $E(\mathrm{Neg})$ the following are equivalent:*

1. *the variety $V(L)$ is amalgamable;*
2. *the class of finitely indecomposable algebras of $V(L)$ is amalgamable.*

We do not know if this statement holds for all extensions of the minimal logic.

As for the restricted interpolation property, it holds

THEOREM 3.5. *For any logic L in $E(\mathrm{J})$ the following are equivalent:*

1. *L has* IPR*;*
2. *$V(L)$ has RAP;*
3. *$V(L)$ has RAP*;*
4. *for any subdirectly irreducible* J*-algebras* $\mathbf{A}$,$\mathbf{B}$,$\mathbf{C}$ *in $V(L)$ having a common opremum Ω, if $\boldsymbol{A}$ is a common subalgebra of* $\mathbf{B}$ *and* $\mathbf{C}$*, then there exist a subdirectly irreducible algebra* $\mathbf{D}$ *in $V(L)$ and monomorphisms $\delta : \mathbf{B} \to \mathbf{D}$, $\varepsilon : \mathbf{C} \to \mathbf{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \boldsymbol{A}$ and $\delta(\Omega)$ is an opremum of* $\mathbf{D}$.

We see that for all varieties of J-algebras, RAP* follows from PBP, so it holds

PROPOSITION 3.6. *For all extensions of the minimal logic,* CIP *implies* PBP *and* PBP *implies* IPR.

We recall some known facts on interpolation properties in J-logics.

LEMMA 3.7. *If a* J*-logic L has* CIP*,* PBP *or* IPR*, then $L_{neg} = L + \bot$ also has the same property.*

PROOF. For CIP and PBP the statement is proved in [11]. The proof for IPR is by analogy. ■

PROPOSITION 3.8. [9] *There exist exactly 16 superintuitionistic logics with the projective Beth property* PBP*; among them exactly 8 logics have* CIP.

The list of superintuitionistic logics with CIP includes the logics Int, LC, LS, Cl and also the trivial logic For. The logic Cl is the greatest among consistent superintuitionistic logics, and the logic LS is the greatest among consistent superintuitionistic logics different from Cl. The logic Cl is characterized by a two-element boolean algebra B_0, the logic LS $=$ Int$+(A \vee (A \to (B \vee \neg B))) + (\neg A \vee \neg\neg A)$ by a three-element linearly ordered Heyting algebra C_1, and the logic LC $=$ Int$+((A \to B) \vee (B \to A))$ by all linearly ordered Heyting algebras. An axiomatization and more detailed description of all the sixteen logics with PBP is given in [9].

Proposition 3.9. [11] *There are exactly seven negative logics with* PBP*, namely:*

(1) Neg,
(2) $\mathrm{NC} = \mathrm{Neg} + ((A \to B) \vee (B \to A))$,
(3) $\mathrm{NE} = \mathrm{Neg} + (A \vee (A \to B))$,
(4) $\mathrm{For} = \mathrm{Neg} + A$,
(5) $\Delta(\mathrm{NC}) = \mathrm{Neg} + (C \vee (C \to (A \to B) \vee (B \to A)))$,
(6) $\Delta(\mathrm{NE}) = \mathrm{Neg} + (C \vee (C \to A \vee (A \to B)))$,
(7) $\mathrm{Neg} + (A \vee (A \to B) \vee (B \to C))$.

The logics (1) – (4) have CIP *and the others do not possess* CIP*.*

The logic NC is characterized by all linearly ordered negative algebras, and the logic NE by a two-element negative algebra. The logic (7) is characterized by a three-element negative algebra. One can find more detail in [11].

Proposition 3.10. *For all superintuitionistic and negative logics, the properties* IPR *and* PBP *are equivalent.*

Proof. For superintuitionistic logics this statement is the main result of [18]. For negative logics, the result immediately follows from the equivalence of these properties in positive logics, which was proved in [15]. ■

All superintuitionistic and negative logics have the weak interpolation property WIP. But it can not be extended to all J-logics [21].

Say that a property P is *decidable* over a logic L if there is an algorithm which, for any finite set Ax of axiom schemes, decides if the logic $L + Ax$ has the property P.

Proposition 3.11. *The properties* CIP, PBP *and* IPR *are decidable over the logics* Int *and* Neg. *The property* WIP *is decidable over* J.

Proof. Decidability of CIP and PBP on the classes of superintuitionistic, positive and negative calculi was proved in [10, 11]. From Proposition 3.10, it follows that IPR is also decidable in these classes. Decidability of WIP in extensions of the logic J is proved in [21]. ■

4. Description of J-logics with WIP

In this section we give more detailed description of J-logics with WIP. Theorem 1.1 reduces consideration of WIP in J-logics to studying extensions of the logic Gl.

Recall denotations of Section 2. For any negative algebra $\mathbf{A}$, we denote by $\mathbf{A}^\Lambda$ a new J-algebra, in which $\bot$ is an opremum and for any $x \in \mathbf{A}^\Lambda$ the condition

$$x \in \mathbf{A} \iff x \leq \bot$$

is satisfied. For any J-logic L we define a class

$$\Lambda(L) = \{\mathbf{A}^\Lambda | \ \mathbf{A} \text{ is a negative algebra and } \mathbf{A}^\Lambda \in V(L)\}.$$

It is easily seen that the following holds:

LEMMA 4.1. *A class $\Lambda(L)$ is empty if and only if L is a negative logic.*

In [20] we have proved that J-logic L has WIP if and only if the class $\Lambda(L)$ is amalgamable. The following proposition shows that classes $\Lambda(L)$ divide the family of Gl-logics into intervals. It gives a useful classification of logics over Gl, which supplies a classification of J-logics given in [25].

PROPOSITION 4.2. [20] *Let a* J*-logic L_0 be generated by the class $\Lambda(L_0)$. Then L_0 contains* Gl *and for any $L \in E(\mathrm{Gl})$ the equivalence holds:*

$$\Lambda(L) = \Lambda(L_0) \iff \mathrm{Neg} \cap L_0 \subseteq L \subseteq L_0.$$

Now we consider some special extensions of Gl. An axiomatization of these logics of the forms $L \uparrow \mathrm{Cl}$ and $L \Uparrow \mathrm{Cl}$, where L is a negative logic, is presented in Proposition 2.7. The logic $\mathrm{Gl} = \mathrm{Neg} \uparrow \mathrm{Cl}$ is characterized by all algebras of the form $\mathbf{A}^\Lambda$, where $\mathbf{A}$ is a negative algebra (see Proposition 2.6).

The key role in description of J-logics with WIP [21] belongs to the following list SL consisting of eight Gl-logics:

$$\mathrm{For},\ \mathrm{Cl},\ (\mathrm{NE} \uparrow \mathrm{Cl}),\ (\mathrm{NC} \uparrow \mathrm{Cl}),\ (\mathrm{Neg} \uparrow \mathrm{Cl}),\ (\mathrm{NE} \Uparrow \mathrm{Cl}),\ (\mathrm{NC} \Uparrow \mathrm{Cl}),\ (\mathrm{Neg} \Uparrow \mathrm{Cl}).$$

Each of these logics L is generated by the class $\Lambda(L)$.

PROPOSITION 4.3. [21] *Let L be any* Gl*-logic of the list SL. Then L has* CIP*, and the classes $V(L)$ and $\Lambda(L)$ are amalgamable.*

Further results of this section are proved in [21]. In that paper all logics over Gl with WIP are described, and an effective criterion is found for verifying WIP in J-logics.

THEOREM 4.4. *For any logic L in $E(\mathrm{J})$ the following are equivalent:*

1. *L has* WIP*;*
2. *the class $\Lambda(L)$ is amalgamable;*

3. $\Lambda(L) = \Lambda(L_0)$ *for some logic* L_0 *in the list SL.*

Proof. Equivalence of (1) and (2) is proved in [20, Theorem 6.2], and equivalence of (1) and (3) in [21]. ■

The property WIP is non-trivial in propositional extensions of the logic Gl. Both the sets of J-logics with WIP and of J-logics without WIP have the cardinality of the continuum. The former set contains all negative logics, i.e. a continual family. The latter one has at least the same cardinality as the set of negative logics different from $\mathrm{Neg}, \mathrm{NC}, \mathrm{NE}, \mathrm{For}$. Nevertheless, we have

Theorem 4.5. [21] *The property* WIP *is decidable over* J, *i.e. there is an algorithm which, for any finite set* Ax *of axiom schemes, decides if the logic* $\mathrm{J} + Ax$ *has* WIP.

5. CIP, IPR and PBP over Gl

The logics of the list SL play a key role in description of Gl-logics possessing WIP. The following theorem is proved in [21, Theorem 8.4].

Theorem 5.1. [21] *A logic* L *over* Gl *has* WIP *if and only if it is representable in a form* $L = L_{neg} \cap L_0$, *where* $L_{neg} = L + \bot$ *is a negative logic and* $L_0 \in SL$.

In this section we find a representation for Gl-logics with CIP, IPR and PBP which is similar to Theorem 5.1.

Proposition 5.2. *Let* L_1 *be a negative logic,* L_2 *any extension of* J. *If* L_1 *and* L_2 *have* CIP, *then* $L_1 \cap L_2$ *also has* CIP.

Proof. Due to Theorem 3.2 it is sufficient to prove amalgamability of the variety $V(L_1 \cap L_2)$, and it is equivalent to existence of an amalgam for any finitely indecomposable algebras of this variety.

Let us take some finitely indecomposable algebras $\mathbf{A}, \mathbf{B}, \mathbf{C} \in V(L_1 \cap L_2)$, where $\mathbf{A}$ is a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$. If none of these algebras is negative, then all of them belong to $V(L_2)$ and there is an amalgam in $V(L_2)$, and so in $V(L_1 \cap L_2)$.

If one of these algebras is negative, then the others are negative too. Note that in this case each of these algebras belongs to $V(L_1) \cup V(L_2 + \bot)$. Each of these two varieties is amalgamble because the logic $L_2 + \bot$ also has CIP. Recall that there are only four amalgamable varieties of negative algebras, and they are comparable with respect to set-theoretic inclusion. Therefore, $V(L_1) \cup V(L_2 + \bot)$ is equal either to $V(L_1)$ or to $V(L_2 + \bot)$, and so there exists an amalgam of the algebras $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $V(L_1 \cap L_2)$. Thence the variety $V(L_1 \cap L_2)$ is amalgamable. ■

Theorem 5.1 gives a convenient representation for Gl-logics with WIP. A similar representation is also useful for other J-logics. We have an easy

LEMMA 5.3. *Any* J*-logic* L *is representable in the form* $L = L_{neg} \cap L_1$ *for a suitable* J*-logic* L_1. *If* $L = L_{neg} \cap L_1$, *then any negative algebra in* $V(L_1)$ *belongs to the variety* $V(L_{neg})$.

PROOF. Evidently, one can take L as L_1. Let $L = L_{neg} \cap L_1$, $\mathbf{A}$ be a negative algebra in $V(L_1)$. Then $\mathbf{A} \in V(L)$ and hence $\mathbf{A} \in V(L_{neg})$. ■

PROPOSITION 5.4. *Let* L *and* L_1 *be* J*-logics, and* $L = L_{neg} \cap L_1$.

(1) *If* L_{neg} *and* L_1 *have* IPR, *then* L *has* IPR.

(2) *If* L_{neg} *and* L_1 *have* PBP, *then* L *has* PBP.

PROOF. (1) Let L_{neg} and L_1 have IPR. We apply Theorem 3.5 and prove that $V(L)$ possesses the property RAP*. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be subdirectly irreducible algebras in $V(L)$ having a common opremum, and $\mathbf{A}$ be a common subalgebra of $\mathbf{B}$ and $\mathbf{C}$. If one of the three algebras is negative, then the others are also negative, and so all the three algebras belong to the variety $V(L_{neg})$, and thus have an amalgam in $V(L_{neg})$ and in $V(L)$.

Let the algebras are not negative. Then they belong to the variety $V(L_1)$, which possesses the property RAP*. Therefore, there is an amalgam in $V(L_1)$ and in $V(L)$.

(2) Let L_{neg} and L_1 have PBP. We apply Theorem 3.3. Then $V(L_{neg})$ and $V(L_1)$ have the property RAP*; by the item (1) $V(L)$ has the same property. Now we show that the class of finite indecomposable algebras in $V(L)$ possesses the property SES. Let $\mathbf{A}, \mathbf{B}$ be finitely indecomposable algebras in $V(L)$, $\mathbf{A}$ be a subalgebra of $\mathbf{B}$ and $b \in \mathbf{B} - \mathbf{A}$.

If $\mathbf{B}$ is a negative algebra, then $\mathbf{A}, \mathbf{B} \in V(L_{neg})$, and the required algebra and monomorphisms exist in $V(L_{neg})$, and so in $V(L)$. If $\mathbf{B}$ is not a negative algebra, then $\mathbf{A}, \mathbf{B} \in V(L_1)$, and the required algebra and monomorphisms exist in $V(L_1)$, and so in $V(L)$. ■

THEOREM 5.5. *Let* L *be an extension of* Gl, *and* $SL = \{\mathrm{For}, \mathrm{Cl}\} \cup \{(L_1 \uparrow \mathrm{Cl}), (L_1 \Uparrow \mathrm{Cl}) \mid L_1 \in \{\mathrm{Neg}, \mathrm{NC}, \mathrm{NE}\}\}$.

1. *L has* CIP *if and only if* $L = L_{neg} \cap L_0$, *where* L_{neg} *has* CIP, $L_0 \in SL$.

2. *L has* IPR *if and only if $L = L_{neg} \cap L_0$, where L_{neg} is a logic with* IPR*, $L_0 \in SL$.*
3. *L has* PBP *if and only if $L = L_{neg} \cap L_0$, where L_{neg} is a logic with* PBP*, $L_0 \in SL$.*

Proof. Let L have CIP, IPR or PBP. Then L_{neg} has the same property by Lemma 3.7. In addition, L has WIP, and by Theorem 5.1 $L = L_{neg} \cap L_0$, where L_0 is some logic in SL.

Conversely, all the logics of the list SL have CIP by Proposition 4.3, and thus they possess also the properties IPR and PBP. Let us take $L_0 \in SL$.

If $L = L_{neg} \cap L_0$ and L_{neg} is a logic with CIP, then L has CIP by Proposition 5.2. If L_{neg} possesses the property IPR or PBP, then L has the same property by Proposition 5.4. ■

Corollary 5.6. *1.* IPR *and* PBP *are equivalent over* Gl*.*

2. There are only finitely many logics with IPR *over* Gl*.*

Proof. (1) In [15] the equivalence of the properties IPR and PBP was proved for positive logics containing the positive fragment Int^+ of the intuitionistic logic.

There is a one-to-one correspondence between the families of positive and of negative logics. For any positive formula A, with the positive logic $\mathrm{Int}^+ + A$ a negative logic $\mathrm{Neg} + A$ is associated. Conversely, for any formula A one can build a positive formula A' by replacing all occurrences of $\perp$ in A by $\top$. Then $\mathrm{Neg} \vdash A \leftrightarrow A'$. Therefore the logics $\mathrm{Neg} + A$ and $\mathrm{Neg} + A'$ coincide, and the positive fragment of $\mathrm{Neg} + A$ is equal to $\mathrm{Int}^+ + A'$. It is easy to see that a logic $\mathrm{Neg} + A$ has CIP, IPR or PBP if and only if its positive fragment has the same property. Thence the equivalence of IPR and PBP in negative logics follows from the equivalence of these properties in positive logics, which was – as we have mentioned – proved in [15].

(2) In Proposition 3.9 all extensions of the logic Neg with the property PBP were listed. By the item (1) we obtain that the logic Neg has exactly seven extensions with the property IPR. By Theorem 5.5 the number of logics with IPR over Gl is finite. ■

Theorem 5.7. *1. A logic L over* Gl *has* CIP*,* PBP *or* IPR *if and only if L has* WIP *and $L_{neg} = L + \perp$ has* CIP*,* PBP *or* IPR*, respectively.*

2. The properties CIP*,* IPR *and* PBP *are decidable over* Gl*.*

Proof. (1) Let L have CIP, PBP or IPR. Then L_{neg} has the same property by Lemma 3.7. In addition, L has WIP.

Conversely, let L have WIP. By Theorem 5.1 it is representable in the form

$$L = L_{neg} \cap L_0$$

for some logic L_0 in the list SL. If $L_{neg} = L + \bot$ has CIP, PBP or IPR, then L has the same property by Theorem 5.5.

(2) Let a logic L be obtained by adding finitely many axiom schemes Ax to Gl. Denote their conjunction by A. By Theorem 4.5 one can check if the logic $L = \mathrm{Gl} + A$ has the property WIP. If it does not possess WIP, then it has neither CIP, nor IPR, nor PBP.

Assume that this logic has WIP. Due to the item (1), it remains to find out if the logic $L_{neg} = \mathrm{Neg} + (A \vee \neg A) + A = \mathrm{Neg} + A$ has the required property CIP, IPR or PBP. Decidability of CIP, PBP and IPR in negative logics is stated in Proposition 3.11. ■

Now we list all the logics with CIP over Gl.

THEOREM 5.8. *A logic L over* Gl *has* CIP *if and only if it is in the list ISL consisting of the following twenty logics:* $\mathrm{For}, \mathrm{NE}, \mathrm{NC}, \mathrm{Neg}, \mathrm{Cl}, \mathrm{NE} \cap \mathrm{Cl}, \mathrm{NC} \cap \mathrm{Cl}, \mathrm{Neg} \cap \mathrm{Cl}, (\mathrm{NE} \Uparrow \mathrm{Cl}), \mathrm{NC} \cap (\mathrm{NE} \Uparrow \mathrm{Cl}), \mathrm{Neg} \cap (\mathrm{NE} \Uparrow \mathrm{Cl}), (\mathrm{NE} \uparrow \mathrm{Cl}), \mathrm{NC} \cap (\mathrm{NE} \uparrow \mathrm{Cl}), \mathrm{Neg} \cap (\mathrm{NE} \uparrow \mathrm{Cl}), (\mathrm{NC} \Uparrow \mathrm{Cl}), \mathrm{Neg} \cap (\mathrm{NC} \Uparrow \mathrm{Cl}), (\mathrm{NC} \uparrow \mathrm{Cl}), \mathrm{Neg} \cap (\mathrm{NC} \uparrow \mathrm{Cl}), (\mathrm{Neg} \Uparrow \mathrm{Cl}), \mathrm{Gl} = (\mathrm{Neg} \uparrow \mathrm{Cl})$.

PROOF. We apply Theorem 5.5. Recall that there are only four negative logics with CIP, namely, For, NE, NC, Neg. Further, if $L = L_{neg} \cap L_0$, then $L_{neg} \subseteq L_{0(neg)}$. Therefore, $L_{neg} \supseteq L_{0(neg)}$ implies $L_{neg} = L_{0(neg)}$ and $L_{neg} \cap L_0 = L_0$. ■

Due to Theorem 5.5 we also can find a list of all Gl-logics with IPR and PBP. An axiomatization of all these logics can be obtained from the axiomatization of negative logics with PBP given in Theorem 3.9 and of logics of the list SL (see Proposition 2.7) by Lemma 2.5(1) on axiomatization of the intersection of two J-logics.

It follows from Theorem 2.1 that the property WIP is preserved by adding the axiom $(A \vee \neg A)$ to any J-logic with WIP. We prove that the properties CIP, PBP and IPR are also preserved.

PROPOSITION 5.9. *Let a* J*-logic L have* CIP, PBP *or* IPR*. Then the logic* $L + (A \vee \neg A)$ *also has the same property.*

PROOF. Denote $L' = L + (A \vee \neg A)$.

Let L have CIP, PBP or IPR. Then L has WIP and by Theorem 1.1 L' also has WIP. By Theorem 5.1 $L' = L'_{neg} \cap L_0$, where L_0 is some logic in SL. Further,

$$L'_{neg} = L + (A \vee \neg A) + \bot = L + \bot = L_{neg}.$$

By Lemma 3.7 the logic L_{neg} also has CIP, PBP or IPR, respectively. Then L' has the same property by Theorem 5.5. ■

6. Conclusions

In the previous section we described all the extensions of the logic Gl with the properties CIP, IPR or PBP. It turned out that there are only finitely many Gl-logics with these properties. Moreover, the properties CIP, IPR and PBP are decidable over Gl. In addition, IPR and PBP are equivalent in all Gl-logics.

As for the whole family of J-logics, we know that the weak interpolation property WIP is decidable, and there is a continuum of J-logics with WIP and a continuum of J-logics without WIP. But the problem of interpolation is not yet solved, and the following problems are still open:

1. How many J-logics have CIP, IPR or PBP?
2. Are these properties decidable over J?
3. Are IPR and PBP equivalent in the family of all J-logics?

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