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Semi-intuitionistic Logic

Abstract. The purpose of this paper is to define a new logic $\mathcal{SI}$ called semi-intuitionistic logic such that the semi-Heyting algebras introduced in [4] by Sankappanavar are the semantics for $\mathcal{SI}$. Besides, the intuitionistic logic will be an axiomatic extension of $\mathcal{SI}$.

Keywords: Intuitionistic logic, Heyting algebras, semi-Heyting algebras, semi-intuitionistic logic.

1. Introduction and preliminaries

In [4], Sankappanavar introduced a new (equational) class of algebras that he called semi-Heyting algebras as an abstraction of Heyting algebras.

Definition 1.1. An algebra $\mathbb{L} = \langle L, \vee, \wedge, \to, 0, 1\rangle$ is a Semi-Heyting algebra if the following conditions hold:

($SH1$) $\langle L, \vee, \wedge, 0, 1\rangle$ *is a lattice with 0 and 1.*

($SH2$) $x \wedge (x \to y) \approx x \wedge y$.

($SH3$) $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$.

($SH4$) $x \to x \approx 1$.

We will denote by $\mathcal{SH}$ the variety of semi-Heyting algebras.

Any semi-Heyting algebra $\mathbb{L}$ is a distributive pseudocomplemented lattice, the congruences on $\mathbb{L}$ are determined by filters and the variety of semi-Heyting algebras is arithmetic, thus extending the corresponding results of Heyting algebras. Besides, semi-Heyting algebras share with Heyting algebras some other strong properties. Since Heyting algebras are the semantics of the intuitionistic propositional calculus, we are led to consider the problem of defining a logic that we will call semi-intuitionistic logic, $\mathcal{SI}$, such that the intuitionistic logic is an expansion of $\mathcal{SI}$ and the variety of semi-Heyting algebras provide the semantics for $\mathcal{SI}$.

Recall [3] that, for a language $\mathcal{L}$ of the zero order, a set of logical axioms for the intuitionistic logic $\mathcal{I}$ consists of all formulas of the form

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(T_1) $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$

(T_2) $\alpha \to (\alpha \vee \beta)$

(T_3) $\beta \to (\alpha \vee \beta)$

(T_4) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma))$

(T_5) $(\alpha \wedge \beta) \to \alpha$

(T_6) $(\alpha \wedge \beta) \to \beta$

(T_7) $(\gamma \to \alpha) \to ((\gamma \to \beta) \to (\gamma \to (\alpha \wedge \beta)))$

(T_8) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma)$

(T_9) $((\alpha \wedge \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma))$

(T_{10}) $(\alpha \wedge \neg\alpha) \to \beta$

(T_{11}) $(\alpha \to (\alpha \wedge \neg\alpha)) \to (\neg\alpha)$

where α, β, γ are any formulas in $\mathcal{L}$ [3]. The only inference rule is *Modus Ponens*: $\mathcal{I} \vdash \phi$ and $\mathcal{I} \vdash \phi \to \gamma$ imply $\mathcal{I} \vdash \gamma$.

Definition 1.2. A theory over $\mathcal{I}$ is a set of formulas. A proof in a theory T is a sequence $\phi_1, \ldots, \phi_n$ of formulas in which each member is either an axiom of $\mathcal{I}$ or a member of T or follows from some preceding member of the sequence using the deduction rule Modus Ponens. $T \vdash \phi$ means that ϕ is provable in T, i.e., is the last member of a proof in T.

Definition 1.3. By an $\mathbb{L}$-*valuation of propositional variables* in a Heyting algebra $\mathbb{L}$ we shall understand any mapping e from the set of variables into $\mathbb{L}$. This mapping can be extended to a *valuation of the set of formulas by defining* $e(\neg\alpha) = e(\alpha) \Rightarrow 0$, $e(\alpha \wedge \beta) = e(\alpha) \wedge e(\beta)$, $e(\alpha \vee \beta) = e(\alpha) \vee e(\beta)$, $e(\alpha \to \beta) = e(\alpha) \Rightarrow e(\beta)$.

A formula α is an $\mathbb{L}$-*tautology* if $e(\alpha) = 1$ for any $\mathbb{L}$-*valuation* e. We say that a formula α is an *H-tautology* if it is an $\mathbb{L}$-tautology for any Heyting algebra $\mathbb{L}$.

The following theorem establishes a relationship between the logic $\mathcal{I}$ and the class of Heyting algebras.

Theorem 1.4. [3] (Completeness). *$\mathcal{I}$ is complete, that is, for each formula ϕ the following are equivalent:*

(a) *ϕ is provable in $\mathcal{I}$.*

(b) *For each Heyting algebra $\mathbb{L}$, ϕ is an $\mathbb{L}$-tautology.*

2. The semi-intuitionistic propositional calculus

Let us introduce now the *semi-intuitionistic logic* $\mathcal{SI}$ by means of the following set of axioms. Axioms ($S1$), ($S2$), ($S3$), ($S4$), ($S5$), ($S6$) and ($S7$) will determine the lattice structure of the corresponding Lindenbaum algebra. Axioms ($S8$) and ($S9$) will establish an algebraic relationship between the implication and the order. As a consequence of axioms ($S10$), ($S11$), ($S12$), ($S13$) and ($S14$) we will have that the algebra of equivalent formulas is a Semi-Heyting algebra, whereas axioms ($S15$) and ($S16$) are needed for a good definition of implication in the Lindenbaum algebra. Finally, the last two axioms describe the behavior of the negation.

(S_1) $(\alpha \to (\alpha\wedge\beta)) \to [(\alpha \to (\alpha\wedge\beta))\wedge[(\beta \to (\beta\wedge\gamma)) \to [(\beta \to (\beta\wedge\gamma))\wedge(\alpha \to (\alpha \wedge \gamma))]]]$

(S_2) $\alpha \to (\alpha \wedge (\alpha \vee \beta))$

(S_3) $\beta \to (\beta \wedge (\alpha \vee \beta))$

(S_4) $(\alpha \to (\alpha \wedge \gamma))) \to [(\alpha \to (\alpha \wedge \gamma)) \wedge [(\beta \to (\beta \wedge \gamma)) \to [(\beta \to (\beta \wedge \gamma)) \wedge ((\alpha \vee \beta) \to ((\alpha \vee \beta) \wedge \gamma))]]]$

(S_5) $(\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge \alpha)$

(S_6) $(\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge \beta)$

(S_7) $(\gamma \to (\gamma\wedge\alpha)) \to [(\gamma \to (\gamma\wedge\alpha))\wedge[(\gamma \to (\gamma\wedge\beta)) \to [(\gamma \to (\gamma\wedge\beta))\wedge(\gamma \to (\gamma \wedge (\alpha \wedge \beta)))]]]$

(S_8) $((\alpha\wedge\beta) \to ((\alpha\wedge\beta)\wedge\gamma)) \to (((\alpha\wedge\beta) \to ((\alpha\wedge\beta)\wedge\gamma))\wedge(\alpha \to (\alpha\wedge(\beta \to (\beta \wedge \gamma)))))$

(S_9) $(\alpha \to (\alpha \wedge (\beta \to (\beta \wedge \gamma)))) \to ((\alpha \to (\alpha \wedge (\beta \to (\beta \wedge \gamma)))) \wedge ((\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge \gamma)))$

(S_{10}) $\alpha \to (\alpha \wedge \alpha)$

(S_{11}) $(\alpha \wedge (\beta \to \gamma)) \to ((\alpha \wedge (\beta \to \gamma)) \wedge (\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))))$

(S_{12}) $(\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))) \to ((\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))) \wedge (\alpha \wedge (\beta \to \gamma)))$

(S_{13}) $(\alpha \wedge (\alpha \to \beta)) \to ((\alpha \wedge (\alpha \to \beta)) \wedge (\alpha \wedge \beta))$

(S_{14}) $(\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge (\alpha \wedge (\alpha \to \beta)))$

(S_{15}) $(\alpha \to (\alpha \wedge \beta)) \to ((\alpha \to (\alpha \wedge \beta)) \wedge ((\beta \to (\beta \wedge \alpha)) \to ((\beta \to (\beta \wedge \alpha)) \wedge ((\alpha \to \gamma) \to ((\alpha \to \gamma) \wedge (\beta \to \gamma))))))$

(S_{16}) $(\alpha \to (\alpha \wedge \beta)) \to ((\alpha \to (\alpha \wedge \beta)) \wedge ((\beta \to (\beta \wedge \alpha)) \to ((\beta \to (\beta \wedge \alpha)) \wedge ((\gamma \to \beta) \to ((\gamma \to \beta) \wedge (\gamma \to \alpha))))))$

(S_{17}) $(\alpha \wedge \neg\alpha) \to ((\alpha \wedge \neg\alpha) \wedge \beta)$

(S_{18}) $(\alpha \to (\alpha \wedge \neg\alpha)) \to ((\alpha \to (\alpha \wedge \neg\alpha)) \wedge (\neg\alpha))$

The rule of inference for $\mathcal{SI}$ will be: $\mathcal{SI} \vdash \phi$ and $\mathcal{SI} \vdash \phi \to (\phi \wedge \gamma)$ imply $\mathcal{SI} \vdash \gamma$. We call this rule *Semi-Modus Ponens*.

The concepts of provability of a formula for a theory on the logic $\mathcal{SI}$, $\mathbb{L}$-valuation in a semi-Heyting algebra and $\mathcal{SH}$-tautology are similar to definitions 1.2 and 1.3.

In what follows we prove some properties of the semi-intuitionistic logic that will be used in Section 3.

LEMMA 2.1. *The following properties hold:*

(a) *If* $\mathcal{SI} \vdash \alpha \to (\alpha \wedge \beta)$ *then* $\mathcal{SI} \vdash (\alpha \wedge \gamma) \to ((\alpha \wedge \gamma) \wedge \beta)$.

(b) *If* $\mathcal{SI} \vdash \alpha \to (\alpha \wedge \beta)$ *then* $\mathcal{SI} \vdash (\alpha \wedge \gamma) \to ((\alpha \wedge \gamma) \wedge (\beta \wedge \gamma))$.

PROOF.

(a) 1. $\mathcal{SI} \vdash \alpha \to (\alpha \wedge \beta)$ by hypothesis.
2. $\mathcal{SI} \vdash (\alpha \wedge \gamma) \to ((\alpha \wedge \gamma) \wedge \alpha)$ by (S_5).
3. $\mathcal{SI} \vdash ((\alpha \wedge \gamma) \to ((\alpha \wedge \gamma) \wedge \alpha)) \to (((\alpha \wedge \gamma) \to ((\alpha \wedge \gamma) \wedge \alpha)) \wedge ((\alpha \to (\alpha \wedge \beta)) \to ((\alpha \to (\alpha \wedge \beta)) \wedge ((\alpha \wedge \gamma) \to ((\alpha \wedge \gamma) \wedge \beta)))))$ by (S_7).
4. $\mathcal{SI} \vdash ((\alpha \to (\alpha \wedge \beta)) \to ((\alpha \to (\alpha \wedge \beta)) \wedge ((\alpha \wedge \gamma) \to ((\alpha \wedge \gamma) \wedge \beta))))$ by SMP in 2 and 3.
5. $\mathcal{SI} \vdash (\alpha \wedge \gamma) \to ((\alpha \wedge \gamma) \wedge \beta)$ by SMP in 1 and 4.

(b) 1. $\mathcal{SI} \vdash \alpha \to (\alpha \wedge \beta)$ by hypothesis.
2. $\mathcal{SI} \vdash (\alpha \wedge \gamma) \to ((\alpha \wedge \gamma) \wedge \beta)$ by (a).
3. $\mathcal{SI} \vdash (\alpha \wedge \gamma) \to ((\alpha \wedge \gamma) \wedge \alpha)$ by (S_5).
4. $\mathcal{SI} \vdash (\alpha \wedge \gamma) \to ((\alpha \wedge \gamma) \wedge (\beta \wedge \gamma))$ by (S_7) and SMP. ■

We shall write $\mathcal{SI} \vdash \alpha \leftrightarrow \beta$ when $\mathcal{SI} \vdash \alpha \to (\alpha \wedge \beta)$ and $\mathcal{SI} \vdash \beta \to (\beta \wedge \alpha)$.

LEMMA 2.2. $\mathcal{SI}$ *proves the following:*

(a) $\mathcal{SI} \vdash \alpha \leftrightarrow (\alpha \wedge \alpha)$.

(b) $\mathcal{SI} \vdash (\alpha \wedge \beta) \leftrightarrow (\beta \wedge \alpha)$.

PROOF.

(a) 1. $\mathcal{SI} \vdash (\alpha \to (\alpha \wedge \alpha)) \to [(\alpha \to (\alpha \wedge \alpha)) \wedge [(\alpha \to (\alpha \wedge \alpha)) \to [(\alpha \to (\alpha \wedge \alpha)) \wedge (\alpha \to (\alpha \wedge (\alpha \wedge \alpha)))]]]$ by (S_9).
2. $\mathcal{SI} \vdash \alpha \to (\alpha \wedge \alpha)$ by (S_{10}).

3. $\mathcal{SI} \vdash [(\alpha \to (\alpha \wedge \alpha)) \to [(\alpha \to (\alpha \wedge \alpha)) \wedge (\alpha \to (\alpha \wedge (\alpha \wedge \alpha)))]]$ by SMP in 1 and 2.
4. $\mathcal{SI} \vdash (\alpha \to (\alpha \wedge (\alpha \wedge \alpha)))$ by SMP in 2 and 3.
5. $\mathcal{SI} \vdash (\alpha \wedge \alpha) \to ((\alpha \wedge \alpha) \wedge \alpha)$ by (S_{10}).
6. $\mathcal{SI} \vdash \alpha \leftrightarrow (\alpha \wedge \alpha)$ by 4 and 5.

(b) 1. $\mathcal{SI} \vdash (\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge \beta)$ by (S_6).
2. $\mathcal{SI} \vdash (\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge \alpha)$ by (S_5).
3. $\mathcal{SI} \vdash (\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge (\beta \wedge \alpha))$ by (S_9) and SMP.
4. $\mathcal{SI} \vdash (\beta \wedge \alpha) \to ((\beta \wedge \alpha) \wedge (\alpha \wedge \beta))$ by a similar argument to 3.
5. $\mathcal{SI} \vdash (\alpha \wedge \beta) \leftrightarrow (\beta \wedge \alpha)$ by 3 and 4. ■

LEMMA 2.3. *If* $\mathcal{SI} \vdash \alpha \leftrightarrow \gamma$ *and* $\mathcal{SI} \vdash \beta \leftrightarrow \delta$ *then* $\mathcal{SI} \vdash (\alpha \to \beta) \leftrightarrow (\gamma \to \delta)$.

PROOF. 1. Since $\mathcal{SI} \vdash \alpha \leftrightarrow \gamma$, $\mathcal{SI} \vdash \alpha \to (\alpha \wedge \gamma)$ and $\mathcal{SI} \vdash \gamma \to (\gamma \wedge \alpha)$.
2. $\mathcal{SI} \vdash (\alpha \to \beta) \to ((\alpha \to \beta) \wedge (\gamma \to \beta))$ by (S_{15}), (S_1) and SMP.
3. Since $\mathcal{SI} \vdash \beta \leftrightarrow \delta$, $\mathcal{SI} \vdash \beta \to (\beta \wedge \delta)$ and $\mathcal{SI} \vdash \delta \to (\delta \wedge \beta)$.
4. $\mathcal{SI} \vdash (\gamma \to \beta) \to ((\gamma \to \beta) \wedge (\gamma \to \delta))$ by (S_{16}), (S_1) and SMP.
5. $\mathcal{SI} \vdash (\alpha \to \beta) \to ((\alpha \to \beta) \wedge (\gamma \to \delta))$ by (S_1) and SMP.
Similarly, $\mathcal{SI} \vdash (\gamma \to \delta) \to ((\gamma \to \delta) \wedge (\alpha \to \beta))$.
Therefore, $\mathcal{SI} \vdash (\alpha \to \beta) \leftrightarrow (\gamma \to \delta)$. ■

LEMMA 2.4. *If* $\mathcal{SI} \vdash \beta$ *then* $\mathcal{SI} \vdash (\alpha \wedge \beta) \leftrightarrow \alpha$.

PROOF. 1. $\mathcal{SI} \vdash (\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge \alpha)$ by (S_5).
2. $\mathcal{SI} \vdash (\beta \wedge \alpha) \to ((\beta \wedge \alpha) \wedge (\alpha \wedge \beta))$ by Lemma 2.2.
3. $\mathcal{SI} \vdash \beta \to (\beta \wedge (\alpha \to (\alpha \wedge (\alpha \wedge \beta))))$ by (S_8) and SMP.
4. $\mathcal{SI} \vdash \beta$ by hypothesis.
5. $\mathcal{SI} \vdash \alpha \to (\alpha \wedge (\alpha \wedge \beta))$ SMP in 3 and 4.
6. $\mathcal{SI} \vdash (\alpha \wedge \beta) \leftrightarrow \alpha$ by 1 and 5. ■

LEMMA 2.5. $\mathcal{SI} \vdash \alpha \to \alpha$.

PROOF. 1. $\mathcal{SI} \vdash (\alpha \wedge \alpha) \leftrightarrow \alpha$ by Lemma 2.2.
2. $\mathcal{SI} \vdash \alpha \to (\alpha \wedge \alpha)$ by (S_{10}).
3. $\mathcal{SI} \vdash \alpha \leftrightarrow \alpha$ by 2.
4. $\mathcal{SI} \vdash (\alpha \to (\alpha \wedge \alpha)) \leftrightarrow (\alpha \to \alpha)$ by Lemma 2.3.
5. $\mathcal{SI} \vdash (\alpha \to (\alpha \wedge \alpha)) \to ((\alpha \to (\alpha \wedge \alpha)) \wedge (\alpha \to \alpha))$ by 4.
6. $\mathcal{SI} \vdash \alpha \to \alpha$ by SMP in 2 and 5. ■

LEMMA 2.6. $\mathcal{SI} \vdash \neg\alpha \leftrightarrow (\alpha \to (\alpha \wedge \neg\alpha))$.

PROOF. 1. $\mathcal{SI} \vdash (\alpha \wedge \neg\alpha) \rightarrow (\alpha \wedge \neg\alpha)$ by Lemma 2.5.
2. $\mathcal{SI} \vdash \neg\alpha \rightarrow (\neg\alpha \wedge (\neg\alpha \wedge ((\alpha \wedge \neg\alpha) \rightarrow (\alpha \wedge \neg\alpha))))$ by Lemma 2.4.
3. $\mathcal{SI} \vdash \neg\alpha \rightarrow (\neg\alpha \wedge (\alpha \wedge \neg\alpha) \rightarrow (\alpha \wedge \neg\alpha))$.
4. $\mathcal{SI} \vdash \neg\alpha \rightarrow (\neg\alpha \wedge ((\alpha \wedge \neg\alpha) \rightarrow (\alpha \wedge \neg\alpha \wedge \neg\alpha)))$.
5. $\mathcal{SI} \vdash \neg\alpha \rightarrow (\neg\alpha \wedge (\alpha \rightarrow (\alpha \wedge \neg\alpha)))$ by (S_{12}) and SMP.
6. $\mathcal{SI} \vdash (\alpha \rightarrow (\alpha \wedge \neg\alpha)) \rightarrow ((\alpha \rightarrow (\alpha \wedge \neg\alpha)) \wedge \neg\alpha)$ by (S_{18}).
7. $\mathcal{SI} \vdash \neg\alpha \leftrightarrow (\alpha \rightarrow (\alpha \wedge \neg\alpha))$ by 5 and 6. ■

LEMMA 2.7. *If* $\mathcal{SI} \vdash \beta \rightarrow (\beta \wedge \alpha)$ *then* $\mathcal{SI} \vdash \neg\alpha \leftrightarrow (\neg\alpha \wedge \neg\beta)$.

PROOF. 1. $\mathcal{SI} \vdash (\alpha \wedge \neg\alpha) \rightarrow ((\alpha \wedge \neg\alpha) \wedge (\neg\beta \wedge \neg\alpha))$ by (S_{17}).
2. $\mathcal{SI} \vdash \beta \rightarrow (\beta \wedge \beta)$ by (S_{10}).
3. $\mathcal{SI} \vdash (\alpha \wedge \neg\alpha \wedge \beta) \rightarrow ((\alpha \wedge \neg\alpha \wedge \beta) \wedge (\neg\beta \wedge \neg\alpha \wedge \beta))$ by Lemma 2.1.
4. By a similar argument to 3, $\mathcal{SI} \vdash (\beta \wedge \neg\beta \wedge \alpha) \rightarrow ((\beta \wedge \neg\beta \wedge \alpha) \wedge (\neg\alpha \wedge \neg\beta \wedge \alpha))$.
5. $\mathcal{SI} \vdash \beta \rightarrow (\beta \wedge \alpha)$ by hypothesis.
6. $\mathcal{SI} \vdash (\beta \wedge \alpha) \rightarrow ((\beta \wedge \alpha) \wedge \beta)$ by (S_5).
7. $\mathcal{SI} \vdash \beta \leftrightarrow (\beta \wedge \alpha)$ by 5 and 6.
8. $\mathcal{SI} \vdash (\beta \wedge \neg\alpha) \leftrightarrow (\beta \wedge \alpha \wedge \neg\alpha)$ by Lemma 2.1.
9. $\mathcal{SI} \vdash (\beta \wedge \beta \wedge \neg\alpha) \leftrightarrow (\beta \wedge \alpha \wedge \neg\alpha)$.
10. $\mathcal{SI} \vdash (\beta \rightarrow (\beta \wedge (\beta \wedge \neg\alpha))) \leftrightarrow (\beta \wedge \beta \wedge \neg\alpha)$ by (S_{13}) and (S_{14}).
11. $\mathcal{SI} \vdash (\beta \rightarrow (\beta \wedge (\beta \wedge \neg\alpha))) \leftrightarrow (\beta \wedge \neg\alpha)$ by (S_1).
12. $\mathcal{SI} \vdash (\beta \wedge ((\beta \wedge \alpha) \rightarrow (\beta \wedge \alpha \wedge \neg\alpha))) \leftrightarrow (\beta \rightarrow (\beta \wedge (\beta \wedge \neg\alpha)))$ by Lemma 2.3 in 7 and 8 and Lemma 2.1.
13. $\mathcal{SI} \vdash (\beta \wedge ((\beta \wedge \alpha) \rightarrow (\beta \wedge \alpha \wedge \neg\alpha))) \leftrightarrow (\beta \wedge \neg\alpha)$ by (S_1).
14. $\mathcal{SI} \vdash (\beta \wedge (\alpha \rightarrow (\alpha \wedge \neg\alpha))) \leftrightarrow (\beta \wedge \neg\alpha)$ by (S_{13}), (S_{14}) and (S_1).
15. $\mathcal{SI} \vdash (\alpha \rightarrow (\alpha \wedge \neg\alpha)) \leftrightarrow \neg\alpha$ by Lemma 2.6.
16. $\mathcal{SI} \vdash (\beta \wedge \neg\beta \wedge (\alpha \rightarrow (\alpha \wedge \neg\alpha))) \leftrightarrow (\neg\beta \wedge \beta \wedge \neg\alpha)$ by Lemma 2.1.
17. $\mathcal{SI} \vdash (\neg\alpha \wedge \neg\beta) \leftrightarrow ((\alpha \rightarrow (\alpha \wedge \neg\alpha)) \wedge (\beta \rightarrow (\beta \wedge \neg\beta)))$ by Lemmas 2.1 and 2.6 and (S_1).
18. $\mathcal{SI} \vdash ((\alpha \rightarrow (\alpha \wedge \neg\alpha)) \wedge (\beta \rightarrow (\beta \wedge \neg\beta))) \leftrightarrow [(\alpha \rightarrow (\alpha \wedge \neg\alpha)) \wedge [(\beta \wedge (\alpha \rightarrow (\alpha \wedge \neg\alpha))) \rightarrow (\beta \wedge \neg\beta \wedge (\alpha \rightarrow (\alpha \wedge \neg\alpha)))]]$ by (S_{13}) and (S_{14}).
19. $\mathcal{SI} \vdash (\neg\alpha \wedge \neg\beta) \leftrightarrow [(\alpha \rightarrow (\alpha \wedge \neg\alpha)) \wedge [(\beta \wedge (\alpha \rightarrow (\alpha \wedge \neg\alpha))) \rightarrow (\beta \wedge \neg\beta \wedge (\alpha \rightarrow (\alpha \wedge \neg\alpha)))]]$ by (S_1).
20. $\mathcal{SI} \vdash [(\alpha \rightarrow (\alpha \wedge \neg\alpha)) \wedge [(\beta \wedge (\alpha \rightarrow (\alpha \wedge \neg\alpha))) \rightarrow (\beta \wedge \neg\beta \wedge (\alpha \rightarrow (\alpha \wedge \neg\alpha)))]] \leftrightarrow [(\alpha \rightarrow (\alpha \wedge \neg\alpha)) \wedge [(\beta \wedge \neg\alpha) \rightarrow (\neg\beta \wedge \beta \wedge \neg\alpha)]]$ by 14, 16 and Lemmas 2.3 and 2.1.
21. $\mathcal{SI} \vdash (\neg\alpha \wedge \neg\beta) \leftrightarrow [(\alpha \rightarrow (\alpha \wedge \neg\alpha)) \wedge [(\beta \wedge \neg\alpha) \rightarrow (\neg\beta \wedge \beta \wedge \neg\alpha)]]$ by (S_1).

22. $\mathcal{SI} \vdash (\neg\alpha \wedge \neg\beta) \leftrightarrow [(\alpha \to (\alpha \wedge \neg\alpha)) \wedge [(\beta \wedge \alpha \wedge \neg\alpha) \to (\neg\beta \wedge \beta \wedge \neg\alpha)]]$ by Lemma 2.3 in 8.
23. $\mathcal{SI} \vdash (\alpha \wedge \neg\alpha) \to ((\alpha \wedge \neg\alpha) \wedge (\neg\beta \wedge \beta \wedge \neg\alpha))$ by (S_{17}).
24. $\mathcal{SI} \vdash (\beta \wedge \alpha \wedge \neg\alpha) \to ((\beta \wedge \alpha \wedge \neg\alpha) \wedge (\neg\beta \wedge \beta \wedge \neg\alpha))$ by Lemma 2.1.
25. Similarly, $\mathcal{SI} \vdash (\neg\alpha \wedge \beta \wedge \neg\beta) \to ((\neg\alpha \wedge \beta \wedge \neg\beta) \wedge (\beta \wedge \alpha \wedge \neg\alpha))$.
26. Thus, from 24 and 25, $\mathcal{SI} \vdash (\beta \wedge \alpha \wedge \neg\alpha) \leftrightarrow (\neg\beta \wedge \beta \wedge \neg\alpha)$.
27. $\mathcal{SI} \vdash (\beta \wedge \alpha \wedge \neg\alpha)$ by Lemma 2.5.
28. $\mathcal{SI} \vdash [(\beta \wedge \alpha \wedge \neg\alpha) \to (\beta \wedge \alpha \wedge \neg\alpha)] \leftrightarrow [(\beta \wedge \alpha \wedge \neg\alpha) \to (\neg\beta \wedge \beta \wedge \neg\alpha)]$ by Lemma 2.3.
29. In particular, $\mathcal{SI} \vdash [(\beta \wedge \alpha \wedge \neg\alpha) \to (\beta \wedge \alpha \wedge \neg\alpha)] \to ([(\beta \wedge \alpha \wedge \neg\alpha) \to (\beta \wedge \alpha \wedge \neg\alpha)] \wedge [(\beta \wedge \alpha \wedge \neg\alpha) \to (\neg\beta \wedge \beta \wedge \neg\alpha)])$.
30. $\mathcal{SI} \vdash [(\beta \wedge \alpha \wedge \neg\alpha) \to (\beta \wedge \alpha \wedge \neg\alpha)]$ by Lemma 2.5.
31. $\mathcal{SI} \vdash [(\beta \wedge \alpha \wedge \neg\alpha) \to (\neg\beta \wedge \beta \wedge \neg\alpha)]$ by SMP in 29 and 30.
32. $\mathcal{SI} \vdash (\neg\alpha \wedge \neg\beta) \leftrightarrow (\alpha \to (\alpha \wedge \neg\alpha))$ by Lemma 2.4 and (S_1) in 22.
33. $\mathcal{SI} \vdash (\neg\alpha \wedge \neg\beta) \leftrightarrow \neg\alpha$ by Lemma 2.6 and (S_1). ■

3. Semi-Heyting algebras: soundness and completeness

In this section we prove that semi-Heyting algebras are the semantics for the semi-intuitionistic logic.

Any semi-Heyting algebras is a pseudocomplemented distributive lattice, with the pseudocomplement given by $x^* = x \to 0$ (see [4]). Nevertheless, the operation $\to$ on semi-Heyting algebras does not enjoy several nice properties of the implication on Heyting algebras or even on BCK-algebras. For example, the order on a semi-Heyting algebra is not determined by the operation of implication. Some of the properties of $\to$ in $\mathcal{SH}$ are listed in the next lemma.

Lemma 3.1. [4] *Let $\mathbb{L} \in \mathcal{SH}$ and $a, b \in \mathbb{L}$.*

(a) *If $a \to b = 1$ then $a \leq b$.*

(b) *If $a \leq b$ then $a \leq a \to b$.*

(c) *$a = b$ if and only if $a \to b = b \to a = 1$.*

(d) *$1 \to a = a$.*

Proof. From $a \to b = 1$ and (SH3), we get $a \wedge 1 = a \wedge b$, that is $a = a \wedge b$, and we have (a). For (b), by (SH3) and since $a \leq b$ it follows that $a = a \wedge (a \to b) \leq a \to b$. Property (c) is clear. To prove (d), observe that $a = 1 \wedge a = 1 \wedge (1 \to a) = 1 \to a$. ■

Now we are going to construct the Lindenbaum algebra for this logic.

Let $Form$ be the set of all formulas of $\mathcal{SI}$. We define a binary relation on $Form$ by:

$$\alpha \preceq \beta \text{ if and only if } \mathcal{SI} \vdash \alpha \to (\alpha \wedge \beta).$$

LEMMA 3.2. *The relation $\preceq$ is reflexive and transitive.*

PROOF. Since $\mathcal{SI} \vdash \alpha \to (\alpha \wedge \alpha)$ (axiom (S_{10})), $\alpha \preceq \alpha$ for every formula α. Let $\alpha, \beta, \gamma \in Form$ be such that $\alpha \preceq \beta$ and $\beta \preceq \gamma$. Then $\mathcal{SI} \vdash \alpha \to (\alpha \wedge \beta)$ and $\mathcal{SI} \vdash \beta \to (\beta \wedge \gamma)$. From (S_1) and SMP we obtain $\mathcal{SI} \vdash \alpha \to (\alpha \wedge \gamma)$. So $\alpha \preceq \gamma$. ■

If we define now on $Form$ the relation: $\alpha \equiv \beta$ if and only if $\alpha \preceq \beta$ and $\beta \preceq \alpha$, then the following lemma is immediate .

LEMMA 3.3. *$\equiv$ is an equivalence relation.*

Let $[\![\alpha]\!]$ denote the equivalence class of α in $Form/\equiv$. If we define $[\![\alpha]\!] \preceq [\![\beta]\!]$ if and only if $\alpha \preceq \beta$, then we obtain the following lemma:

LEMMA 3.4. *$\langle Form/\equiv, \preceq\rangle$ is a partially ordered set.*

LEMMA 3.5. *$\langle Form/\equiv, \preceq\rangle$ is a lattice.*

PROOF. By (S_2), $\mathcal{SI} \vdash \alpha \to (\alpha \wedge (\alpha \vee \beta))$, so $[\![\alpha]\!] \preceq [\![\alpha \vee \beta]\!]$. In a similar way, from (S_3) we obtain $[\![\beta]\!] \preceq [\![\alpha \vee \beta]\!]$. Let $\gamma \in Form$ be such that $[\![\alpha]\!] \preceq [\![\gamma]\!]$ and $[\![\beta]\!] \preceq [\![\gamma]\!]$. Then $\mathcal{SI} \vdash \alpha \to (\alpha \wedge \gamma)$ and $\mathcal{SI} \vdash \beta \to (\beta \wedge \gamma)$. By (S_4) and SMP, $\mathcal{SI} \vdash (\alpha \vee \beta) \to ((\alpha \vee \beta) \wedge \gamma)$ and consequently, $[\![\alpha \vee \beta]\!] \preceq [\![\gamma]\!]$. Hence $[\![\alpha]\!] \vee [\![\beta]\!] = [\![\alpha \vee \beta]\!]$. In a similar way it can be proved that $[\![\alpha]\!] \wedge [\![\beta]\!] = [\![\alpha \wedge \beta]\!]$. Therefore $\langle Form/\equiv, \preceq\rangle$ is a lattice. ■

LEMMA 3.6. *Let α be any formula. Then $\langle Form/\equiv, \wedge, \vee, [\![\alpha \wedge \neg\alpha]\!], [\![\alpha \to \alpha]\!]\rangle$ is a bounded lattice.*

PROOF. By Lemma 3.5, it is enough to see that $[\![\alpha \wedge \neg\alpha]\!]$ and $[\![\alpha \to \alpha]\!]$ are the bottom and top element of $Form$ respectively. Let $\beta \in Form$. From Lemma 2.5, $\mathcal{SI} \vdash \alpha \to \alpha$ and from Lemma 2.4, $\mathcal{SI} \vdash \beta \to (\beta \wedge (\alpha \to \alpha))$. Thus $[\![\beta]\!] \preceq [\![\alpha \to \alpha]\!]$. Condition $[\![\alpha \wedge \neg\alpha]\!] \preceq [\![\beta]\!]$ follows immediately from (S_{17}). ■

Now we define an implication operation on $Form/\equiv$ by: $[\![\alpha]\!] \Rightarrow [\![\beta]\!] = [\![\alpha \to \beta]\!]$.

THEOREM 3.7. *$\langle Form/\equiv, \wedge, \vee, \Rightarrow, [\![\alpha \wedge \neg\alpha]\!], [\![\alpha \to \alpha]\!]\rangle$ is a semi-Heyting algebra.*

Proof. By lemma 2.3, $\Rightarrow$ is well defined. Let us see that the axioms of semi-Heyting algebra are satisfied. By definition, $[\![\alpha]\!] \Rightarrow [\![\alpha]\!] = [\![\alpha \to \alpha]\!]$. From (S_{13}) and (S_{14}) we have $[\![\alpha]\!] \wedge ([\![\alpha]\!] \Rightarrow [\![\beta]\!]) = [\![\alpha]\!] \wedge [\![\alpha \to \beta]\!] = [\![\alpha \wedge (\alpha \to \beta)]\!] = [\![\alpha \wedge \beta]\!] = [\![\alpha]\!] \wedge [\![\beta]\!]$. Finally, from (S_{11}) and (S_{12}), $[\![\alpha]\!] \wedge ([\![\beta]\!] \Rightarrow [\![\gamma]\!]) = [\![\alpha]\!] \wedge [\![\beta \to \gamma]\!] = [\![\alpha \wedge (\beta \to \gamma)]\!] = [\![\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))]\!] = [\![\alpha]\!] \wedge [\![(\alpha \wedge \beta) \to (\alpha \wedge \gamma)]\!] = [\![\alpha]\!] \wedge ([\![\alpha \wedge \beta]\!] \Rightarrow [\![\alpha \wedge \gamma]\!]) = [\![\alpha]\!] \wedge (([\![\alpha]\!] \wedge [\![\beta]\!]) \Rightarrow ([\![\alpha]\!] \wedge [\![\gamma]\!]))$. ■

Our next goal is to prove that the semi-intuitionistic logic $\mathcal{SI}$ is sound and complete with respect to the class of semi-Heyting algebras.

First we prove some useful lemmas.

Lemma 3.8. *Let* $\mathbb{L} = \langle L, \wedge, \vee, \Rightarrow, 0, 1\rangle$ *be a semi-Heyting algebra. For* $a, b, c \in L$, $a \wedge b \leq c$ *if and only if* $a \leq b \Rightarrow (b \wedge c)$.

Proof. Let $a, b, c \in L$. Suppose that $a \wedge b \leq c$. Then $a \wedge (b \Rightarrow (b \wedge c)) = a \wedge [(a \wedge b) \Rightarrow (a \wedge b \wedge c)] = a \wedge [(a \wedge b) \Rightarrow (a \wedge b)] = a \wedge 1 = a$.

For the converse, suppose that $a \leq b \Rightarrow (b \wedge c)$. Then $a \wedge b \wedge c = a \wedge b \wedge b \wedge c = a \wedge b \wedge (b \Rightarrow (b \wedge c)) = a \wedge b$. Consequently, $a \wedge b \leq c$. ■

Observe that in a semi-Heyting algebra the order is not determined by the implication $\Rightarrow$.

The following corollary is immediate.

Corollary 3.9. *Let* $\mathbb{L} = \langle L, \wedge, \vee, \Rightarrow, 0, 1\rangle$ *be a semi-Heyting algebra. For* $a, b \in L$, $a \leq b$ *if and only if* $1 = a \Rightarrow (a \wedge b)$.

We move on to soundness. First we are going to prove that if ϕ is one of the axioms (S_1), (S_4), (S_7), (S_{15}), (S_{16}), (S_8) or (S_9) then ϕ is an $\mathbb{L}$-tautology for each semi-Heyting algebra $\mathbb{L}$.

Lemma 3.10. *Let* $\mathbb{L} = \langle L, \wedge, \vee, \Rightarrow, 0, 1\rangle$ *be a semi-Heyting algebra.*

$\mathbb{L} \models x \Rightarrow (x \wedge y) \leq (y \Rightarrow (y \wedge z)) \Rightarrow [(y \Rightarrow (y \wedge z)) \wedge (x \Rightarrow (x \wedge z))]$.

Proof. $(a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge c)) \wedge (a \Rightarrow (a \wedge c)) = (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge c)) \wedge ((a \wedge b) \Rightarrow (a \wedge b \wedge c)) = (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge c)) \wedge ((a \wedge b \wedge c) \Rightarrow (a \wedge b \wedge c)) = (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge c))$. The result follows from Lemma 3.8. ■

Lemma 3.11. *Let* $\mathbb{L} = \langle L, \wedge, \vee, \Rightarrow, 0, 1\rangle$ *be a semi-Heyting algebra.*

$\mathbb{L} \models (x \Rightarrow (x \wedge z)) \Rightarrow [(x \Rightarrow (x \wedge z)) \wedge [(y \Rightarrow (y \wedge z)) \Rightarrow [(y \Rightarrow (y \wedge z)) \wedge ((x \vee y) \Rightarrow ((x \vee y) \wedge z))]]] \approx 1$.

Proof. $(a \Rightarrow (a \wedge c)) \wedge [(a \vee b) \Rightarrow ((a \vee b) \wedge c)] = (a \Rightarrow (a \wedge c)) \wedge [[(a \vee b) \wedge (a \Rightarrow (a \wedge c))] \Rightarrow [(a \vee b) \wedge c \wedge (a \Rightarrow (a \wedge c))]] = (a \Rightarrow (a \wedge c)) \wedge [[(a \wedge c) \vee (b \wedge (a \Rightarrow (a \wedge c)))] \Rightarrow ((a \vee b) \wedge c)]$. Then $(a \Rightarrow (a \wedge c)) \wedge (b \Rightarrow (b \wedge c)) \wedge [(a \vee b) \Rightarrow ((a \vee b) \wedge c)] = (a \Rightarrow (a \wedge c)) \wedge (b \Rightarrow (b \wedge c)) \wedge [[(a \wedge c) \vee (b \wedge (a \Rightarrow (a \wedge c)))] \Rightarrow ((a \vee b) \wedge c)] = (a \Rightarrow (a \wedge c)) \wedge (b \Rightarrow (b \wedge c)) \wedge [((a \wedge c) \vee (b \wedge c)) \Rightarrow ((a \vee b) \wedge c)] = (a \Rightarrow (a \wedge c)) \wedge (b \Rightarrow (b \wedge c)) \wedge 1 = (a \Rightarrow (a \wedge c)) \wedge (b \Rightarrow (b \wedge c))$. Hence $(a \Rightarrow (a \wedge c)) \wedge (b \Rightarrow (b \wedge c)) \leq (a \vee b) \Rightarrow ((a \vee b) \wedge c)$. From Lemma 3.8, $a \Rightarrow (a \wedge c) \leq (b \Rightarrow (b \wedge c)) \Rightarrow [(b \Rightarrow (b \wedge c)) \wedge ((a \vee b) \Rightarrow ((a \vee b) \wedge c))]$, and by Corollary 3.9, $1 = (a \Rightarrow (a \wedge c)) \Rightarrow [(a \Rightarrow (a \wedge c)) \wedge [(b \Rightarrow (b \wedge c)) \Rightarrow [(b \Rightarrow (b \wedge c)) \wedge ((a \vee b) \Rightarrow ((a \vee b) \wedge c))]]]$. ■

Lemma 3.12. *Let $\mathbb{L} = \langle L, \wedge, \vee, \Rightarrow, 0, 1\rangle$ be a semi-Heyting algebra.*

$\mathbb{L} \models (z \to (z \wedge x)) \to [(z \to (z \wedge x)) \wedge [(z \to (z \wedge y)) \to [(z \to (z \wedge y)) \wedge (z \Rightarrow (z \wedge (x \wedge y)))]]] \approx 1.$

Proof. $(c \Rightarrow (c \wedge a)) \wedge (c \Rightarrow (c \wedge (a \wedge b))) \wedge (c \Rightarrow (c \wedge b)) = (c \Rightarrow (c \wedge a)) \wedge ((c \wedge a) \Rightarrow ((c \wedge a) \wedge b)) \wedge (c \Rightarrow (c \wedge b)) = (c \Rightarrow (c \wedge a)) \wedge ((c \wedge a \wedge b) \Rightarrow (c \wedge a \wedge b)) \wedge (c \Rightarrow (c \wedge b)) = (c \Rightarrow (c \wedge a)) \wedge (c \Rightarrow (c \wedge b))$. Then $(c \Rightarrow (c \wedge a)) \wedge (c \Rightarrow (c \wedge b)) \leq c \Rightarrow (c \wedge (a \wedge b))$. The result follows from Lemma 3.8. ■

Lemma 3.13. *Let $\mathbb{L} = \langle L, \wedge, \vee, \Rightarrow, 0, 1\rangle$ be a semi-Heyting algebra.*

$\mathbb{L} \models (x \Rightarrow (x \wedge y)) \Rightarrow [(x \Rightarrow (x \wedge y)) \wedge [(y \Rightarrow (y \wedge x)) \Rightarrow [(y \Rightarrow (y \wedge x)) \wedge [(x \Rightarrow z) \Rightarrow ((x \Rightarrow z) \wedge (y \Rightarrow z))]]]] \approx 1.$

Proof. $(a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a)) \wedge (a \Rightarrow c) \wedge (b \Rightarrow c) = (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a)) \wedge ((a \wedge b) \Rightarrow (c \wedge (a \Rightarrow (a \wedge b))) \wedge ((b \wedge a) \Rightarrow (c \wedge (a \Rightarrow (a \wedge b))) = (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a)) \wedge [(a \wedge (a \Rightarrow (a \wedge b))) \Rightarrow (c \wedge (a \Rightarrow (a \wedge b)))] \wedge [(b \wedge (b \Rightarrow (b \wedge a))) \Rightarrow (c \wedge (b \Rightarrow (b \wedge a)))] = (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a)) \wedge (a \Rightarrow c) \wedge [(a \wedge b) \Rightarrow (c \wedge (b \Rightarrow (b \wedge a)))] = (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a)) \wedge (a \Rightarrow c) \wedge [(a \wedge (a \Rightarrow (a \wedge b))) \Rightarrow (c \wedge (b \Rightarrow (b \wedge a))) = (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a)) \wedge (a \Rightarrow c) \wedge [(a \wedge (a \Rightarrow (a \wedge b))) \wedge (b \Rightarrow (b \wedge a))) \Rightarrow (c \wedge (b \Rightarrow (b \wedge a)))] = (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a)) \wedge (a \Rightarrow c) \wedge [(a \wedge (a \Rightarrow (a \wedge b))) \wedge (b \Rightarrow (b \wedge a))) \Rightarrow (c \wedge (b \Rightarrow (b \wedge a)) \wedge (a \Rightarrow (a \wedge b)))] = (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a)) \wedge (a \Rightarrow c) \wedge (a \Rightarrow c) = (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a)) \wedge (a \Rightarrow c)$. Thus $(a \Rightarrow (a \wedge b)) \wedge [(b \Rightarrow (b \wedge a)) \Rightarrow [(b \Rightarrow (b \wedge a)) \wedge [(a \Rightarrow c) \Rightarrow ((a \Rightarrow c) \wedge (b \Rightarrow c))]]] = (a \Rightarrow (a \wedge b)) \wedge 1 = (a \Rightarrow (a \wedge b))$. The result follows from Lemma 3.8. ■

Lemma 3.14. *Let $\mathbb{L} = \langle L, \wedge, \vee, \Rightarrow, 0, 1\rangle$ be a semi-Heyting algebra.*

$\mathbb{L} \models (x \Rightarrow (x \wedge y)) \Rightarrow [(x \Rightarrow (x \wedge y)) \wedge [(y \Rightarrow (y \wedge x)) \Rightarrow [(y \Rightarrow (y \wedge x)) \wedge ((z \Rightarrow y) \Rightarrow ((z \Rightarrow y) \wedge (z \Rightarrow x)))]]] \approx 1.$

PROOF. $(a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a)) \wedge (c \Rightarrow b) \wedge (c \Rightarrow a) = [[c \wedge (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a))] \Rightarrow [a \wedge (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a))]] \wedge (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a)) \wedge (c \Rightarrow b) = [[c \wedge (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a))] \Rightarrow (a \wedge b)] \wedge (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a)) \wedge (c \Rightarrow b) = [[c \wedge (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a))] \Rightarrow [b \wedge (b \Rightarrow (b \wedge a))]] \wedge (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a)) \wedge (c \Rightarrow b) = (c \Rightarrow b) \wedge (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a)) \wedge (c \Rightarrow b) = (c \Rightarrow b) \wedge (a \Rightarrow (a \wedge b)) \wedge (b \Rightarrow (b \wedge a))$. The identity follows from Corollary 3.9. ■

LEMMA 3.15. *Let* $\mathbb{L} = \langle L, \wedge, \vee, \Rightarrow, 0, 1\rangle$ *be a semi-Heyting algebra.*

(a) $\mathbb{L} \models ((x \wedge y) \Rightarrow ((x \wedge y) \wedge z)) \Rightarrow (((x \wedge y) \Rightarrow ((x \wedge y) \wedge z)) \wedge (x \Rightarrow (x \wedge (y \Rightarrow (y \wedge z))))) \approx 1$

(b) $\mathbb{L} \models (x \Rightarrow (x \wedge (y \Rightarrow (y \wedge z)))) \Rightarrow ((x \Rightarrow (x \wedge (y \Rightarrow (y \wedge z)))) \wedge ((x \wedge y) \Rightarrow ((x \wedge y) \wedge z))) \approx 1$

PROOF.

(a) By Corollary 3.9 we have to prove that $(a \wedge b) \Rightarrow ((a \wedge b) \wedge c) \leq a \Rightarrow (a \wedge (b \Rightarrow (b \wedge c)))$. $a \wedge [(a \wedge b) \Rightarrow ((a \wedge b) \wedge c)] = a \wedge [b \Rightarrow (b \wedge c)]$ and $a \wedge (b \Rightarrow (b \wedge c)) \wedge ((a \wedge b) \Rightarrow ((a \wedge b) \wedge c)) = a \wedge (b \Rightarrow (b \wedge c))$. Consequently $((a \wedge b) \Rightarrow ((a \wedge b) \wedge c)) \wedge [a \Rightarrow (a \wedge (b \Rightarrow (b \wedge c)))] = ((a \wedge b) \Rightarrow ((a \wedge b) \wedge c)) \wedge 1 = ((a \wedge b) \Rightarrow ((a \wedge b) \wedge c))$.

(b) By Corollary 3.9 we have to prove that $a \Rightarrow (a \wedge (b \Rightarrow (b \wedge c))) \leq (a \wedge b) \Rightarrow ((a \wedge b) \wedge c)$. $a \wedge b \wedge [a \Rightarrow (a \wedge (b \Rightarrow (b \wedge c)))] = b \wedge a \wedge (b \Rightarrow (b \wedge c)) = a \wedge b \wedge c$ and $a \wedge b \wedge c \wedge [a \Rightarrow (a \wedge (b \Rightarrow (b \wedge c)))] = a \wedge b \wedge c \wedge c = a \wedge b \wedge c$. Then $[a \Rightarrow (a \wedge (b \Rightarrow (b \wedge c)))] \wedge [(a \wedge b) \Rightarrow ((a \wedge b) \wedge c)] = [a \Rightarrow (a \wedge (b \Rightarrow (b \wedge c)))] \wedge 1 = a \Rightarrow (a \wedge (b \Rightarrow (b \wedge c)))$. ■

By an $\mathbb{L}$-*valuation of propositional variables* in a semi-Heyting algebra $\mathbb{L}$ we shall understand any mapping e from the set of variables into $\mathbb{L}$. This mapping can be extended to a *valuation* of the set of formulas by defining $e(\neg\alpha) = e(\alpha) \Rightarrow 0$, $e(\alpha \wedge \beta) = e(\alpha) \wedge e(\beta)$, $e(\alpha \vee \beta) = e(\alpha) \vee e(\beta)$, $e(\alpha \to \beta) = e(\alpha) \Rightarrow e(\beta)$.

A formula α is an $\mathbb{L}$-*tautology* if $e(\alpha) = 1$ for any $\mathbb{L}$-*valuation* e. We say that a formula α is an $\mathcal{SH}$-*tautology* if it is an $\mathbb{L}$-tautology for any semi-Heyting algebra $\mathbb{L}$. The following theorem proves that the logic $\mathcal{SI}$ is sound with respect to $\mathbb{L}$-tautologies.

THEOREM 3.16 (Soundness). *If* ϕ *is provable in* $\mathcal{SI}$, *then* ϕ *is an* $\mathbb{L}$-*tautology for each semi-Heyting algebra* $\mathbb{L}$.

Proof. Let ϕ be an axiom of the logic $\mathcal{SI}$ and $\mathbb{L} \in \mathcal{SH}$. If ϕ is (S_1), (S_4), (S_7), (S_{15}), (S_{16}), (S_8) or (S_9) the result follows from Lemmas 3.8, 3.10, 3.11, 3.12, 3.13, 3.14 and 3.15, respectively.

Let e be an $\mathbb{L}$-valuation. Then

(S_2): $e(\alpha \to (\alpha \wedge (\alpha \vee \beta))) = e(\alpha) \Rightarrow e(\alpha \wedge (\alpha \vee \beta)) = e(\alpha) \Rightarrow (e(\alpha) \wedge (e(\alpha) \vee e(\beta))) = e(\alpha) \Rightarrow e(\alpha) = 1$.

(S_3): Similarly $e(\beta \to (\beta \wedge (\beta \vee \alpha))) = 1$.

(S_5) $e((\alpha\wedge\beta) \to ((\alpha\wedge\beta)\wedge\alpha)) = e(\alpha\wedge\beta) \Rightarrow e(\alpha\wedge\beta\wedge\alpha) = (e(\alpha)\wedge e(\beta)) \Rightarrow (e(\alpha) \wedge e(\beta)) = 1$.

(S_6): In a similar way $e((\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge \beta)) = 1$.

(S_{10}): $e(\alpha \to (\alpha\wedge\alpha)) = e(\alpha) \Rightarrow e(\alpha) = 1$. $e[(\alpha\wedge(\beta \to \gamma)) \to ((\alpha\wedge(\beta \to \gamma))\wedge(\alpha\wedge((\alpha\wedge\beta) \to (\alpha\wedge\gamma))))] = (e(\alpha)\wedge(e(\beta) \Rightarrow e(\gamma))) \Rightarrow ((e(\alpha)\wedge(e(\beta) \Rightarrow e(\gamma)))\wedge(e(\alpha)\wedge((e(\alpha)\wedge e(\beta)) \Rightarrow (e(\alpha)\wedge e(\gamma))))) = (e(\alpha)\wedge(e(\beta) \Rightarrow e(\gamma))) \Rightarrow ((e(\alpha) \wedge (e(\beta) \Rightarrow e(\gamma))) \wedge (e(\alpha) \wedge (e(\beta) \Rightarrow e(\gamma)))) = 1$.

(S_{12}): $e[(\alpha\wedge((\alpha\wedge\beta) \to (\alpha\wedge\gamma))) \to ((\alpha\wedge((\alpha\wedge\beta) \to (\alpha\wedge\gamma)))\wedge(\alpha\wedge(\beta \to \gamma)))] = 1$. $e[(\alpha \wedge (\alpha \to \beta)) \to ((\alpha \wedge (\alpha \to \beta)) \wedge (\alpha \wedge \beta))] = (e(\alpha) \wedge (e(\alpha) \to e(\beta))) \Rightarrow ((e(\alpha) \wedge (e(\alpha) \to e(\beta))) \wedge (e(\alpha) \wedge e(\beta)))] = (e(\alpha) \wedge e(\beta)) \Rightarrow (e(\alpha) \wedge e(\beta) \wedge e(\alpha) \wedge e(\beta)) = 1$.

(S_{14}): In a similar way $e[(\alpha \wedge \beta) \to ((\alpha \wedge \beta) \wedge (\alpha \wedge (\alpha \to \beta)))] = 1$.

(S_{17}): $e[(\alpha \wedge \neg\alpha) \to ((\alpha \wedge \neg\alpha) \wedge \beta)] = (e(\alpha) \wedge e(\neg\alpha)) \Rightarrow ((e(\alpha) \wedge e(\neg\alpha)) \wedge e(\beta)) = (e(\alpha) \wedge (e(\alpha) \Rightarrow 0)) \Rightarrow (e(\alpha) \wedge (e(\alpha) \Rightarrow 0) \wedge e(\beta)) = (e(\alpha) \wedge 0) \Rightarrow (e(\alpha) \wedge 0 \wedge e(\beta)) = 0 \Rightarrow 0 = 1$ y $e[(\alpha \to (\alpha \wedge \neg\alpha)) \to ((\alpha \to (\alpha \wedge \neg\alpha)) \wedge (\neg\alpha))] = ((e(\alpha)) \Rightarrow 0) \Rightarrow (((e(\alpha)) \Rightarrow 0) \wedge ((e(\alpha)) \Rightarrow 0)) = 1$.

So we have proved that every axiom is an $\mathcal{SH}$-tautology.

Suppose that ϕ and $\phi \to (\phi \wedge \alpha)$ are $\mathcal{SH}$-tautologies. Then $e(\phi) = e(\phi \to (\phi \wedge \alpha)) = 1$. Thus we have that $e(\phi) \geq e(\phi) \wedge e(\alpha) = e(\phi) \wedge (e(\phi) \Rightarrow (e(\phi) \wedge e(\alpha))) = e(\phi) \wedge e(\phi \to (\phi \wedge \alpha)) = 1 \wedge 1 = 1$. Hence $e(\alpha) = 1$. So α is an $\mathcal{SH}$-tautology. ∎

Theorem 3.17 (Completeness). *$\mathcal{SI}$ is complete: for each formula ϕ the following are equivalent:*

(a) *ϕ is provable in $\mathcal{SI}$.*

(b) *For each semi-Heyting algebra $\mathbb{L}$, ϕ is an $\mathbb{L}$-tautology.*

Proof. (a) implies (b) is an immediate consequence of Theorem 3.16. Suppose that ϕ is an $\mathbb{L}$-tautology for every semi-Heyting algebra $\mathbb{L}$. Consider the Lindenbaum algebra $\langle Form/\!\equiv, \wedge, \vee, \Rightarrow, [\![\alpha \wedge \neg\alpha]\!], [\![\alpha \to \alpha]\!]\rangle$. By Theorem 3.7, $\langle Form/\!\equiv, \wedge, \vee, \Rightarrow, [\![\alpha \wedge \neg\alpha]\!], [\![\alpha \to \alpha]\!]\rangle$ is a semi-Heyting algebra. Now, for each propositional variable p we define $e(p) = [\![p]\!]$ and we extend this valuation to the set of all formulas. By hypothesis, $e(\phi) = [\![\alpha \to \alpha]\!]$.

So $[\![\phi]\!] = [\![\alpha \to \alpha]\!]$ and, consequently, $\mathcal{SI} \vdash (\alpha \to \alpha) \to ((\alpha \to \alpha) \wedge \phi)$. From Lemmas 2.5 and 2.4, $\mathcal{SI} \vdash \phi$. Thus ϕ is provable in the logic $\mathcal{SI}$. ■

The following is a variant of the *deduction theorem.*

THEOREM 3.18. (Deduction Theorem) *Let T be a theory and let ϕ, ψ be formulas. $T \cup \{\phi\} \vdash \psi$ if and only if there is a natural number n such that $T \vdash \phi^n \to (\phi^n \wedge \psi)$ (where ϕ^n is $\phi \wedge \ldots \wedge \phi$, n factors). Consequently, since $\mathcal{SI} \vdash \phi \wedge \phi \leftrightarrow \phi$, $T \cup \{\phi\} \vdash \psi$ if and only if $T \vdash \phi \to (\phi \wedge \psi)$*

In order to prove the theorem we need two lemmas.

LEMMA 3.19. *If $\mathcal{SH} \vdash \phi \to (\phi \wedge \gamma)$ and $\mathcal{SH} \vdash \beta \to (\beta \wedge (\gamma \to (\gamma \wedge \psi)))$ then $\mathcal{SH} \vdash (\phi \wedge \beta) \to ((\phi \wedge \beta) \wedge \psi)$.*

PROOF.

1. $\mathcal{SI} \vdash \phi \to (\phi \wedge \gamma)$ by hypothesis
2. $\mathcal{SI} \vdash (\phi \wedge \beta) \to ((\phi \wedge \beta) \wedge \gamma)$ by Lemma 2.1
3. $\mathcal{SI} \vdash \beta \to (\beta \wedge (\gamma \to (\gamma \wedge \psi)))$ by hypothesis
4. $\mathcal{SI} \vdash (\gamma \wedge \beta) \to ((\gamma \wedge \beta) \wedge (\gamma \to (\gamma \wedge \psi)))$ by Lemma 2.1
5. $\mathcal{SI} \vdash (\phi \wedge \beta) \to ((\phi \wedge \beta) \wedge \gamma \wedge (\gamma \to (\gamma \wedge \psi)))$ by transitivity in 2 and 4
6. $\mathcal{SI} \vdash (\gamma \wedge (\gamma \to (\gamma \wedge \psi))) \to ((\gamma \wedge (\gamma \to (\gamma \wedge \psi))) \wedge (\gamma \wedge \psi))$ by (S_{13})
7. $\mathcal{SI} \vdash (\phi \wedge \beta) \to ((\phi \wedge \beta) \wedge (\gamma \wedge \psi))$ by transitivity in 5 and 6
8. $\mathcal{SI} \vdash (\gamma \wedge \psi) \to (\gamma \wedge \psi \wedge \psi)$ by axiom (S_6)
9. $\mathcal{SI} \vdash (\phi \wedge \beta) \to ((\phi \wedge \beta) \wedge \psi)$ by transitivity in 7 and 8. ■

LEMMA 3.20. *Let T be a theory over $\mathcal{SI}$. Let ϕ, ψ be formulas. If $T \vdash \phi \to (\phi \wedge \psi)$ then $T \cup \{\phi\} \vdash \psi$.*

PROOF. Clearly $T \cup \{\phi\} \vdash \phi$. As $T \vdash \phi \to (\phi \wedge \psi)$ by hypothesis, then $T \cup \{\phi\} \vdash \phi \to (\phi \wedge \psi)$. By SMP, $T \cup \{\phi\} \vdash \psi$. ■

PROOF. (Deduction Theorem) Suppose that there is an n such that $T \vdash \phi^n \to (\phi^n \wedge \psi)$. If $n = 1$, $T \vdash \phi \to (\phi \wedge \psi)$. From Lemma 3.20, $T \cup \{\phi\} \vdash \psi$. If $n > 1$, $T \vdash \phi \wedge \phi^{n-1} \to (\phi \wedge \phi^{n-1} \wedge \psi)$. By ($S_8$) and SMP, $T \vdash \phi \to (\phi \wedge (\phi^{n-1} \to (\phi^{n-1} \wedge \psi)))$ and consequently, $T \cup \{\phi\} \vdash \phi \to (\phi \wedge (\phi^{n-1} \to (\phi^{n-1} \wedge \psi)))$. As $T \cup \{\phi\} \vdash \phi$ by SMP, $T \cup \{\phi\} \vdash \phi^{n-1} \to (\phi^{n-1} \wedge \psi)$. Applying SMP $n - 1$ times we have $T \cup \{\phi\} \vdash \psi$.

Suppose now that $T \cup \{\phi\} \vdash \psi$. Let $\gamma_1, \ldots, \gamma_k$ a proof of ψ in the theory $T \cup \{\phi\}$. If $k = 1$, ψ is an axiom in the logic $\mathcal{SH}$, $\psi \in T$ or $\psi = \phi$. Since

$T \cup \{\phi\} \vdash \phi$ we have $T \cup \{\phi\} \vdash \phi \to (\phi \wedge \psi)$ (Lemma 2.4). If $k > 1$, then γ_k is obtained by applying semi-modus ponens to $\gamma_i, \gamma_i \to (\gamma_i \wedge \gamma_k)$. By inductive hypothesis, there exist $n, m \in \mathbb{N}$ such that $T \vdash \phi^n \to (\phi^n \wedge \gamma_i)$ and $T \vdash \phi^m \to (\phi^m \wedge (\gamma_i \to (\gamma_i \wedge \gamma_k)))$. From Lemma 3.19, $T \vdash (\phi^n \wedge \phi^m) \to ((\phi^n \wedge \phi^m) \wedge \gamma_k)$. Hence $T \vdash \phi^{n+m} \to (\phi^{n+m} \wedge \gamma_k)$. ■

Definition 3.21. A theory T is contradictory (or inconsistent) if $T \vdash \alpha \wedge \neg\alpha$ for some formula α. Otherwise T is consistent.

Lemma 3.22. *T is inconsistent if and only if $T \vdash \phi$ for each ϕ.*

Proof. If T is inconsistent, $T \vdash \alpha \wedge \neg\alpha$ for some formula α. Let ϕ be a formula. By axiom (S_{17}), $T \vdash (\alpha \wedge \neg\alpha) \to ((\alpha \wedge \neg\alpha) \wedge \phi)$. Finally, by SMP, $T \vdash \phi$. The converse is immediate. ■

Definition 3.23. Let T be a fixed theory over $\mathcal{SI}$. For each formula ϕ, let $[\phi]_T$ be the set of all formulas ψ such that $T \vdash \phi \leftrightarrow \psi$ (formulas T-provably equivalent to ϕ). Let L_T denote the set of all classes $[\phi]_T$. We define on L_T: $0 := [\alpha \wedge \neg\alpha]_T$, $1 := [\alpha \to \alpha]_T$, $[\phi]_T \wedge [\phi']_T = [\phi \wedge \phi']_T$, $[\phi]_T \vee [\phi']_T = [\phi \vee \phi']_T$ and $[\phi]_T \Rightarrow [\phi']_T = [\phi \to \phi']_T$. This algebra is denoted by $\mathbb{L}_T$.

As in Theorem 3.7 it can be proved that:

Lemma 3.24. *$\mathbb{L}_T$ is a semi-Heyting algebra.*

According to [2, Def. 2.4.1] we adopt the following definition of completeness.

Definition 3.25. A theory T is complete if for each pair ϕ, ψ of formulas, $T \vdash (\phi \vee (\phi \to \psi)) \to (\phi \to \psi)$ or $T \vdash \psi \to (\phi \wedge \psi)$.

We say that $\mathbb{L} \in \mathcal{SH}$ is a *semi-Heyting chain* if the lattice reduct of $\mathbb{L}$ is totally ordered.

Theorem 3.26. [1] *An equational basis for the variety generated by semi-Heyting chains relative to $\mathcal{SH}$ is given by*

$$((x \vee (x \Rightarrow y)) \Rightarrow (x \Rightarrow y)) \vee (y \Rightarrow (x \wedge y)) \approx 1$$

Lemma 3.27. *A theory T is complete if and only if the algebra $\mathbb{L}_T$ is a semi-Heyting chain.*

Proof. Suppose that T is complete. Let $[\phi]_T, [\psi]_T \in \mathbb{L}_T$. By hypothesis, $T \vdash (\phi \vee (\phi \to \psi)) \to (\phi \to \psi)$ or $T \vdash \psi \to (\phi \wedge \psi)$. Then $[(\phi \vee (\phi \to \psi)) \to$

$(\phi \to \psi)]_T = 1$ or $[\psi \to (\phi \wedge \psi)]_T = 1$. So $\mathbb{L}_T \models ((x \vee (x \to y)) \to (x \to y)) \vee (y \to (x \wedge y)) \approx 1$. Hence, by theorem 3.26 $\mathbb{L}_T$ is a semi-Heyting chain. The converse in immediate since $\mathbb{L}_T \models ((x \vee (x \to y)) \to (x \to y)) \vee (y \to (x \wedge y)) \approx 1$ whenever $\mathbb{L}_T$ is a semi-Heyting chain. ■

DEFINITION 3.28. (a) An axiom schema given by a formula $\Phi(p_1, \dots, p_n)$ is the set of all formulas $\Phi(\phi_1, \dots, \phi_n)$ resulting by the substitution of ϕ_i for p_i $(i = 1, \dots, n)$ in $\Phi(p_1, \dots, p_n)$.

(b) A logical calculus $\mathcal{C}$ is a schematic extension of $\mathcal{SI}$ if it results from $\mathcal{SI}$ by adding some (finitely or infinitely many) axiom schemata to its axioms (semi-modus ponens remains the only inference rule).

(c) Let $\mathcal{C}$ be a schematic extension of $\mathcal{SI}$ and let $\mathbb{L}$ be a semi-Heyting algebra. $\mathbb{L}$ is a $\mathcal{C}$-algebra if all axioms of $\mathcal{C}$ are $\mathbb{L}$-tautologies.

Observe that the algebra $\mathbb{L}_C$ of classes of $\mathcal{C}$-equivalent formulas is itself a $\mathcal{C}$-algebra:
if $\Phi(\phi_1, \dots, \phi_n)$ is an instance of the axiom schema $\Phi(p_1, \dots, p_n)$ and $e(p_i) = [\psi_i]_C$ then $e(\Phi(\phi_1, \dots, \phi_n)) = [\Phi(\phi'_1, \dots, \phi'_n)]_C$, where ϕ' results from ϕ_i by substituting ψ_i for p_i. Thus $\Phi(\phi'_1, \dots, \phi'_n)$ is also an instance of the schema and therefore $[\Phi(\phi'_1, \dots, \phi'_n)]_C = [1]_C$.

Then, as in the proof of theorem 3.17 we can establish the next result:

THEOREM 3.29 (Completeness). *Let $\mathcal{C}$ be a schematic extension of $\mathcal{SI}$ and let ϕ be a formula. The following are equivalent:*

(a) *$\mathcal{C}$ proves ϕ.*

(b) *ϕ is an $\mathbb{L}$-tautology for each $\mathcal{C}$-algebra $\mathbb{L}$.*

4. Relationship between the logics $\mathcal{I}$ and $\mathcal{SI}$

In this section we prove that the intuitionistic logic is an axiomatic extension of the semi-intuitionistic logic.

LEMMA 4.1. *The logic $\mathcal{I}$ satisfies the following properties:*

(a) $\mathcal{I} \vdash (\alpha \to \beta) \to (\alpha \to (\alpha \wedge \beta))$.

(b) *If* $\mathcal{I} \vdash \alpha \to \beta$ *then* $\mathcal{I} \vdash (\alpha \wedge \gamma) \to (\beta \wedge \gamma)$.

(c) $\mathcal{I} \vdash \alpha \to (\alpha \wedge \alpha)$.

Proof.

(a) 1. $\mathcal{I} \vdash (\alpha \to \alpha) \to ((\alpha \to \beta) \to (\alpha \to (\alpha \wedge \beta)))$ by (T_7).
2. $\mathcal{I} \vdash (\alpha \wedge \alpha) \to \alpha$ by (T_5).
3. $\mathcal{I} \vdash (((\alpha \wedge \alpha) \to \alpha) \wedge \alpha) \to \alpha$ by (T_5)
4. $\mathcal{I} \vdash [(((\alpha\wedge\alpha) \to \alpha)\wedge\alpha) \to \alpha] \to [((\alpha\wedge\alpha) \to \alpha) \to (\alpha \to \alpha)]$ by (T_9).
5. $\mathcal{I} \vdash ((\alpha \wedge \alpha) \to \alpha) \to (\alpha \to \alpha)$ by MP in 3 and 4.
6. $\mathcal{I} \vdash \alpha \to \alpha$ by MP in 2 and 5.
7. $\mathcal{I} \vdash (\alpha \to \beta) \to (\alpha \to (\alpha \wedge \beta))$ by MP in 1 and 6.

(b) 1. $\mathcal{I} \vdash (\alpha \wedge \gamma) \to \alpha$ by (T_5).
2. $\mathcal{I} \vdash \alpha \to \beta$ by hypothesis.
3. $\mathcal{I} \vdash (\alpha \wedge \gamma) \to \beta$ by (T_1) and MP.
4. $\mathcal{I} \vdash (\alpha \wedge \gamma) \to \gamma$ by (T_6).
5. $\mathcal{I} \vdash (\alpha \wedge \gamma) \to (\beta \wedge \gamma)$ by (T_7) and MP.

(c) 1. $\mathcal{I} \vdash (\alpha \to \alpha) \to (\alpha \to (\alpha \wedge \alpha))$ by(a).
2. $\mathcal{I} \vdash \alpha \to \alpha$.
3. $\mathcal{I} \vdash \alpha \to (\alpha \wedge \alpha)$ by MP. ∎

The following lemma states that the logic $\mathcal{I}$ proves all the axioms of the logic $\mathcal{SI}$.

Lemma 4.2. *If ϕ is an axiom of the logic $\mathcal{SI}$ then $\mathcal{I} \vdash \phi$.*

Proof. Let ϕ an axiom of the logic $\mathcal{SI}$. So ϕ is provable in the logic $\mathcal{SI}$. Let $\mathbb{L}$ be a Heyting algebra. In particular $\mathbb{L}$ is a semi-Heyting algebra. By Theorem 3.17, ϕ is an $\mathbb{L}$-tautology. Hence ϕ is provable in the logic $\mathcal{I}$ [3]. ∎

In the following lemma we prove that Modus Ponens coincides with Semi-Modus Ponens in the logic $\mathcal{I}$.

Lemma 4.3. *The following conditions are equivalent:*

(a) *If $\mathcal{I} \vdash \alpha$ and $\mathcal{I} \vdash \alpha \to \beta$ then $\mathcal{I} \vdash \beta$.*

(b) *If $\mathcal{I} \vdash \alpha$ and $\mathcal{I} \vdash \alpha \to (\alpha \wedge \beta)$ then $\mathcal{I} \vdash \beta$.*

Proof. (a) $\Rightarrow$ (b)

1. $\mathcal{I} \vdash \alpha$ by hypothesis.
2. $\mathcal{I} \vdash \alpha \to (\alpha \wedge \beta)$ by hypothesis.
3. $\mathcal{I} \vdash (\alpha \wedge \beta) \to \beta$ by (T_6).
4. $\mathcal{I} \vdash \alpha \to \beta$ by (T_1) and MP.
5. $\mathcal{I} \vdash \beta$ by (a).

(b) $\Rightarrow$ (a)

1. $\mathcal{I} \vdash \alpha$ by hypothesis.
2. $\mathcal{I} \vdash \alpha \to \beta$ by hypothesis.
3. $\mathcal{I} \vdash (\alpha \wedge \alpha) \to (\alpha \wedge \beta)$ by Lemma 4.1.
4. $\mathcal{I} \vdash \alpha \to (\alpha \wedge \alpha)$ by Lemma 4.1.
5. $\mathcal{I} \vdash \beta$ by (b). ■

From Lemmas 4.2 and 4.3 we can state that the logic $\mathcal{I}$ is an axiomatic extension of the logic $\mathcal{SI}$. The following lemma shows that $\mathcal{I}$ does not coincide with $\mathcal{SI}$.

LEMMA 4.4. *There exists a formula ϕ such that $\mathcal{I} \vdash \phi$ and $\mathcal{SI} \nvdash \phi$.*

PROOF. Let ϕ be the formula $(\alpha \wedge \neg\alpha) \to (\alpha \to \alpha)$. Clearly $\mathcal{I} \vdash (\alpha \wedge \neg\alpha) \to (\alpha \to \alpha)$. Let $\mathbb{L}$ be the two-element semi-Heyting chain with $0 \to 1 \approx 0$ and consider a valuation $e : Form \to \mathbb{L}$. Suppose that $\mathcal{SI} \vdash (\alpha \wedge \neg\alpha) \to (\alpha \to \alpha)$. By Theorem 3.17, ϕ is an $\mathbb{L}$-tautology. Hence $e(\phi) = 1$. But $1 = e(\phi) = e((\alpha \wedge \neg\alpha) \to (\alpha \to \alpha)) = e(\alpha \wedge \neg\alpha) \Rightarrow e(\alpha \to \alpha) = 0 \Rightarrow 1 = 0$, a contradiction. So $\mathcal{SI} \nvdash (\alpha \wedge \neg\alpha) \to (\alpha \to \alpha)$. ■

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References

[1] ABAD, M., J. M. CORNEJO, and J. P. DÍAZ VARELA, 'The Variety Generated by Semi-Heyting Chains', to appear in *Soft Computing*.

[2] HÁJEK, P., *Metamathematics of Fuzzy Logic*, Trends in Logic, Studia Logica Library, Dordrecht, Boston, London, Vol. 4, 1998.

[3] RASIOWA, H., and R. SIKORSKI, *The Mathematics of Metamathematics*, Polska Akademia Nauk, Monografie Matematyczne, Tom 41, Warszawa, 1963.

[4] SANKAPPANAVAR, H. P., 'Semi-Heyting Algebras: An Abstraction From Heyting Algebras', *Actas del IX Congreso Dr. Antonio A. R. Monteiro*, 33–66, 2007.

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