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Priestley Style Duality for Distributive Meet-semilattices

Abstract. We generalize Priestley duality for distributive lattices to a duality for distributive meet-semilattices. On the one hand, our generalized Priestley spaces are easier to work with than Celani's DS-spaces, and are similar to Hansoul's Priestley structures. On the other hand, our generalized Priestley morphisms are similar to Celani's meet-relations and are more general than Hansoul's morphisms. As a result, our duality extends Hansoul's duality and is an improvement of Celani's duality.

Keywords: Distributive meet-semilattices, distributive lattices, duality theory.

1. Introduction

In the study of algebras related to non-classical logics, semilattices are always present in the background. Each of residuated lattices, MV-algebras, Heyting algebras, Boolean algebras, and modal algebras has a semilattice reduct; often the semilattice reduct is distributive. Our aim is to give a Priestley style duality for distributive semilattices.

Topological representation of distributive semilattices goes back to Stone's pioneering work [13]. For distributive join-semilattices with bottom it was worked out in detail in Grätzer [7, Sec. II.5, Thm. 8]. A full duality between meet-semilattices with top (which are dual to join-semilattices with bottom) and certain ordered spectral-like topological spaces was developed by Celani [3]. The main novelty of [3] was the characterization of meet-semilattice homomorphisms preserving top by means of certain binary relations. But the ordered topological spaces of [3] are not necessarily Hausdorff and so are difficult to work with. On the other hand, Hansoul [8, 9] developed rather nice Priestley style duals of bounded join-semilattices. He also gave a dual characterization of join-semilattice homomorphisms preserving all existing finite meets, but had no dual analogue of all join-semilattice homomorphisms.

In this paper we develop a full Priestley style duality for distributive meet-semilattices and meet-semilattice homomorphisms. We obtained our

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duality independently from Hansoul, but it turns out that our generalized Priestley spaces are equal to his Priestley structures. In addition, our generalized Priestley morphisms are similar to Celani's meet-relations, and provide dual description of meet-semilattice homomorphisms. In our setting, meet-semilattice homomorphisms preserving all existing finite joins are characterized by means of functional generalized Priestley morphisms. Thus, our duality is an improvement of Celani's duality and extends Hansoul's duality. Since our work is in the dual setting of distributive meet-semilattices and our approach is different from Hansoul's, we present our results on generalized Priestly spaces in full detail. However, we give a detailed comparison with Hansoul's and Celani's work at the end of the paper.

The paper is organized as follows. In Section 2 we recall some basic facts about distributive meet-semilattices. We also discuss filters and ideals of distributive meet-semilattices, and recall the basics of Priestley duality for bounded distributive lattices. In Section 3 we introduce the distributive envelope $D(L)$ of a distributive meet-semilattice L, and relate filters and ideals of L to filters and ideals of $D(L)$. We also introduce sup-homomorphisms, which are dual to Hansoul's homomorphisms, and provide an abstract characterization of $D(L)$. In Section 4 we introduce some of the main ingredients of our duality such as Frink ideals and optimal filters of L, and give a detailed account of their main properties. The introduction of optimal filters is one of the crucial points of our development. They correspond to prime filters of $D(L)$. It is optimal filters and *not* prime filters that serve as points of the dual space of L, which allows us to prove tha t the Priestley-like topology on the dual of L is compact, thus providing an improvement of [7, 3], where the dual of L was constructed by means of prime filters of L. In Section 5 we introduce generalized Priestley spaces, prove their main properties, and provide a representation theorem for bounded distributive meet-semilattices by means of generalized Priestley spaces. In Section 6 we introduce generalized Priestley morphisms and show that the category of bounded distributive meet-semilattices and meet-semilattice homomorphisms is dually equivalent to the category of generalized Priestley spaces and generalized Priestley morphisms. In Section 7 we show that the subclasses of generalized Priestley morphisms which dually correspond to sup-homomorphisms can be characterized by means of special functions between generalized Priestley spaces, which we call strong Priestley morphisms. In Section 8 we show how our duality works by giving dual descriptions of F rink ideals, ideals, filters, and 1-1 and onto homomorphisms. In Section 9 we show how to adjust our technique to handle the non-bounded case. Finally, in Section 10 we briefly compare our work with that of Grätzer, Celani, and Hansoul.

The main findings of this paper were reported at the International Workshop on Topological Methods in Logic, June 3–5, 2008, Tbilisi, Georgia. The paper is a condensed version of [1], which incorporates our dualities for distributive meet-semilattices as well as for implicative semilattices, but is too long for a journal publication. Our duality for implicative semilattices can be found in [2]. In order to keep the paper relatively short, we opted to skip some of the proofs and refer the interested reader to [1].

2. Preliminaries

A *meet-semilattice* is a commutative idempotent semigroup $\langle S, \cdot \rangle$. As usual, we denote $\cdot$ by $\wedge$, and consider the partial order $\leq$ on S given by $a \leq b$ iff $a = a \wedge b$. Then $a \wedge b$ is the greatest lower bound of $\{a, b\}$ and each nonempty finite subset of S has a greatest lower bound. Below we will be interested in meet-semilattices with the greatest element $\top$, i.e., in commutative idempotent monoids $\langle M, \wedge, \top \rangle$. Let M denote the category of meet-semilattices with $\top$ and meet-semilattice homomorphisms preserving $\top$.

Meet-semilattices serve as a natural generalization of lattices. Similarly, distributive meet-semilattices serve as a natural generalization of distributive lattices. A meet-semilattice L is *distributive* if for each $a, b_1, b_2 \in L$ with $b_1 \wedge b_2 \leq a$, there exist $c_1, c_2 \in L$ such that $b_1 \leq c_1$, $b_2 \leq c_2$, and $a = c_1 \wedge c_2$. As follows from [7, Sec. II.5, Lem. 1], a lattice $\langle L, \wedge, \vee \rangle$ is distributive iff the meet-semilattice $\langle L, \wedge \rangle$ is distributive. Let DM denote the category of distributive meet-semilattices with $\top$ and meet-semilattice homomorphisms preserving $\top$. Obviously $\mathsf{DM} \subset \mathsf{M}$. However, DM is not a variety and the variety generated by DM is M. In fact, M is generated by the two element meet-semilattice $\mathbf{2} = \{\bot, \top\}$ [10].

For a poset P and $A \subseteq P$ let

$$\uparrow A = \{x \in P : \exists a \in A \text{ with } a \leq x\} \text{ and } \downarrow A = \{x \in P : \exists a \in A \text{ with } x \leq a\}.$$

If A is the singleton $\{a\}$, then we write $\uparrow a$ and $\downarrow a$ instead of $\uparrow \{a\}$ and $\downarrow \{a\}$, respectively. We call A an *upset* (resp. *downset*) if $A = \uparrow A$ (resp. $A = \downarrow A$).

Let L be a meet-semilattice. A nonempty subset F of L is a *filter* if (i) it is an upset and (ii) $a, b \in F$ implies $a \wedge b \in F$. We call a filter F of L *proper* if $F \neq L$. Similar to lattices, we have that L is a filter of L, and if L has a top element, then an arbitrary intersection of filters of L is again a filter of L. Therefore, for each $X \subseteq L$, there exists a least filter containing X, which we call the *filter generated by* X and denote by $[X)$. It is obvious that

$$a \in [X) \text{ iff there exists a finite } Y \subseteq X \text{ such that } \bigwedge Y \leq a.$$

In particular, the filter generated by $x \in L$ is the upset $\uparrow x$. We also point out that if $X = \emptyset$, then $\bigwedge X = \top$, and so $[X) = \{\top\}$.

Let $\mathsf{F}(L)$ denote the set of filters of L. Obviously the structure $\langle \mathsf{F}(L), \cap, \vee \rangle$ forms a lattice, where $F_1 \vee F_2 = [F_1 \cup F_2)$. In particular, $\uparrow a \vee \uparrow b = \uparrow(a \wedge b)$. A meet-semilattice L is distributive iff the lattice $\langle \mathsf{F}(L), \cap, \vee \rangle$ is distributive (c.f. [7, Sec. II.5, Lem. 1]).

A proper filter F of a meet-semilattice L is *meet-prime* if it is a prime element of the lattice $\mathsf{F}(L)$; that is, if for any two filters F_1, F_2 of L with $F_1 \cap F_2 \subseteq F$, we have $F_1 \subseteq F$ or $F_2 \subseteq F$. Meet-prime filters serve as an obvious generalization of prime filters of a lattice because a filter of a lattice L is prime iff it is meet-prime. From now on we will call meet-prime filters of a meet-semilattice simply *prime*.

Since in a meet-semilattice L the join of two elements of L may not exist, the notion of an ideal of L needs to be adjusted appropriately. There are several notions of an ideal of a poset in the literature. We will use the following one: A nonempty subset I of a meet-semilattice L is an *ideal* if it is a downset which is *updirected* (if $a, b \in I$, then there exists $c \in I$ with $a, b \leq c$). An ideal I is *proper* if $I \neq L$. In lattices, ideals coincide with lattice ideals.

For a subset A of a meet-semilattice L, let A^u denote the set of *upper bounds* of A, and let A^l denote the set of *lower bounds* of A. Then a nonempty subset I of L is an ideal iff for each $a, b \in L$ we have $a, b \in I$ iff $\{a, b\}^u \cap I \neq \emptyset$; this last condition is equivalent to saying that for each $a, b \in L$ we have $a, b \in I$ iff $(\uparrow a \cap \uparrow b) \cap I \neq \emptyset$. We note that if L has top, then L itself is always an ideal. However, unlike the case with filters, a nonempty intersection of a family of ideals may not be an ideal as the following example shows.

Example 2.1. Let L be the meet-semilattice shown in Fig. 1. Then each $\downarrow c_n$ is an ideal of L, but $\bigcap_{n \in \omega} \downarrow c_n = \{\bot, a, b\}$ is not an ideal of L.

Nevertheless, we have the following analogue of the prime filter lemma for distributive meet-semilattices. For a proof we refer to [7, Sec. II.5, Lem. 2] or [3, Thm. 8].

Lemma 2.2 (Prime Filter Lemma). *Suppose that L is a distributive meet-semilattice. If F is a filter and I is an ideal of L with $F \cap I = \emptyset$, then there exists a prime filter P of L such that $F \subseteq P$ and $P \cap I = \emptyset$.*

As a corollary, it immediately follows that each proper filter F of a distributive meet-semilattice L is the intersection of the prime filters of L containing F.

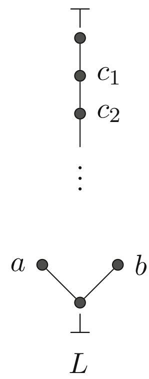

Fig. 1

We call an ideal I of L *prime* if it is proper and for each $a, b \in L$ with $a \wedge b \in I$, either $a \in I$ or $b \in I$. We have the following analogue of a well-known theorem for lattices.

PROPOSITION 2.3. *A subset F of a meet-semilattice L is a prime filter iff $I = L - F$ is a prime ideal.*

PROOF. If F is a prime filter of L, then it is nonempty and proper. Therefore $I = L - F \neq \emptyset, L$. Since F is an upset, I is a downset. To show that I is updirected, suppose that $a, b \in I$. If $(\uparrow a \cap \uparrow b) \cap I = \emptyset$, then $\uparrow a \cap \uparrow b \subseteq F$. Since F is prime, $\uparrow a \subseteq F$ or $\uparrow b \subseteq F$, so either $a \notin I$ or $b \notin I$, a contradiction. Thus, $(\uparrow a \cap \uparrow b) \cap I \neq \emptyset$, and so I is an ideal. Finally, to show that I is prime, suppose that $a \wedge b \in I$. Then $a \wedge b \notin F$. Since F is a filter, either $a \notin F$ or $b \notin F$. Thus, $a \in I$ or $b \in I$. Conversely, if $I = L - F$ is a prime ideal, then $I \neq \emptyset, L$, and so $F \neq \emptyset, L$. Since I is a downset, F is an upset. Moreover, if $a, b \in F$, then $a, b \notin I$, and since I is prime, $a \wedge b \notin I$. Thus, $a \wedge b \in F$, and so F is a filter. Finally, to show that F is prime, suppose that $F_1 \cap F_2 \subseteq F$. If $F_1 \not\subseteq F$ and $F_2 \not\subseteq F$, then $F_1 \cap I \neq \emptyset$ and $F_2 \cap I \neq \emptyset$. Therefore, there exist $a_1 \in F_1 \cap I$ and $a_2 \in F_2 \cap I$. Since I is updirected, there is $c \in I$ with $a_1, a_2 \leq c$. It follows that $c \in F \cap I$, a contradiction. Thus, either $F_1 \subseteq F$ or $F_2 \subseteq F$, and so F is prime. ■

We conclude this preliminary section by a brief overview of Priestley duality for bounded distributive lattices. We recall that a *Priestley space* is an ordered topological space $X = \langle X, \tau, \leq \rangle$ which is compact and satisfies the *Priestley separation axiom*: if $x \not\leq y$, then there is a *clopen* (closed and open) upset U of X such that $x \in U$ and $y \notin U$. It follows from the Priestley separation axiom that X is in fact Hausdorff and that clopen sets form a basis for the topology. Thus, each Priestley space is a *Stone space* (compact, Hausdorff, and zero-dimensional).

For two Priestley spaces X and Y, a morphism $f : X \to Y$ is a *Priestley morphism* if f is continuous and order-preserving. We denote the category of Priestley spaces and Priestley morphisms by PS. Let also BDL denote the category of bounded distributive lattices and bounded lattice homomorphisms. Then we have that BDL is dually equivalent to PS [11]. We recall that the functors $(-)_* : \mathsf{DL} \to \mathsf{PS}$ and $(-)^* : \mathsf{PS} \to \mathsf{DL}$ establishing the dual equivalence are constructed as follows. If L is a bounded distributive lattice, then $L_* = \langle X, \tau, \leq \rangle$, where X is the set of prime filters of L, $\leq$ is set-theoretic inclusion, and τ is the topology generated by the subbasis $\{\varphi(a) : a \in L\} \cup \{\varphi(b)^c : b \in L\}$, where $\varphi(a) = \{x \in X : a \in x\}$ is the Stone map and $(-)^c$ denotes set-theoretic complement. If $h \in \hom(L, K)$, then $h_* = h^{-1}$. If X is a Priestley space, then X^* is the lattice of clopen upsets of X, and if $f \in \hom(X, Y)$, then $f^* = f^{-1}$. It follows from [11, 12] that the functors $(-)_*$ and $(-)^*$ are well-defined, and that they establish a dual equivalence of BDL and PS.

3. The Distributive envelope

Let L be a meet-semilattice and let $\mathrm{Pr}(L)$ denote the set of prime filters of L. We define the map $\sigma : L \to \mathcal{P}(\mathrm{Pr}(L))$ by $\sigma(a) = \{x \in \mathrm{Pr}(L) : a \in x\}$ for each $a \in L$. When convenient we will write σ_L. The next theorem goes back to Stone [13] and we leave the proof to the interested reader (but see [1, Thm. 4.1]).

THEOREM 3.1. *If L is a meet-semilattice, then $\sigma : L \to \mathcal{P}(\mathrm{Pr}(L))$ is a meet-semilattice homomorphism. If L has top, then σ preserves top, and if L has bottom, then σ preserves bottom. In addition, if L is distributive, then σ is a meet-semilattice embedding.*

The map σ has the following important property.

LEMMA 3.2. *For a distributive meet-semilattice L and $a_1, \dots, a_n, b \in L$ we have:*

$$\bigcap_{i=1}^{n} {\uparrow} a_i \subseteq {\uparrow} b \quad \textit{iff} \quad \sigma(b) \subseteq \bigcup_{i=1}^{n} \sigma(a_i) \quad \textit{iff} \quad \bigcap_{i=1}^{n} {\uparrow}\sigma(a_i) \subseteq {\uparrow}\sigma(b).$$

PROOF. We only prove the first equivalence because the second one is obvious. First suppose that $\bigcap_{i=1}^{n} {\uparrow} a_i \subseteq {\uparrow} b$ and $x \in \sigma(b)$. Then $\bigcap_{i=1}^{n} {\uparrow} a_i \subseteq x$, and as x is a prime filter of L, there exists $1 \leq i \leq n$ such that ${\uparrow} a_i \subseteq x$. Therefore, $x \in \bigcup_{i=1}^{n} \sigma(a_i)$, and so $\sigma(b) \subseteq \bigcup_{i=1}^{n} \sigma(a_i)$. Conversely, suppose that

$\sigma(b) \subseteq \bigcup_{i=1}^n \sigma(a_i)$ and $c \in \bigcap_{i=1}^n {\uparrow} a_i$. Then $\sigma(a_i) \subseteq \sigma(c)$ for each $1 \leq i \leq n$. Therefore, $\bigcup_{i=1}^n \sigma(a_i) \subseteq \sigma(c)$, and so $\sigma(b) \subseteq \sigma(c)$. Thus, $c \in {\uparrow} b$, and we conclude that $\bigcap_{i=1}^n {\uparrow} a_i \subseteq {\uparrow} b$. ■

Let $D(L)$ denote the sublattice of the lattice $\mathrm{Up}(\mathrm{Pr}(L))$ of upsets of $\mathrm{Pr}(L)$ generated by $\sigma[L]$. Since $\sigma[L]$ is closed under finite intersections, for each $A \in \mathrm{Up}(\mathrm{Pr}(L))$ we have:

$$A \in D(L) \quad \text{iff} \quad A = \bigcup_{i=1}^{n} \sigma(a_i) \text{ for some } a_i \in L.$$

It follows that the pair $(D(L), \sigma)$ has the following three properties: (i) $D(L)$ is a distributive lattice, (ii) $\sigma[L]$ is join-dense in $D(L)$, and (iii) $\sigma : L \to D(L)$ is a meet-semilattice embedding with the property stated in Lemma 3.2. We will see that these three properties give an abstract characterization of $D(L)$. Whenever convenient we will identify a distributive meet-semilattice L with $\sigma[L]$ and consider L as a join-dense $\wedge$-subalgebra of $D(L)$.

Definition 3.3. For a distributive meet-semilattice L, we call $D(L)$ the *distributive envelope* of L.

Let L and K be distributive meet-semilattices and let $h : L \to K$ be a meet-semilattice homomorphism. If there exist $a, b \in L$ such that $a \vee b$ exists in L and $h(a) \vee h(b)$ exists in K, it is not necessary that $h(a \vee b) = h(a) \vee h(b)$. Therefore, h may not be extended to a lattice homomorphism from $D(L)$ to $D(K)$. We introduce a stronger notion of a homomorphism between distributive meet-semilattices, which we call a sup-homomorphism. We show that sup-homomorphisms preserve all existing finite joins and can be extended to lattice homomorphisms between the corresponding distributive envelopes. We also give an abstract characterization of the distributive envelope by means of sup-homomorphisms, and prove that the category of distributive lattices and lattice homomorphisms is a reflective subcategory of the category of distributive meet-semilattices and sup-homomorphisms. The notion of sup-homomorphism is dual to Hansoul's notion [8, 9] of morphism for distributive join-semilattices, which is a join-semilattice homomorphism preserving all existing finite meets.

Definition 3.4. Let L and K be distributive meet-semilattices. A meet-semilattice homomorphism $h : L \to K$ is a *sup-homomorphism* if for each $a_1, \ldots, a_n, b \in L$ we have:

$$\bigcap_{i=1}^{n} {\uparrow} a_i \subseteq {\uparrow} b \text{ implies } \bigcap_{i=1}^{n} {\uparrow} h(a_i) \subseteq {\uparrow} h(b). \tag{1}$$

By Lemma 3.2, $\sigma : L \to D(L)$ is a sup-homomorphism.

PROPOSITION 3.5. *Let L and K be distributive meet-semilattices and let $h : L \to K$ be a meet-semilattice homomorphism. Then h is a sup-homomorphism iff h preserves all existing finite joins.*

PROOF. Let h be a sup-homomorphism and let $a_1, \ldots, a_n \in L$ be such that $a_1 \vee \ldots \vee a_n$ exists in L. Then $\bigcap_{i=1}^n {\uparrow} a_i = {\uparrow}(a_1 \vee \ldots \vee a_n)$. Since h is order-preserving, by the definition of sup-homomorphisms, $\bigcap_{i=1}^n {\uparrow} h(a_i) = {\uparrow} h(a_1 \vee \ldots \vee a_n)$. Therefore, $h(a_1 \vee \ldots \vee a_n)$ is the join of $h(a_1), \ldots, h(a_n)$ in K. Thus, h preserves all existing finite joins. Conversely, suppose that h preserves all existing finite joins. Let $a_1, \ldots, a_n, b \in L$ with $\bigcap_{i=1}^n {\uparrow} a_i \subseteq {\uparrow} b$. Then, in the lattice of filters of L, $(\bigcap_{i=1}^n {\uparrow} a_i) \vee {\uparrow} b = {\uparrow} b$. Since the lattice of filters of L is distributive, $\bigcap_{i=1}^n ({\uparrow} a_i \vee {\uparrow} b) = {\uparrow} b$. From ${\uparrow} a_i \vee {\uparrow} b = {\uparrow}(a_i \wedge b)$ it follows that $\bigcap_{i=1}^n {\uparrow}(a_i \wedge b) = {\uparrow} b$. This implies that b is the join of $a_1 \wedge b, \ldots, a_n \wedge b$ in L. Therefore, since h preserves all existing finite joins, the join of $h(a_1 \wedge b), \ldots, h(a_n \wedge b)$ exists in K and is $h(b)$. Thus, $\bigcap_{i=1}^n {\uparrow} h(a_i \wedge b) = {\uparrow} h(b)$, which means that

$$ {\uparrow} h(b) = \bigcap_{i=1}^{n} {\uparrow}(h(a_i) \wedge h(b)) = \bigcap_{i=1}^{n} ({\uparrow} h(a_i) \vee {\uparrow} h(b)). $$

Using the distributivity of the lattice of filters of K, we obtain

$$ {\uparrow} h(b) = {\uparrow} h(b) \vee \bigcap_{i=1}^{n} {\uparrow} h(a_i). $$

Consequently, $\bigcap_{i=1}^n {\uparrow} h(a_i) \subseteq {\uparrow} h(b)$, and so h is a sup-homomorphism. ■

It is easy to see that the composition of sup-homomorphisms is a sup-homomorphism, and that the identity map is a sup-homomorphism. Let $\mathsf{DM^S}$ denote the category of distributive meet-semilattices and sup-homomorphisms. If L and K are distributive lattices and $h : L \to K$ is a lattice homomorphism, then $h : L \to K$ is a sup-homomorphism. Therefore, we have the forgetful functor $U : \mathsf{DLat} \to \mathsf{DM^S}$ that forgets $\vee$. As follows from the next proposition (whose proof we skip and refer the interested reader to [1, Prop. 5.6]), the map $D(-)$ sending a distributive meet-semilattice L to its distributive envelope $D(L)$ extends to a functor $D : \mathsf{DM^S} \to \mathsf{DLat}$, which is left adjoint to U. Consequently, $U(\mathsf{DLat})$ is a reflective subcategory of $\mathsf{DM^S}$.

PROPOSITION 3.6. *Let L and K be distributive meet-semilattices. If $h : L \to K$ is a sup-homomorphism, then there is a unique lattice homomorphism*

$D(h) : D(L) \to D(K)$ such that $D(h) \circ \sigma_L = \sigma_K \circ h$. Moreover, if h is 1-1, then so is $D(h)$.

Noting that if K is a distributive lattice, then $D(K)$ is (isomorphic to) K, the following is an immediate consequence of Proposition 3.6.

Corollary 3.7. *Let L be a distributive meet-semilattice and D be a distributive lattice. If $h : L \to D$ is a sup-homomorphism, then there is a unique lattice homomorphism $D(h) : D(L) \to D$ such that $D(h) \circ \sigma = h$. Moreover, if h is 1-1, then so is $D(h)$.*

Next theorem (whose proof can be found in [1, Thm. 5.8]) provides an abstract characterization of the distributive envelope of a distributive meet-semilattice by a universal property. A similar characterization can also be found in Hansoul [8, 9] for distributive join-semilattices.

Theorem 3.8. *Let L be a distributive meet-semilattice. The distributive envelope $D(L)$ of L is up to isomorphism the unique distributive lattice E for which there is a 1-1 sup-homomorphism $e : L \to E$ such that for each distributive lattice D and a 1-1 sup-homomorphism $h : L \to D$, there is a unique 1-1 lattice homomorphism $k : E \to D$ with $k \circ e = h$.*

Next theorem gives yet another abstract characterization of distributive envelopes.

Theorem 3.9. *Let L be a distributive meet-semilattice. The distributive envelope $D(L)$ of L is up to isomorphism the unique distributive lattice E for which there is a 1-1 sup-homomorphism $e : L \to E$ such that $e[L]$ is join-dense in E.*

Proof. Let E be a distributive lattice and $e : L \to E$ be a 1-1 sup-homomorphism such that $e[L]$ is join-dense in E. For a distributive lattice D and a 1-1 sup-homomorphism $h : L \to D$, we prove that there is a unique 1-1 lattice homomorphism $k : E \to D$ with $k \circ e = h$. Let $a_1, \ldots, a_n, b_1 \ldots, b_m \in L$ be such that $e(a_1) \vee \ldots \vee e(a_n) = e(b_1) \vee \ldots \vee e(b_m)$. Then $e(a_i) \leq e(b_1) \vee \ldots \vee e(b_m)$ for each i. Therefore, $\bigcap_{i=1}^{m} {\uparrow} e(b_j) \subseteq {\uparrow} e(a_i)$ for each i. Since e is a 1-1 sup-homomorphism, $\bigcap_{j=1}^{m} {\uparrow} b_j \subseteq {\uparrow} a_i$. As h is a sup-homomorphism, $\bigcap_{j=1}^{m} {\uparrow} h(b_j) \subseteq {\uparrow} h(a_i)$ for each i. Thus, $h(a_1) \vee \ldots \vee h(a_n) \leq h(b_1) \vee \ldots \vee h(b_m)$. By a similar argument we obtain the other inequality. Define $k : E \to D$ by

$$k(c) = h(a_1) \vee \ldots \vee h(a_n),$$

where $a_1, \ldots, a_n \in L$ are such that $c = e(a_1) \vee \ldots e(a_n)$; they exist because $e[L]$ is join-dense in E. Then it is obvious that $k : E \to D$ is a unique lattice homomorphism such that $k \circ e = h$. ■

Next lemma establishes basic relations between filters and ideals of L and $D(L)$.

LEMMA 3.10. *Let L be a distributive meet-semilattice and let $D(L)$ be the distributive envelope of L. Then:*

1. *If F is a filter of L, then $\uparrow_{D(L)}\sigma[F]$ is a filter of $D(L)$.*
2. *If P is a prime filter of L, then $\uparrow_{D(L)}\sigma[F]$ is a prime filter of $D(L)$.*
3. *If I is an ideal of L, then $\downarrow_{D(L)}\sigma[I]$ is an ideal of $D(L)$.*
4. *If I is a prime ideal of L, then $\downarrow_{D(L)}\sigma[I]$ is a prime ideal of $D(L)$.*
5. *If L has top and F is a filter of $D(L)$, then $\sigma^{-1}(F)$ is a filter of L.*

PROOF. (1) Let F be a filter of L. Since $\uparrow_{D(L)}\sigma[F]$ is an upset of $D(L)$, it is enough to show that $\uparrow_{D(L)}\sigma[F]$ is closed under binary intersections. Let $A, B \in \uparrow_{D(L)}\sigma[F]$. Then there exist $a, b \in F$ such that $\sigma(a) \subseteq A$ and $\sigma(b) \subseteq B$. Therefore, $\sigma(a \wedge b) = \sigma(a) \cap \sigma(b) \subseteq A \cap B$. Since F is a filter of L, we have $a \wedge b \in F$. Thus, $A \cap B \in \uparrow_{D(L)}\sigma[F]$.

(2) Let P be a prime filter of L and let $A \cup B \in \uparrow_{D(L)}\sigma[P]$. Let $a_1, \ldots, a_n, b_1, \ldots, b_m \in L$ be such that $A = \sigma(a_1) \cup \ldots \cup \sigma(a_n)$ and $B = \sigma(b_1) \cup \ldots \cup \sigma(b_m)$. Then $\sigma(a_1) \cup \ldots \cup \sigma(a_n) \cup \sigma(b_1) \cup \ldots \cup \sigma(a_m) \in \uparrow_{D(L)}\sigma[P]$. Therefore, there exists $c \in P$ such that $\sigma(c) \subseteq \sigma(a_1) \cup \ldots \cup \sigma(a_n) \cup \sigma(b_1) \cup \ldots \cup \sigma(a_m)$. Thus, $\bigcap_{i=1}^n \uparrow a_i \cap \bigcap_{j=1}^m \uparrow b_j \subseteq \uparrow c \subseteq P$. As P is prime, $\uparrow a_i \subseteq P$ for some i or $\uparrow b_j \subseteq P$ for some j. Since by assumption $\sigma(a_i) \subseteq A$ and $\sigma(b_j) \subseteq B$, it follows that $A \in \uparrow_{D(L)}\sigma[P]$ or $B \in \uparrow_{D(L)}\sigma[P]$.

(3) Let I be an ideal of L. Since $\downarrow_{D(L)}\sigma[I]$ is a downset of $D(L)$, we need to show that $\downarrow_{D(L)}\sigma[I]$ is closed under binary unions. Let $A, B \in \downarrow_{D(L)}\sigma[I]$. Then there exist $a, b \in I$ such that $A \subseteq \sigma(a)$ and $B \subseteq \sigma(b)$. Since I is an ideal of L, there exists $e \in \{a, b\}^u \cap I$. Thus, $A, B \subseteq \sigma(e)$, implying that $A \cup B \in \downarrow_{D(L)}\sigma[I]$.

(4) Let I be a prime ideal of L. By (3), $\downarrow_{D(L)}\sigma[I]$ is an ideal of $D(L)$. To show that it is prime, let $A = \bigcup_{i=1}^n \sigma(a_i)$, $B = \bigcup_{j=1}^m \sigma(b_j)$, and $A \cap B \in \downarrow_{D(L)}\sigma[I]$. Then $\sigma(a_i) \cap \sigma(b_j) \in \downarrow_{D(L)}\sigma[I]$, and so $a_i \wedge b_j \in I$ for each i, j. Since I is prime, either $a_i \in I$ or $b_j \in I$. We look at $a_1 \wedge b_1, \ldots, a_1 \wedge b_m$. If $a_1 \notin I$, then $b_1, \ldots, b_m \in I$, and so there exists $c \in \bigcap_{j=1}^m \uparrow b_j \cap I$. Therefore, $B = \bigcup_{j=1}^m \sigma(b_j) \subseteq \sigma(c) \in \sigma[I]$, so $B \in \downarrow_{D(L)}\sigma[I]$, and so without loss of generality we may assume that $a_1 \in I$. Now we look at $a_2 \wedge b_1, \ldots, a_2 \wedge b_m$. If $a_2 \notin I$, then $b_1, \ldots, b_m \in I$, so again $B \in \downarrow_D\sigma[I]$. Thus, without loss of generality we may assume that $a_1, a_2 \in I$. Going through all $a_1, \ldots, a_n$ we

obtain that either $B \in {\downarrow_{D(L)}}\sigma[I]$ or $A \in {\downarrow_{D(L)}}\sigma[I]$. It follows that ${\downarrow_{D(L)}}\sigma[I]$ is a prime ideal of $D(L)$.

(5) Let F be a filter of $D(L)$. Since $\top \in L$, we have $\mathrm{Pr}(L) = \sigma(\top) \in D(L)$, so $\sigma(\top) = \mathrm{Pr}(L) \in F$, and so $\top \in \sigma^{-1}(F)$. Thus, $\sigma^{-1}(F)$ is nonempty. Suppose that $a \in \sigma^{-1}(F)$ and $a \leq b$. Then $\sigma(a) \in F$ and $\sigma(a) \subseteq \sigma(b)$. Since F is an upset of $D(L)$, it follows that $\sigma(b) \in F$. Therefore, $b \in \sigma^{-1}(F)$. For $a, b \in \sigma^{-1}(F)$ we have $\sigma(a), \sigma(b) \in F$. Since F is a filter of $D(L)$, we have $\sigma(a \wedge b) = \sigma(a) \cap \sigma(b) \in F$. Thus, $a \wedge b \in \sigma^{-1}(F)$, and so $\sigma^{-1}(F)$ is a filter of L. ■

Next example shows that there exist ideals I of $D(L)$ such that $\sigma^{-1}(I)$ is not an ideal of L.

EXAMPLE 3.11. Consider the distributive meet-semilattice L shown in Fig. 1. The ordered set $\langle \mathrm{Pr}(L), \subseteq \rangle$ of prime filters of L together with the distributive envelope $D(L)$ of L is shown in Fig. 2. We have that $I = \{\emptyset, \sigma(a), \sigma(b), \sigma(a) \cup \sigma(b)\}$ is an ideal of $D(L)$, but that $\sigma^{-1}(I) = \{\bot, a, b\}$ is not an ideal of L.

LEMMA 3.12. *Let L be a distributive meet-semilattice and I an ideal of $D(L)$. Then I is the ideal of $D(L)$ generated by $\sigma[\sigma^{-1}(I)] = I \cap \sigma[L]$.*

PROOF. Let J be the ideal of $D(L)$ generated by $\sigma[\sigma^{-1}(I)] = I \cap \sigma[L]$. Obviously $J \subseteq I$. On the other hand, if $A \in I$, then as $A = \bigcup_{i=1}^{n} \sigma(b_i)$ for some $b_i \in L$, we have $\sigma(b_i) \in I \cap \sigma[L]$ for each $i \leq n$. Thus, $A \in J$. ■

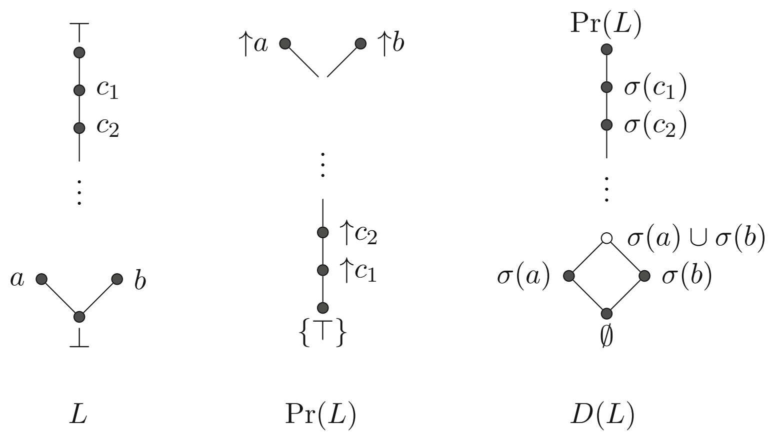

Fig. 2

As a consequence, we obtain that for a distributive meet-semilattice L and ideals I, J of $D(L)$, the following three conditions are equivalent: (i) $I = J$, (ii) $\sigma^{-1}(I) = \sigma^{-1}(J)$, and (iii) $I \cap \sigma[L] = J \cap \sigma[L]$.

The situation with filters is different because there exist filters F of $D(L)$ such that F is not generated by $\sigma[\sigma^{-1}(F)] = F \cap \sigma[L]$, as the next example shows.

Example 3.13. Consider the distributive meet-semilattice L and its distributive envelope $D(L)$ shown in Fig. 2. Then $F = \{\sigma(a) \cup \sigma(b), \sigma(c_n), \Pr(L) : n \in \omega\}$ is a filter of $D(L)$ which is not generated by $\sigma[\sigma^{-1}(F)] = F \cap \sigma[L] = \{\sigma(c_n), \Pr(L) : n \in \omega\}$.

4. Frink ideals and optimal filters

Let L be a distributive meet-semilattice. If $\sigma^{-1}(I)$ were an ideal of L for each ideal I of $D(L)$, then Lemma 3.12 would imply that there is a 1-1 correspondence between ideals of L and $D(L)$. However, $\sigma^{-1}(I)$ is not necessarily an ideal of L as we saw in Example 3.11. This forces us to introduce a weaker notion of an ideal of L, first considered by Frink [5, p. 227] for posets.

Definition 4.1. Let L be a meet-semilattice. We call a nonempty subset I of L a *Frink ideal* (*F-ideal* for short) if for each finite subset A of I, we have $A^{ul} \subseteq I$. Equivalently, I is a Frink ideal if for each $a_1, \ldots, a_n \in I$ and $c \in L$, whenever $\bigcap_{i=1}^{n} {\uparrow} a_i \subseteq {\uparrow} c$, we have $c \in I$. We call an F-ideal I of L *proper* if $I \neq L$, and *prime* if it is proper and $a \wedge b \in I$ implies $a \in I$ or $b \in I$.

It is easy to verify that ${\downarrow} a$ is an F-ideal for each $a \in L$. Moreover, unlike the case with ideals, a nonempty intersection of a family of F-ideals is again an F-ideal. Therefore, for each nonempty $X \subseteq L$, there exists a least F-ideal containing X. We call it the *F-ideal generated by* X, and denote it by $(X]$. It is easy to see that

$$
\begin{aligned}
(X] &= \{a \in L : \exists \text{ finite } A \subseteq X \text{ with } a \in A^{ul}\} \\
&= \{a \in L : \exists a_1, \ldots, a_n \in X \text{ with } \textstyle\bigcap_{i=1}^{n} {\uparrow} a_i \subseteq {\uparrow} a\}.
\end{aligned}
$$

Next lemma is easy to prove (see [1, Lem. 4.11]).

Lemma 4.2. *Let L be a meet-semilattice. Then each ideal of L is an F-ideal and each F-ideal of L is a downset. Moreover, if L is a lattice, then the two notions coincide with the usual notion of an ideal of a lattice.*

In particular, if a meet-semilattice L is finite and has a top element, then L is a lattice, and so each F-ideal of L is an ideal. On the other hand, there exist meet-semilattices for which not every F-ideal is an ideal. For example, if L is the lattice shown in Fig. 1, then $I = \{\bot, a, b\}$ is an F-ideal which is not an ideal. The next lemma is useful in obtaining a 1-1 correspondence between F-ideals of a distributive meet-semilattice L and ideals of its distributive envelope $D(L)$.

THEOREM 4.3. *Let L be a distributive meet-semilattice and let $D(L)$ be its distributive envelope. Then $I \subseteq L$ is a (prime) F-ideal of L iff there is a (prime) ideal J of $D(L)$ such that $I = \sigma^{-1}(J)$.*

PROOF. Let I be an F-ideal of L and J be the ideal of $D(L)$ generated by $\sigma[I] = \{\sigma(a) : a \in I\}$. It is clear that $I \subseteq \sigma^{-1}(J)$. To prove the other inclusion, let $b \in \sigma^{-1}(J)$. Then $\sigma(b) \in J$, so $\sigma(b) \subseteq \bigcup_{i=1}^n \sigma(a_i)$ for some $a_1, \ldots, a_n \in I$. By Lemma 3.2, $\bigcap_{i=1}^n {\uparrow} a_i \subseteq {\uparrow} b$, and since I is an F-ideal, $b \in I$. Conversely, let $I = \sigma^{-1}(J)$ for J an ideal of $D(L)$. Then $I \neq \emptyset$ and if $a_1, \ldots, a_n \in I$ and $c \in L$ are such that $\bigcap_{i=1}^n {\uparrow} a_i \subseteq {\uparrow} c$, then, by Lemma 3.2, we have $\sigma(c) \subseteq \bigcup_{i=1}^n \sigma(a_i)$. Therefore, $\sigma(c) \in J$, and so $c \in I$. Thus, I is an F-ideal of L.

Let I be a prime F-ideal of L and J be the ideal of $D(L)$ generated by $\sigma[I]$. Then $I = \sigma^{-1}(J)$. To show that J is prime, suppose that $A \cap B \in J$. Then $A = \bigcup_{i=1}^n \sigma(a_i)$ and $B = \bigcup_{j=1}^m \sigma(b_j)$ for some $a_1, \ldots, a_n, b_1, \ldots, b_m \in L$. Therefore, $\bigcup_{i,j} (\sigma(a_i) \cap \sigma(b_j)) \in J$, and so $\sigma(a_i) \cap \sigma(b_j) \in \sigma[I]$ for all i, j. Since I is prime, $a_i \in I$ or $b_j \in I$. We look at $a_1 \wedge b_1, \ldots, a_1 \wedge b_m$. If $a_1 \notin I$, then $b_1, \ldots, b_m \in I$, so $B = \bigcup_{j=1}^m \sigma(b_j) \in J$. Therefore, without loss of generality we may assume that $a_1 \in I$. Now we look at $a_2 \wedge b_1, \ldots, a_2 \wedge b_m$. If $a_2 \notin I$, then $b_1, \ldots, b_m \in I$, so again $B = \bigcup_{j=1}^m \sigma(b_j) \in J$. Thus, without loss of generality we may assume that $a_1, a_2 \in I$. Going through all $a_1, \ldots, a_n$ we obtain that either $B = \bigcup_{j=1}^m \sigma(b_j) \in J$ or $A = \bigcup_{i=1}^n \sigma(a_i) \in J$. It follows that J is a prime ideal of $D(L)$. The converse implication is easy to prove. ■

COROLLARY 4.4. *Let L be a distributive meet-semilattice. The ordered set of F-ideals of L is isomorphic to the ordered set of ideals of $D(L)$, and the ordered set of prime F-ideals of L is isomorphic to the ordered set of prime ideals of $D(L)$.*

DEFINITION 4.5. Let L be a distributive meet-semilattice. A filter F of L is said to be *optimal* if there exists a prime filter P of $D(L)$ such that $F = \sigma^{-1}[P]$. We denote the set of optimal filters of L by $\mathrm{Opt}(L)$. Clearly each optimal filter is proper.

REMARK 4.6. Optimal filters of L correspond to weakly prime ideals of the dual L^d of L, which are the key ingredients of the duality developed in [8, 9].

Moreover, as was pointed out in [2, Rem. 3.2], the optimal filters of L are the pseudoprime elements [6, Def. I-3.24] of the lattice of filters of L.

LEMMA 4.7 (Optimal Filter Lemma). *Let L be a distributive meet-semilattice. If F is a filter and I is an F-ideal of L with $F \cap I = \emptyset$, then there exists an optimal filter G of L such that $F \subseteq G$ and $G \cap I = \emptyset$.*

PROOF. Let F be a filter and I be an F-ideal of L with $F \cap I = \emptyset$. Let also ∇ be the filter and Δ be the ideal of $D(L)$ generated by $\sigma[F]$ and $\sigma[I]$, respectively. Suppose that there exists $A \in \nabla \cap \Delta$. Then there are $a_1, \dots, a_n \in I$ and $b \in F$ such that $A = \bigcup_{i=1}^n \sigma(a_i)$ and $\sigma(b) \subseteq A$. Therefore, $\sigma(b) \subseteq \bigcup_{i=1}^n \sigma(a_i)$. By Lemma 3.2, $\bigcap_{i=1}^n {\uparrow} a_i \subseteq {\uparrow} b$. Since I is an F-ideal, we obtain $b \in I$, a contradiction. Thus, $\nabla \cap \Delta = \emptyset$, and so there is a prime filter P of $D(L)$ such that $\nabla \subseteq P$ and $P \cap \Delta = \emptyset$. It follows that $F \subseteq \sigma^{-1}[P]$ and $\sigma^{-1}[P] \cap I = \emptyset$. If we set $G = \sigma^{-1}[P]$, then G is the desired optimal filter. ■

It immediately follows that each proper filter F of a distributive meet-semilattice is the intersection of the optimal filters containing F.

PROPOSITION 4.8. *Let L be a distributive meet-semilattice and let F be a filter of L. Then the following conditions are equivalent:*

1. *F is an optimal filter.*
2. *$L - F$ is an F-ideal.*
3. *There is an F-ideal I of L such that $F \cap I = \emptyset$ and F is maximal among the filters of L with this property.*

PROOF. To prove the implication (1)⇒(2), let F be an optimal filter of L, P be a prime filter of $D(L)$ such that $F = \sigma^{-1}[P]$, and $I = L - F$. Then F is a proper filter of L, and so I is nonempty. For $a_1, \dots, a_n \in I$ and $c \in L$ with $\bigcap_{i=1}^n {\uparrow} a_i \subseteq {\uparrow} c$, Lemma 3.2 implies that $\sigma(c) \subseteq \bigcup_{i=1}^n \sigma(a_i)$. If $c \notin I$, then $c \in F$, and so $\sigma(c) \in P$. Therefore, $\bigcup_{i=1}^n \sigma(a_i) \in P$. Since P is prime, $\sigma(a_i) \in P$ for some $a_i \in I$. Thus, $a_i \in F \cap I$, which is a contradiction. It follows that $c \in I$, and so I is an F-ideal. The implication (2)⇒(3) is obvious. Finally, to prove the implication (3)⇒(1), suppose that F is a filter and I is an F-ideal of L such that $F \cap I = \emptyset$ and F is maximal among the filters of L with this property. By the optimal filter lemma, there is an optimal filter G of L such that $F \subseteq G$ and $G \cap I = \emptyset$. Since F is a maximal filter with this property, $F = G$. Thus, F is optimal. ■

In particular, if L is a finite distributive meet-semilattice with top, then L is a finite distributive lattice, and so each optimal filter of L is prime. On the other hand, there exist distributive meet-semilattices in which not every optimal filter is prime. For example, in the distributive meet-semilattice L shown in Fig. 1 one can easily check that $F = L - \{\bot, a, b\}$ is an optimal filter of L, but that it is not prime.

It follows from Corollary 4.4 and Proposition 4.8 that optimal filters of a distributive meet-semilattice L are in a 1-1 correspondence with prime filters of the distributive envelope $D(L)$ of L. In fact, we have the following relations between optimal filters of L and prime filters of $D(L)$ (for a proof we refer to [1, Prop. 4.21]).

PROPOSITION 4.9. *Let L be a distributive meet-semilattice.*

1. *If $P \in \mathrm{Pr}(D(L))$, then $\sigma^{-1}(P) \in \mathrm{Opt}(L)$ and $\uparrow_{D(L)}(P \cap \sigma[L]) = P$.*
2. *If $F \in \mathrm{Opt}(L)$, then $\uparrow_{D(L)}\sigma[F] \in \mathrm{Pr}(D(L))$.*
3. *For a filter F of $D(L)$, $F \in \mathrm{Pr}(D(L))$ iff there is $G \in \mathrm{Opt}(L)$ such that $F = \uparrow_{D(L)}\sigma[G]$.*
4. *If $P, Q \in \mathrm{Pr}(D(L))$, then the following conditions are equivalent:*
   (a) *$P \subseteq Q$.*
   (b) *$\sigma^{-1}(P) \subseteq \sigma^{-1}(Q)$.*
   (c) *$\uparrow_{D(L)}(P \cap \sigma[L]) \subseteq \uparrow_{D(L)}(Q \cap \sigma[L])$.*

Thus, for a distributive meet-semilattice L, we have that F-ideals of L correspond to ideals of $D(L)$, that prime F-ideals of L correspond to prime ideals of $D(L)$, and that optimal filters of L correspond to prime filters of $D(L)$.

Let L be a distributive meet-semilattice. It is easy to see that the map $\varphi : L \to \mathrm{Up}(\mathrm{Opt}(L))$ defined by

$$\varphi(a) = \{x \in \mathrm{Opt}(L) : a \in x\}$$

is a meet-semilattice homomorphism, that it preserves top whenever L has a top, and that it preserves bottom whenever L has a bottom. It also follows from the optimal filter lemma that φ is 1-1. Thus, we obtain:

PROPOSITION 4.10. *Let L be a distributive meet-semilattice. Then L is isomorphic to the meet-semilattice $\langle\{\varphi(a) : a \in L\}, \cap\rangle$, and so the meet-semilattices $\langle\{\sigma(a) : a \in L\}, \cap\rangle$ and $\langle\{\varphi(a) : a \in L\}, \cap\rangle$ are isomorphic.*

LEMMA 4.11. *Let L be a distributive meet-semilattice and let $a, b_1, \ldots b_n \in L$. Then*

$$\bigcap_{i=1}^{n} \uparrow b_i \subseteq \uparrow a \quad \textit{iff} \quad \varphi(a) \subseteq \bigcup_{i=1}^{n} \varphi(b_i) \quad \textit{iff} \quad \bigcap_{i=1}^{n} \uparrow\varphi(b_i) \subseteq \uparrow\varphi(a).$$

Proof. We only prove the first equivalence. First suppose that $\varphi(a) \subseteq \bigcup_{i=1}^{n} \varphi(b_i)$. If $c \in \bigcap_{i=1}^{n} \uparrow b_i$ and $c \notin \uparrow a$, then $b_i \leq c$ for each $i \leq n$ and $a \not\leq c$. By the optimal filter lemma, there exists an optimal filter x of L such that $a \in x$ and $c \notin x$. But then $b_i \notin x$ for each $i \leq n$. Therefore, $x \in \varphi(a)$ but $x \notin \bigcup_{i=1}^{n} \varphi(b_i)$, a contradiction. Thus, $a \leq c$, so $c \in \uparrow a$, and so $\bigcap_{i=1}^{n} \uparrow b_i \subseteq \uparrow a$. Now suppose that $\bigcap_{i=1}^{n} \uparrow b_i \subseteq \uparrow a$. If $x \in \varphi(a)$ and $x \notin \bigcup_{i=1}^{n} \varphi(b_i)$, then $a \in x$ and $b_i \notin x$ for each $i \leq n$. Since x is an optimal filter, by Proposition 4.8, $L - x$ is an F-ideal, and $b_i \in L - x$ for each $i \leq n$ and $\bigcap_{i=1}^{n} \uparrow b_i \subseteq \uparrow a$ imply $a \in L - x$, a contradiction. Thus, $x \in \bigcup_{i=1}^{n} \varphi(b_i)$, and so $\varphi(a) \subseteq \bigcup_{i=1}^{n} \varphi(b_i)$. ■

It follows that closing $\varphi[L]$ under nonempty finite unions is isomorphic to $D(L)$, thus providing one more concrete realization of $D(L)$.

5. Generalized Priestley spaces

In this section we introduce the first main concept of the paper, that of *generalized Priestley space*. We show how to construct the generalized Priestley space L_* from a bounded distributive meet-semilattice L, and conversely, how a generalized Priestley space X gives rise to the bounded distributive meet-semilattice X^*. We further prove that a bounded distributive meet-semilattice L is isomorphic to ${L_*}^*$, thus providing a representation theorem for bounded distributive meet-semilattices. Furthermore, we show that a generalized Priestley space X is order-homeomorphic to ${X^*}_*$.

Let L be a bounded distributive meet-semilattice and let $D(L)$ be its distributive envelope. Then $D(L)$ is a bounded distributive lattice. We let

$$L_* = \mathrm{Opt}(L) \text{ and } L_+ = \mathrm{Pr}(L).$$

As we have already seen, $L_+ \subseteq L_*$ and $\langle L_* \subseteq\rangle \cong \langle \mathrm{Pr}(D(L)), \subseteq\rangle$. Define $\varphi_D : D(L) \to \mathrm{Up}(\mathrm{Pr}(D(L)))$ by $\varphi_D(A) = \{x \in D(L)_* : A \in x\}$. It is easy to see that the map $\varphi : L \to \mathrm{Up}(L_*)$ satisfies

$$\varphi(a) = \{\sigma^{-1}(x) : x \in \varphi_D(\sigma(a))\}.$$

Let $\mathfrak{B}_L = \varphi[L]$. Then the map $h : \mathfrak{B}_L \to \{\varphi_D(\sigma(a)) : a \in L\}$ given by

$$h(\varphi(a)) = \varphi_D(\sigma(a))$$

is a bounded meet-semilattice isomorphism. Consequently, the meet-semilattices L, $\mathfrak{B}_L$, and $\{\varphi_D(\sigma(a)) : a \in L\}$ are isomorphic to each other.

We recall that $\{\varphi_D(A)-\varphi_D(B) : A, B \in D(L)\}$ is a basis for the Priestley topology τ_P on $\mathrm{Pr}(D(L))$. This easily implies that the Priestley topology on $\mathrm{Pr}(D(L))$ has $\{\varphi_D(\sigma(a)) : a \in L\} \cup \{\varphi_D(\sigma(b))^c : b \in L\}$ as a subbasis, because every $A \in D(L)$ is of the form $\bigcup_{i=1}^n \sigma(a_i)$ for some $a_1, \ldots, a_n \in L$. This fact motivates defining a topology τ on L_* by letting $\{\varphi(a) : a \in L\} \cup \{\varphi(b)^c : b \in L\}$ be a subbasis for τ. Since $\langle \mathrm{Opt}(L), \subseteq\rangle$ and $\langle \mathrm{Pr}(D(L)), \subseteq\rangle$ are order-isomorphic, we obtain that $\langle L_*, \tau, \subseteq\rangle$ is a Priestley space order-homeomorphic to the Priestley dual $\langle \mathrm{Pr}(D(L)), \tau_P, \subseteq\rangle$ of $D(L)$.

LEMMA 5.1. *Let L be a bounded distributive meet-semilattice. Then L_+ is dense in $\langle L_*, \tau\rangle$ and for each $x \in L_*$, there is $y \in L_+$ such that $x \subseteq y$.*

PROOF. Since $\mathcal{S} = \{\varphi(a) : a \in L\} \cup \{\varphi(b)^c : b \in L\}$ is a subbasis for τ and $\{\varphi(a) : a \in L\}$ is closed under finite intersections, an element of the basis for τ that $\mathcal{S}$ generates has the form $\varphi(a) \cap \varphi(b_1)^c \cap \ldots \cap \varphi(b_n)^c$ for some $a, b_1, \ldots, b_n \in L$. If $\varphi(a) \cap \varphi(b_1)^c \cap \ldots \cap \varphi(b_n)^c \neq \emptyset$, then $\varphi(a) \not\subseteq \bigcup_{i=1}^n \varphi(b)$. Therefore, by Lemma 4.11, $\bigcap_{i=1}^n {\uparrow} b_i \not\subseteq {\uparrow} a$. Thus, by Lemma 3.2, $\sigma(a) \not\subseteq \bigcup_{i=1}^n \sigma(b_i)$. Hence, there is $y \in L_+$ such that $a \in y$ and $b_1, \ldots, b_n \notin y$. It follows that $\varphi(a) \cap \varphi(b_1)^c \cap \ldots \varphi(b_n)^c \cap L_+ \neq \emptyset$, and so L_+ is dense in L_*. To conclude the proof, let $x \in L_* = \mathrm{Opt}(L)$. Then $L - x \neq \emptyset$. Let $a \in L - x$. Then $x \cap {\downarrow} a = \emptyset$, and by the prime filter lemma, there is $y \in \mathrm{Pr}(L)$ such that $x \subseteq y$ and $a \notin y$. ■

LEMMA 5.2. *Each open upset of L_* is a union of elements of $\mathfrak{B}_L$.*

PROOF. Let U be an open upset of L_* and let $x \in U$. It is sufficient to find $a \in L$ such that $x \in \varphi(a) \subseteq U$. For each $y \notin U$ we have $x \not\subseteq y$. Therefore, there is $a_y \in L$ such that $a_y \in x$ and $a_y \notin y$. Thus, $\bigcap\{\varphi(a_y) : y \notin U\} \cap U^c = \emptyset$. This by compactness of L_* implies that there exist $a_1, \ldots, a_n \in L$ such that $\varphi(a_1) \cap \cdots \cap \varphi(a_n) \cap U^c = \emptyset$. Moreover, $x \in \varphi(a_i)$ for each $i \leq n$. Therefore, $x \in \varphi(a_1 \wedge \cdots \wedge a_n) \subseteq U$, and so there exists $a = a_1 \wedge \cdots \wedge a_n$ in L with $x \in \varphi(a) \subseteq U$. ■

Let $D(\mathfrak{B}_L)$ denote the distributive lattice generated (in $\mathrm{Up}(L_*)$) by $\mathfrak{B}_L$. Then $A \in D(\mathfrak{B}_L)$ iff A is a finite union of elements of $\mathfrak{B}_L$. It follows easily that $D(\mathfrak{B}_L)$ is isomorphic to $D(L)$. Let $\mathfrak{CU}(L_*)$ denote the lattice of clopen upsets of L_*. This lattice is isomorphic to the lattice of clopen upsets of the Priestley space of $D(L)$. Moreover, by Lemma 5.2, each element of $\mathfrak{CU}(L_*)$ is a union of elements of $\mathfrak{B}_L$, and since each element of $\mathfrak{CU}(L_*)$ is compact, it is a finite union of elements of $\mathfrak{B}_L$. Thus, $D(\mathfrak{B}_L) = \mathfrak{CU}(L_*)$, and so the lattice of clopen upsets of L_* is also isomorphic to $D(L)$.

Let $\langle X, \tau, \leq\rangle$ be a Priestley space and let X_0 be a dense subset of X. For a clopen subset U of X, we say that X_0 is *cofinal in* U if $\max(U) \subseteq X_0$. We call a clopen upset U of X X_0-*admissible* if X_0 is cofinal in U^c. We note that a clopen upset U is X_0-admissible iff $\max(U^c) \subseteq X_0$, which happens iff $U^c = {\downarrow}(X_0 - U)$. We denote by X^* the set of all X_0-admissible clopen upsets of X and for $x \in X$ we let

$$\mathcal{I}_x = \{U : x \notin U \text{ and } U \text{ is an } X_0\text{-admissible clopen upset of } X\}.$$

PROPOSITION 5.3. *Let L be a bounded distributive meet-semilattice and let U be a clopen upset of L_*. Then U is L_+-admissible iff there exists $a \in L$ such that $U = \varphi(a)$.*

PROOF. Suppose that $U = \varphi(a)$ with $a \in L$. Since $L_+ \subseteq L_*$ and $\varphi(a)$ is an upset, $L_* - \varphi(a)$ is a downset, so ${\downarrow}(L_+ - \varphi(a)) \subseteq L_* - \varphi(a)$. Conversely, if $x \in L_* - \varphi(a)$, then $x \notin \varphi(a)$. Therefore, $a \notin x$. So $x \cap {\downarrow}a = \emptyset$, and by the prime filter lemma, there is $y \in \mathrm{Pr}(L)$ such that $x \subseteq y$ and $a \notin y$. Thus, $x \subseteq y$ and $y \in L_+ - \varphi(a)$, implying that $x \in {\downarrow}(L_+ - \varphi(a))$. It follows that $L_* - \varphi(a) = {\downarrow}(L_+ - \varphi(a))$. Now if $x \in \max(L_* - \varphi(a))$. Then $x \in {\downarrow}(L_+ - \varphi(a))$, and as x is a maximal point of $L_* - \varphi(a)$, we have $x \in L_+ - \varphi(a) \subseteq L_+$. Thus, U is L_+-admissible. Conversely, let U be a L_+-admissible clopen upset of L_*. Then $\max(L_* - U) \subseteq L_+$ and there exist $a_1, \dots a_n \in L$ such that $U = \varphi(a_1) \cup \dots \cup \varphi(a_n)$. Let F be the filter $\bigcap_{i=1}^n {\uparrow}a_i$ and let I be the Frink ideal generated by $a_1, \dots, a_n$. If $F \cap I = \emptyset$, then, by the optimal filter lemma, there exists an optimal filter x of L such that $F \subseteq x$ and $x \cap I = \emptyset$. Since $a_i \in I$, we have $a_i \notin x$ for each $i \le n$, so $x \notin \varphi(a_1) \cup \dots \cup \varphi(a_n) = U$. Therefore, $x \in L_* - U$. Since $\langle L_*, \tau, \subseteq\rangle$ is a Priestley space and $L_* - U$ is a closed subset of L_*, there exists $y \in \max(L_* - U)$ such that $x \subseteq y$. But then $y \in L_+$ and $y \notin U$. Moreover, $\bigcap_{i=1}^n {\uparrow}a_i \subseteq x \subseteq y$ and as y is prime, there is $i \le n$ such that ${\uparrow}a_i \subseteq y$. Hence, $y \in \varphi(a_i) \subseteq U$, which is a contradiction. Therefore, there is $a \in F \cap I$. Thus, $a \in \bigcap_{i=1}^n {\uparrow}a_i$ and $\bigcap_{i=1}^n {\uparrow}a_i \subseteq {\uparrow}a$. So $\bigcap_{i=1}^n {\uparrow}a_i = {\uparrow}a$, $a = a_1 \vee \dots \vee a_n$, and so $\varphi(a) = \varphi(a_1) \cup \dots \cup \varphi(a_n) = U$. ■

It follows from Proposition 5.3 that for each clopen downset V of L_*, the following three conditions are equivalent: (a) $V = \varphi(a)^c$ for some $a \in L$, (b) $V = {\downarrow}(L_+ \cap V)$, and (c) $\max(V) \subseteq L_+$. We also have that

$$\mathfrak{B}_L = \{U \in \mathfrak{CU}(L_*) : U \text{ is } L_+\text{-admissible}\}.$$

PROPOSITION 5.4. *Let L be a bounded distributive meet-semilattice. Then $x \in L_+$ iff $\langle \mathcal{I}_x, \subseteq\rangle$ is updirected.*

PROOF. Let $x \in L_+$ and let $\varphi(a), \varphi(b) \in \mathcal{I}_x$. Then $x \notin \varphi(a), \varphi(b)$. Therefore, $a, b \notin x$. Since x is a prime filter of L, it follows that ${\uparrow}a \cap {\uparrow}b \not\subseteq x$. Thus, there exists $c \in {\uparrow}a \cap {\uparrow}b$ such that $c \notin x$. Consequently, $\varphi(a) \cup \varphi(b) \subseteq \varphi(c)$

and $x \notin \varphi(c)$. So $\varphi(c) \in \mathcal{I}_x$, and so $\mathcal{I}_x$ is updirected. Conversely, suppose that $\mathcal{I}_x$ is updirected. We show that $x \in L_+$. If not, then there exist filters F_1 and F_2 of L such that $F_1 \cap F_2 \subseteq x$ but $F_1 \not\subseteq x$ and $F_2 \not\subseteq x$. Let $a_1 \in F_1 - x$ and $a_2 \in F_2 - x$. Then $x \notin \varphi(a_1), \varphi(a_2)$, and so $\varphi(a_1), \varphi(a_2) \in \mathcal{I}_x$. Since $\mathcal{I}_x$ is updirected, there exists $\varphi(a) \in \mathcal{I}_x$ such that $\varphi(a_1) \cup \varphi(a_2) \subseteq \varphi(a)$. From $\varphi(a) \in \mathcal{I}_x$ it follows that $a \notin x$, and from $\varphi(a_1) \cup \varphi(a_2) \subseteq \varphi(a)$ it follows that $a \in {\uparrow}a_1 \cap {\uparrow}a_2 \subseteq F_1 \cap F_2 \subseteq x$. The obtained contradiction proves that $x \in L_+$. ■

The results we have established about the dual space of a bounded distributive meet-semilattice L justify the following definition of a generalized Priestley space. Let $\langle X, \tau, \leq \rangle$ be a Priestley space and X_0 be a dense subset of X. We let X^* denote the set of X_0-admissible clopen upsets of X.

DEFINITION 5.5. A quadruple $X = \langle X, \tau, \leq, X_0 \rangle$ is a *generalized Priestley space* if:

1. $\langle X, \tau, \leq \rangle$ is a Priestley space.
2. X_0 is a dense subset of X.
3. For each $x \in X$ there is $y \in X_0$ such that $x \leq y$.
4. $x \in X_0$ iff $\mathcal{I}_x$ is updirected.
5. For all $x, y \in X$, we have $x \leq y$ iff $(\forall U \in X^*)(x \in U \Rightarrow y \in U)$.

REMARK 5.6. Condition (3) of Definition 5.5 is equivalent to $\max X \subseteq X_0$, which means that $\emptyset$ is X_0-admissible. Also, when in a generalized Priestley space X we have $X_0 = X$, then $X^* = \mathfrak{CU}(X)$, so conditions (2)–(5) of Definition 5.5 become redundant, and so X becomes a Priestley space. Thus, the notion of a generalized Priestley space generalizes that of a Priestley space.

PROPOSITION 5.7. *Let L be a bounded distributive meet-semilattice L. Then the quadruple $L_* = \langle L_*, \tau, \subseteq, L_+ \rangle$ is a generalized Priestley space.*

PROOF. For optimal filters x and y we clearly have that $x \subseteq y$ iff $x \in \varphi(a)$ implies $y \in \varphi(a)$ for each $a \in L$. Now apply Lemma 5.1 and Proposition 5.4. ■

Since for each bounded distributive meet-semilattice L we have ${L_*}^* = \varphi[L]$, we immediately obtain:

THEOREM 5.8 (Representation Theorem). *For each bounded distributive meet-semilattice L, there exists a generalized Priestley space X such that L is isomorphic to X^*.*

Proposition 5.9. *Let X be a generalized Priestley space. Then $X^* = \langle X^*, \cap, X, \emptyset\rangle$ is a bounded distributive meet-semilattice.*

Proof. First we show that X^* is closed under $\cap$. If $U, V \in X^*$, then $\max((U \cap V)^c) = \max(U^c \cup V^c) \subseteq \max(U^c) \cup \max(V^c) \subseteq X_0$. Thus, $U \cap V \in X^*$. Next $\max(X^c) = \max(\emptyset) = \emptyset \subseteq X_0$, so $X \in X^*$. Also, by condition (3) of Definition 5.5, $\max(\emptyset^c) = \max(X) \subseteq X_0$, and so $\emptyset \in X^*$. Lastly we show that $\langle X^*, \cap\rangle$ is distributive. Let $U, V, W \in X^*$ with $U \cap V \subseteq W$. Then $W^c \subseteq U^c \cup V^c$. For each $x \in \max(W^c)$ we have that $x \in U^c$ or $x \in V^c$. Therefore, $W \in \mathcal{I}_x$ and either $U \in \mathcal{I}_x$ or $V \in \mathcal{I}_x$. Since $x \in X_0$, by condition (4) of Definition 5.5, from $W, U \in \mathcal{I}_x$ it follows that there exists $U_x \in \mathcal{I}_x$ such that $W \cup U \subseteq U_x$; and from $W, V \in \mathcal{I}_x$ it follows th at there exists $V_x \in \mathcal{I}_x$ such that $W \cup V \subseteq V_x$. Thus, $W^c = \bigcup\{K_x : x \in \max(W^c)\}$, where $K_x = {U_x}^c$ or $K_x = {V_x}^c$. Since W^c is compact and each K_x is open, there exist finite subsets A, B of $\max(W^c)$ such that $W^c = \bigcup\{{U_x}^c : x \in A\} \cup \bigcup\{{V_x}^c : x \in B\}$. Let $U' = \bigcap\{U_x : x \in A\}$ and $V' = \bigcap\{V_x : x \in B\}$. Clearly $U \subseteq U'$, $V \subseteq V'$, and $U, V \in X^*$. Moreover, $W^c = \bigcup\{{U_x}^c : x \in A\} \cup \bigcup\{{V_x}^c : x \in B\}$ implies $W = U' \cap V'$. Thus, $\langle X^*, \cap\rangle$ is distributive. Consequently, $\langle X^*, \cap, X, \emptyset\rangle$ is a bounded distributive meet-semilattice. ∎

The proof of the next proposition can be found in [1, Prop. 6.15, Lem. 6.17, and Cor. 6.16 and 6.18].

Proposition 5.10. *Let X be a generalized Priestley space. Then:*

1. *The closure of X^* under finite unions is $\mathfrak{CU}(X)$.*
2. *The family $X^* \cup \{U^c : U \in X^*\}$ is a subbasis for the topology on X.*
3. *Let $U, U_1, \ldots, U_n \in X^*$. Then $\bigcap_{i=1}^n {\uparrow}U_i \subseteq {\uparrow}U$ iff $U \subseteq \bigcup_{i=1}^n U_i$.*
4. *$\mathfrak{CU}(X)$ is isomorphic to $D(X^*)$.*

Let X be a generalized Priestley space. We define $\varepsilon : X \to {X^*}_*$ by

$$\varepsilon(x) = \{U \in X^* : x \in U\}.$$

First we show that ε is well-defined.

Proposition 5.11. *Let X be a generalized Priestley space. For each $x \in X$, we have $\varepsilon(x)$ is an optimal filter of X^*. Moreover, if $x \in X_0$, then $\varepsilon(x)$ is a prime filter of X^*.*

PROOF. Let X be a generalized Priestley space. By Proposition 5.9, $\langle X^*, \cap, X, \emptyset\rangle$ is a bounded distributive meet-semilattice. Clearly $\varepsilon(x)$ is a filter of X^* and $X^* - \varepsilon(x)$ is nonempty. We show that $X^* - \varepsilon(x)$ is an F-ideal of X^*. Suppose that $U_1, \ldots, U_n \in X^* - \varepsilon(x)$, $U \in X^*$, and $\bigcap_{i=1}^n \uparrow U_i \subseteq \uparrow U$. By Proposition 5.10, $U \subseteq \bigcup_{i=1}^n U_i$. Since $x \notin \bigcup_{i=1}^n U_i$, it follows that $x \notin U$. Therefore, $U \in X^* - \varepsilon(x)$, so $X^* - \varepsilon(x)$ is an F-ideal, and so, by Proposition 4.8, $\varepsilon(x)$ is an optimal filter. Now suppose that $x \in X_0$ and that $\varepsilon(x)$ is not a prime filter of X^*. Then there exist filters F_1 and F_2 of X^* such that $F_1 \cap F_2 \subseteq \varepsilon(x)$, but $F_1 \not\subseteq \varepsilon(x)$ and $F_2 \not\subseteq \varepsilon(x)$. Let $U_1 \in F_1 - \varepsilon(x)$ and $U_2 \in F_2 - \varepsilon(x)$. Then $x \notin U_1, U_2$, and so $U_1, U_2 \in \mathcal{I}_x$. By condition (4) of Definition 5.5, there exists $V \in \mathcal{I}_x$ such that $U_1 \cup U_2 \subseteq V$. Hence, $V \in \uparrow U_1 \cap \uparrow U_2 \subseteq F_1 \cap F_2 \subseteq \varepsilon(x)$. Thus, $x \in V$, a contradiction. We conclude that $\varepsilon(x)$ is a prime filter of X^*. ■

PROPOSITION 5.12. *The map $\varepsilon : X \to X^*{}_*$ is 1-1 and onto. Moreover, if P is a prime filter of X^*, then $\varepsilon^{-1}(P) \in X_0$.*

PROOF. It follows from condition (5) of Definition 5.5 that ε is 1-1. We show that ε is onto. Suppose that P is an optimal filter of X^*. Let $I = X^* - P$. By Proposition 4.8, I is an F-ideal of X^*. Let G be the filter of $\mathfrak{CU}(X)$ generated by P and J be the ideal of $\mathfrak{CU}(X)$ generated by I. We claim that $G \cap J = \emptyset$. If not, then there exist $V \in \mathfrak{CU}(X)$, $U \in P$, and $U_1, \ldots, U_n \in I$ such that $U \subseteq V$ and $V \subseteq U_1 \cup \ldots \cup U_k$. By Proposition 5.10, $\bigcap_{i=1}^n \uparrow U_i \subseteq \uparrow V \subseteq \uparrow U$. Since I is an F-ideal, we have $U \in I$, so $U \notin P$, a contradiction. Thus, by the prime filter lemma, there is a prime filter F of $\mathfrak{CU}(X)$ such that $G \subseteq F$ and $F \cap J = \emptyset$. By Priestley duality, there exists $x \in X$ such that $F = \{U \in \mathfrak{CU}(X) : x \in U\}$. We show that $P = F \cap X^*$. It is clear that $P \subseteq F \cap X^*$. Conversely, if $U \in F \cap X^*$ and $U \notin P$, then $U \in I$, which is a contradiction since F is disjoint from I. Thus, $P = F \cap X^* = \{U \in X^* : x \in U\}$. Consequently $\varepsilon(x) = P$, and so ε is onto.

Now suppose that P is a prime filter of X^*. Since ε is onto, there exists $x \in X$ such that $\varepsilon(x) = P$. If $x \notin X_0$, then by condition (4) of Definition 5.5, $\mathcal{I}_x$ is not updirected, so there exist $U, V \in \mathcal{I}_x$ such that for no $W \in \mathcal{I}_x$ we have $U \cup V \subseteq W$. Therefore, for each $W \in X^*$, from $W \in \uparrow U \cap \uparrow V$ it follows that $x \in W$, and so $W \in \varepsilon(x)$. Thus, $\uparrow U \cap \uparrow V \subseteq \varepsilon(x)$, and as $\varepsilon(x)$ is a prime filter of X^*, $\uparrow U \subseteq \varepsilon(x)$ or $\uparrow V \subseteq \varepsilon(x)$. Consequently, $U \in \varepsilon(x)$ or $V \in \varepsilon(x)$, and so $x \in U$ or $x \in V$, a contradiction. Thus, $x \in X_0$. ■

THEOREM 5.13. *For a generalized Priestley space X, the map $\varepsilon : X \to X^*{}_*$ is an order-homeomorphism. Moreover, $\varepsilon[X_0] = X^*{}_+$.*

PROOF. It follows from Propositions 5.11 and 5.12 that ε is a bijection and that $\varepsilon[X_0] = X^*{}_+$. Condition (5) of Definition 5.5 implies that for each

$x, y \in X$, we have $x \leq y$ iff $\varepsilon(x) \subseteq \varepsilon(y)$. Thus, ε is an order-isomorphism. We show that ε is a homeomorphism. By Proposition 5.10, $X^* \cup \{U^c : U \in X^*\}$ is a subbasis for the topology on X, and $\{\varphi(U) : U \in X^*\} \cup \{\varphi(U)^c : U \in X^*\}$ is a subbasis for the topology on $X^*{}_*$. For $U \in X^*$ we have:

$$x \in \varepsilon^{-1}(\varphi(U)) \text{ iff } \varepsilon(x) \in \varphi(U) \text{ iff } U \in \varepsilon(x) \text{ iff } x \in U.$$

Thus, $\varepsilon^{-1}(\varphi(U)) = U$ and $\varepsilon^{-1}(\varphi(U)^c) = U^c$. It follows that ε is continuous. Now since ε is a continuous map between compact Hausdorff spaces, ε is a homeomorphism. ■

6. Categorical equivalences

In this section we introduce the second main concept of the paper, that of *generalized Priestley morphism*, and extend our representation theorem of the previous section to a full duality between the categories of bounded distributive meet-semilattices and generalized Priestley spaces.

Let X and Y be nonempty sets. Given a relation $R \subseteq X \times Y$, for each $A \subseteq Y$ we define

$$\Box_R A = \{x \in X : (\forall y \in Y)(xRy \Rightarrow y \in A)\} = \{x \in X : R[x] \subseteq A\}.$$

It is easy to verify that $\Box_R(Y) = X$ and that $\Box_R(A \cap B) = \Box_R A \cap \Box_R B$ for each $A, B \subseteq Y$.

Let L and K be bounded distributive meet-semilattices and let $h : L \to K$ be a meet-semilattice homomorphism preserving top. We define $R_h \subseteq K_* \times L_*$ by

$$xR_h y \text{ iff } h^{-1}(x) \subseteq y$$

for each $x \in K_*$ and $y \in L_*$. We call R_h the *dual* of h.

PROPOSITION 6.1. *Let L and K be bounded distributive meet-semilattices and let $h : L \to K$ be a meet-semilattice homomorphism preserving top. Then:*

1. *If $x \not R_h y$, then there is $a \in L$ such that $y \notin \varphi(a)$ and $R_h[x] \subseteq \varphi(a)$.*
2. $\varphi(h(a)) = \Box_{R_h} \varphi(a)$.

PROOF. To prove (1) suppose that $x \not R_h y$. Then $h^{-1}(x) \not\subseteq y$, so there is $a \in L$ such that $a \in h^{-1}(x)$ and $a \notin y$. Therefore, $y \notin \varphi(a)$, and if $xR_h z$, then $a \in z$. Thus, $R_h[x] \subseteq \varphi(a)$. To prove (2) let $x \in \varphi(h(a))$, then $a \in h^{-1}(x)$. Therefore, for each $z \in L_*$ with $xR_h z$, we have $a \in z$. Thus, $R_h[x] \subseteq \varphi(a)$, and so $\varphi(h(a)) \subseteq \Box_{R_h} \varphi(a)$. Conversely, if $x \in \Box_{R_h} \varphi(a)$, then

$R_h[x] \subseteq \varphi(a)$. If $x \notin \varphi(h(a))$, then $a \notin h^{-1}(x)$. So $h^{-1}(x) \cap {\downarrow}a = \emptyset$, and by the prime filter lemma, there exists $y \in L_+ \subseteq L_*$ such that $h^{-1}(x) \subseteq y$ and $a \notin y$. But $h^{-1}(x) \subseteq y$ implies $y \in R_h[x]$, so $a \in y$, which is a contradiction. We conclude that $x \in \varphi(h(a))$. Thus, $\Box_{R_h}\varphi(a) \subseteq \varphi(h(a))$, and so $\varphi(h(a)) = \Box_{R_h}\varphi(a)$. ■

Definition 6.2. Let X and Y be generalized Priestley spaces. A relation $R \subseteq X \times Y$ is called a *generalized Priestley morphism* if the following two conditions are satisfied:

1. If $x \not R y$, then there is $U \in Y^*$ such that $y \notin U$ and $R[x] \subseteq U$.
2. If $U \in Y^*$, then $\Box_R U \in X^*$.

The proof of the next lemma can be found in [1, Lem. 8.3].

Lemma 6.3. *Let X and Y be generalized Priestley spaces and $R \subseteq X \times Y$ be a generalized Priestley morphism. Then $(\leq_X \circ R) \subseteq R$ and $(R \circ \leq_Y) \subseteq R$.*

Remark 6.4. It is easy to verify that the first condition of Lemma 6.3 is equivalent to saying that for each $B \subseteq Y$ we have $R^{-1}[B]$ is a downset of X, that the second one is equivalent to saying that for each $A \subseteq X$ we have $R[A]$ is an upset of Y, and that together they are equivalent to $(\leq_X \circ R \circ \leq_Y) \subseteq R$.

Given a generalized Priestley morphism $R \subseteq X \times Y$, we define the map $h_R : Y^* \to X^*$ by

$$h_R(U) = \Box_R U$$

for each $U \in Y^*$.

Lemma 6.5. *If $R \subseteq X \times Y$ is a generalized Priestley morphism, then $h_R : Y^* \to X^*$ is a meet-semilattice homomorphism preserving top.*

Proof. Let $U, V \in Y^*$. Then $h_R(U \cap V) = \Box_R(U \cap V) = \Box_R U \cap \Box_R V = h_R(U) \cap h_R(V)$. Moreover, $h_R(Y) = \Box_R Y = X$. ■

Proposition 6.6. *Let L and K be bounded distributive meet-semilattices and let $h : L \to K$ be a meet-semilattice homomorphism preserving top. Then for each $a \in L$ we have $\varphi(h(a)) = h_{R_h}(\varphi(a))$.*

Proof. We have $x \in \varphi(h(a))$ iff $h(a) \in x$ iff $a \in h^{-1}(x)$, and $x \in h_{R_h}\varphi(a)$ iff $x \in \Box_{R_h}\varphi(a)$ iff $(\forall y \in L_*)(x R_h y \Rightarrow a \in y)$ iff $(\forall y \in L_*)(h^{-1}(x) \subseteq y \Rightarrow a \in y)$. Now either $h^{-1}(x) = L$ or $h^{-1}(x)$ is a proper filter of L. If $h^{-1}(x) = L$, then for all $y \in L_*$ we have $h^{-1}(x) \not\subseteq y$. Therefore, both $a \in h^{-1}(x)$ and $(\forall y \in L_*)(h^{-1}(x) \subseteq y \Rightarrow a \in y)$ are trivially true, and so $a \in h^{-1}(x)$ iff $(\forall y \in L_*)(h^{-1}(x) \subseteq y \Rightarrow a \in y)$. On the other hand, if $h^{-1}(x)$ is a proper filter of L, then by the optimal filter lemma, $h^{-1}(x)$ is the intersection of all the optimal filters of L containing $h^{-1}(x)$. Hence, $a \in h^{-1}(x)$ iff $(\forall y \in L_*)(h^{-1}(x) \subseteq y \Rightarrow a \in y)$. Thus, in either case we have $\varphi(h(a)) = h_{R_h}(\varphi(a))$. ∎

Proposition 6.7. *Let $R \subseteq X \times Y$ be a generalized Priestley morphism. Then for each $x \in X$ and $y \in Y$ we have xRy iff $\varepsilon(x)R_{h_R}\varepsilon(y)$.*

Proof. Let xRy. We show $\varepsilon(x)R_{h_R}\varepsilon(y)$. If $U \in h_R^{-1}(\varepsilon(x))$, then $h_R(U) \in \varepsilon(x)$, so $x \in h_R(U)$, and so $R[x] \subseteq U$. Therefore, $y \in U$, so $U \in \varepsilon(y)$, and so $h_R^{-1}(\varepsilon(x)) \subseteq \varepsilon(y)$. Thus, $\varepsilon(x)R_{h_R}\varepsilon(y)$. Now let $x \not R y$. Then, by condition (1) of Definition 6.2, there is $U \in Y^*$ such that $y \notin U$ and $R[x] \subseteq U$. Therefore, $y \notin U$ and $x \in h_R(U)$. Thus, we have $U \notin \varepsilon(y)$ and $h_R(U) \in \varepsilon(x)$. It follows that $h_R^{-1}(\varepsilon(x)) \not\subseteq \varepsilon(y)$. Consequently, xRy iff $\varepsilon(x)R_{h_R}\varepsilon(y)$. ∎

Unfortunately, the usual set-theoretic composition of two generalized Priestley morphisms may not be a generalized Priestley morphism. Therefore, we introduce the composition of two generalized Priestley morphisms as follows. Let X, Y, and Z be generalized Priestley spaces, and $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be generalized Priestley morphisms. Define $S{*}R \subseteq X \times Z$ by

$$x(S{*}R)z \quad \text{iff} \quad (\forall U \in Z^*)((S \circ R)[x] \subseteq U \Rightarrow z \in U),$$

where $S \circ R$ is the usual set-theoretic composition of R and S. Note that $S \circ R \subseteq S{*}R$, and if $S \circ R$ is already a generalized Priestley morphism, then $S{*}R = S \circ R$. We also have that

$$x(S{*}R)z \quad \text{iff} \quad (\forall U \in Z^*)(x \in \Box_R \Box_S U \Rightarrow z \in U)$$

and that

$$x(S{*}R)z \quad \text{iff} \quad \varepsilon(x)R_{(h_R \circ h_S)}\varepsilon(z).$$

Moreover,

$$\Box_R \Box_S U = (h_R \circ h_S)(U) = \Box_{(S*R)} U.$$

To see this suppose that $x \in (h_R \circ h_S)(U)$. Then $(h_R \circ h_S)(U) \in \varepsilon(x)$ and so $U \in (h_R \circ h_S)^{-1}[\varepsilon(x)]$, which means that $(\forall z \in Z)(\varepsilon(x)R_{(h_R \circ h_S)}\varepsilon(z) \Rightarrow z \in U)$. Therefore, $(\forall z \in Z)(x(S{*}R)z \Rightarrow z \in U)$. Thus, $(S{*}R)[x] \subseteq U$ and $x \in \Box_{(S*R)}U$. Reasoning backwards we also obtain that if $x \in \Box_{(S*R)}U$, then $x \in (h_R \circ h_S)(U)$.

Lemma 6.8. *If X, Y, and Z are generalized Priestley spaces and $R \subseteq X \times Y$, $S \subseteq Y \times Z$ are generalized Priestley morphisms, then $S{*}R \subseteq X \times Z$ is a generalized Priestley morphism.*

PROOF. To prove that condition (1) of Definition 6.2 holds, let $(x, y) \notin S{*}R$. Then there is $U \in Z^*$ such that $x \in \Box_R \Box_S U$ and $z \notin U$. By the above observation, we have $(S{*}R)[x] \subseteq U$ and $z \notin U$. To see that condition (2) of Definition 6.2 is also satisfied, let $U \in Z^*$. Then $(h_R \circ h_S)(U) \in X^*$, which, by the above observation, means that $\Box_{(S*R)} U \in X^*$. Thus, $S{*}R$ is a generalized Priestley morphism. ■

It is not difficult to see that the composition operation $*$ is associative and that for each generalized Priestley space X, the relation $\leq_X \subseteq X \times X$ is a generalized Priestley morphism such that for each generalized Priestley space Y and each generalized Priestley morphisms $R \subseteq X \times Y$ and $S \subseteq Y \times X$, we have $R \circ \leq_X = R$ and $\leq_X \circ S = S$.

As an immediate consequence of these facts and Lemma 6.8 we obtain that generalized Priestley spaces and generalized Priestley morphisms form a category, in which $*$ is the composition of two morphisms and $\leq_X$ is the identity morphism for each object X. We denote this category by GPS. Let also BDM denote the category of bounded distributive meet-semilattices and meet-semilattice homomorphisms preserving top.

We are ready to prove one of the main theorems of the paper, that BDM is dually equivalent to GPS.

THEOREM 6.9. *The category* BDM *is dually equivalent to the category* GPS.

PROOF. Define the functors $(-)_* : \mathsf{BDM} \to \mathsf{GPS}$ and $(-)^* : \mathsf{GPS} \to \mathsf{BDM}$ as follows. For a bounded distributive meet-semilattice L, set $L_* = \langle L_*, \tau, \subseteq, L_+ \rangle$, and for a meet-semilattice homomorphism h preserving top, set $h_* = R_h$; also for a generalized Priestley space X, let X^* be the bounded distributive meet-semilattice of X_0-admissible clopen upsets of X, and for a generalized Priestley morphism R, let $R^* = h_R$.

In order to prove that the functors $(-)_*$ and $(-)^*$ establish a dual equivalence of BDM and GPS, we define the natural transformations from the identity functor $\mathrm{id}_{\mathsf{BDM}} : \mathsf{BDM} \to \mathsf{BDM}$ to the functor $(-)_*{}^* : \mathsf{BDM} \to \mathsf{BDM}$ and from the identity functor $\mathrm{id}_{\mathsf{GPS}} : \mathsf{GPS} \to \mathsf{GPS}$ to the functor $(-)^*{}_* : \mathsf{GPS} \to \mathsf{GPS}$.

The first natural transformation associates with each object L of BDM the isomorphism $\varphi_L : L \to L_*{}^*$; and the second natural transformation associates with each object X of GPS the generalized Priestley morphism $R_{\varepsilon_X} \subseteq X \times X^*{}_*$ given by

$$x R_{\varepsilon_X} \varepsilon(y) \quad \text{iff} \quad \varepsilon_X(x) \subseteq \varepsilon_X(y)$$

for each $x, y \in X$. The result follows. ■

Now we turn our attention to meet-semilattice homomorphisms preserving bottom and to sup-homomorphisms.

LEMMA 6.10. *Let L and K be bounded distributive meet-semilattices and let $h : L \to K$ be a meet-semilattice homomorphism preserving top. Then:*

1. *h preserves bottom iff ${R_h}^{-1}[L_*] = K_*$.*
2. *h is a sup-homomorphism iff $R_h[x]$ has a least element for each $x \in K_*$.*

PROOF. (1) First note that $h(\bot) = \bot$ iff $\varphi(h(\bot)) = \varphi(\bot)$, and by Proposition 6.6, the last condition holds iff $h_{R_h}(\varphi(\bot)) = \varphi(\bot)$, which holds iff $h_{R_h}(\emptyset) = \emptyset$. This last condition is equivalent to $R_h^{-1}[L_*] = K_*$.

(2) For each $x \in K_*$, we show that $R_h[x]$ has a least element iff $h^{-1}[x] \in L_*$. If $h^{-1}[x] \in L_*$, then it is clear that $h^{-1}[x]$ is the least element of $R_h[x]$. Conversely, let y be the least element of $R_h[x]$. By the optimal filter lemma, $h^{-1}[x] = \bigcap\{z \in L_* : h^{-1}[x] \subseteq z\} = \bigcap R_h[x] = y$. Therefore, $h^{-1}[x] \in L_*$. By Proposition 3.5 (see also [1, Prop. 5.2]), h is a sup-homomorphism iff $h^{-1}[x] \in L_*$ for each $x \in K_*$. Thus, h is a sup-homomorphism iff $R_h[x]$ has a least element for each $x \in K_*$. ■

DEFINITION 6.11. Let X and Y be generalized Priestley spaces and let $R \subseteq X \times Y$ be a generalized Priestley morphism.

1. We call R *total* if $R^{-1}[Y] = X$.
2. We call R *functional* if for each $x \in X$ there is $y \in Y$ such that $R[x] = {\uparrow}y$.

Obviously R is functional iff $R[x]$ has a least element. It is also clear that each functional generalized Priestley morphism is total. As an immediate consequence of Theorem 6.9 and Lemma 6.10, we obtain:

COROLLARY 6.12. *Let X and Y be generalized Priestley spaces and $R \subseteq X \times Y$ be a generalized Priestley morphism. Then:*

1. *h_R preserves bottom iff R is total.*
2. *h_R is a sup-homomorphism iff R is functional.*

In particular, it follows that each sup-homomorphism preserves bottom. It is not difficult to show that if X, Y, and Z are generalized Priestley spaces, and $R \subseteq X \times Y$, $S \subseteq Y \times Z$ are total generalized Priestley morphisms, then so is $S{*}R$; and if R, S are functional, then so is $S{*}R$. In this last case, the set-theoretic composition $S \circ R$ is already a functional generalized Priestley morphism, and so $S * R = S \circ R$.

Let GPS^T denote the category of generalized Priestley spaces and total generalized Priestley morphisms. This category is obviously a proper subcategory of GPS. Let also GPS^F denote the category of generalized Priestley spaces and functional generalized Priestley morphisms, which is clearly a proper subcategory of GPS^T.

We let BDM^B denote the category of bounded distributive meet-semilattices and bounded meet-semilattice homomorphisms, and BDM^S denote the category of bounded distributive meet-semilattices and sup-homomorphisms. Similarly, we have that BDM^S is a proper subcategory of BDM^B, and that BDM^B is a proper subcategory of BDM.

By restricting the functors $(-)_*$ and $(-)^*$ to the appropriate categories, from the results obtained above it immediately follows that BDM^B is dually equivalent to GPS^T and BDM^S is dually equivalent to GPS^F.

7. Functional morphisms

In this section we show that functional generalized Priestley morphisms can be characterized by means of special functions between generalized Priestley spaces, which we call strong Priestley morphisms. Let X and Y be Priestley spaces. We recall that a map $f : X \to Y$ is a Priestley morphism if f is continuous and order-preserving.

Definition 7.1. Let X and Y be generalized Priestley spaces. We call a map $f : X \to Y$ a *strong Priestley morphism* if f is order-preserving and $U \in Y^*$ implies $f^{-1}(U) \in X^*$.

Since $X^* \cup \{U^c : U \in X^*\}$ and $Y^* \cup \{V^c : V \in Y^*\}$ form subbases for the Priestley topologies on X and Y, respectively, and $f^{-1}(V^c) = f^{-1}(V)^c$ for each $V \subseteq Y$, it follows that each strong Priestley morphism is a continuous function, hence a Priestley morphism. We note that the composition of strong Priestley morphisms is again a strong Priestley morphism, and that the identity map $\mathrm{id}_X : X \to X$ is a strong Priestley morphism. Therefore, generalized Priestley spaces and strong Priestley morphisms form a category in which composition is the usual set-theoretic composition of functions and the identity morphism is the usual identity function. We denote this category by GPS^S.

Let X and Y be generalized Priestley spaces and $R \subseteq X \times Y$ be a functional generalized Priestley morphism. We define $f^R : X \to Y$ by

$$f^R(x) = \text{the least element of } R[x].$$

It is not difficult to check that if X and Y are generalized Priestley spaces and $R \subseteq X \times Y$ is a functional generalized Priestley morphism, then $f^R : X \to Y$

is a strong Priestley morphism, and that if X, Y, and Z are generalized Priestley spaces and $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ are functional generalized Priestley morphisms, then $f^{S*R} = f^S \circ f^R$ (see [1, Lem. 9.2]).

Now let X and Y be generalized Priestley spaces and $f : X \to Y$ be a strong Priestley morphism. Define $R^f \subseteq X \times Y$ by

$$xR^f y \text{ iff } f(x) \leq y.$$

It is also not difficult to check that if X and Y are generalized Priestley spaces and $f : X \to Y$ is a strong Priestley morphism, then R^f is a functional generalized Priestley morphism, and that if X, Y, and Z are generalized Priestley spaces and $f : X \to Y$ and $g : Y \to Z$ are strong Priestley morphisms, then $R^{g \circ f} = R^g * R^f$ (see [1, Lem. 9.3]).

Moreover, it is also easy to see that if X and Y are generalized Priestley spaces, $R \subseteq X \times Y$ is a functional generalized Priestley morphism, and $f : X \to Y$ is a strong Priestley morphism, then $R^{f^R} = R$ and $f^{R^f} = f$ (see [1, Lem. 9.4]). Therefore we obtain:

Proposition 7.2. *The categories* GPS^F *and* GPS^S *are isomorphic.*

This together with the duality between BDM^S and GPS^F immediately implies that the categories BDM^S and GPS^S are dually equivalent, thus providing us with the distributive meet-semilattice version of Hansoul's duality for distributive join-semilattices and join-semilattice homomorphisms preserving all existing finite meets [8, 9]. An explicit construction of the functors from BDM^S to GPS^S and vice versa can be obtained based on the following observation (see [1, Lem. 9.7]).

Lemma 7.3.

1. *Let X and Y be generalized Priestley spaces and $f : X \to Y$ be a strong Priestley morphism. Then for each $U \in Y^*$ we have $h_{R^f}(U) = f^{-1}(U)$.*
2. *Let L and K be bounded distributive meet-semilattices and $h : L \to K$ be a sup-homomorphism. Then $f^{R_h}(y) = h^{-1}(y)$ for each $y \in K_*$.*

We can define the functors $(-)_\star : \mathsf{BDM}^\mathsf{S} \to \mathsf{GPS}^\mathsf{S}$ and $(-)^\star : \mathsf{GPS}^\mathsf{S} \to \mathsf{BDM}^\mathsf{S}$ explicitly as follows: If L is a bounded distributive meet-semilattice, then $L_\star = L_*$ and if $h : L \to K$ is a sup-homomorphism, then $h_\star = h^{-1}$; also, if X is a generalized Priestley space, then $X^\star = X^*$, and if $f : X \to Y$ is a strong Priestley morphism, then $f^\star = f^{-1}$. Therefore, the functors $(-)_\star : \mathsf{BDM}^\mathsf{S} \to \mathsf{GPS}^\mathsf{S}$ and $(-)^\star : \mathsf{GPS}^\mathsf{S} \to \mathsf{BDM}^\mathsf{S}$ behave exactly like the Priestley functors $(-)_* : \mathsf{BDL} \to \mathsf{PS}$ and $(-)^* : \mathsf{PS} \to \mathsf{BDL}$.

Priestley duality between BDL and PS follows from the duality between BDM^S and GPS^S. Indeed, if L is a bounded distributive lattice, then $L_* =$

L_+, and so $\langle L_*, \tau, \subseteq, L_+\rangle = \langle L_+, \tau, \subseteq\rangle$ is a Priestley space. Conversely, if $X = \langle X, \tau, \leq\rangle$ is a Priestley space, then $X^* = \mathfrak{CU}(X)$. Moreover, given two Priestley spaces X and Y, a map $f : X \to Y$ is a strong Priestley morphism iff f is order-preserving and $V \in \mathfrak{CU}(Y)$ implies $f^{-1}(V) \in \mathfrak{CU}(X)$. Because $\mathfrak{CU}(X) \cup \{U^c : U \in \mathfrak{CU}(X)\}$ and $\mathfrak{CU}(Y) \cup \{V^c : V \in \mathfrak{CU}(Y)\}$ are subbases for the Priestley topologies on X and Y, respectively, the last condition is equivalent to f being continuous. Thus, the notions of a strong Priestley morphism and of a Priestley morphism coincide. If L and K are bounded distributive lattices and $h : L \to K$ is a bounded meet-semilattice homomorphism, then h preserves $\vee$ iff h is a sup-homomorphism. Thus, lattice homomorphisms and sup-homomorphisms coincide. Priestley duality is now an immediate consequence of these observations and the duality between $\mathsf{BDM^S}$ and $\mathsf{GPS^S}$.

We conclude this section by generalizing Priestley duality to cover homomorphisms which do not necessarily preserve $\vee$. These results are easy consequences of our dualities established in the previous section. Let $\mathsf{BDL}^{\wedge,\top}$ and $\mathsf{BDL}^{\wedge,\top,\bot}$ denote the categories of bounded distributive lattices and meet-semilattice homomorphisms preserving top and of bounded distributive lattices and bounded meet-semilattice homomorphisms, respectively. Clearly BDL is a proper subcategory of $\mathsf{BDL}^{\wedge,\top,\bot}$ and $\mathsf{BDL}^{\wedge,\top,\bot}$ is a proper subcategory of $\mathsf{BDL}^{\wedge,\top}$. Let $\mathsf{PS^R}$, $\mathsf{PS^T}$, and $\mathsf{PS^F}$ denote the categories of Priestley spaces as objects and generalized Priestley morphisms, total generalized Priestley morphisms, and functional generalized Priestley morphisms as morphisms, respectively. Clearly $\mathsf{PS^F}$ is a proper subcategory of $\mathsf{PS^T}$ and $\mathsf{PS^T}$ is a proper subcategory of PS. Moreover, $\mathsf{BDL}^{\wedge,\top}$ is dually equivalent to $\mathsf{PS^R}$, $\mathsf{BDL}^{\wedge,\top,\bot}$ is dually equivalent to $\mathsf{PS^T}$, and BDL is dually equivalent to $\mathsf{PS^F}$, which is isomorphic to PS. Thus, we obtain the following table of dual equivalences we have established. For two categories C and D, $C \overset{\mathrm{d}}{\sim} D$ means that C is dually equivalent to D, and $C \cong D$ means that C is isomorphic to D.

$$\begin{array}{rcccl}
\mathsf{BDM} & \overset{\mathrm{d}}{\sim} & \mathsf{GPS} & & \\
\mathsf{BDM^B} & \overset{\mathrm{d}}{\sim} & \mathsf{GPS^T} & & \\
\mathsf{BDM^S} & \overset{\mathrm{d}}{\sim} & \mathsf{GPS^F} & \cong & \mathsf{GPS^S}
\end{array}
\qquad\qquad
\begin{array}{rcccl}
\mathsf{BDL}^{\wedge,\top} & \overset{\mathrm{d}}{\sim} & \mathsf{PS^R} & & \\
\mathsf{BDL}^{\wedge,\top,\bot} & \overset{\mathrm{d}}{\sim} & \mathsf{PS^T} & & \\
\mathsf{BDL} & \overset{\mathrm{d}}{\sim} & \mathsf{PS^F} & \cong & \mathsf{PS}
\end{array}$$

8. Duality at work

In this section we show how the duality developed in the previous sections works by establishing dual descriptions of a number of algebraic concepts that play an important role in the theory of distributive meet-semilattices.

8.1. Dual description of Frink ideals, ideals, and filters

We start by recalling that for a bounded distributive lattice L and its Priestley space X, there is a lattice isomorphism between the lattice of ideals of L and the lattice of open upsets of X given by

$$I \mapsto U(I) = \bigcup\{\varphi(a) : a \in I\}.$$

The inverse of this isomorphism is given by

$$U \mapsto I(U) = \{a \in L : \varphi(a) \subseteq U\}.$$

Moreover, there is a lattice isomorphism between the lattice of filters of L (ordered by $\supseteq$) and the lattice of closed upsets of X which is given by

$$F \mapsto C(F) = \bigcap\{\varphi(a) : a \in F\}$$

and its inverse isomorphism by

$$C \mapsto F(C) = \{a \in L : C \subseteq \varphi(a)\}.$$

Then we have $I \subseteq J$ iff $U(I) \subseteq U(J)$, $I = I(U(I))$, and $U(I(U)) = U$; and $F \supseteq G$ iff $C(F) \subseteq C(G)$, $F = F(C(F))$, and $C(F(C)) = C$. Now we show how these correspondences work for Frink ideals, ideals, and filters of bounded distributive meet-semilattices.

Let L be a bounded distributive meet-semilattice and let $D(L)$ be its distributive envelope. Let also $X = \langle X, \tau, \leq, X_0\rangle$ be the generalized Priestley space of L. We know that $\langle X, \tau, \leq\rangle$ is order-homeomorphic to the Priestley space of $D(L)$. Since the lattice of Frink ideals of L is isomorphic to the lattice of ideals of $D(L)$, we immediately obtain from the above:

PROPOSITION 8.1. *Let L be a bounded distributive meet-semilattice and let X be its generalized Priestley space. Then the maps $I \mapsto U(I)$ and $U \mapsto I(U)$ defined as above set an isomorphism of the lattice of Frink ideals of L with the lattice of open upsets of X.*

In particular, prime F-ideals of L correspond to the open upsets of X of the form $({\downarrow}x)^c$ for $x \in X$. Now we give the dual description of ideals of L. Since each ideal of L is an F-ideal, ideals correspond to special open upsets of X.

THEOREM 8.2. *Let L be a bounded distributive meet-semilattice and let X be its generalized Priestley space. Then the maps $I \mapsto U(I)$ and $U \mapsto I(U)$ defined as above set an isomorphism of the ordered set of ideals of L with the ordered set of open upsets U of X such that $X - U = {\downarrow}(X_0 - U)$.*

Proof. First we show that if I is an ideal of L, then $X - U(I) = {\downarrow}(X_0 - U(I))$. The inclusion ${\downarrow}(X_0 - U(I)) \subseteq X - U(I)$ is trivial. To prove the other inclusion, let $x \in X - U(I)$. Then $x \cap I = \emptyset$. By the prime filter lemma, there is a prime filter y of L such that $x \subseteq y$ and $y \cap I = \emptyset$. Thus, $y \in X_0 - U(I)$, and so $x \in {\downarrow}(X_0 - U(I))$.

Now we prove that if U is an open upset of X such that $X - U = {\downarrow}(X_0 - U)$, then $I(U)$ is an ideal of L. Suppose that U is an open upset of X. It follows from Proposition 8.1 that $I(U)$ is an F-ideal of L. Let $a, b \in I(U)$ with ${\uparrow}a \cap {\uparrow}b \cap I(U) = \emptyset$. By the optimal filter lemma, there exists $x \in X$ such that ${\uparrow}a \cap {\uparrow}b \subseteq x$ and $x \cap I(U) = \emptyset$. Therefore, $x \notin U$, so $x \in X - U$, and so there exists $y \in X_0 - U$ such that $x \leq y$. It follows that ${\uparrow}a \cap {\uparrow}b \subseteq y$, and as y is a prime filter, we have ${\uparrow}a \subseteq y$ or ${\uparrow}b \subseteq y$. Thus, $a \in y$ or $b \in y$, which is a contradiction because $a, b \in I(U)$. Consequently, ${\uparrow}a \cap {\uparrow}b \cap I(U) \neq \emptyset$, and so $I(U)$ is an ideal of L. Now apply Proposition 8.1. ■

Our next task is to give the dual description of prime ideals of L.

Lemma 8.3. *Let L be a bounded distributive meet-semilattice and let X be its generalized Priestley space. Then I is a prime ideal of L iff $U(I) = ({\downarrow}x)^c$ for some $x \in X_0$.*

Proof. Let I be a prime ideal of L and let $x = L - I$. By Proposition 2.3, $x \in X_0$. Moreover, we have $y \in U(I)$ iff $y \cap I \neq \emptyset$ iff $y \not\subseteq x$ iff $y \in ({\downarrow}x)^c$. Thus, $U(I) = ({\downarrow}x)^c$. Conversely, suppose that $U(I) = ({\downarrow}x)^c$ for some $x \in X_0$. Then, by Theorem 8.2, $I = I(U(I))$ is an ideal because $\max(U(I)^c) = \max({\downarrow}x) = \{x\} \subseteq X_0$. We show that it is prime. Let $a \wedge b \in I(U(I))$. Then $\varphi(a) \cap \varphi(b) = \varphi(a \wedge b) \subseteq U(I)$. So $\varphi(a) \cap \varphi(b) \subseteq ({\downarrow}x)^c$, and so ${\downarrow}x \subseteq \varphi(a)^c \cup \varphi(b)^c$. Therefore, $x \in \varphi(a)^c \cup \varphi(b)^c$, which implies that $x \in \varphi(a)^c$ or $x \in \varphi(b)^c$. Thus, ${\downarrow}x \subseteq \varphi(a)^c$ or ${\downarrow}x \subseteq \varphi(b)^c$, so $\varphi(a) \subseteq ({\downarrow}x)^c$ or $\varphi(b) \subseteq ({\downarrow}x)^c$. It follows that $\varphi(a) \subseteq U(I)$ or $\varphi(b) \subseteq U(I)$, so $a \in I(U(I)) = I$ or $b \in I(U(I)) = I$, and so I is a prime ideal. ■

Putting Theorem 8.2 and Lemma 8.3 together, we obtain:

Proposition 8.4. *Let L be a bounded distributive meet-semilattice and let X be its generalized Priestley space. Then the maps $I \mapsto U(I)$ and $U \mapsto I(U)$ defined as above set an isomorphism of the ordered set of prime ideals of L with the ordered set of open upsets of X of the form $({\downarrow}x)^c$ for some $x \in X_0$.*

Now we give the dual description of filters of L. Since there are less filters in L than in $D(L)$, not every closed upset of X corresponds to a filter of L.

THEOREM 8.5. *Let L be a bounded distributive meet-semilattice and let X be its generalized Priestley space. Then the maps $F \mapsto C(F)$ and $C \mapsto F(C)$ defined as above set an isomorphism of the lattice of filters of L (ordered by reverse inclusion) and the lattice of closed upsets C of X satisfying the condition $X - C = {\downarrow}(X_0 - C)$.*

PROOF. First we prove that if F is a filter of L, then $X - C(F) = {\downarrow}(X_0 - C(F))$. The inclusion ${\downarrow}(X_0 - C(F)) \subseteq X - C(F)$ is trivial. For the other inclusion, let $x \in X - C(F)$. Then $x \notin C(F)$, and so there exists $a \in F$ such that $a \notin x$. By the prime filter lemma, there is $y \in X_0$ such that $x \subseteq y$ and $a \notin y$. Thus, $y \notin C(F)$, so $y \in X_0 - C(F)$, and so $x \in {\downarrow}(X_0 - C(F))$. Next, it is easy to see that if C is a closed upset of X, then $F(C)$ is a filter of L. We show that $C = C(F(C))$ iff $X - C = {\downarrow}(X_0 - C)$. Let $C = C(F(C))$. Since $F(C)$ is a filter of L, we have $X - C(F(C)) = {\downarrow}(X_0 - C(F(C)))$. From $C = C(F(C))$ and the last equality we get $X - C = {\downarrow}(X_0 - C)$. Conversely, suppose that $X - C = {\downarrow}(X_0 - C)$. We show that $C = C(F(C))$. Since $C(F(C)) = \bigcap\{\varphi(a) : C \subseteq \varphi(a)\}$, it is obvious that $C \subseteq C(F(C))$. For the converse, suppose that $x \notin C$. Then there exists $y \in X_0 - C$ such that $x \leq y$. Since C is a closed upset of X, C is the intersection of clopen upsets of X containing C. Therefore, from $y \notin C$ it follows that there is a clopen upset U of X such that $C \subseteq U$ and $y \notin U$. As each clopen upset of X is a finite union of elements of $\varphi[L]$, there exist $a_1, \ldots, a_n \in L$ such that $U = \varphi(a_1) \cup \ldots \cup \varphi(a_n)$. Thus, $y \notin \varphi(a_1) \cup \ldots \cup \varphi(a_n)$, and so $a_1, \ldots, a_n \notin y$. Since y is a prime filter of L, we have $\bigcap_{i=1}^n {\uparrow} a_i \not\subseteq y$. Therefore, there exists $a \in \bigcap_{i=1}^n {\uparrow} a_i$ such that $a \notin y$. But then $\varphi(a) \supseteq \varphi(a_1) \cup \ldots \cup \varphi(a_n) = U \supseteq C$ and $y \notin \varphi(a)$. Consequently, $C \subseteq \varphi(a)$ and $x \notin \varphi(a)$, implying that $x \notin C(F(C))$. The theorem follows. ∎

In particular, since there is a 1-1 correspondence between prime filters and prime ideals of L, we obtain that prime filters of L correspond to closed upsets of X of the form ${\uparrow}x$ for $x \in X_0$. Also, optimal filters of L correspond to closed upsets of X of the form ${\uparrow}x$ for $x \in X$.

8.2. Dual description of 1-1 and onto homomorphisms

Our next task is to give the dual description of 1-1 and onto homomorphisms.

LEMMA 8.6. *Let X and Y be generalized Priestley spaces and let $R \subseteq X \times Y$ be a generalized Priestley morphism.*

1. *If F is a closed subset of X, then $R[F]$ is a closed upset of Y.*
2. *If G is a closed subset of Y, then $R^{-1}[G]$ is a closed downset of X.*

Proof. (1) Suppose that F is a closed subset of X. It follows from Lemma 6.3 that $R[F]$ is an upset of Y. We show that $R[F]$ is closed in Y. Let $y \notin R[F]$. Then for each $x \in F$ we have $x \not R y$. By condition (1) of Definition 6.2, there is $U_x \in Y^*$ such that $R[x] \subseteq U_x$ and $y \notin U_x$. Thus, $x \in \Box_R U_x$ and by condition (2) of Definition 6.2, $\Box_R U_x \in X^*$, so $\Box_R U_x$ is clopen. Then we have $F \subseteq \bigcup\{\Box_R U_x : x \in F\}$. Since F is a closed subset of a compact space, F is compact. Therefore, there are $x_1, \dots, x_n \in F$ such that $F \subseteq \bigcup_{i=1}^n \Box_R U_{x_i}$. We claim that $U_{x_1}^c \cap \dots \cap U_{x_n}^c \cap R[F] = \emptyset$. If not, then there exists $z \in U_{x_1}^c \cap \dots \cap U_{x_n}^c \cap R[F]$. Thus, there is $u \in F$ such that uRz. But then $u \in \Box_R U_{x_i}$, so $z \in U_{x_i}$ for some $i \leq n$, which is a contradiction. It follows that there is an open neighborhood $U_{x_1}^c \cap \dots \cap U_{x_n}^c$ of y missing $R[F]$, so $R[F]$ is closed in Y.

(2) Suppose that G is a closed subset of Y. It follows from Lemma 6.3 that $R^{-1}[G]$ is a downset of X. We show that $R^{-1}[G]$ is closed in X. Let $x \notin R^{-1}[G]$. Then for each $y \in G$ we have $x \not R y$. So, by condition (1) of Definition 6.2, there is $U_y \in Y^*$ such that $x \in \Box_R U_y$ and $y \notin U_y$. Therefore, $G \subseteq \bigcup\{U_y^c : y \in G\}$, and as G is compact, there are $y_1, \dots, y_n \in G$ such that $G \subseteq \bigcup_{i=1}^n U_{y_i}^c$. We claim that $\Box_R U_{y_1} \cap \dots \cap \Box_R U_{y_n} \cap R^{-1}[G] = \emptyset$. If not, then there is $z \in \Box_R U_{y_1} \cap \dots \cap \Box_R U_{y_n} \cap R^{-1}[G]$. So $R[z] \subseteq U_{y_1} \cap \dots \cap U_{y_n}$ and $z \in R^{-1}[G]$. Thus, there is $u \in G$ such that zRu. But then $u \in U_{y_1} \cap \dots \cap U_{y_n} \cap G$, which is a contradiction. Consequently, there is an open neighborhood $\Box_R U_{y_1} \cap \dots \cap \Box_R U_{y_n}$ of x missing $R^{-1}[G]$, so $R^{-1}[G]$ is closed in X. ■

Definition 8.7. Let X and Y be generalized Priestley spaces and let $R \subseteq X \times Y$ be a generalized Priestley morphism.

1. We call R *onto* if for each $y \in Y$ there is $x \in X$ such that $R[x] = {\uparrow} y$.
2. We call R *1-1* if for each $x \in X$ and $U \in X^*$ with $x \notin U$, there is $V \in Y^*$ such that $R[U] \subseteq V$ and $R[x] \not\subseteq V$.

Let X and Y be generalized Priestley spaces and let $R \subseteq X \times Y$ be a generalized Priestley morphism. We observe that if R is 1-1, then R is total. Indeed, if R is not total, then there exists $x \in X$ such that $R[x] = \emptyset$. Therefore, for each $V \in Y^*$ we have $R[x] \subseteq V$. Thus, R can not be 1-1. We also observe that using condition (5) of Definition 5.5, it is easy to verify that if a generalized Priestley morphism R is 1-1, then $x \not\leq y$ implies $R[y] \not\subseteq R[x]$, and $x \notin U$ implies $R[x] \not\subseteq R[U]$ for each $x, y \in X$ and $U \in X^*$. However, these two conditions do not imply that R is 1-1 (see [1, Ex. 11.14]).

Lemma 8.8. *Let L and K be bounded distributive meet-semilattices and let $h : L \to K$ be a meet-semilattice homomorphism preserving top. Then for $x \in K_*$ and $y \in L_*$ we have $R_h[x] = {\uparrow}y$ iff $h^{-1}(x) = y$.*

Proof. First suppose that $R_h[x] = {\uparrow}y$. Then xR_hy, so $h^{-1}(x) \subseteq y$, and so $h^{-1}(x)$ is a proper filter of L. Thus, by the optimal filter lemma, $h^{-1}(x) = \bigcap\{z \in L_* : h^{-1}(x) \subseteq z\} = \bigcap\{z \in L_* : xR_hz\} = \bigcap R_h[x] = \bigcap {\uparrow}y = y$. Now suppose that $h^{-1}(x) = y$. Then $R_h[x] = \{z \in L_* : xR_hz\} = \{z \in L_* : h^{-1}(x) \subseteq z\} = \{z \in L_* : y \subseteq z\} = {\uparrow}y$. ■

Theorem 8.9. *Let L and K be bounded distributive meet-semilattices and let $h : L \to K$ be a meet-semilattice homomorphism preserving top.*

1. *h is 1-1 iff R_h is onto.*
2. *h is onto iff R_h is 1-1.*

Proof. (1) Suppose that h is 1-1. We show that R_h is onto. Let $y \in L_*$. Since h preserves $\wedge$, we have ${\uparrow}h[y]$ is a filter of K. Let J be the F-ideal generated by $h[L - y]$. We claim that ${\uparrow}h[y] \cap J = \emptyset$. If ${\uparrow}h[y] \cap J \neq \emptyset$, then there exist $a \in y$, $a_1, \dots, a_n \in L - y$, and $b \in K$ such that $h(a) \leq b$ and $\bigcap_{i=1}^n {\uparrow}h(a_i) \subseteq {\uparrow}b$. Therefore, $\bigcap_{i=1}^n {\uparrow}h(a_i) \subseteq {\uparrow}h(a)$. Since h is 1-1, we have $\bigcap_{i=1}^n {\uparrow}a_i \subseteq {\uparrow}a$. As y is an optimal filter of L, we have $L - y$ is an F-ideal of L, so $a \in L - y$, a contradiction. Thus, ${\uparrow}h[y] \cap J = \emptyset$, and by the optimal filter lemma, there is $x \in K_*$ such that ${\uparrow}h[y] \subseteq x$ and $x \cap J = \emptyset$. It follows that $h^{-1}(x) = y$, and so $R_h[x] = {\uparrow}y$ by Lemma 8.8. Now suppose that R_h is onto. For $a, b \in L$ with $a \neq b$, we may assume without loss of generality that $a \not\leq b$. Then ${\uparrow}a \cap {\downarrow}b = \emptyset$, and so by the prime filter lemma, there is $y \in L_+ \subseteq L_*$ such that $a \in y$ and $b \notin y$. Since R_h is onto, there is $x \in K_*$ such that $R_h[x] = {\uparrow}y$. This, by Lemma 8.8, implies that $h^{-1}(x) = y$. Thus, $h(a) \in x$ and $h(b) \notin x$, and so $h(a) \not\leq h(b)$. It follows that h is 1-1.

(2) Suppose that h is onto. We show that R_h is 1-1. Let $x \in K_*$, $b \in K$, and $x \notin \varphi(b)$. Since h is onto, there is $a \in L$ such that $h(a) = b$. By Proposition 6.6, $\Box_{R_h}(\varphi(a)) = \varphi(b)$. So $R_h[\varphi(b)] \subseteq \varphi(a)$ and $R_h[x] \not\subseteq \varphi(a)$. Thus, R_h is 1-1. Now suppose that R_h is 1-1. Let $b \in K$. For each $x \in K_*$ such that $b \notin x$, we have $x \notin \varphi(b)$. Since R_h is 1-1, there exists $a_x \in L$ such that $R_h[\varphi(b)] \subseteq \varphi(a_x)$ and $R_h[x] \not\subseteq \varphi(a_x)$. Then $\varphi(b) \subseteq \Box_{R_h}\varphi(a_x)$ and $x \notin \Box_{R_h}\varphi(a_x)$. Therefore, $\bigcap\{\Box_{R_h}\varphi(a_x) : x \notin \varphi(b)\} \cap \varphi(b)^c = \emptyset$. Since X is compact, there exist $x_1, \dots x_n \notin \varphi(b)$ such that $\bigcap_{i=1}^n \Box_{R_h}\varphi(a_{x_i}) \cap \varphi(b)^c = \emptyset$. Thus, $\Box_{R_h}\varphi(a_{x_1} \wedge \dots \wedge a_{x_n}) \cap \varphi(b)^c = \emptyset$, so $\varphi(b) = \Box_{R_h}\varphi(a_{x_1} \wedge \dots \wedge a_{x_n}) = \varphi(h(a_{x_1} \wedge \dots \wedge a_{x_n}))$, and so $b = h(a_{x_1} \wedge \dots \wedge a_{x_n})$. It follows that h is onto. ■

Proposition 8.10. *Let X and Y be generalized Priestley spaces and let $R \subseteq X \times Y$ be a generalized Priestley morphism.*

1. *$R \subseteq X \times Y$ is onto iff $R_{h_R} \subseteq {X^*}_* \times {Y^*}_*$ is onto.*
2. *$R \subseteq X \times Y$ is 1-1 iff $R_{h_R} \subseteq {X^*}_* \times {Y^*}_*$ is 1-1.*

Proof. Apply Theorem 8.9 and Propositions 6.6 and 6.7. ■

Theorem 8.11. *Let X and Y be generalized Priestley spaces and let $R \subseteq X \times Y$ be a generalized Priestley morphism.*

1. *R is 1-1 iff h_R is onto.*
2. *R is onto iff h_R is 1-1.*

Proof. It follows from Theorem 8.9 and Proposition 8.10. ■

Thus, we obtain that 1-1 homomorphisms of bounded distributive meet-semilattices preserving top correspond to onto generalized Priestley morphisms, and that bounded 1-1 homomorphisms correspond to total onto generalized Priestley morphisms. Moreover, onto homomorphisms of bounded distributive meet-semilattices preserving top coincide with bounded onto homomorphisms (which is easy to see either algebraically or by recalling that each 1-1 generalized Priestley morphism is total) and correspond to 1-1 generalized Priestley morphisms.

Our results above imply the well-known dual description of 1-1 and onto homomorphisms of bounded distributive lattices. We skip the details and refer the interested reader to [1, Sec. 11.2.1].

9. Non-bounded case

The duality we have developed for bounded distributive meet-semilattices can be modified accordingly to obtain a duality for non-bounded distributive meet-semilattices. In this section we discuss briefly the main ideas of the modification.

First we deal with the case of distributive meet-semilattices with top but possibly without bottom. Let $L \in \mathsf{DM}$. If L does not have bottom, then we have to add L to the set of optimal filters of L, and so then for each $a \in L$, we have $L \in \varphi(a)$. As a result, L is the greatest element of L_*, and so $\max(L_*)$ is not contained in L_+. Thus, we have to drop condition (3) of Definition 5.5. Moreover, for $x = L$ we have $\mathcal{I}_x = \emptyset$, so $\mathcal{I}_x$ is trivially updirected, although $L \notin L_+$. Thus, we have to modify condition (4) of Definition 5.5 as follows: $x \in L_+$ iff $\mathcal{I}_x$ is nonempty and updirected. This suggests the following modification of the definition of a generalized Priestley space.

Definition 9.1. A quadruple $X = \langle X, \tau, \leq, X_0 \rangle$ is called a $*$-*generalized Priestley space* if (i) $\langle X, \tau, \leq \rangle$ is a Priestley space, (ii) X_0 is a dense subset of X, (iii) $x \in X_0$ iff $\mathcal{I}_x$ is nonempty and updirected, and (iv) for all $x, y \in X$, we have $x \leq y$ iff $(\forall U \in X^*)(x \in U \Rightarrow y \in U)$.

Clearly each generalized Priestley space is a $*$-generalized Priestley space, so the concept of $*$-generalized Priestley space generalizes that of generalized Priestley space. Moreover, a $*$-generalized Priestley space is a generalized Priestley space iff $\max(X) \subseteq X_0$ iff X^* has a bottom element. Therefore, a $*$-generalized Priestley space is a generalized Priestley space iff it satisfies condition (3) of Definition 5.5. Let GPS^* denote the category of $*$-generalized Priestley spaces and generalized Priestley morphisms. Then we immediately obtain the following theorem, which generalizes the duality we obtained for BDM to DM.

Theorem 9.2. *The category* DM *is dually equivalent to the category* GPS^**.*

If L is a distributive meet-semilattice without top but with bottom, then two cases are possible: either $D(L)$ has top or $D(L)$ does not have top. If $D(L)$ has top, then we obtain the dual of L in exactly the same way as in the bounded case. But in this case L will be realized as ${L_*}^* - \{L_*\}$. If $D(L)$ does not have top, then again we construct the dual of L as before, however in this case the space we obtain is locally compact but not compact. We can handle this as the case for distributive lattices [12, Sec. 10] by adding a new top to L. If $L^\top$ is the resulting meet-semilattice, then the dual space of $L^\top$ is the one-point compactification of the dual of L. Moreover, the new point of $(L^\top)_*$ is the smallest optimal filter $\{\top\}$ of $L^\top$, which is below every point of L_*.

This way we can handle all possible situations; that is, when L has $\top$, but lacks $\bot$; when L has $\bot$, but lacks $\top$; or the most general case, when L lacks both $\top$ and $\bot$.

10. Comparison with the relevant work

In this final section we compare our duality with the existing dualities for distributive meet-semilattices. The first representation of distributive meet-semilattices is already present in the pioneering work of Stone [13]. It was made more explicit in Grätzer [7], where with each join-semilattice L with bottom is associated the space $S(L)$ of prime ideals of L. The topology on $S(L)$ is generated by the basis consisting of the sets $r(a) = \{I \in S(L) : a \notin I\}$. The space $S(L)$ is not Hausdorff in general, and it is compact iff L has a top. In [7] it is shown that such spaces can be characterized as topological spaces $\langle X, \tau \rangle$ satisfying (i) X is T_0, (ii) the compact open subsets of X form a basis for τ, and (iii) for each closed subset F of X and each nonempty downdirected family of compact open subsets $\{U_i : i \in I\}$ of X, from $F \cap U_i \neq \emptyset$ for each $i \in I$, it follows that $F \cap \bigcap\{U_i : i \in I\} \neq \emptyset$. Erné [4] showed that these spaces are exactly the compactly based sober spaces, where we recall that a subset A of a topological space X is *irreducible* if $A = F \cup G$ implies $A = F$ or $A = G$ for any closed subsets F, G of X, that X is *sober* if each closed irreducible subset of X is the closure of a unique point $x \in X$, and that X is *compactly based* if it has a basis of compact open sets.

It follows from Celani [3] that this 1-1 correspondence between distributive join-semilattices with bottom and compactly based sober spaces extends to full duality. Celani chose to work with meet-semilattices instead of join-semilattices, hence his building blocks for the dual space were prime filters instead of prime ideals. To be more specific, let us recall that DM denotes the category of distributive meet-semilattices with top as objects and meet-semilattice homomorphisms preserving top as morphisms. Celani's dual category has (in the terminology of [3]) DS-spaces as objects and meet-relations between two DS-spaces as morphisms. We recall from [2, Sec. 6] that *DS-spaces* are simply compactly based sober spaces. For a DS-space X, let $\mathcal{E}(X)$ denote the set of compact open subsets of X, and let $D_X = \{U \subseteq X : U^c \in \mathcal{E}(X)\}$. Let X and Y be two DS-spaces and let $R \subseteq X \times Y$ be a binary relation. We call R a *meet-relation* if (i) $U \in D_Y$ implies $\Box_R U \in D_X$ and (ii) $R[x]$ is closed for each $x \in X$. Let DS denote the category of DS-spaces as objects and meet-relations as morphisms.

Although not addressed in [3], the composition of two meet-relations is not the usual set-theoretic composition. Rather, similar to the case of GPS, we have that for DS-spaces X, Y, and Z and meet-relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, the composition $S * R \subseteq X \times Z$ is given by

$$x(S * R)z \text{ iff } (\forall U \in D_Z)(x \in \Box_R \Box_S(U) \Rightarrow z \in U)$$

for each $x \in X$ and $z \in Z$.

Celani's functors $(-)_+ : \mathsf{DM} \to \mathsf{DS}$ and $(-)^+ : \mathsf{DS} \to \mathsf{DM}$ are defined as follows. If $L \in \mathsf{DM}$, then $L_+ = \langle L_+, \tau \rangle$, where L_+ is the set of prime filters of L and τ is the topology generated by the basis $\{\sigma(a)^c : a \in L\}$; if $h : L \to K$ is a meet-semilattice homomorphism preserving $\top$, then $h_+ = R_h \subseteq K_+ \times L_+$ is defined by xR_hy iff $h^{-1}(x) \subseteq y$. If X is a DS-space, then $X^+ = \langle D_X, \cap, X \rangle$; if X and Y are DS-spaces and $R \subseteq X \times Y$ is a meet-relation, then $R^+ = h_R : D_Y \to D_X$ is defined by $h_R(U) = \Box_R U$. Then it follows from [3] that the functors $(-)_+$ and $(-)^+$ are well-defined, and that they establish a dual equivalence of the categories DM and DS.

The bounded distributive meet-semilattices are exactly the objects of DM whose dual spaces are compact. Indeed, if L is a bounded distributive meet-semilattice, then $\sigma(\bot) = \emptyset$, so $\sigma(a)^c = L_+$, and so L_+ is compact as $\{\sigma(a)^c : a \in L\} = \mathcal{E}(L_+)$. Conversely, if L_+ is compact, then $L_+ = \sigma(a)^c$ for some $a \in L$, so a is the bottom of L, and so L is bounded. It follows that the full subcategory BDM of DM whose objects are bounded distributive meet-semilattices is dually equivalent to the full subcategory CDS of DS whose objects are compact DS-spaces. Now putting Celani's duality together with ours, we obtain that CDS is equivalent to GPS. In fact, as follows from [1, Prop. 13.3 and 13.4], for a generalized Priestley space $X = \langle X, \tau, \leq, X_0\rangle$, the space $X_0 = \langle X_0, \tau_0\rangle$ is a compact DS-space, where τ_0 is the topology generated by the basis $\{X_0 - U : U \in X^*\}$; also, for generalized Priestley spaces X and Y and a generalized Priestley morphism $R \subseteq X \times Y$, the relation $R_0 = R \cap (X_0 \times Y_0)$ is a meet-relation between the compact DS-spaces X_0 and Y_0.

We conclude the paper by comparing our work to that of Hansoul [8, 9]. Like Grätzer, Hansoul prefers to work with distributive join-semilattices. But unlike both Grätzer and Celani, he tries to build a Priestley-like dual of a bounded distributive join-semilattice. Thus, his work is the closest to ours. We recall the main definition of [8, 9]. A *Priestley structure* is a tuple $X = \langle X, \tau, \leq, X_0\rangle$, where:

1. $\langle X, \tau, \leq\rangle$ is a Priestley space.
2. X_0 is a dense subset of X.
3. If $x, y \in X$ with $x \not\leq y$, then there is $z \in X_0$ such that $x \not\leq z$ and $y \leq z$.
4. X_0 is the set of elements of X for which the family of clopen downsets U that contain x and have the property that $U \cap X_0$ is cofinal in U is a basis of clopen downset neighborhoods of x.
5. For each $x \in X$ there exists $y \in X_0$ such that $x \leq y$.

Hansoul constructs the dual X of a bounded distributive join-semilattice L by taking *weakly prime ideals* of L as points of X and by taking prime ideals of L as points of the dense subset X_0 of X. The weakly prime ideals of L are exactly the optimal filters of the dual L^d of L and the prime ideals of L are exactly the prime filters of L^d. Thus, Hansoul's construction is dual to ours. It turns out that Hansoul's Priestley structures are equal to our generalized Priestley spaces (see [1, Prop. 13.6]).

In [8, 9] Hansoul provided duality for the category whose objects are bounded join-semilattices and whose morphisms correspond to our sup-homomorphisms. Consequently, Hansoul's duality is a particular case of our duality.

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