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Classical Modal De Morgan Algebras

Abstract. In this note we introduce the variety $\mathcal{CDM}_{\Box}$ of classical modal De Morgan algebras as a generalization of the variety $\mathcal{TMA}$ of Tetravalent Modal algebras studied in [11]. We show that the variety $\mathcal{V}_0$ defined by H. P. Sankappanavar in [13], and the variety $\boldsymbol{S}$ of Involutive Stone algebras introduced by R. Cignoli and M. S de Gallego in [5], are examples of classical modal De Morgan algebras. We give a representation theory, and we study the regular filters, i.e., lattice filters closed under an implication operation. Finally we prove that the variety $\mathcal{TMA}$ has the Amalgamation Property and the Superamalgamation Property.

Keywords: modal operator, De Morgan algebras, Amalgamation property.

1. Preliminaries

In this paper we define and study the variety $\mathcal{CDM}_{\Box}$ of classical modal Morgan algebras, or classical $\Box$-De Morgan algebras. This variety is an interesting generalization of the variety $\mathcal{TMA}$ of Tetravalent Modal algebras introduced by A. Monteiro and mainly studied by I. Loureiro [11] and J. M. Font and M. Rius [6]. A classical $\Box$-De Morgan algebra is a De Morgan algebra endowed with an interior operator $\Box$ satisfying an additional condition. Moreover, the variety $\mathcal{CMD}_{\Box}$ is intimately connected with some well-known varieties of De Morgan algebras with pseudocomplementation, like the variety $\mathcal{V}_0$ introduced by H. P. Sankappanavar [13] (see also [7]), or the variety $\boldsymbol{S}$ of Involutive Stone algebras introduced by R. Cignoli and M. S de Gallego in [5].

The paper is organized as follows. In the remaining part of this section we review some results on distributive lattices with modal operators, distributive pseudocomplemented algebras and De Morgan algebras. In Section 2 we define the variety $\mathcal{CDM}_{\Box}$ of classical $\Box$-De Morgan algebras, and we introduce some examples. In Section 3 we give an equational characterization of the variety of De Morgan-Stone algebras as De Morgan algebras with an additional modal operator Δ. This characterization is similar to the definition of the Involutive Stone algebras. As a consequence of this

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characterization we obtain a new equational definition of the variety $\mathcal{V}_0$. In Section 4 we study the representation of classical $\Box$-De Morgan algebras. In Section 5 we study the class of filters that are closed under $\Box$, called regular filters. We prove that these filters can be characterized as a lattice filter closed under an implication operation. We characterized the prime regular filters, and in the case of Δ-De Morgan algebras we can give a characterization of the maximal regular filters in terms of minimal prime filters. In Section 6 we prove that the variety $\mathcal{TMA}$ has the Amalgamation Property and the Superamalgamation Property.

Let us consider a poset $\langle X, \leq \rangle$, i.e., a set X endowed with a reflexive, antisymmetric and transitive binary relation $\leq$. A subset $U \subseteq X$ is said to be *increasing* if for all $x, y \in X$ such that $x \in U$ and $x \leq y$, we have $y \in U$. The set of all increasing subsets of X is denoted by $\mathcal{P}_i(X)$. It is clear that $\langle \mathcal{P}_i(X), \cup, \cap, \emptyset, X \rangle$ is a bounded distributive lattice. For each $Y \subseteq X$, the increasing set (decreasing set) generated by Y is $[Y) = \{x \in X \mid \exists y \in Y\,(y \leq x)\}$ $((Y] = \{x \in X \mid \exists y \in Y\,(x \leq y)\})$. Let Y be a subset of a set X. The theoretical complement of Y is denoted by $Y^c = X - Y$.
Let R be a binary relation on a set X. For each $x \in X$, let us consider the subset $R(x) = \{y \in X \mid (x, y) \in R\}$. For each $U \subseteq X$ define the set $\Box_R(U) = \{x \in X \mid R(x) \subseteq U\}$.

Let $A = \langle A, \vee, \wedge, 0, 1 \rangle$ be a bounded distributive lattice and let $X(A)$ be the set of all prime filters of A. It is known that the map $\sigma : A \to \mathcal{P}_i(X(A))$ given by $\sigma(a) = \{P \in X(A) \mid a \in P\}$ is an injective lattice homomorphism. The filter (ideal) generated by a subset $H \subseteq A$ will be denoted by $[H)$ $((H])$. The lattice of all filters of A is denoted by $Fi(A)$.
A $\Box$-*lattice* is a pair $\langle A, \Box \rangle$ where A is a distributive lattice and $\Box$ is a unary operator such that $\Box 1 = 1$, $\Box(a \wedge b) = \Box a \wedge \Box b$, for all $a, b \in A$.
Similarly, a $\Diamond$-*lattice* is an algebra $\langle A, \Diamond \rangle$ where A is a distributive lattice and $\Diamond$ is a unary operator such that $\Diamond 0 = 0$, $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$, for all $a, b \in A$. A *modal lattice*, or $\Box\Diamond$-*lattice*, is an algebra $\langle A, \Box, \Diamond \rangle$ such that $\langle A, \Box \rangle$ is a $\Box$-lattice, and $\langle A, \Diamond \rangle$ is a $\Diamond$-lattice.
Let $\langle A, \Box \rangle$ be a $\Box$-lattice. We define binary relation R_A on $X(A)$ as

$$(P, Q) \in R_A \text{ iff } \Box^{-1}(P) \subseteq Q,$$

for $P, Q \in X(A)$. For the proof of the following result see [4], or [2].

LEMMA 1.1. *Let $\langle A, \Box \rangle$ be a $\Box$-lattice. Let $P \in X(A)$ and $a \in A$. Then $\Box a \in P$ iff for every $Q \in X(A)$ such that $(P, Q) \in R_A$, then $a \in Q$.*

A *De Morgan algebra* is a pair $\langle A, \sim\rangle$ where A is a bounded distributive lattice and $\sim$ is a unary operation on A satisfying the following identities:

1. $\sim (x \vee y) = \sim x \wedge \sim y$,
2. $\sim\sim x = x$,
3. $\sim 0 = 1$.

A *pseudocomplemented distributive lattice* (or *p-algebra*) is a pair $\langle A, ^\circ \rangle$ where A is a bounded distributive lattice and $^\circ$ is a unary operation on A satisfying the following identities:

SP1. $a \wedge (a \wedge b)^\circ = a \wedge b^\circ$,
SP2. $a \wedge 0^\circ = a$,
SP3. $0^{\circ\circ} = 0$.

A triple $\langle A, ^\circ, ^+ \rangle$ is a double p-algebra if $\langle A, ^\circ \rangle$ is a p-algebra and its dual $\langle A, ^+ \rangle$ is also a p-algebra. A *Stone algebra* is a p-algebra $\langle A, ^\circ \rangle$ such that satisfies the identity $a^\circ \vee a^{\circ\circ} = 1$. Let us recall that a p-algebra A is a Stone algebra iff $(a \wedge b)^\circ = a^\circ \vee b^\circ$ for all $a, b \in A$. A *double Stone algebra* is a double p-algebra $\langle A, ^\circ, ^+, \rangle$ such that $a^\circ \vee a^{\circ\circ} = 1$ and $a^+ \wedge a^{++} = 0$.
An algebra $\langle A, ^\circ, \sim\rangle$ is a *pseudocomplemented De Morgan* algebra, or *p-De Morgan algebra*, if $\langle A, ^\circ \rangle$ is a p-algebra and $\langle A, \sim\rangle$ is a De Morgan algebra. A p-De Morgan algebra A is also a double p-algebra where the dual pseudocomplemented is defined by $a^+ = \sim (\sim a)^\circ$. A *De Morgan-Stone algebra* is a p-De Morgan algebra $\langle A, \sim, ^\circ \rangle$ such that $\langle A, ^\circ \rangle$ is a Stone algebra. We note that a De Morgan-Stone algebra is also a double Stone algebra.
An important variety of p-De Morgan algebras introduced by Sankappanavar in [13] is the variety $\mathcal{V}_0$ of p-De Morgan algebras satisfying the identity $a \wedge (\sim a)^\circ = (\sim (a \wedge (\sim a)^\circ))^\circ$.

2. Classical $\Box$-De Morgan algebras

DEFINITION 2.1. A *modal De Morgan algebra, or* $\Box$-De Morgan algebra, is a triple $\boldsymbol{A} = \langle A, \Box, \sim\rangle$ such that $\langle A, \sim\rangle$ is a De Morgan algebra, and $\langle A, \Box\rangle$ is a $\Box$-lattice.

We note that a modal De Morgan algebra $\boldsymbol{A}$ is also a modal lattice, because the operator $\Diamond$ is defined by $\Diamond a = \sim \Box \sim a$. Now we define the principal class of modal De Morgan algebras that we will study in this paper.

DEFINITION 2.2. A *classical* $\Box$*-De Morgan algebra* is a $\Box$-De Morgan algebra $\boldsymbol{A}$ such that:

1. $\Box a \leq a$,
2. $\Box a \leq \Box\Box a$,
3. $\Box a \vee \sim \Box a = 1$, for all $a \in A$.

The variety of classical $\Box$-De Morgan algebras is denoted by $\mathcal{CDM}_{\Box}$.

LEMMA 2.3. *Let $\boldsymbol{A}$ be a $\Box$-De Morgan algebra such that $\Box a \vee \sim \Box a = 1$, for all $a \in A$. Then $a \vee \sim \Box a = 1$ iff $\Box a \leq a$, for all $a \in A$.*

PROOF. $\Rightarrow$) Let $a \in A$. Then

$$\begin{array}{rcl} \Box a & = & \Box a \wedge (a \vee \sim \Box a) \quad = \quad (\Box a \wedge a) \vee (\Box a \wedge \sim \Box a) \\ & = & (\Box a \wedge a) \vee 0 \qquad\quad\;\; = \quad \Box a \wedge a. \end{array}$$

$\Leftarrow$) Let $a \in A$. Then, $1 = \Box a \vee \sim \Box a \leq a \vee \sim \Box a$. So, $a \vee \sim \Box a = 1$. ■

Now, we shall give some examples that will show that some known varieties of algebras are also classical $\Box$-De Morgan algebras.

EXAMPLE 2.4. A $\Box$-De Morgan algebra $\boldsymbol{A}$ is a *Tetravalent Modal algebra* (see [11] or [6]), if the operator $\Box$ satisfies the conditions:

1. $a \vee \sim \Box a = 1$, and
2. $\sim a \wedge a = a \wedge \sim \Box a$, for all $a, b \in A$.

These algebras were introduced by A. Monteiro in 1978. They provide an interesting generalization of the three-valued Lukasiewicz algebras. We note that if $\boldsymbol{A}$ satisfies the identity $\Box(a \vee b) = \Box a \vee \Box b$, for all $a, b \in A$, then $\boldsymbol{A}$ is a three-valued Lukasiewicz algebra *(*see [11] or [6]). The variety of Tetravalent Modal algebras is denoted by $\mathcal{TMA}$.

EXAMPLE 2.5. An algebra $\langle A, \Box, \sim \rangle$ is an *involutive Stone algebra* if $\langle A, \sim \rangle$ is a De Morgan algebra and the operator $\Box$ satisfies the conditions $\Box 1 = 1$, $\Box a \leq a$, $\Box a \vee \sim \Box a = 1$, and $\Box(a \vee b) = \Box a \vee \Box b$, for all $a, b \in A$. The variety $\boldsymbol{S}$ of involutive Stone algebras was introduced in [5] by R. Cignoli and M. S. de Gallego. It is easy to see that every algebra of $\boldsymbol{S}$ is also in $\mathcal{CDM}_{\Box}$.

3. De Morgan-Stone algebras

Let us recall that a De Morgan-Stone algebra is a p-De Morgan algebra $\langle A, \sim, ^{\circ} \rangle$ such that $\langle A, ^{\circ} \rangle$ is a Stone algebra. We note that a De Morgan-Stone algebra is also a double Stone algebra. We shall give an equational characterization of the variety of De Morgan-Stone algebras as De Morgan algebras with an additional modal operator Δ. As a consequence we have also obtained a characterization of the variety $\mathcal{V}_0$ studied by [13] and [7].

DEFINITION 3.1. A Δ-*De Morgan algebra* is an algebra $\boldsymbol{A} = \langle A, \sim, \Delta \rangle$ such that $\langle A, \sim \rangle$ is a De Morgan algebra and Δ is a unary operator $\Delta : A \to A$ satisfying the following identities:

Δ1. $\Delta 0 = 0$.
Δ2. $\Delta 1 = 1$.
Δ3. $a \wedge \Delta \sim a = 0$.
Δ4. $\Delta (a \vee b) = \Delta a \vee \Delta b$.

PROPOSITION 3.2. *Let $\boldsymbol{A}$ be a Δ-De Morgan algebra. Then the following properties are valid:*

(1) $\sim a \wedge \Delta a = 0$, $a \vee \sim \Delta a = 1$,
(2) $\Delta a \vee \sim \Delta^2 a = 1$, $\sim \Delta a \wedge \Delta^2 a = 0$,
(3) $\Delta \Delta a \leq a$,
(4) $\Delta \sim a \wedge \Delta \sim \Delta \sim a = 0$,
(5) $\Delta \sim a \vee \Delta \sim \Delta \sim a = 1$.

PROOF. Items (1) and (2) follows by **Δ3.** We see (3). As $a \vee \sim \Delta a = 1$, we get

$$\begin{aligned}\Delta^2 a = \Delta^2 a \wedge (a \vee \sim \Delta a) &= \left(\Delta^2 a \wedge a\right) \vee \left(\Delta^2 a \wedge \sim \Delta a\right) \\ &= \left(\Delta^2 a \wedge a\right) \vee 0 = \Delta^2 a \wedge a.\end{aligned}$$

Thus, $\Delta^2 a \leq a$.
(4) follows from **Δ3**. We show (5). Let $a \in A$. Then, $\Delta \sim a \vee \Delta \sim \Delta \sim a = \Delta (\sim a \vee \sim \Delta \sim a) = \Delta 1 = 1$. ■

THEOREM 3.3. *Let $\langle A, \sim \rangle$ be a De Morgan algebra. Then $\langle A, ^{\circ}, \sim \rangle$ is a De Morgan-Stone algebra iff there exists a unary operator $\Delta : A \to A$ such that $\langle A, \sim, \Delta \rangle$ is a Δ-De Morgan algebra.*

PROOF. $\Rightarrow$) Let $\langle A, ^{\circ}, \sim \rangle$ be a De Morgan-Stone algebra. Let us define the operator $\Delta a = (\sim a)^{\circ}$. Then it is clear that the operation Δ satisfies the identities **Δ1-Δ4**.
$\Leftarrow$) Suppose that $\langle A, \sim, \Delta \rangle$ is a Δ-De Morgan algebra. We prove that A is a p-algebra under the operation $^{\circ}$ defined by $a^{\circ} = \Delta \sim a$. We need to prove that $\langle A, ^{\circ} \rangle$ satisfies the identities SP1, SP2 and SP3 given in the introduction.

Let $a, b \in A$. Then:

$$\begin{aligned} a \wedge (a \wedge b)^{\circ} &= a \wedge \Delta \sim (a \wedge b) \\ &= a \wedge \Delta (\sim a \vee \sim b) \\ &\overset{\mathbf{\Delta 4}}{=} a \wedge (\Delta \sim a \vee \Delta \sim b) \\ &= (a \wedge \Delta \sim a) \vee (a \wedge \Delta \sim b) \\ &\overset{\mathbf{\Delta 3}}{=} 0 \vee (a \wedge \Delta \sim b) = a \wedge b^{\circ}. \end{aligned}$$

$$a \wedge 0^{\circ} = a \wedge \Delta \sim 0 = a \wedge \Delta 1 \overset{\mathbf{\Delta 2}}{=} a \wedge 1 = a$$
$$0^{\circ\circ} = \Delta \sim \Delta \sim 0 = \Delta 0 \overset{\mathbf{\Delta 1}}{=} 0$$

Thus, $\langle A, ^{\circ} \rangle$ is a p-algebra. We prove that A satisfies de Stone identity. Let $a \in A$. Then:

$$\begin{aligned} a^{\circ} \vee a^{\circ\circ} &= \Delta \sim a \vee \Delta \sim \Delta \sim a \overset{\mathbf{\Delta 4}}{=} \Delta (\sim a \vee \sim \Delta \sim a) \\ &= \Delta (\sim (a \wedge \Delta \sim a)) \overset{\mathbf{\Delta 3}}{=} \Delta \sim 0 \overset{\mathbf{\Delta 2}}{=} 1. \end{aligned}$$

■

COROLLARY 3.4. [13] *Let* $\boldsymbol{A}$ *be a* Δ*-De Morgan algebra. Let* $a^{\circ} = \Delta \sim a.$*, for each* $a \in A$*. Then:*

1. $\langle A, \sim, ^{\circ} \rangle \in \mathcal{V}_0$ *iff* $\boldsymbol{A}$ *satisfies the identity:*

   Δ5. $\Delta (a \wedge \Delta a) = a \wedge \Delta a$*, all* $a \in A$*.*

2. $\langle A, \sim, ^{\circ} \rangle \in \mathbf{S}$ *if and only if* $\langle A, \sim, ^{\circ} \rangle \in \mathcal{V}_0$*, and*

   Δ6. $\Delta a \wedge a = \Delta a$ *for, all* $a \in A$*.*

PROOF. It is inmediate. ■

COROLLARY 3.5. *Let* $\langle A, \sim, \Delta \rangle \in \mathcal{V}_0$*. Then* $\langle A, \sim, \Box \rangle \in \mathcal{CDM}_{\Box}$*, under the operation* $\Box$ *defined by* $\Box a = a \wedge \Delta a$*, for each* $a \in A$*.*

PROOF. It is clear that $\Box a \leq a$, for each $a \in A$. Moreover,

$$\begin{aligned} \Box\Box a &= \Box(a \wedge \Delta a) &&= a \wedge \Delta a \wedge \Delta(a \wedge \Delta a) \\ &= a \wedge \Delta a \wedge \Delta a \wedge \Delta\Delta a &&= a \wedge \Delta a \wedge \Delta\Delta a \\ &= a \wedge \Delta a &&= \Box a. \end{aligned}$$

By Properties 3.2 we get that

$$\begin{aligned} \Box a \vee \sim \Box a &= (a \wedge \Delta a) \vee \sim a \vee \sim \Delta a \\ &= (a \vee \sim a \vee \sim \Delta a) \wedge (\Delta a \vee \sim a \vee \sim \Delta a) \\ &= 1 \wedge 1 = 1 \end{aligned}$$

Thus, $\langle A, \sim, \Box \rangle \in \mathcal{CDM}_{\Box}$. ■

If $\boldsymbol{A} \in \mathbf{S}$, then $\Box a = \Delta a$, for all $a \in A$. Then we get the following result.

Corollary 3.6. *A classical* $\Box$*-De Morgan algebra* $\boldsymbol{A}$ *is an involutive Stone algebra iff* $\Box (a \vee b) = \Box a \vee \Box b$, *for all* $a, b \in A$.

4. Representation by sets

A *De Morgan poset* is a structure $\langle X, \leq, g \rangle$ where $\langle X, \leq \rangle$ is a poset and $g : X \to X$ is a function such that $g(g(x)) = x$, and if $x \leq y$, then $g(y) \leq g(x)$, for all $x, y \in X$. If A is a De Morgan algebra, then $\langle X(A), \subseteq, g \rangle$ is a De Morgan poset, where $g : X(A) \to X(A)$ is the involution defined by $g(P) = A - (\sim P)$, where $\sim P = \{\sim a \mid a \in P\}$.

A structure $\mathcal{F} = \langle X, \leq, g, R \rangle$ is a *De Morgan frame* if $\langle X, \leq, g \rangle$ is a De Morgan poset, and R is a binary relation on X such that $(\leq \circ R) \subseteq R$, where $\circ$ is the composition of relations. If $\mathcal{F}$ is a De Morgan frame, then the condition $(\leq \circ R) \subseteq R$ ensures that $\Box_R (U) \in \mathcal{P}_i (X)$, for all $U \in \mathcal{P}_i (X)$ (see [3], or [4] for more details). Thus, $A(\mathcal{F}) = \langle \mathcal{P}_i(X), \sim, \Box_R \rangle$ is a $\Box$-De Morgan algebra. Any subalgebra of $A(\mathcal{F})$ is called a *complex algebra.* Now we study under what conditions $A(\mathcal{F})$ is a classical $\Box$-De Morgan algebra.

Proposition 4.1. *Let* $\langle X, \leq, g, R \rangle$ *be a De Morgan frame. Then*
(1) *For all* $U \in \mathcal{P}_i (X)$, $\sim U \cap \Box_R (U) = \emptyset$ *iff* $g(x) \in R(x)$, *for all* $P \in X$.
(2) *For all* $U \in \mathcal{P}_i (X)$, $\sim \Box_R (U) \cup \Box_R (U) = X$ *iff* $R(x) = R(g(x))$, *for all* $x \in X$.
(3) *For all* $U \in \mathcal{P}_i (X)$, $\Box_R(U) \subseteq \Box_R \Box_R (U)$ *iff* R *is transitive.*

Proof. (1) $\Rightarrow$) Suppose that there exists $x \in X$ such that $g(x) \notin R(x)$. Let us consider the increasing subset $U = R(x)$. Then, $g(x) \notin U$ iff $x \in \sim U$. So $x \notin \Box_R (U)$, i.e., $R(x) \nsubseteq U = R(x)$, which is a contradiction. The other direction is easy.
(2) $\Rightarrow$) Suppose that there exists $x \in X$ such that $R(x) \nsubseteq R(g(x))$. Let us consider the increasing subset $U = R(g(x))$. Then $x \notin \Box_R (U)$. So, $x \in \sim \Box_R(U)$, i.e., $R(g(x)) \nsubseteq U = R(g(x))$, which is impossible. Thus, $R(x) \subseteq R(g(x))$. The proof of the inclusion $R(g(x)) \subseteq R(x)$ is similar. The other direction is easy.
For (3) see [2] or [4]. ■

We shall say that a De Morgan frame $\mathcal{F} = \langle X, \leq, g, R \rangle$ is *classical* if R is transitive, reflexive and $R(x) = R(g(x))$, for all $x \in X$. We note that the condition of $g(x) \in R(x)$, for all $x \in X$, is equivalent to the condition $x \in R(x)$, for all $x \in X$.

Let $\boldsymbol{A}$ be a $\Box$-De Morgan algebra. The structure $\mathcal{F}_{\boldsymbol{A}} = \langle X(A), \subseteq, g, R_A\rangle$ is called the *frame* of $\boldsymbol{A}$. By the previous proposition it is easy to prove the following representation theorem.

THEOREM 4.2. *Every $\Box$-De Morgan algebra $\boldsymbol{A}$ is isomorphic to a subalgebra of $A(\mathcal{F}_{\boldsymbol{A}})$.*

PROOF. Clearly, $\mathcal{F}_{\boldsymbol{A}}$ is a De Morgan frame. By Proposition 4.1 $A(\mathcal{F}_{\boldsymbol{A}})$ is a $\Box$-De Morgan algebra of sets. Let us consider the map $\sigma : \boldsymbol{A} \to A(\mathcal{F}_{\boldsymbol{A}})$ defined by $\sigma(a) = \{P \in X(A) \mid a \in P\}$. It is clear that σ is an injective De Morgan homomorphism. From Corollary 1.1 it is follows that $\sigma(\Box a) = \Box_{R_A}(\sigma(a))$, for each $a \in A$. Thus, $\boldsymbol{A}$ is isomorphic to a subalgebra of $A(\mathcal{F}_{\boldsymbol{A}})$. ■

In the following result we prove that in a classical $\Box$-De Morgan algebra $\boldsymbol{A}$, the relation R_A can be defined by means of the operator $\Diamond$.

PROPOSITION 4.3. *Let $\boldsymbol{A}$ be a classical $\Box$-De Morgan algebra. Then for all $P, Q \in X(A)$, $\Box^{-1}(P) \subseteq Q$ iff $Q \subseteq \Diamond^{-1}(P)$.*

PROOF. Let $P, Q \in X(A)$. Suppose that $(P, Q) \in R_A$ and that $Q \nsubseteq \Diamond^{-1}(P)$. Then there exists $a \in Q$ and $\Diamond a = \sim \Box \sim a \notin P$. Since $\Box \sim a \vee \sim \Box \sim a = 1 \in P$, $\Box \sim a \in P$. As $\Box \sim a \leq \Box\Box \sim a \in P$, and $\Box^{-1}(P) \subseteq Q$, $\Box \sim a \in Q$. Then $0 = a \wedge \Box \sim a \in Q$, which is an absurd. Thus, $Q \subseteq \Diamond^{-1}(P)$. The other direction is similar. ■

PROPOSITION 4.4. *Let $\boldsymbol{A}$ be a $\Box$-De Morgan algebra. Then:*
(1) *For all $a \in A$, $\sim a \wedge \Box a = 0$ iff $g(P) \in R_A(P)$, for all $P \in X(A)$.*
(2) *For all $a \in A$, $\Box a \vee \sim \Box a = 1$ iff $R_A(P) = R_A(g(P))$, for all $P \in X(A)$.*
(3) *For all $a \in A$, $\Box a \leq \Box\Box a$ iff R_A is transitive.*

PROOF. (1) $\Rightarrow$) Let $P \in X(A)$. If $(P, g(P)) \notin R_A$, then there exists $a \in A$ such that $\Box a \in P$ and $a \notin g(P)$. So, $\Box a \wedge \sim a = 0 \in P$, which is a contradiction.

The direction $\Leftarrow$) follows by Proposition 4.1 taking into account that $\sigma[A]$ is a subalgebra of $\mathcal{P}_i(X(A))$.
(2) $\Rightarrow$) Let $P, Q \in X(A)$ such that $Q \in R_A(P)$. If $Q \notin R_A(g(P))$, then there exists $a \in A$ such that $\Box a \in g(P)$ and $a \notin Q$. So, $\sim \Box a \notin P$, and by hypothesis, $\Box a \in P$. As $(P, Q) \in R_A$, $a \in Q$, which is a contradiction. Thus, $Q \in R_A(g(P))$.
The direction $\Leftarrow$) follows by Proposition 4.1. Item (3) is well-known. ■

Now we study the representation of Tetravalent Modal algebras.

Proposition 4.5. *Let* $\boldsymbol{A}$ *be a* $\Box$*-De Morgan algebra satisfying the identities* $\Box a \vee \sim \Box a = 1$ *and* $\Box a \leq a$*. Then the following conditions are equivalent:*
(1) $\sim a \wedge a = a \wedge \sim \Box a$, *for all* $a \in A$.
(2) *If* $(P, Q) \in R_A$, *then* $P \subseteq Q$ *or* $g(P) \subseteq Q$, *for* $P, Q \in X(A)$.

Proof. (1) $\Rightarrow$ (2) Let $P, Q \in X(A)$ such that $(P, Q) \in R_A$. Suppose that $P \nsubseteq Q$ and $g(P) \nsubseteq Q$. Then $P \cap g(P) \nsubseteq Q$, because Q is a prime filter. So there exists $a \in P \cap g(P)$ and $a \notin Q$. Then $\sim a \notin P$. As $\sim a \wedge a = a \wedge \sim \Box a$, and $a \in P$, $\sim \Box a \notin P$. So, $\Box a \in g(P)$. Since $\Box a \wedge \sim \Box a = 0$, $\sim \Box a \notin g(P)$, i.e., $\Box a \in P$. So, $a \in Q$, because $(P, Q) \in R_A$, which is a contradiction.
(2) $\Rightarrow$ (1). As $\Box a \leq a$, we get that $\sim a \wedge a \leq a \wedge \sim \Box a$, for all $a \in A$. Suppose that there exists $a \in A$ such that $a \wedge \sim \Box a \nleq \sim a \wedge a$. Then there exists $P \in X(A)$, such that $a \wedge \sim \Box a \in P$, and $\sim a \notin P$. So, $a \in g(P)$. As $\sim \Box a \in P$, $\Box a \notin g(P)$. Then there exists $Q \in X(A)$ such that $(g(P), Q) \in R_A$, and $a \notin Q$. By hypothesis, $g(P) \subseteq Q$ or $P \subseteq Q$. As $a \in g(P)$, $g(P) \nsubseteq Q$, and consequently $P \subseteq Q$. But from $a \in P$, we get $a \in Q$, which is a contradiction. Thus, $\sim a \wedge a = a \wedge \sim \Box a$, for all $a \in A$. ■

Corollary 4.6. *Let* $\boldsymbol{A} \in \mathcal{CDM}_\Box$*. The following conditions are equivalent:*
(1) $\boldsymbol{A} \in \mathcal{TMA}$.
(2) $\forall P, Q \in X(A)$, $((P, Q) \in R_A$ *iff* $P \subseteq Q$ *or* $g(P) \subseteq Q)$.

Proof. (1) $\Rightarrow$ (2). If $\boldsymbol{A}$ *is a Tetravalent Modal algebra, then by Proposition 4.5 we get that when* $(P, Q) \in R_A$ we have $P \subseteq Q$ or $g(P) \subseteq Q$. Assume $P \subseteq Q$ or $g(P) \subseteq Q$. If $P \subseteq Q$, then $\Box^{-1}(P) \subseteq P \subseteq Q$, i.e., $(P, Q) \in R_A$. If $g(P) \subseteq Q$, $(g(P), Q) \in R_A$. As $R_A(P) = R_A(g(P))$, we get $(P, Q) \in R_A$.
(2) $\Rightarrow$ (1). It follows by Proposition *4.5.* ■

We end this section with a topological representation theorem which generalizes a celebrated result valid for closure algebras (see [12]).

Let $\boldsymbol{A} \in \mathcal{CDM}_\Box$. We consider the map $\sigma : \boldsymbol{A} \to A(\mathcal{F}_{\boldsymbol{A}})$ defined in the proof of Theorem 4.2. As $\sigma(\Box 1) = \sigma(1) = X(A)$, and $\sigma(\Box a) \cap \sigma(\Box b) = \sigma(\Box a \wedge \Box b) = \sigma(\Box(a \wedge b))$, the family $\{\sigma(\Box a) \mid a \in A\}$ is a basis for a topology τ on $X(A)$. The ordered topological space $T(A) = \langle X(A), \subseteq, g, \tau \rangle$ is called the *McKinsey-Tarski topological space* of $\boldsymbol{A}$. The interior operator of $T(A)$ is denoted by I.

Proposition 4.7. *Let* $\boldsymbol{A} \in \mathcal{CDM}_\Box$*. Then* $\boldsymbol{A}(T(A)) = \langle \mathcal{P}_i(X(A)), \cup, \cap, \sim, \mathrm{I} \rangle$ *is a classical* $\Box$*-De Morgan algebra, and the mapping* $\sigma : \boldsymbol{A} \to \boldsymbol{A}(T(A))$ *is an embedding of classical* $\Box$*-De Morgan algebras.*

PROOF. We know that $\langle \mathcal{P}_i(X(A)), \cup, \cap, \sim\rangle$ is a De Morgan algebra, where $\sim U = g(U)^c$, for each $U \in \mathcal{P}_i(X(A))$. As $\{\sigma(\Box a) \mid a \in A\}$ is a basis for τ, the interior of a subset $U \subseteq X(A)$ is

$$\mathrm{I}(U) = \bigcup \{\sigma(\Box a) \mid \sigma(a) \subseteq U\}.$$

As each subset $\sigma(\Box a)$ is an increasing subset, we have that $\mathrm{I}(U)$ is an increasing subset. Thus, $\mathrm{I}(U) \in \mathcal{P}_i(X(A))$, for each $U \in \mathcal{P}_i(X(A))$. We prove that $\sim \mathrm{I}(U) \in \mathcal{P}_i(X(A))$. From the identity $\Box a \vee \sim \Box a = 1$, it is easy to see that $\Box a \in P$ iff $\Box a \in g(P)$, for each $P \in X(A)$. Thus, $\sigma(\Box a) = g(\sigma(\Box a))$. As consequence of this fact we get that $g(\mathrm{I}(U)) = \mathrm{I}(U)$. So, $g(\mathrm{I}(U))^c = \sim \mathrm{I}(U) \in \mathcal{P}_i(X(A))$, and thus, $\boldsymbol{A}(T(A)) \in \mathcal{CDM}_\Box$.

It is clear that σ is an embedding of De Morgan algebras. As $\sigma(\Box a) = \mathrm{I}(\sigma(a))$, we get that σ is an embedding of classical $\Box$-De Morgan algebras. ∎

5. Regular filters

Let $\boldsymbol{A}$ be a $\Box$-De Morgan algebra. A filter F of $\boldsymbol{A}$ is *regular* if $\Box a \in F$, whenever $a \in F$. The lattice of regular filters of $\boldsymbol{A}$ is denoted by $F_r(\boldsymbol{A})$. A regular filter F is *$\Box$-prime* if $\Box a \vee \Box b \in F$, then $\Box a \in F$ or $\Box b \in F$, for any $a, b \in A$. The set of maximal filters of $\boldsymbol{A}$ (also called ultrafilters) is denoted by $\max X(A)$. A *maximal regular filter* F is a regular filter maximal in $F_r(A)$. For each filter F of A, let $\hat{F} = \{P \in X(A) \mid F \subseteq P\}$. Recall that a set $Y \subseteq X(A)$ is *involutive* if $g(P) \in Y$, for each $P \in Y$.

We define an implication connective $\to$ by $a \to b = \sim \Box a \vee \Box b$, for each $a, b \in A$. If $\boldsymbol{A}$ satisfies the identity $\Box a \vee \sim \Box a = 1$, then it is easy to see that $a \to b = 1$ iff $\Box a \leq \Box b$, for all $a, b \in A$. Also, if $\boldsymbol{A} \in \mathcal{CDM}_\Box$, then $a \to b = 1$ iff $\Box a \leq b$, for all $a, b \in A$.

PROPOSITION 5.1. *Let $\boldsymbol{A} \in \mathcal{CDM}_\Box$. Let F be a filter of $\boldsymbol{A}$. F is a regular filter iff for all $a, b \in A$, if a, $a \to b \in F$, then $b \in F$.*

PROOF. $\Rightarrow$) Let a, $a \to b \in F$. Then, $\Box a \in F$, and

$$\begin{aligned} \Box a \wedge (a \to b) &= \Box a \wedge (\sim \Box a \vee \Box b) &&= (\Box a \wedge \sim \Box a) \vee (\Box a \wedge \Box b) \\ &= \Box a \wedge \Box b &&\leq \Box b \leq b \in F. \end{aligned}$$

$\Leftarrow$) Let $a \in F$. Since, $a \to \Box a = \sim \Box a \vee \Box a = 1 \in F$, we get $\Box a \in F$. ∎

PROPOSITION 5.2. *Let $\boldsymbol{A} \in \mathcal{CDM}_\Box$. Let F be a regular filter and let I be an ideal. If $F \cap I = \emptyset$, then there exists a $\Box$-prime filter P such that $F \subseteq P$ and $P \cap I = \emptyset$.*

PROOF. Consider the subset $\mathcal{F} = \{H \in F_r(\boldsymbol{A}) \mid F \subseteq H \text{ and } H \cap I = \emptyset\} \subseteq Fr(\boldsymbol{A})$. Since $F \in \mathcal{F}$, $\mathcal{F} \neq \emptyset$. It is clear that the union of a chain of elements of $\mathcal{F}$ is also in $\mathcal{F}$. So, by Zorn's lemma, there exists a regular filter P maximal in $\mathcal{F}$. We prove that P is $\Box$-prime. Let $a, b \in A$ such that $\Box a \vee \Box b \in P$. Suppose that $\Box a \notin P$ and $\Box b \notin P$. Let us consider the filters $P_a = \{x \in A \mid \exists p \in P(\Box p \wedge \Box a \leq x)\}$ and $P_b = \{x \in A \mid \exists p \in P(\Box p \wedge \Box b \leq x)\}$. It is easy to prove that P_a and P_b are regular filters, and $P \subset P_a \cap P_b$. Then, $P_a, P_b \notin \mathcal{F}$. Thus, $P_a \cap I \neq \emptyset$ and $P_b \cap I \neq \emptyset$. It follows that there exist $x, y \in I$ such that $a \to x \in P$ and $b \to y \in P$. Since I is an ideal, $x \vee y = c \in I$. So, $a \to x \leq a \to c \in P$ and $b \to y \leq b \to c \in P$. Thus, $(a \to c) \wedge (b \to c) = (\sim \Box a \vee \Box c) \wedge (\sim \Box b \vee \Box c) = \sim (\Box a \vee \Box b) \vee \Box c \in P$. Since $\Box a \vee \Box b \in P$, we deduce that $\Box c \leq c \in P$, which is a contradiction. Thus, we conclude that P is $\Box$-prime. ■

COROLLARY 5.3. *Let $\boldsymbol{A} \in \mathcal{CDM}_{\Box}$, $a, b \in A$, if $a \to b \neq 1$, then there exists a $\Box$-prime filter P of $\boldsymbol{A}$ such that $a \in P$ and $b \notin P$.*

PROOF. Assume that $a \to b \neq 1$. Let us consider the filter $[\Box a)$. So it is easy to see that $[\Box a) \cap (b] = \emptyset$. By Proposition 5.2, there exists a $\Box$-prime filter P of A such that $\Box a \leq a \in P$ and $b \notin P$. ■

PROPOSITION 5.4. *Let $\boldsymbol{A}$ be a $\Box$-De Morgan algebra such that $\Box a \vee \sim \Box a = 1$, and $\sim a \wedge \Box a = 1$, for all $a \in A$. Let $F \in F_r(\boldsymbol{A})$. Then*
(1) *$\hat{F} = \{P \in X(A) \mid F \subseteq P\}$ is an involutive set.*
(2) *If $\boldsymbol{A} \in \mathcal{CDM}_{\Box}$, then F is $\Box$-prime if and only if there exists $P \in X(A)$ such that $F = \Box^{-1}(P)$.*
(3) *If F is a maximal regular filter of A, then there exists $P \in \max X(A)$ such that $F = \Box^{-1}(P)$.*
(4) $F = \bigcap \{\Box^{-1}(P) \mid F \subseteq P \in X(A)\}$.
(5) *For each $a \notin F$, there exists a maximal regular filter M such that $F \subseteq M$, and $a \notin M$.*

PROOF. (1) Let $P \in X(A)$ such that $F \subseteq P$. Suppose that there exists $a \in F$ such that $a \notin g(P)$. Then $\sim a \in P$, and since $\Box a \in F \subseteq P$, we deduce that $\sim a \wedge \Box a = 0 \in P$, which is a contradiction. Thus, $F \subseteq g(P)$. So, $\hat{F}$ is an involutive set.
(2) Suppose that F is $\Box$-prime. We note that $\Box(\Box a \vee \Box b) = \Box a \vee \Box b$, for all $a, b \in A$, because $\Box a \leq a$ and $\Box a \leq \Box\Box a$, for all $a \in A$.
Let us consider the set $\Box(F^c) = \{\Box a \mid a \notin F\}$. This set is closed under disjuntions, because if $a, b \notin F$, then $\Box a \vee \Box b \notin F$, since F is $\Box$-prime. Then $\Box(\Box a \vee \Box b) = \Box a \vee \Box b \in \Box(F^c)$. Let $(\Box(F^c)]$ the ideal generated

by $\Box(F^c)$. We note that $F \cap (\Box(F^c)] = \emptyset$. Otherwise, there exists $f \in F$ and there exist $c_1, ..., c_n \notin F$ such that $f \leq \Box c_1 \vee \Box c_2 \vee \ldots \vee \Box c_n$. So, $\Box c_1 \vee \Box c_2 \vee \ldots \vee \Box c_n \in F$ and since F is $\Box$-prime, $\Box c_i \leq c_i \in F$ for some $1 \leq i \leq n$, which is a contradiction. So, by the Prime Filter Theorem there exists $P \in X(A)$ such that $F \subseteq P$ and $P \cap \Box(F^c) = \emptyset$. Thus, $F = \Box^{-1}(P)$. The converse direction is easy.

(3) Let F be a maximal regular filter. Since all filter is contained in a prime filter, and every prime filter is contained in an ultrafilter, we have that there exists $P \in \max X(A)$ such that $F \subseteq P$. It follows that $F \subseteq \Box^{-1}(P)$, because F is a regular filter. Also, it is easy to check that $\Box^{-1}(P)$ is a regular filter. As F is a maximal regular filter, $F = \Box^{-1}(P)$.

(4) and (5) are easy and left to the reader. ■

For Tetravalent Modal algebras the regular $\Box$-prime filters admit a nice characterization.

Lemma 5.5. *Let $\boldsymbol{A} \in \mathcal{TMA}$. Then $\Box^{-1}(P) = P \cap g(P)$, for all $P \in X(A)$.*

Proof. Let $P \in X(A)$. It is clear that the set $\Box^{-1}(P)$ is a regular $\Box$-prime filter. By item (1) of Proposition 5.4 we deduce that $\Box^{-1}(P) \subseteq P \cap g(P)$. Suppose that there exists $a \in P \cap g(P)$ but $\Box a \notin P$. Then $\sim \Box a \in P$. As $a \in g(P)$, $\sim a \notin P$. So, $a \wedge \sim \Box a = \sim a \wedge a \notin P$, but as $a \in P$, we get that $\sim a \in P$, i.e., $a \notin g(P)$, which is a contradiction. Thus, $\Box^{-1}(P) = P \cap g(P)$. ■

Let $\boldsymbol{A}$ be a Δ-De Morgan algebra. Let us define the operator $\Box$ by $\Box a = \Delta a \wedge \Delta^2 a$, for each $a \in A$. Then it is easy to see that $\langle A, \Box, \sim \rangle$ is $\Box$-De Morgan algebra satisfying the conditions $\Box a \vee \sim \Box a = 1$, and $\sim a \wedge \Box a = 1$, for all $a \in A$. For the case of Δ-De Morgan algebras we can give a characterization of maximal regular filters in terms of minimal prime filters. Moreover, it is immediate to check that a filter F of A is regular iff $\Delta a \in F$, for each $a \in F$.

Theorem 5.6. *Let $\boldsymbol{A}$ be a Δ-De Morgan algebra. If F is a maximal regular filter then there exists $U_1, U_2 \in \max X(A)$ such that $F = g(U_1) \cap g(U_2)$.*

Proof. Let F be a maximal regular filter. Since all filter is contained in a prime filter, and every prime filter is contained in an ultrafilter, we have that there exists $U_1 \in \max X(A)$ such that $F \subseteq U_1$. As F is regular, $F \subseteq g(U_1)$. As $g(U_1)$ is a minimal prime filter, there exists an ultrafilter U_2 such that $F \subseteq g(U_1) \subseteq U_2$. Again, as F is regular, we have that $F \subseteq g(U_2)$, Thus, $F \subseteq g(U_1) \cap g(U_2)$.

Now, we prove that $g(U_1) \cap g(U_2)$ is a regular filter. We recall $\boldsymbol{A}$ can be defined as a De Morgan-Stone algebra $\langle A, ^\circ, \sim \rangle$, where $a^\circ = \Delta \sim a$, for each $a \in A$. Let $a \in g(U_1) \cap g(U_2)$. Then $\sim a \notin U_1$, and as U_1 is an ultrafilter, and $(\sim a)^\circ = \Delta a \in U_1$. Similarly we can to prove that $\Delta a \in U_2$. Suppose that $\Delta a \notin g(U_1)$. From Proposition 3.2, $\Delta a \vee \Delta \sim \Delta a = 1 \in g(U_1)$. So, $\Delta \sim \Delta a \in g(U_1)$, and as $g(U_1) \subseteq U_2$, we get that $\Delta \sim \Delta a \in U_2$. Thus, $\Delta a \wedge \Delta \sim \Delta a = 0 \in U_2$, which is a contradiction. Thus, $\Delta a \in g(U_1)$. Similarly we can to see that $\Delta a \in g(U_2)$. Thus, $g(U_1) \cap g(U_2)$ is a regular filter, and as $F \subseteq g(U_1) \cap g(U_2)$, and F is maximal, $F = g(U_1) \cap g(U_2)$. ■

We finish with a result that is needed in the next section. We recall the following property valid in any Tetravalent Modal algebra $\boldsymbol{A}$ (see Lemma 2.4 of [11]):

$$\text{For all } P, Q \in X(A), \text{if } Q \subseteq P, \text{ then } Q = P \text{ or } g(Q) = P. \qquad (\mathbf{P})$$

THEOREM 5.7. *Let $\boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathcal{TMA}$. Suppose that $\boldsymbol{A}_0$ is a subalgebra of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$. Let $a \in \boldsymbol{A}_1$, $b \in \boldsymbol{A}_2$ and let us suppose that there exists not $c \in A_0$ such that $a \to_1 c = 1$ and $c \to_2 b = 1$. Then there exists $P \in \max X(A_1)$, and $Q \in \max X(A_2)$ such that*

$$\Box a \in P,\ b \notin Q \quad \textit{and} \quad P \cap A_0 = Q \cap A_0.$$

PROOF. Let $\leq_i$ the order of $\boldsymbol{A}_i$, with $i \in \{1, 2, 3\}$. Let us consider the filters $[\Box a)_{A_1}$ in $\boldsymbol{A}_1$ and $\big[[\Box a)_{A_1} \cap A_0\big)_{A_2}$ in $\boldsymbol{A}_2$. It is easy to check that $\big[[\Box a)_{A_1} \cap A_0\big)_{A_2}$ is a regular filter of $\boldsymbol{A}_2$. We note that $b \notin \big[[\Box a)_{A_1} \cap A_0\big)_{A_2}$, because otherwise there exists $c \in A_0$ such that $\Box a \leq_1 c \leq_2 b$, and this implies that $a \to_1 c = 1$, and $c \to_2 b = 1$, which is a contradiction. Then from (4) of Proposition 5.4, we deduce that there exists $Q \in X(A_2)$ that such that

$$[\Box a)_{A_1} \cap A_0 \subseteq \Box^{-1}(Q) \text{ and } b \notin Q. \qquad (5.1)$$

From Lemma 5.5 we get that $\Box^{-1}(Q) = Q \cap g(Q)$. It follows that $[\Box a)_{A_1} \cap A_0 \subseteq Q \cap g(Q)$. Let F be the filter in $\boldsymbol{A}_1$ generated by $\{\Box a\} \cup \big(\Box^{-1}(Q) \cap A_0\big)$ and let I be the ideal in $\boldsymbol{A}_1$ generated by the set $Q^c \cap A_0$. We prove that $F \cap I = \emptyset$. We suppose the contrary. So, there exist elements $x \in A_1$, $y \in \Box^{-1}(Q) \cap A_0$, and $z \in Q^c \cap A_0$ such that $\Box a \wedge y \leq_1 x \leq_1 z$. This implies that $\Box a \leq_1 y \to_1 z \in [\Box a)_{A_1} \cap A_0$. Now, by (5.1) $y \to_1 z = \sim \Box y \vee \Box z \in \Box^{-1}(Q)$, and since $\Box y \in \Box^{-1}(Q)$, we get $(\sim \Box y \vee \Box z) \wedge \Box y = \Box z \wedge \Box y \leq \Box z \leq z \in \Box^{-1}(Q) \subseteq Q$. So, $z \in Q$, which is a contradiction. Thus, there exists $P \in X(A_1)$ such that $\Box a \in P$, $\Box^{-1}(Q) \cap A_0 \subseteq P$ and $P \cap A_0 \subseteq Q$.

From $\Box^{-1}(Q) \cap A_0 \subseteq P$, we get that $\Box^{-1}(Q) \cap A_0 \subseteq \Box^{-1}(P) \cap A_0$. From $A_0 \cap P \subseteq Q$, we get that $\Box^{-1}(P) \cap A_0 \subseteq \Box^{-1}(Q) \cap A_0$. Thus, $\Box^{-1}(Q) \cap A_0 = \Box^{-1}(P) \cap A_0$. As $\Box^{-1}(Q) = Q \cap g(Q)$, we deduce that $Q \cap g(Q) \cap A_0 = P \cap g(P) \cap A_0$. Since, $Q \cap g(Q) \cap A_0 \subseteq P \cap A_0$, we deduce by the property **(P)** that $Q \cap A_0 = P \cap A_0$ or $g(Q) \cap A_0 = P \cap A_0$. In any case we have the result. ■

6. Amalgamation Property

Let $\mathcal{V}$ be a variety of Tetravalent Modal algebras. We shall say that $\mathcal{V}$ has the *Amalgamation Property* if for any triple $\boldsymbol{A}_0$, $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathcal{V}$ and injective homomorphisms $f_1 : \boldsymbol{A}_0 \to \boldsymbol{A}_1$ and $f_2 : \boldsymbol{A}_0 \to A_2$ there exists $\boldsymbol{B} \in \mathcal{CDM}_\Box$ and injective homomorphisms $j_1 : \boldsymbol{A}_1 \to \boldsymbol{B}$ and $j_2 : \boldsymbol{A}_2 \to \boldsymbol{B}$ such that $j_1 \circ f_1 = j_2 \circ f_2$. We shall say that $\mathcal{V}$ has the *Superamalgamation Property* if $\mathcal{V}$ has the Amalgamation Property and in addition the maps j_1 and j_2 above have the following property:

For all $(a, b) \in A_1 \times A_2$ such that $j_1(a) \leq j_2(b)$ implies that there exists $c \in A_0$ such that $a \to_1 f_1(c) = 1$ and $f_2(c) \to_2 b = 1$, where $\to_1$ and $\to_2$ denote the implication in $\boldsymbol{A}_1$ and in $\boldsymbol{A}_2$, respectively.

In the definition above we can consider the algebra $\boldsymbol{A}_0$ as a subalgebra of $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$, because $\boldsymbol{A}_0$ is isomorphic to the subalgebra $f_1(A_0) = \{f_1(c) \mid c \in A_0\}$ of $\boldsymbol{A}_1$ and $\boldsymbol{A}_0$ is isomorphic to the subalgebra $f_2(A_0) = \{f_2(c) \mid c \in A_0\}$ of $\boldsymbol{A}_2$.

THEOREM 6.1. *The variety $\mathcal{TMA}$ has the Amalgamation Property.*

PROOF. Let us consider $\boldsymbol{A}_0$, $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathcal{TMA}$ and suppose that $\boldsymbol{A}_0$ is a subalgebra of $\boldsymbol{A}_1$ and of $\boldsymbol{A}_2$. Let us consider the set

$$\boldsymbol{X} = \{(P, Q) \in X(A_1) \times X(A_2) \mid P \cap A_0 = Q \cap A_0\}.$$

Let $P \in X(A_1)$. Since $P \cap A_0 \in X(A_0)$, by known results of lattice theory there exists $Q \in X(A_2)$ such that $P \cap A_0 = Q \cap A_0$, i.e., $(P, Q) \in \boldsymbol{X}$. Thus, $\boldsymbol{X} \neq \emptyset$. The map $g : \boldsymbol{X} \to \boldsymbol{X}$ defined by $g(P, Q) = (g(P), g(Q))$ is an involution on $\boldsymbol{X}$. Consider the structure $A(\boldsymbol{X}) = \langle \mathcal{P}(\boldsymbol{X}), \sim, \Box \rangle$, where $\Box$ is defined by $\Box U = U \cap g(U)$, for $U \in \mathcal{P}(\boldsymbol{X})$. It is easy to see that $A(\boldsymbol{X})$ is a Tetravalent Modal algebra. Let us consider the maps $j_1 : A_1 \to A(\boldsymbol{X})$ and $j_2 : A_2 \to A(\boldsymbol{X})$ defined by:

$$j_1(d) = \{(P, Q) \in X \mid d \in P\}$$

and

$$j_2(d) = \{(P, Q) \in X \mid d \in Q\},$$

respectively. We prove that j_1 and j_2 are injective homomorphisms. To see that j_1 is injective, let $a, b \in A_1$ such that $a \nleq_1 b$. Then there exists $P \in X(A_1)$ such that $a \in P$ and $b \notin P$. We know that for each $P \in X(A_1)$ there exists $Q \in X(A_2)$ such that $(P,Q) \in X$. So, $(P,Q) \in j_1(a)$ and $(P,Q) \notin j_1(b)$, i.e., $j_1(a) \nsubseteq j_1(b)$. Thus, j_1 is injective. It is clear that j_1 is a De Morgan homomorphism.

We prove that j_1 preserves the operation $\Box$. Let $(P,Q) \in X$. Then we have the following chain of equivalences:

$$\begin{aligned} (P,Q) \in j_1(\Box a) &\Leftrightarrow \Box a \in P \Leftrightarrow a \in P \text{ and } a \in g(P) \\ &\Leftrightarrow (P,Q) \in j_1(a) \cap g(j_1(a)) = \Box j_1(a). \end{aligned}$$

In a like manner, we can prove that j_2 is an injective homomorphism.

We prove that $j_1(c) = j_2(c)$ for any $c \in A_0$. Let $c \in A_0$ and $(P,Q) \in X$. Then, $(P,Q) \in j_1(c)$ iff $c \in P$ iff $c \in P \cap A_0 = Q \cap A_0$ iff $c \in Q$ iff $(P,Q) \in j_2(c)$. Thus, the variety $\mathcal{TMA}$ has the Amalgamation Property. ■

Theorem 6.2. *The variety $\mathcal{TMA}$ has the Superamalgamation Property.*

Proof. The proof of the Superamalgamation Property is actually analogous to the previous one. Let us consider $\boldsymbol{A}_0$, $\boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathcal{TMA}$. Suppose that $\boldsymbol{A}_0$ is a subalgebra of $\boldsymbol{A}_1$ and of $\boldsymbol{A}_2$. Let us consider the set $\boldsymbol{X}$ and algebra $A(\boldsymbol{X})$ of the proof above. We prove that *f*or all $(a,b) \in A_1 \times A_2$ such that $j_1(a) \subseteq j_2(b)$ there exists $c \in A_0$ such that $a \to_1 c = 1$ and $c \to_2 b = 1$, where $\to_1$ and $\to_2$ denote the implication in $\boldsymbol{A}_1$ and in $\boldsymbol{A}_2$, respectively. If suppose that there exists no $c \in A_0$ such that $a \to_1 c = 1$ and $c \to_2 b = 1$, then by Theorem 5.7 there exist $P \in X(A_1)$ and $Q \in X(A_2)$ such that $\Box a \in P$, $b \notin Q$ and $P \cap A_0 = Q \cap A_0$, i.e., $(P,Q) \in j_1(\Box a)$ and $(P,Q) \notin j_2(b)$. Thus, $j_1(\Box a) \nsubseteq j_2(b)$, and since $\Box a \leq_1 a$ we get $j_1(a) \nsubseteq j_2(b)$. Therefore, $\mathcal{TMA}$ has the Superamalgamation Property. ■

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