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$\in_K$: a Non-Fregean Logic of Explicit Knowledge

Abstract. We present a new logic-based approach to the reasoning about knowledge which is independent of possible worlds semantics. $\in_K$ (Epsilon-K) is a non-Fregean logic whose models consist of propositional universes with subsets for true, false and known propositions. Knowledge is, in general, not closed under rules of inference; the only valid epistemic principles are the knowledge axiom $K_i\varphi \rightarrow \varphi$ and some minimal conditions concerning common knowledge in a group. Knowledge is explicit and all forms of the logical omniscience problem are avoided. Various stronger epistemic properties such as positive and/or negative introspection, the K-axiom, closure under logical connectives, etc. can be restored by imposing additional semantic constraints. This yields corresponding sublogics for which we present sound and complete axiomatizations. As a useful tool for general model constructions we study abstract versions of some 3-valued logics in which we interpret truth as knowledge. We establish a connection between $\in_K$ and the well-known syntactic approach to explicit knowledge proving a result concerning equi-expressiveness. Furthermore, we discuss some self-referential epistemic statements, such as the knower paradox, as relaxations of variants of the liar paradox and show how these epistemic "paradoxes" can be solved in $\in_K$. Every specific $\in_K$-logic is defined as a certain extension of some underlying classical abstract logic.

Keywords: epistemic logic, non-Fregean Logic, logical omniscience, explicit knowledge, classical abstract logic, self-reference, epistemic paradox, liar paradox.

1. Introduction

The program of Non-Fregean Logic was introduced by R. Suszko as a proposal to develop logic without the Fregean Axiom (see [16, 17, 18]), i.e. without the assumption that sentences denote nothing but truth values: the True and the False. The basic propositional logic that relies on the principles of non-Fregean Logic is the *Sentential Calculus with Identity* SCI, investigated by Bloom and Suszko [3]. Semantics of SCI is defined in such a way that a formula not only denotes a truth value but also an object of the given model-theoretic universe of "situations". In our context we call such entities "propositions" (also "facts"). Propositions (facts) have a truth value and represent a semantic content which is determined by the ambient model and a semantic function that maps formulas to propositions. In the epistemic

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non-Fregean logic $\in_K$, propositions (facts) are not only the bearers of truth values but also the objects of knowledge, i.e. what an agent knows are propositions. If we say that "formula φ is true" or "agent i knows φ", then we mean that φ denotes a true proposition, and agent i knows the proposition denoted by φ, respectively. Our approach to semantics relies on the works of W. Sträter [15] and P. Zeitz [20] on $\in_T$-Logic. Sträter introduced $\in_T$ as a logic with classical connectives, an idenitity connective, a total truth predicate and first-order quantification over propositions. $\in_T$ is a non-Fregean logic and can be seen as an extension of SCI. However, Sträter's approach to semantics differs essentially from the algebraic semantics presented by Bloom and Suszko for SCI [3]. Sträter's "non-algebraic" semantics was simplified and further developed by Zeitz [20] and partially by the author [8]. We develop $\in_K$-logic as an epistemic extension of $\in_T$-Logic without quantifiers. Within this quantifier-free framework we will define a non-algebraic semantics in the style of Sträter as well as an algebraic semantics and will show that both approaches are equivalent. The idea for an epistemic extension of $\in_T$ without possible worlds semantics was already formulated by Zeitz (p. 130 in [20]) but was not further investigated. Our presentation relies in some technical aspects on [11] where an intuitionistic (and quantifier-free) extension of classical $\in_T$-Logic is presented. We consider $\in_K$ as an extension of an underlying classical abstract logic $\mathcal{L}$. The epistemic and semantic predicates of the $\in_K$-extension then apply in particular to the formulas of the logic $\mathcal{L}$. Such extensions were first investigated in [20] and [8] within the framework of $\in_T$. We will work here with a more sophisticated notion of abstract logic studied in [9, 10, 12]. This approach has some parallels to works on abstract logics due to Suszko, Bloom and Brown (see, e.g., [2, 4]) where abstract logics are defined in terms of closure spaces. In [9, 12] we study abstract logics as meet semi-lattices which are generated by a minimal meet-dense subset: the set of totally prime theories, which forms the bottom of a hierarchy of κ-prime theories ($\kappa \geq \omega$ a cardinal) defined in [12]. In the case of classical logics this minimal generator set is precisely the set of maximal theories. The following basic notions and results about abstract logics will be useful for our purposes. For more details we refer the reader to [12]. We work here with a "reduced" definition of classical abstract logic considering only the connectives of negation and implication (compare the notion presented in [10, 12] which derives from the definition of intuitionistic abstract logic).

Definition 1.1. $\mathcal{L} = (Expr_{\mathcal{L}}, Th_{\mathcal{L}}, \{\sim, \rightarrowtail\})$ is a classical abstract logic if the following conditions hold true:

(i) $Expr_{\mathcal{L}}$ is a set whose elements are called expressions (or formulas);

(ii) $Th_{\mathcal{L}}$, called the set of theories, is a subset of the power set of $Expr_{\mathcal{L}}$ such that $\mathcal{T} \subseteq Th_{\mathcal{L}}$ and $\mathcal{T} \neq \varnothing$ implies $\bigcap \mathcal{T} \in Th_{\mathcal{L}}$;

(iii) the set $MTh_{\mathcal{L}} \subseteq Th_{\mathcal{L}}$ of maximal theories (with respect to inclusion) is the minimal generator set $\mathcal{G}_{\mathcal{L}}$ (i.e., every theory is the intersection of a non-empty subset of $MTh_{\mathcal{L}} = \mathcal{G}_{\mathcal{L}}$, and $\mathcal{G}_{\mathcal{L}}$ is minimal with this property);

(iv) the consequence relation defined by $A \Vdash_{\mathcal{L}} a :\Leftrightarrow a \in \bigcap\{T \in Th_{\mathcal{L}} \mid A \subseteq T\}$ is compact (or finitary), that is, $A \Vdash_{\mathcal{L}} a$ implies the existence of a finite $A' \subseteq A$ with $A' \Vdash_{\mathcal{L}} a$;

(v) $\{\sim, \rightarrowtail\}$ is a set of operations on the set of formulas, called (abstract) connectives, such that for all $a, b \in Expr_{\mathcal{L}}$ and for all $T \in \mathcal{G}_{\mathcal{L}}$ the following holds:

- $\sim a \in T \Longleftrightarrow a \notin T$
- $a \rightarrowtail b \in T \Longleftrightarrow a \notin T$ or $b \in T$

A set of expressions A is said to be $\mathcal{L}$-consistent if A is contained in some theory of $\mathcal{L}$; otherwise, A is $\mathcal{L}$-inconsistent. If the context is clear, we say "consistent" and "inconsistent" instead of "$\mathcal{L}$-consistent" and "$\mathcal{L}$-inconsistent", respectively.

Definition 1.2. Let $\mathcal{L} = (Expr_{\mathcal{L}}, Th_{\mathcal{L}})$ be any structure satisfying (i) and (ii) of Definition 1.1. Let $\alpha > 0$ be an ordinal. An α-chain of theories $(T_i \mid i < \alpha)$ is a linear ordered sequence of theories T_i, $i < \alpha$, i.e. $T_i \subseteq T_j$ iff $i < j < \alpha$. We say that $\mathcal{L}$ is closed under chains if for every ordinal $\alpha > 0$ the union of an α-chain of theories is a theory.

One can do without condition (iii) in Definition 1.1: The compactness of the consequence relation and the existence of an inconsistent formula is equivalent with the condition that the logic is closed under chains (see, e.g., Theorem 2.17 [12]). A logic closed under chains has a minimal generator set $\mathcal{G}_{\mathcal{L}}$ (Corollary 2.12 [12]) which, by the first item of condition (v), must be the set of maximal theories. — From the definitions also follows that a set of expressions T is a theory iff T is consistent and deductively closed (i.e. T is contained in some theory, and $T \Vdash_{\mathcal{L}} a$ implies $a \in T$). — We introduce conjunction and disjunction as abbreviations: $a \curlywedge b :=\sim (a \rightarrowtail \sim b)$, $a \curlyvee b :=\sim a \rightarrowtail b$.

2. Syntax

Let $\mathcal{L} = (Expr_{\mathcal{L}}, Th_{\mathcal{L}}, \{\sim, \rightarrowtail\})$ be a classical abstract logic and $MTh_{\mathcal{L}} \subseteq Th_{\mathcal{L}}$ its set of complete (i.e. maximal) theories. Expressions and theories of $\mathcal{L}$ are also called $\mathcal{L}$-expressions and $\mathcal{L}$-theories, respectively. $1, \ldots, n$ is a collection of agents. By a group of agents we always mean a non-empty subset $G \subseteq \{1, \ldots, n\}$. $\mathcal{C}$ is a set of constant symbols (for propositions) containing at least the symbols $\top$ and $\bot$ (for a true, a false proposition, respectively). $\mathcal{C}$ possibly contains symbols $\top_i$ $(1 \leq i \leq n)$ to ensure that agent i knows at least one proposition, which is denoted by $\top_i$. $V = \{x_0, x_1, \ldots\}$ is an infinite set of propositional variables.

Definition 2.1. The set $Expr(\mathcal{C}, \mathcal{L}, n)$ of expressions (or formulas) is the smallest set that contains $Expr_{\mathcal{L}}$, $\mathcal{C}$ and V and is closed under the following condition: if φ and ψ are expressions, G is a group and $1 \leq i \leq n$ is an agent, then $(\varphi : true)$, $(\varphi : false)$, $(\varphi \rightarrow \psi)$, $(\neg\varphi)$, $(\varphi \equiv \psi)$, $(\varphi < \psi)$, $(K_i\varphi)$, $(C_G\varphi)$ are expressions. Variables, constant symbols and $\mathcal{L}$-expressions are called atomic. The set of sentences, denoted by $Sent(\mathcal{C}, \mathcal{L}, n)$, is the set of variable-free expressions.

We may omit outermost parentheses. Parentheses may also be omitted in accordance with the following order of descending priority of connectives and operators: $\neg, : true, : false, \equiv, <, K_i, C_G, \rightarrow$.

We take $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \leftrightarrow \psi$ as abbreviations for $\neg(\varphi \rightarrow \neg\psi)$, $\neg\varphi \rightarrow \psi$ and $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, respectively. $\varphi_1 \wedge \varphi_2 \wedge \ldots \wedge \varphi_m$ stands for $(\ldots(\varphi_1 \wedge \varphi_2) \wedge \varphi_3) \ldots \wedge \varphi_m)$. Similarly for disjunction.

The notion of subexpression is defined as usual. $sub(\varphi)$ denotes the set of all subexpressions of φ. We write $\varphi \prec \psi$, if φ is a proper subexpression of ψ. Let $var(\varphi) = sub(\varphi) \cap V$ be the set of variables occurring in φ, $at(\varphi) = sub(\varphi) \cap (\mathcal{C} \cup Expr_{\mathcal{L}} \cup V)$ the set of atomic subexpressions of φ, and $con(\varphi) = sub(\varphi) \cap \mathcal{C}$ the set of all constant symbols occurring in φ.

Definition 2.2. A substitution is a function

$$\sigma : \mathcal{C} \cup Expr_{\mathcal{L}} \cup V \rightarrow Expr(\mathcal{C}, \mathcal{L}, n).$$

If $A \subseteq \mathcal{C} \cup Expr_{\mathcal{L}} \cup V$ and $\sigma(u) = u$ for all $u \in (\mathcal{C} \cup Expr_{\mathcal{L}} \cup V) \smallsetminus A$, then we write $\sigma : A \rightarrow Expr(\mathcal{C}, \mathcal{L}, n)$. If σ is a substitution, $u_0, \ldots, u_m \in \mathcal{C} \cup Expr_{\mathcal{L}} \cup V$ and $\varphi_0, \ldots, \varphi_m \in Expr(\mathcal{C}, \mathcal{L}, n)$, then the substitution $\sigma[u_0 := \varphi_0, \ldots, u_m := \varphi_m]$ is defined by:

$$\sigma[u_0 := \varphi_0, \ldots, u_m := \varphi_m](u) = \begin{cases} \varphi_i & \text{if } u = u_i, \text{ for some } i \leq m \\ \sigma(u) & \text{else} \end{cases}$$

The substitution $u \mapsto u$ is denoted by ε. Instead of $\varepsilon[u_0 := \varphi_0, \ldots, u_m := \varphi_m]$ we write $[u_0 := \varphi_0, \ldots, u_m := \varphi_m]$. A substitution σ extends in the canonical way to a function $[\sigma] : Expr(\mathcal{C}, \mathcal{L}, n) \to Expr(\mathcal{C}, \mathcal{L}, n)$ (we use postfix notation for $[\sigma]$). The composition of two substitutions σ and τ is the substitution $(\sigma \circ \tau)$ defined by $(\sigma \circ \tau)(u) = \sigma(u)[\tau]$, for $u \in \mathcal{C} \cup Expr_{\mathcal{L}} \cup V$.

Lemma 2.3. *Suppose that φ, ψ are expression and σ, τ are substitutions.*

(i) *If $\sigma(u) = \tau(u)$ for all $u \in at(\varphi)$, then $\varphi[\sigma] = \varphi[\tau]$.*
(ii) *$\varphi[\sigma \circ \tau] = \varphi[\sigma][\tau]$.*
(iii) *If $\varphi \prec \psi$, then $\varphi[\sigma] \prec \psi[\sigma]$.*

Lemma 2.4. *Let φ, ψ be expressions. The following statements are equivalent:*
(i) There is some expression ψ', a variable $x \in var(\psi')$ and a substitution σ such that $\psi'[\sigma[x := \varphi]] = \psi$.
(ii) φ is a subexpression of ψ.

Corollary 2.5. *Let φ, ψ be expressions. Then $\varphi \prec \psi$ if and only if there is an expression ψ' and a variable $x \in var(\psi')$ with $x \neq \psi'$ such that $\psi'[x := \varphi] = \psi$.*

Every system having *modus ponens* among its rules and comprising the axioms of the following list will be an extension of SCI, the Sentential Calculus with Identity [3]. Recall that $\mathcal{C}$ possibly contains some of the symbols $\top_1, \ldots, \top_n$.

Definition 2.6. By $\mathbf{Ax}_n$ we denote the following list of axioms.

(i) $\sim a \to \neg a$ and $\neg a \to \sim a$, where $a \in Expr_{\mathcal{L}}$
(ii) $\varphi \to (\varphi : true)$ and $\varphi : true \to \varphi$
(iii) $\neg\varphi \to (\varphi : false)$ and $\varphi : false \to \neg\varphi$
(iv) $\varphi \equiv \varphi$
(v) $\varphi \equiv \psi \to (\varphi \to \psi)$
(vi) $\psi \equiv \psi' \to \chi[x := \psi] \equiv \chi[x := \psi']$
(vii) $\varphi < \psi$, whenever $\varphi \prec \psi$
(viii) $\varphi < \psi \to (\psi < \chi \to \varphi < \chi)$
(ix) $\bot \to \varphi$
(x) $\varphi \to \top$
(xi) $\varphi \to (\psi \to \varphi)$
(xii) $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
(xiii) $\neg\varphi \to (\varphi \to \psi)$
(xiv) $(\varphi \to \psi) \to ((\neg\varphi \to \psi) \to \psi)$

(xv) $K_i\top_i$, whenever $\top_i$ belongs to the language
(xvi) $K_i\varphi \rightarrow \varphi$
(xvii) $C_G\varphi \rightarrow K_i\varphi$, whenever $i \in G$
(xviii) $C_G\varphi \rightarrow C_GK_i\varphi$, whenever $i \in G$
(xix) $C_G\varphi \rightarrow C_{G'}\varphi$, whenever $G' \subseteq G$

We will work with only one rule, namely *modus ponens*. One recognizes that the epistemic axioms (xvi)-(xix) together with the axioms of classical propositional logic form a fragment of the modal epistemic system T_n^C (see, e.g., [13] for an axiomatization). In the last section we will add several stronger principles of knowledge. The resulting extensions are fragments, though only weak ones, of such well-known systems of epistemic logic as $S4_n^C$ or $S5_n^C$.

3. Semantics

We work with the language $Expr(\mathcal{C}, \mathcal{L}, n)$.

DEFINITION 3.1. An $\in_K$-structure (or model)

$$\mathcal{M} = (M, TRUE, (TRUE_i)_{1\leq i\leq n}, FALSE, <^{\mathcal{M}}, \Gamma, T)$$

with respect to T is given by the following:

(1) $M = TRUE \cup FALSE$ is the non-empty propositional universe consisting of true and false propositions. We require $TRUE \cap FALSE = \varnothing$. For $1 \leq i \leq n$ the set $TRUE_i \subseteq TRUE$ is the set of propositions that agent i (explicitly) knows.
(2) $<^{\mathcal{M}} \subseteq M \times M$ is called the reference relation between propositions.
(3) $T \in MTh_{\mathcal{L}}$ is a complete theory of the underlying logic $\mathcal{L}$.
(4) $\Gamma : Expr(\mathcal{C}, \mathcal{L}, n) \times M^V \rightarrow M$ is a so-called semantic function that maps an expression φ to its denotation: a proposition $\Gamma(\varphi, \gamma) \in M$. Γ depends on assignments $\gamma : V \rightarrow M$ of propositions to variables. If $\gamma \in M^V$ and δ is a substitution, then $\gamma\delta \in M^V$ denotes the assignment defined by $\gamma\delta(x) = \Gamma(\delta(x), \gamma)$.

Γ satisfies the following **structure conditions**:

(i) For all $x \in V$ and all assignments $\gamma \in M^V$, $\Gamma(x, \gamma) = \gamma(x)$ (Extension Property (EP));
(ii) If φ is an expression and $\gamma, \gamma' \in M^V$ are assignments with $\gamma(x) = \gamma'(x)$ for all $x \in var(\varphi)$, then $\Gamma(\varphi, \gamma) = \Gamma(\varphi, \gamma')$ (Coincidence Property (CP))[1];

[1] If $var(\varphi) = \varnothing$, then (CP) justifies to write $\Gamma(\varphi)$ instead of $\Gamma(\varphi, \gamma)$.

(iii) If φ is an expression, $\gamma \in M^V$ an assignment and $\sigma : V \to Expr(\mathcal{C}, \mathcal{L}, n)$ a substitution, then $\Gamma(\varphi[\sigma], \gamma) = \Gamma(\varphi, \gamma\sigma)$ (Substitution Property (SP));
(iv) If $\varphi \prec \psi$, then $\Gamma(\varphi, \gamma) <^{\mathcal{M}} \Gamma(\psi, \gamma)$, for all expressions φ, ψ and all assignments γ (Reference Property (RP)).

Γ satisfies the following **truth conditions**. For all expressions φ, ψ, for all assignments γ, for all agents $i \in \{1, \ldots, n\}$, and for all groups G of agents:
(i) $\Gamma(a) \in TRUE \Leftrightarrow a \in T$, for all $a \in Expr_{\mathcal{L}}$ (Correspondence Property)
(ii) $\Gamma(\top) \in TRUE$ and $\Gamma(\bot) \in FALSE$ and $\Gamma(\top_i) \in TRUE_i$
(iii) $\Gamma(\varphi : true, \gamma) \in TRUE \Leftrightarrow \Gamma(\varphi, \gamma) \in TRUE$
(iv) $\Gamma(\varphi : false, \gamma) \in TRUE \Leftrightarrow \Gamma(\varphi, \gamma) \in FALSE$
(v) $\Gamma(\neg\varphi, \gamma) \in TRUE \Leftrightarrow \Gamma(\varphi, \gamma) \in FALSE$
(vi) $\Gamma(\varphi \to \psi, \gamma) \in TRUE \Leftrightarrow \Gamma(\varphi, \gamma) \in FALSE$ or $\Gamma(\psi, \gamma) \in TRUE$
(vii) $\Gamma(\varphi \equiv \psi, \gamma) \in TRUE \Leftrightarrow \Gamma(\varphi, \gamma) = \Gamma(\psi, \gamma)$
(viii) $\Gamma(\varphi < \psi, \gamma) \in TRUE \Leftrightarrow \Gamma(\varphi, \gamma) <^{\mathcal{M}} \Gamma(\psi, \gamma)$
(ix) $\Gamma(K_i\varphi, \gamma) \in TRUE \Leftrightarrow \Gamma(\varphi, \gamma) \in TRUE_i$
(x) $\Gamma(C_G\varphi, \gamma) \in TRUE \Rightarrow \Gamma(K_i\varphi, \gamma) \in TRUE$ and $\Gamma(C_G K_i\varphi, \gamma) \in TRUE$ for all $i \in G$
(xi) $\Gamma(C_G\varphi, \gamma) \in TRUE \Rightarrow \Gamma(C_{G'}\varphi, \gamma) \in TRUE$, whenever $G' \subseteq G$.
$\mathcal{M}$ is said to be transitive if the reference relation $<^{\mathcal{M}}$ is transitive on M.

We refer to the semantic function Γ also as the Gamma-function. The structure conditions of the Gamma-function were established in similar ways by Sträter [15] and Zeitz [20], with exception of the Reference Property (RP), which was introduced together with the reference connective $<$ in [10, 11]. The truth conditions guarantee the intended meaning of the connectives, the truth predicate and the epistemic operators. In the last section we will impose some additional truth conditions in order to establish stronger properties of knowledge.

Definition 3.2. Let $\mathcal{M} = (M, TRUE, (TRUE_i)_{1 \le i \le n}, FALSE, <^{\mathcal{M}}, \Gamma, T)$ be a model and G a group. A set $X \subseteq \bigcap_{i \in G} TRUE_i$ is said to be closed under G if for any expression φ and any assignment $\gamma : V \to M$,

$$\Gamma(\varphi, \gamma) \in X \Rightarrow \Gamma(K_i\varphi, \gamma) \in X \text{ for every } i \in G.$$

Let $COMMON_G := \{\Gamma(\varphi, \gamma) \mid \Gamma(C_G\varphi, \gamma) \in TRUE, \varphi \in Expr(\mathcal{C}, \mathcal{L}, n), \gamma \in M^V\}$, and let $GREATEST_G$ be the union of all sets which are closed under G.

The *Substitution Principle* (Theorem 3.7 below) ensures that closure under G and $COMMON_G$ are well-defined concepts.

By a sequence of agents $i_1, \dots, i_k \in G$ we mean a k-tuple $(i_1, ..., i_k) \in G^k$. In the following we consider formulas $K_{i_1} K_{i_2} \dots K_{i_k}\varphi$ where $i_1, \dots, i_k$ is a sequence of agents of a given G. If $k = 0$, then we put $K_{i_1} \dots K_{i_k}\varphi := \varphi$.

LEMMA 3.3. *Suppose $\mathcal{M} = (M, TRUE, (TRUE_i)_{1\le i\le n}, FALSE, <^{\mathcal{M}}, \Gamma, T)$ is a model. For any G, $COMMON_G$ is closed under G and $GREATEST_G$ is the greatest set that is closed under G. In particular,*

$$COMMON_G \subseteq GREATEST_G \subseteq \bigcap_{i\in G} TRUE_i.$$

Furthermore, for any expression φ and any assignment γ,

$$\Gamma(C_G\varphi, \gamma) \in TRUE \Rightarrow \Gamma(K_{i_1} K_{i_2} \dots K_{i_k}\varphi, \gamma) \in TRUE, \quad \text{for all } k \ge 0 \text{ and for all sequences } i_1, \dots, i_k \in G.$$

The following conditions (a) and (b) are equivalent:
(a) $COMMON_G = GREATEST_G$.
(b) For any φ and any assignment γ:
$\Gamma(C_G\varphi, \gamma) \in TRUE \Leftrightarrow \Gamma(K_{i_1} K_{i_2} \dots K_{i_k}\varphi, \gamma) \in TRUE$, for all $k \ge 0$ and for all sequences $i_1, \dots, i_k \in G$.

PROOF. The first assertions follow easily from the definitions and truth condition (x). We prove the equivalence of (a) and (b). Suppose (a) is true. Let φ be any formula and γ any assignment such that the right side of the biconditional of (b) holds true. Let $Y = \{\Gamma(K_{i_1} K_{i_2} \dots K_{i_k}\varphi, \gamma) \mid k \ge 0$ and $i_1, \dots, i_k \in G\}$. Since Y is closed under G, $Y \subseteq GREATEST_G$ and in particular $\Gamma(\varphi, \gamma) \in GREATEST_G = COMMON_G$. Hence, $\Gamma(C_G\varphi, \gamma) \in TRUE$. Now suppose (b) is true. Let $m \in GREATEST_G$. There is some $x \in V$ and some assignment γ such that $\Gamma(x, \gamma) = \gamma(x) = m$. Since $GREATEST_G$ is closed, $\Gamma(K_{i_1} K_{i_2} \dots K_{i_k} x, \gamma) \in GREATEST_G \subseteq TRUE$, for every $k \ge 0$ and for every sequence $i_1, \dots, i_k \in G$. Thus, $\Gamma(C_G x, \gamma) \in TRUE$ and $m = \Gamma(x, \gamma) \in COMMON_G$. ■

REMARK 3.4. *If φ is any formula and G is any group, then the infinite collection of all formulas $K_{i_1} K_{i_2} \dots K_{i_k}\varphi$, where $k \ge 0$ and $i_1, \dots, i_k \in G$ is any sequence, can be seen as a formalization of: "the proposition denoted by φ is common knowledge in group G". Note that under the equivalent conditions (a) and (b) of Lemma 3.3 the single formula $C_G\varphi$ captures this notion precisely. Also observe that under the assumptions (a), (b) one may replace "$\Rightarrow$" by "$\Leftrightarrow$" in truth condition (x).*

Definition 3.5. If $\mathcal{M}$ is a model w.r.t. T and $\gamma : V \to M$ is an assignment, then $(\mathcal{M}, \gamma)$ is called an interpretation w.r.t. T. $(\mathcal{M}, \gamma)$ is called a transitive interpretation if $\mathcal{M}$ is transitive. — The satisfaction relation between interpretations and expressions is defined by: $(\mathcal{M}, \gamma) \vDash \varphi :\Leftrightarrow \Gamma(\varphi, \gamma) \in TRUE$. If $(\mathcal{M}, \gamma) \vDash \varphi$, then $(\mathcal{M}, \gamma)$ is called a model of φ. If φ is a sentence, then we may write $\mathcal{M} \vDash \varphi$. The satisfaction relation extends in the usual way to sets of expressions (of sentences). The set of all formulas satisfied by a given interpretation $(\mathcal{M}, \gamma)$ is called the theory of $(\mathcal{M}, \gamma)$.

Definition 3.6. The classical abstract logic generated by the theories of all *transitive* $\in_K$-interpretations is called the $\in_K$-extension of the underlying logic $\mathcal{L}$ (w.r.t. the set of constant symbols $\mathcal{C}$ and set of agents $1, \ldots, n$). We denote this logic by $\mathcal{L}^n_{\mathcal{C}}$ and its consequence relation by $\Vdash$. Any logic generated by a non-empty subclass of the class of all transitive $\in_K$-interpretations is called a $\in_K$-sublogic of $\mathcal{L}^n_{\mathcal{C}}$.

Note that we consider only *transitive* models. Also note that we define the notion of a "sublogic" in a model-theoretic sense; a valid formula is also valid in any sublogic. The existence of models is shown in section 5. The *Substitution Lemma* for $\in_I$-Logic found in [11] can easily be adopted to the case of $\in_K$-Logic. It implies the following *Substitution Principle*.

Theorem 3.7 (Substitution Principle). *Let $\mathcal{M}$ be a model with semantic function Γ. Suppose $\varphi_1, \varphi_2, \psi_1, \psi_2$ are expressions and $\gamma, \gamma' : V \to M$ are assignments such that $\Gamma(\varphi_1, \gamma) = \Gamma(\psi_1, \gamma')$ and $\Gamma(\varphi_2, \gamma) = \Gamma(\psi_2, \gamma')$. Then $\Gamma(\varphi_1 : true, \gamma) = \Gamma(\psi_1 : true, \gamma')$, $\Gamma(\varphi_1 : false, \gamma) = \Gamma(\psi_1 : false, \gamma')$, $\Gamma(\neg\varphi_1, \gamma) = \Gamma(\neg\psi_1, \gamma')$, $\Gamma(\varphi_1 \equiv \varphi_2, \gamma) = \Gamma(\psi_1 \equiv \psi_2, \gamma')$, $\Gamma(\varphi_1 < \varphi_2, \gamma) = \Gamma(\psi_1 < \psi_2, \gamma')$, $\Gamma(\varphi_1 \to \varphi_2, \gamma) = \Gamma(\psi_1 \to \psi_2, \gamma')$, $\Gamma(K_i\varphi_1, \gamma) = \Gamma(K_i\psi_1, \gamma')$ $\Gamma(C_G\varphi_1, \gamma) = \Gamma(C_G\psi_1, \gamma')$.*

Corollary 3.8. *For all expressions ψ, ψ', χ and variables $x \in V$:*

$$\Vdash \psi \equiv \psi' \to \chi[x := \psi] \equiv \chi[x := \psi'].$$

Note that Corollary 3.8 also follows directly from the structure properties (SP) and (CP) of a model.

Definition 3.9. Let $\mathcal{M}$ be a model and let $\gamma : V \to M$ be an assignment. The interpretation $(\mathcal{M}, \gamma)$ is called intensional[2] if for all expressions φ, ψ the following two conditions hold:

[2]Note that we use the term "intensional" in a rather strong sense, which deviates from its ordinary use in modal logic. What we mean is that in an intensional model the denotation of a formula is in one-to-one correspondence with its syntactical form so that different formulas cannot denote the same proposition (see also the discussions in [11]).

(i) $(\mathcal{M},\gamma) \vDash \varphi \equiv \psi \Rightarrow \varphi = \psi$, and
(ii) $(\mathcal{M},\gamma) \vDash \varphi < \psi \Rightarrow \varphi \prec \psi$.
$(\mathcal{M},\gamma)$ is called $\equiv$-intensional (or injective) if for all expressions φ, ψ condition (i) is satisfied. $(\mathcal{M},\gamma)$ is called $<$-intensional if for all expressions φ, ψ: if $(\mathcal{M},\gamma) \vDash \varphi < \psi$, then there are expressions φ', ψ' such that $\varphi' \prec \psi'$ and $\Gamma(\varphi',\gamma) = \Gamma(\varphi,\gamma)$ and $\Gamma(\psi',\gamma) = \Gamma(\psi,\gamma)$. If the above conditions remain true replacing all occurrences of "expressions" by "sentences", then the structure $\mathcal{M}$ is called intensional, $\equiv$-intensional, $<$-intensional, respectively.

We apply the notion of standard interpretation (standard model) in order to identify the intended structures.

Definition 3.10. An interpretation $(\mathcal{M},\gamma)$ is called surjective if every proposition is denoted by some expression, i.e. for every $m \in M$ there is some expression φ such that $\Gamma(\varphi,\gamma) = m$. A model $\mathcal{M}$ is called surjective if every proposition is the denotation of some sentence, i.e. for every $m \in M$ there is some sentence φ with $\Gamma(\varphi) = m$. The interpretation $(\mathcal{M},\gamma)$ is said to be a standard interpretation if the following hold:
(i) $COMMON_G = GREATEST_G$, for all groups G;
(ii) $(\mathcal{M},\gamma)$ is surjective;[3]
(iii) $(\mathcal{M},\gamma)$ is $<$-intensional.
The structure $\mathcal{M}$ is called a standard model if $COMMON_G = GREATEST_G$ (for all G), and $\mathcal{M}$ is surjective and $<$-intensional.

The notion of $<$-intensionality is weaker than the notion given by item (ii) in Definition 3.9. However, both notions coincide if the structure is $\equiv$-intensional.

Lemma 3.11. *Let $\mathcal{M}$ be a model and let $\gamma : V \to M$ be an assignment. $(\mathcal{M},\gamma)$ is intensional if and only if $(\mathcal{M},\gamma)$ is both $\equiv$-intensional and $<$-intensional. If $(\mathcal{M},\gamma)$ is surjective and $<$-intensional, then $\mathcal{M}$ is transitive. Thus, all standard interpretations are transitive. The assertions remain true if we consider a model $\mathcal{M}$ instead of an interpretation $(\mathcal{M},\gamma)$.*

Proof. The proof of the first assertion is an easy exercise. Suppose that $(\mathcal{M},\gamma)$ is surjective and $<$-intensional. Let $p, q, r \in M$ such that $p <^{\mathcal{M}} q$ and $q <^{\mathcal{M}} r$. There are expressions φ, ψ, χ with $\Gamma(\varphi,\gamma) = p$, $\Gamma(\psi,\gamma) = q$,

[3] Because of the condition of surjectivity one referee suggested to consider the term "nominalistic" instead of "standard". Although there are certain philosophical reasons for such a designation we prefer here the more neutral term "standard model". Standard models are the intended ones; they are in particular transitive (see the next lemma).

$\Gamma(\chi,\gamma) = r$. Thus, $\Gamma(\varphi,\gamma) <^{\mathcal{M}} \Gamma(\psi,\gamma) <^{\mathcal{M}} \Gamma(\chi,\gamma)$. $<$-intensionality implies the existence of expressions $\varphi', \psi', \psi'', \chi'$ such that $\Gamma(\varphi',\gamma) = \Gamma(\varphi,\gamma)$, $\Gamma(\psi',\gamma) = \Gamma(\psi'',\gamma) = \Gamma(\psi,\gamma)$ and $\Gamma(\chi',\gamma) = \Gamma(\chi,\gamma)$ and $\varphi' \prec \psi'$, $\psi'' \prec \chi'$. Note that not necessarily $\psi' = \psi''$. By Corollary 2.5, there is an expression χ_0 with $x \in var(\chi_0)$ and $\chi_0[x := \psi''] = \chi'$. By Corollary 3.8, $\Gamma(\chi',\gamma) = \Gamma(\chi_0[x := \psi''],\gamma) = \Gamma(\chi_0[x := \psi'],\gamma)$. Again by Corollary 2.5, $\varphi' \prec \psi' \prec \chi_0[x := \psi']$. Transitivity of $\prec$ yields $\varphi' \prec \chi_0[x := \psi']$. By (RP), $\Gamma(\varphi',\gamma) <^{\mathcal{M}} \Gamma(\chi_0[x := \psi'],\gamma) = \Gamma(\chi',\gamma)$. ■

We now present an alternative way to look at models. A model can be seen as an algebra of propositions where the functions of this algebra are interpreted symbols of the language. This is the way in which models of SCI are defined as algebras in [3]. We will see that the non-algebraic approach and the algebraic approach to semantics are equivalent, and we will make use of both points of view, depending on the situation. One advantage of the non-algebraic approach seems to be that the language can be extended immediately by quantifiers (see [15, 20]). In a recent paper K. Robering [14] presents an "algebraic" interpretation of quantifiers within the framework of $\in_T$-logic. He then shows that the resulting logic is proof-theoretically equivalent with original $\in_T$.

Definition 3.12. Let $\Sigma = \{\rightarrow, \equiv, <, : true, : false, \neg\} \cup \{K_i \mid 1 \leq i \leq n\} \cup \{C_G \mid G \text{ a group}\} \cup \mathcal{C} \cup Expr_{\mathcal{L}}$. An $\in_K$-algebra w.r.t. $Expr(\mathcal{C}, \mathcal{L}, n)$

$$\mathcal{M} = (M, (f_s)_{s\in\Sigma})$$

is given by a non-empty universe M, binary functions $f_{\rightarrow}$, $f_{\equiv}$, $f_{<}$, unary functions $f_{\neg}, f_{:true}$, $f_{:false}$, f_{K_i} $(1 \leq i \leq n)$, f_{C_G} (for each group G) on M, and elements f_c (for every $c \in \mathcal{C}$), f_a (for every $a \in Expr_{\mathcal{L}}$) of M. A valuation of $\mathcal{M}$ is a function $h : Expr(\mathcal{C}, \mathcal{L}, n) \rightarrow M$ such that for all expressions φ, ψ the following holds:

$h(\top) = f_{\top}$, $h(\bot) = f_{\bot}$, $h(\top_i) = f_{\top_i}$ $(1 \leq i \leq n)$, $h(c) = f_c$ $(c \in C)$, $h(a) = f_a$ $(a \in Expr_{\mathcal{L}})$;

$h(\varphi : true) = f_{:true}(h(\varphi))$ $h(\varphi : false) = f_{:false}(h(\varphi))$; $h(\neg\varphi) = f_{\neg}(h(\varphi))$;

$h(\varphi \rightarrow \psi) = f_{\rightarrow}(h(\varphi), h(\psi))$;$h(\varphi \equiv \psi) = f_{\equiv}(h(\varphi), h(\psi))$; $h(\varphi < \psi) = f_{<}(h(\varphi), h(\psi))$; $h(K_i\varphi) = f_{K_i}(h(\varphi))$; $h(C_G\varphi) = f_{C_G}(h(\varphi))$.

Definition 3.13. Let $\mathcal{M} = (M, (f_s)_{s\in\Sigma})$ be an $\in_K$-algebra. Suppose $TRUE$, $TRUE_i$ $(1 \leq i \leq n)$ and $FALSE$ are sets such that $M = TRUE \cup FALSE$,

$TRUE \cap FALSE = \varnothing$ and $TRUE_i \subseteq TRUE$. Let $T \in MTh_{\mathcal{L}}$ be a complete $\mathcal{L}$-theory. Furthermore, $<^{\mathcal{M}} \subseteq M \times M$. Then

$$\mathcal{M} = (M, (f_s)_{s\in\Sigma}, TRUE, (TRUE_i)_{1\leq i\leq n}, FALSE, <^{\mathcal{M}}, T)$$

is an adequate $\in_K$-algebra if the following **truth conditions** (i)-(xi) and the **structure condition** (xii) hold. For all $m, m' \in M$ and all $a \in Expr_{\mathcal{L}}$,
(i) $f_a \in TRUE \Leftrightarrow a \in T$, for every $a \in Expr_{\mathcal{L}}$ (Correspondence Property)
(ii) $f_\top \in TRUE$, $f_{\top_i} \in TRUE_i$ $(1 \leq i \leq n)$, $f_\bot \in FALSE$
(iii) $f_{:true}(m) \in TRUE \Leftrightarrow m \in TRUE$
(iv) $f_{:false}(m) \in TRUE \Leftrightarrow m \in FALSE$
(v) $f_\neg(m) \in TRUE \Leftrightarrow m \in FALSE$
(vi) $f_\rightarrow(m, m') \in TRUE \Leftrightarrow m \in FALSE$ or $m' \in TRUE$
(vii) $f_\equiv(m, m') \in TRUE \Leftrightarrow m = m'$
(viii) $f_<(m, m') \in TRUE \Leftrightarrow m <^{\mathcal{M}} m'$
(ix) $f_{K_i}(m) \in TRUE \Leftrightarrow m \in TRUE_i$
(x) $f_{C_G}(m) \in TRUE \Rightarrow f_{K_i}(m) \in TRUE$ and $f_{C_G}(f_{K_i}(m)) \in TRUE$, for all $i \in G$
(xi) $f_{C_G}(m) \in TRUE \Rightarrow f_{C_{G'}}(m) \in TRUE$, whenever $G' \subseteq G$
(xii) The relation $<^{\mathcal{M}}$ is transitive on M, and for all unary $s \in \Sigma$ and all binary $t \in \Sigma$ the following hold: $m <^{\mathcal{M}} f_s(m)$, $m <^{\mathcal{M}} f_t(m, m')$ and $m <^{\mathcal{M}} f_t(m', m)$.

The language of $\in_K$-Logic itself is an $\in_K$-algebra which is generated over the set of variables V. Thus, a valuation h of an $\in_K$-algebra $\mathcal{M}$ is an homomorphism between $\in_K$-algebras. If h, h' are valuations of any $\in_K$-algebra $\mathcal{M}$ such that $h(x) = h'(x)$ for all $x \in V$, then it follows that $h = h'$. That is, each assignment $\gamma : V \rightarrow M$ is uniquely extended into a valuation h_γ such that $h_\gamma(x) = \gamma(x)$ for all $x \in V$.

Theorem 3.14. (i) *Let* $\mathcal{M} = (M, TRUE, (TRUE_i)_{1\leq i\leq n}, FALSE, <^{\mathcal{M}}, \Gamma, T)$ *be a transitive model. Then*

$$\mathcal{M}' = (M, (f_s)_{s\in\Sigma}, TRUE, (TRUE_i)_{1\leq i\leq n}, FALSE, <^{\mathcal{M}}, T)$$

is an adequate $\in_K$*-algebra such that for each assignment* $\gamma : V \rightarrow M$ *the function* $h_\gamma : Expr(\mathcal{C}, \mathcal{L}, n) \rightarrow M$ *defined by* $h_\gamma(\varphi) = \Gamma(\varphi, \gamma)$ *is a valuation of* $\mathcal{M}'$.
(ii) *Let* $\mathcal{M}' = (M, (f_s)_{s\in\Sigma}, TRUE, (TRUE_i)_{1\leq i\leq n}, FALSE, <^{\mathcal{M}'}, T)$ *be an adequate* $\in_K$*-algebra and for each assignment* $\gamma : V \rightarrow M$ *let* h_γ *be the unique valuation of* $\mathcal{M}'$ *that extends* γ*. Then*

$$\mathcal{M} = (M, TRUE, (TRUE_i)_{1\leq i\leq n}, FALSE, <^{\mathcal{M}'}, \Gamma, T)$$

is a transitive model, where $\Gamma(\varphi, \gamma) := h_\gamma(\varphi)$.

PROOF. Let $\mathcal{M} = (M, TRUE, (TRUE_i)_{1\leq i\leq n}, FALSE, <^{\mathcal{M}}, \Gamma, T)$ be a transitive $\in_K$-model. We define the constants by $f_\top = \Gamma(\top)$, $f_{\top_i} = \Gamma(\top_i)$, $f_\bot = \Gamma(\bot)$ and $f_c = \Gamma(c)$, $f_a = \Gamma(a)$, for $c \in \mathcal{C}$ and $a \in Expr_{\mathcal{L}}$, respectively. The function $f_\rightarrow : M \times M \rightarrow M$ is defined as follows. Let $m, m' \in M$. We choose an assignment $\gamma : V \rightarrow M$ and expressions φ, ψ such that $\Gamma(\varphi, \gamma) = m$ and $\Gamma(\psi, \gamma) = m'$ (consider, for instance, variables x, y and any γ with $\gamma(x) = m$ and $\gamma(y) = m'$). We put $f_\rightarrow(m, m') = f_\rightarrow(\Gamma(\varphi, \gamma), \Gamma(\psi, \gamma)) := \Gamma(\varphi \rightarrow \psi, \gamma)$. Note that by Theorem 3.7 this definition is independent of the particular choice of the assignment γ and the expressions φ, ψ. The remaining functions are defined similarly. Now one easily shows that for any assignment γ, the function h_γ, $\varphi \mapsto \Gamma(\varphi, \gamma)$, is a valuation, and (i)-(xii) of an adequate $\in_K$-algebra are satisfied. Condition (xii) follows from (RP).
Now let $\mathcal{M}' = (M, (f_s)_{s\in\Sigma}, TRUE, (TRUE_i)_{1\leq i\leq n}, FALSE, <^{\mathcal{M}'}, T)$ be an adequate $\in_K$-algebra. $\Gamma(\varphi, \gamma) := h_\gamma(\varphi)$ satisfies the structure conditions of an $\in_K$-model: (EP) is obvious. (CP) follows from the fact that a valuation is uniquely determined by its restriction to V. Recall that for an assignment γ and a substitution of variables σ, $\gamma\sigma$ denotes the assignment $x \mapsto \Gamma(\sigma(x), \gamma) = h_\gamma(\sigma(x))$. By induction on φ it follows that $\Gamma(\varphi[\sigma], \gamma) = h_\gamma(\varphi[\sigma]) = h_{\gamma\sigma}(\varphi) = \Gamma(\varphi, \gamma\sigma)$, thus (SP) holds. Let us show (RP). Suppose $\varphi \prec \psi$. There is a chain $\varphi_0 \prec \varphi_1 \prec \cdots \prec \varphi_k \prec \varphi_{k+1}$ of maximal length, where $\varphi_0 = \varphi$ and $\varphi_{k+1} = \psi$. That is, for every $i \leq k$, φ_i is an immediate proper subexpression of φ_{i+1}. Let $\gamma : V \rightarrow M$ be any assignment. Then for each $i \leq k$: $\Gamma(\varphi_{i+1}, \gamma) = f_s(\Gamma(\varphi_i, \gamma))$ for some unary function f_s, or $\Gamma(\varphi_{i+1}, \gamma) = f_t(\Gamma(\varphi_i, \gamma), m)$ or $\Gamma(\varphi_{i+1}, \gamma) = f_t(m, \Gamma(\varphi_i, \gamma))$ for some binary function f_t and some $m \in M$. By truth condition (xii), we get a chain of propositions $m_0 <^{\mathcal{M}} m_1 <^{\mathcal{M}} \cdots <^{\mathcal{M}} m_{k+1}$, where $\Gamma(\varphi_i, \gamma) = m_i$, for $i \leq k+1$. By transitivity of $<^{\mathcal{M}}$, $\Gamma(\varphi, \gamma) = m_0 <^{\mathcal{M}} m_{k+1} = \Gamma(\psi, \gamma)$. Thus, (RP) holds. Finally, the truth conditions of an $\in_K$-model follow from the truth conditions (i)-(xi) of the adequate $\in_K$-algebra $\mathcal{M}'$. ■

4. Soundness and Completeness

Let $\mathcal{L}$ be a classical abstract logic. We suppose that a sound and complete relation of derivability $\vdash_{\mathcal{L}}$ for logic $\mathcal{L}$ is given by means of some calculus whose nature needs not to be specified: $A \Vdash_{\mathcal{L}} b \Leftrightarrow A \vdash_{\mathcal{L}} b$, for all $A \cup \{b\} \subseteq Expr_{\mathcal{L}}$. We show that under these assumptions our list of axioms $\mathbf{Ax}_n$ together with the rule of modus ponens gives rise to a notion of derivability $\vdash$ which is sound and complete w.r.t. our $\in_K$-logic $\mathcal{L}^n_{\mathcal{C}}$. The validity of axiom (vi) follows from Corollary 3.8. The soundness of the remaining axioms and modus ponens follows readily from the truth conditions of transitive models. It remains to prove completeness.

Definition 4.1. A set $A \subseteq Expr_{\mathcal{L}}$ is said to be $\vdash_{\mathcal{L}}$-consistent if there is some $b \in Expr_{\mathcal{L}}$ such that $A \nvdash_{\mathcal{L}} b$; otherwise, A is $\vdash_{\mathcal{L}}$-inconsistent. For $A \subseteq Expr_{\mathcal{L}}$ we put $A^{\vdash_{\mathcal{L}}} := \{b \in Expr_{\mathcal{L}} \mid A \vdash_{\mathcal{L}} b\}$. — For $\Phi \subseteq Expr(\mathcal{C}, \mathcal{L}, n)$ let $\Phi^{\vdash}$ be the smallest set of expressions that contains $\Phi \cup \mathbf{Ax}_n \cup (\Phi \cap Expr_{\mathcal{L}})^{\vdash_{\mathcal{L}}}$ and is closed under modus ponens. If $\varphi \in \Phi^{\vdash}$, then we write $\Phi \vdash \varphi$ and say that φ is derivable from Φ, or Φ proves φ. If some expression is not derivable from Φ, then we say that Φ is $\vdash$-consistent; otherwise Φ is called $\vdash$-inconsistent. Φ is maximally $\vdash$-consistent if Φ is $\vdash$-consistent and is not the proper subset of a $\vdash$-consistent set.

The proofs of the following facts are folklore.

Proposition 4.2. *Every $\vdash$-consistent set extends to a maximally $\vdash$-consistent set. If $\Phi \nvdash \varphi$, then there is a maximally $\vdash$-consistent superset $\Phi' \supseteq \Phi$ such that $\Phi' \nvdash \varphi$. If Φ is maximally $\vdash$-consistent, then for all expressions φ, ψ the following holds:*
(i) Either $\varphi \in \Phi$ or $\neg\varphi \in \Phi$;
(ii) $\Phi \vdash \varphi$ iff $\varphi \in \Phi$;
(iii) $\varphi \rightarrow \psi \in \Phi$ iff $\varphi \notin \Phi$ or $\psi \in \Phi$.

Proposition 4.3. *Let $\Phi \subseteq Expr(\mathcal{C}, \mathcal{L}, n)$ be a set of expressions and let $T = \Phi \cap Expr_{\mathcal{L}}$. If Φ is $\vdash$-consistent, then T is $\vdash_{\mathcal{L}}$-consistent. If Φ is maximally $\vdash$-consistent, then T is a maximal $\mathcal{L}$-theory, i.e. $T \in MTh_{\mathcal{L}}$.*

Proof. Suppose T is $\vdash_{\mathcal{L}}$-inconsistent. Then in particular $T \vdash_{\mathcal{L}} a$ and $T \vdash_{\mathcal{L}} \sim a$. Then $\Phi \vdash a$ and by (i) of $\mathbf{Ax}_n$, $\Phi \vdash \neg a$. By axiom (xiii), all formulas are derivable. Thus, Φ is $\vdash$-inconsistent. Now suppose Φ is maximally $\vdash$-consistent. We have shown that T is $\vdash_{\mathcal{L}}$-consistent. By completeness of $\vdash_{\mathcal{L}}$, T is $\mathcal{L}$-consistent, that is, T is contained in some $\mathcal{L}$-theory. Suppose $T \Vdash_{\mathcal{L}} a$. Then $T \vdash_{\mathcal{L}} a$ and $\Phi \vdash a$, thus $a \in \Phi$ and therefore $a \in T$. That is, T is $\mathcal{L}$-consistent and deductively closed and therefore an $\mathcal{L}$-theory. Now suppose there is an $\mathcal{L}$-theory $T' \supseteq T$ and $a \in T' \smallsetminus T$. Then $a \notin \Phi$ and $\neg a \in \Phi$. Thus $\sim a \in \Phi$ and $\sim a \in T \subseteq T'$. This contradicts the consistency of T'. Thus, T is a maximal $\mathcal{L}$-theory. ■

Definition 4.4. Let Φ be maximally $\vdash$-consistent. $\varphi \approx_{\Phi} \psi :\Leftrightarrow \varphi \equiv \psi \in \Phi$.

Lemma 4.5. *Let Φ be maximally $\vdash$-consistent. Put $\approx = \approx_{\Phi}$. Then*
(i) $\approx$ is an equivalence relation on the set of expressions such that $\varphi \approx \varphi'$ and $\psi \approx \psi'$ implies: $\varphi : true \approx \varphi' : true, \varphi : false \approx \varphi' : false, \neg\varphi \approx$

$\neg\varphi', K_i\varphi \approx K_i\varphi', C_G\varphi \approx C_G\varphi', (\varphi < \psi) \approx (\varphi' < \psi'), (\varphi \equiv \psi) \approx (\varphi' \equiv \psi'), (\varphi \rightarrow \psi) \approx (\varphi' \rightarrow \psi')$.
(ii) If $\varphi \approx \psi$, then $\varphi \in \Phi \Leftrightarrow \psi \in \Phi$, and $K_i\varphi \in \Phi \Leftrightarrow K_i\psi \in \Phi$.

PROOF. (i): Reflexivity of $\approx$ is obvious by axiom (iv). Suppose $\varphi \approx \psi$. Let $\chi := (x \equiv \varphi)$, where $x \notin var(\varphi)$. Then $\chi[x := \varphi] \approx \chi[x := \psi]$, by axiom (vi). Since $\Phi \vdash \chi[x := \varphi]$, axiom (v) yields $\Phi \vdash \chi[x := \psi]$. Thus, $\psi \approx \varphi$ and $\approx$ is symmetric. Transitivity follows similarly. Now suppose $\varphi \approx \varphi'$ and $\psi \approx \psi'$. Let $x \in V \smallsetminus var(\psi')$ and $y \in V \smallsetminus var(\varphi)$. Then by axiom (vi), $\varphi < \psi = (\varphi < y)[y := \psi] \approx (\varphi < y)[y := \psi'] = \varphi < \psi' = (x < \psi')[x := \varphi] \approx (x < \psi')[x := \varphi'] = \varphi' < \psi'$. Now apply transitivity of $\approx$. Similarly for the remaining cases.
(ii): This follows from axiom (v), symmetry of $\approx$ and item (i). ■

THEOREM 4.6. *Every $\vdash$-consistent set has a transitive model.*

PROOF. Let Φ' be $\vdash$-consistent and let $\Phi \supseteq \Phi'$ be a maximally $\vdash$-consistent extension. We consider the propositional universe $M = \{\overline{\varphi} \mid \varphi \in Expr(\mathcal{C}, \mathcal{L}, n)\}$, where $\overline{\varphi}$ denotes the equivalence class of φ modulo $\approx_\Phi$. Let Σ be the set of syntactic operators as given in Definition 3.12. For all $s \in \Sigma$ we define functions f_s as follows: $f_\top = \overline{\top}$, $f_\bot = \overline{\bot}$, $f_c = \overline{c}$ $(c \in \mathcal{C})$, $f_a = \overline{a}$ $(a \in Expr_\mathcal{L})$, $f_{:true}(\overline{\varphi}) = \overline{\varphi : true}$, $f_\equiv(\overline{\varphi}, \overline{\psi}) = \overline{\varphi \equiv \psi}$, etc. We put $TRUE = \{\overline{\varphi} \mid \varphi \in \Phi\}$, $FALSE = \{\overline{\varphi} \mid \varphi \notin \Phi\}$ and $TRUE_i = \{\overline{\varphi} \mid K_i\varphi \in \Phi\}$. We define the reference relation $<^\mathcal{M}$ on M by $\overline{\varphi} <^\mathcal{M} \overline{\psi} :\Leftrightarrow \varphi < \psi \in \Phi$. All these things are well-defined by Lemma 4.5. From axiom (viii) follows that $<^\mathcal{M}$ is transitive. By Proposition 4.3, $T = \Phi \cap Expr_\mathcal{L}$ is a complete $\mathcal{L}$-theory. Applying axioms of $\mathbf{Ax}_n$ one shows that M together with the subsets $TRUE, FALSE, TRUE_i, 1 \leq i \leq n$, the functions $f_s, s \in \Sigma$, the relation $<^\mathcal{M}$ and the complete $\mathcal{L}$-theory T forms an adequate $\in_K$-algebra. Then $\mathcal{M} := (TRUE, FALSE, (TRUE_i)_{1 \leq i \leq n}, <^\mathcal{M}, \Gamma, T)$, where $\Gamma(\varphi, \gamma) = h_\gamma(\varphi)$ for every assignment $\gamma : V \rightarrow M$, is a transitive $\in_K$-model (Theorem 3.14). Now let $\beta : V \rightarrow M$ be the assignment defined by $x \rightarrow \overline{x}$ for $x \in V$. It follows that $h_\beta(\varphi) = \overline{\varphi}$, for all expressions φ. Thus, $(\mathcal{M}, \beta) \vDash \varphi \Leftrightarrow \Gamma(\varphi, \beta) \in TRUE \Leftrightarrow \overline{\varphi} \in TRUE \Leftrightarrow \varphi \in \Phi$. In particular, the interpretation $(\mathcal{M}, \beta)$ is a transitive model of $\Phi' \subseteq \Phi$. ■

COROLLARY 4.7. *For all sets of expressions $\Phi \cup \{\varphi\}$, $\Phi \Vdash \varphi \Leftrightarrow \Phi \vdash \varphi$.*

5. Model constructions

In the first part of this section we construct finite models. Finite models have the property that they satisfy many equations $\varphi \equiv \psi$. We are in particular interested in equations that assert self-referential propositions, that is, equations of the form $\varphi \equiv \psi$ with $\varphi \prec \psi$, such as $c \equiv (c : true)$ which asserts a truth-teller. It is in general not hard to find such finite models.[4] The problem is that finite models satisfy also many "undesired" equations. The question arises whether there are infinite standard models that satisfy only a few specific equations. A partial answer is given in [11] where an infinite standard model with exactly two self-referential propositions (a true and a false truth teller) is constructed. One of our main results (Theorem 5.4) provides a more general solution. It says that for every satisfiable set of equations (with some additional properties) one can find a transitive model that satisfies only the given set of equations and a certain closure set of equations. As a corollary we get the existence of intensional models. These results are also used in order to prove a fact concerning equi-expressiveness with respect to the well-known syntactical approach to explicit knowledge. Finally, in the last subsection we present constructions of intensional standard models with specific epistemic properties.

5.1. Some finite models

Let T be a complete $\mathcal{L}$-theory. As usual, $\top, \bot \in \mathcal{C}$. Suppose $\{\top_1, \ldots, \top_n\} \subseteq \mathcal{C}$. Our propositional universe is $M = \{f_\top, f_{\top_1}, \ldots, f_{\top_n}, f_\bot\}$. The propositions $f_\top, f_{\top_1}, \ldots, f_{\top_n}$ are not necessarily pairwise distinct. That is, agents i, j know the same proposition iff $f_{\top_i} = f_{\top_j}$. In particular, $f_\top = f_{\top_1} = \cdots = f_{\top_n}$ is possible. In this case we get a two-element model. Of course, we require that $f_\bot$ is distinct from the other propositions. We put $TRUE_i = \{f_{\top_i}\}$, $FALSE = \{f_\bot\}$, $TRUE = \{f_\top\} \cup \bigcup_{1 \leq i \leq n} TRUE_i$. Then $M = TRUE \cup FALSE$. Furthermore, $<^{\mathcal{M}} := \{(f_{\top_i}, f_\top) \mid 1 \leq i \leq n\} \cup \{(f_{\top_i}, f_\bot) \mid 1 \leq i \leq n\} \cup \{(f_\top, f_\top), (f_\top, f_\bot), (f_\bot, f_\top), (f_\bot, f_\bot)\}$. In order to guarantee the Correspondence Property of the Gamma-function we assume a function $g : \mathcal{C} \cup Expr_{\mathcal{L}} \to M$ with the property: $g(a) \in TRUE$ iff $a \in T$, for $a \in Expr_{\mathcal{L}}$; and $g(\top) = f_\top$, $g(\bot) = f_\bot$, $g(\top_i) = f_{\top_i}$, $i = 1, \ldots, n$. Let $\gamma : V \to M$ be any assignment. Then we define:

$$\Gamma(x, \gamma) = \gamma(x), \text{ for } x \in V.$$
$$\Gamma(u, \gamma) = g(u), \text{ for } u \in Expr_{\mathcal{L}} \cup \mathcal{C}.$$

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[4] An extreme case is a two-element model where the denotation of a formula can be identified with its truth value. Such a model is called Fregean or extensional. If there were only Fregean models, then the Fregean Axiom would hold.

$$\Gamma(\varphi : true, \gamma) = \begin{cases} f_\top, \text{ if } \Gamma(\varphi, \gamma) \in TRUE \\ f_\perp, \text{ else.} \end{cases}$$

$$\Gamma(\varphi : false, \gamma) = \Gamma(\neg\varphi, \gamma) = \begin{cases} f_\top, \text{ if } \Gamma(\varphi, \gamma) = f_\perp \\ f_\perp, \text{ else.} \end{cases}$$

$$\Gamma(\varphi \to \psi, \gamma) = \begin{cases} f_\top, \text{ if } \Gamma(\varphi, \gamma) \in FALSE \text{ or } \Gamma(\psi, \gamma) \in TRUE \\ f_\perp, \text{ else.} \end{cases}$$

$$\Gamma(\varphi \equiv \psi, \gamma) = \begin{cases} f_\top, \text{ if } \Gamma(\varphi, \gamma) = \Gamma(\psi, \gamma) \\ f_\perp, \text{ else.} \end{cases}$$

$$\Gamma(\varphi < \psi, \gamma) = \begin{cases} f_\top, \text{ if } \Gamma(\varphi, \gamma) <^{\mathcal{M}} \Gamma(\psi, \gamma) \\ f_\perp, \text{ else.} \end{cases}$$

$$\Gamma(K_i\varphi, \gamma) = \begin{cases} f_\top, \text{ if } \Gamma(\varphi, \gamma) = f_{\top_i} \\ f_\perp, \text{ else.} \end{cases}$$

$$\Gamma(C_G\varphi, \gamma) = \begin{cases} f_\top, \text{ if } \Gamma(\varphi, \gamma) = f_{\top_{i_1}} = \cdots = f_{\top i_k} = f_\top, G = \{i_1, \ldots, i_k\} \\ f_\perp, \text{ else.} \end{cases}$$

We claim that $\mathcal{M} = (M, TRUE, (TRUE_i)_{1 \le i \le n}, FALSE, <^{\mathcal{M}}, \Gamma, T)$ is a standard model. (EP) is obvious. (CP) and (SP) follow by induction on expressions. (RP) as well as the truth conditions are clear by the construction. Suppose $G = \{i_1, \ldots, i_k\}$. Then $COMMON_G = GREATEST_G = \{f_\top\}$, if $f_{\top_{i_1}} = \cdots = f_{\top i_k} = f_\top$. Otherwise, $COMMON_G = GREATEST_G = \varnothing$. Obviously, every proposition is the denotation of some sentence. It is also easy to check that $\mathcal{M}$ is $<$-intensional. Thus, $\mathcal{M}$ is a standard model.

THEOREM 5.1 (Existence of models). *For every complete $\mathcal{L}$-theory T there exist standard models with respect to T.*

5.2. Construction of models that satisfy specific equations

In the following we extend a construction method developed in [11].

Definition 5.2. An expression of the form $\varphi \equiv \psi$ is called an equation, an expression of the form $\varphi < \psi$ is called a reference. A formula is called equation-free (reference-free) if no subformula is an equation (a reference). Sometimes we write equations $\varphi \equiv \psi$ as tuples (φ, ψ) and call such a tuple an identification. We say that an identification (φ, ψ) is simple if φ and ψ are equation- and reference-free. A set E of identifications is called simple if all elements of E are simple identifications. A set E of identifications is called consistent if there is a model that satisfies all identifications of E, i.e., all equations $\varphi \equiv \psi$ with $(\varphi, \psi) \in E$. If E is a set of identifications, then let E^* be the smallest equivalence relation that contains E and is closed under the following conditions: if $(\varphi, \psi) \in E^*$ and $(\varphi', \psi') \in E^*$, then $(\varphi : true, \psi : true) \in E^*, (\varphi : false, \psi : false) \in E^*, (\neg\varphi, \neg\psi) \in E^*, (K_i\varphi, K_i\psi) \in E^*, (C_G\varphi, C_G\psi) \in E^*, (\varphi \equiv \varphi', \psi \equiv \psi') \in E^*, (\varphi < \varphi', \psi < \psi') \in E^*, (\varphi \rightarrow \varphi', \psi \rightarrow \psi') \in E^*$. We call this condition the congruence property of E^*, and we call E^* the congruence generated by E.

Lemma 5.3. *If E is a consistent set of identifications, then E^* is a consistent set of identifications.*

Proof. This follows from the congruence property of E^* and the Substitution Principle (Theorem 3.7). ■

Theorem 5.4. *If E is a simple and consistent set of identifications, then there is a transitive interpretation $(\mathcal{M}^E, \beta^E)$ that satisfies exactly the identifications of the generated congruence E^*, i.e. $(\mathcal{M}^E, \beta^E) \vDash \varphi \equiv \psi \Leftrightarrow (\varphi, \psi) \in E^*$.*

Proof. Suppose E is a simple and consistent set of identifications. Let

$$\mathcal{M} = (M, TRUE, (TRUE_i)_{1 \leq i \leq n}, FALSE, <^{\mathcal{M}}, \Gamma, T)$$

be a model and let $\beta : V \rightarrow M$ be an assignment such that $(\mathcal{M}, \beta) \vDash \varphi \equiv \psi$, for all $(\varphi, \psi) \in E$. By Theorem 3.7, $(\mathcal{M}, \beta) \vDash \varphi \equiv \psi$, for all $(\varphi, \psi) \in E^*$. We construct a transitive interpretation that satisfies *exactly* the identifications of E^*. Let φ^E denote the equivalence class of $\varphi \in Expr(\mathcal{C}, \mathcal{L}, n)$ modulo E^*. The propositional universe is $M^E := \{\varphi^E \mid \varphi \in Expr(\mathcal{C}, \mathcal{L}, n)\}$. The reference relation $<^E$ on M^E is given by $\varphi^E <^E \psi^E :\Leftrightarrow$ there are $\varphi' \in \varphi^E, \psi' \in \psi^E$ such that $\varphi' \prec \psi'$. For an assignment $\gamma : V \rightarrow M^E$ let $\delta_\gamma : V \rightarrow Expr(\mathcal{C}, \mathcal{L}, n)$ be a choice function that selects an element $\delta_\gamma(x) \in \gamma(x)$

for each $x \in V$. Note that δ_γ is a substitution. We define the new Gamma-function by $\Gamma^E(\varphi, \gamma) := \varphi[\delta_\gamma]^E$. The structure conditions (EP), (CP), (SP), (RP) now follow exactly in the same way as in the construction in [11], section 4.4. Recall that $COMMON_G$ is given as in Definition 3.2 with respect to $\mathcal{M}$. Now we define $TRUE^E$, $FALSE^E$ inductively as the smallest sets such that for all atomic expressions u and for all expressions φ, ψ the following conditions hold:

- $u^E \in TRUE^E$, if $\Gamma(u, \beta) \in TRUE$;
  $u^E \in FALSE^E$, if $\Gamma(u, \beta) \in FALSE$
- $(\varphi \equiv \psi)^E \in TRUE^E$, if $\varphi^E = \psi^E$; $(\varphi \equiv \psi)^E \in FALSE^E$, if $\varphi^E \neq \psi^E$
- $(\varphi < \psi)^E \in TRUE^E$, if $\varphi^E <^E \psi^E$; $(\varphi < \psi)^E \in FALSE^E$, if not $\varphi^E <^E \psi^E$
- $(\varphi : true)^E \in TRUE^E$, if $\varphi^E \in TRUE^E$; $(\varphi : true)^E \in FALSE^E$, if $\varphi^E \in FALSE^E$
- $(\varphi : false)^E \in TRUE^E$, if $\varphi^E \in FALSE^E$; $(\varphi : false)^E \in FALSE^E$, if $\varphi^E \in TRUE^E$
- $(\neg\varphi)^E \in TRUE^E$, if $\varphi^E \in FALSE^E$; $(\neg\varphi)^E \in FALSE^E$, if $\varphi^E \in TRUE^E$
- $(\varphi \to \psi)^E \in TRUE^E$, if $\varphi^E \in FALSE^E$ or $\psi^E \in TRUE^E$; $(\varphi \to \psi)^E \in FALSE^E$, if $\varphi^E \in TRUE^E$ and $\psi^E \in FALSE^E$
- $(K_i\varphi)^E \in TRUE^E$, if $\varphi^E \in TRUE^E$ and $\Gamma(\varphi, \beta) \in TRUE_i$; $(K_i\varphi)^E \in FALSE^E$, if $\varphi^E \in FALSE^E$ or $\Gamma(\varphi, \beta) \notin TRUE_i$
- $(C_G\varphi)^E \in TRUE^E$, if $\varphi^E \in TRUE^E$ and $\Gamma(\varphi, \beta) \in COMMON_G$. $(C_G\varphi)^E \in FALSE^E$, if $\varphi^E \in FALSE^E$ or $\Gamma(\varphi, \beta) \notin COMMON_G$.

Finally, for $i = 1, \dots, n$ we put $TRUE_i^E := \{\varphi^E \mid (K_i\varphi)^E \in TRUE^E\}$, and for each group G we put $COMMON_G^E := \{\varphi^E \mid (C_G\varphi)^E \in TRUE^E\}$. Then the following is not hard to prove:

(i) $M^E = TRUE^E \cup FALSE^E$ and $TRUE^E \cap FALSE^E = \varnothing$
(ii) $TRUE_i^E \subseteq TRUE^E$
(iii) $(K_i\varphi)^E \in TRUE^E \Leftrightarrow \varphi^E \in TRUE_i^E$
(iv) $COMMON_G^E \subseteq \bigcap_{i \in G} TRUE_i^E$
(v) $\varphi^E \in COMMON_G^E \Rightarrow (K_i\varphi)^E \in COMMON_G^E$ for all $i \in G$

Claim1: If no subformula of φ is an equation or a reference, then

$\varphi^E \in TRUE^E \Rightarrow \Gamma(\varphi, \beta) \in TRUE$,
$\varphi^E \in FALSE^E \Rightarrow \Gamma(\varphi, \beta) \in FALSE$,
$\varphi^E \in TRUE_i^E \Rightarrow \Gamma(\varphi, \beta) \in TRUE_i$,

$\varphi^E \in COMMON_G^E \Rightarrow \Gamma(\varphi, \beta) \in COMMON_G$.
Claim 1 follows by induction on φ.

Claim2: $TRUE^E$, $FALSE^E$, $TRUE_i^E$ and $COMMON_G^E$ are well-defined.

Proof of Claim 2. Suppose $\varphi^E \in TRUE^E$ and let ψ be any formula with $(\varphi, \psi) \in E^*$. We show that the construction implies $\psi^E \in TRUE^E$. First we suppose $(\varphi, \psi) \in E$. Since E is simple, Claim 1 yields $\Gamma(\varphi, \beta) \in TRUE$. But $(\mathcal{M}, \beta) \vDash \varphi \equiv \psi$, thus $\Gamma(\psi, \beta) \in TRUE$. Then Claim 1 implies $\psi^E \in TRUE^E$. The cases $\varphi^E \in FALSE^E$, $\varphi^E \in TRUE_i^E$ and $\varphi^E \in COMMON_G^E$ follow similarly. The general case $(\varphi, \psi) \in E^*$ now follows by induction on the inductive definition of E^*.

One easily shows that $\mathcal{M}^E := (M^E, TRUE^E, (TRUE^E)_{1 \leq i \leq n}, FALSE^E, <^E \Gamma^E, T)$ satisfies the truth conditions of a model. Let $\beta^E : V \to M^E$ be the assignment $x \mapsto x^E$. By induction on φ, using the congruence property of E^*, it follows that $\varphi[\delta_{\beta^E}]^E = \varphi^E$. Then $\Gamma^E(\varphi, \beta^E) = \varphi[\delta_{\beta^E}]^E = \varphi^E$, for any φ. In particular,

$(\mathcal{M}^E, \beta^E) \vDash \varphi \equiv \psi \Leftrightarrow \Gamma^E(\varphi \equiv \psi, \beta^E) \in TRUE^E \Leftrightarrow \Gamma^E(\varphi, \beta^E) = \Gamma^E(\psi, \beta^E) \Leftrightarrow \varphi^E = \psi^E \Leftrightarrow (\varphi, \psi) \in E^*$, and

$(\mathcal{M}^E, \beta^E) \vDash \varphi < \psi \Leftrightarrow \Gamma^E(\varphi < \psi, \beta^E) \in TRUE^E \Leftrightarrow \Gamma^E(\varphi, \beta^E) <^E \Gamma^E(\psi, \beta^E) \Leftrightarrow \varphi^E <^E \psi^E \Leftrightarrow \varphi' \prec \psi'$, for appropriate expressions $\varphi' \in \varphi^E$ and $\psi' \in \psi^E$. Thus, $(\mathcal{M}^E, \beta^E)$ is $<$-intensional. Obviously, $(\mathcal{M}^E, \beta^E)$ is surjective. By Lemma 3.11, $\mathcal{M}^E$ is transitive. ∎

REMARK 5.5. *If for every G, $COMMON_G = GREATEST_G$ in the underlying model $\mathcal{M}$ in the proof of the previous theorem, then it follows that $COMMON_G^E$ is maximal with the properties (iv) and (v), i.e., it is the greatest set that is closed under G. In this case, $(\mathcal{M}^E, \beta^E)$ is a standard interpretation.*

COROLLARY 5.6. *For every $T \in MTh_{\mathcal{L}}$ there are intensional standard interpretations w.r.t. T.*

PROOF. The empty set of identifications $E = \varnothing$ is both simple and consistent, and $E^* = \{(\varphi, \varphi) \mid \varphi \in Expr(\mathcal{C}, \mathcal{L}, n)\}$. By Theorem 5.1, there exists a standard model $\mathcal{M}$ w.r.t. T. Of course, $\mathcal{M}$ satisfies E. The previous theorem yields:

$$(\mathcal{M}^E, \beta^E) \vDash \varphi \equiv \psi \Leftrightarrow (\varphi, \psi) \in E^* \Leftrightarrow \varphi = \psi,$$
$$(\mathcal{M}^E, \beta^E) \vDash \varphi < \psi \Leftrightarrow \varphi \prec \psi.$$

By Remark 5.5, $(\mathcal{M}^E, \beta^E)$ is a standard interpretation. ∎

Corollary 5.7. *If* $(\mathcal{M}, \beta)$ *is an interpretation w.r.t.* T*, then there is an intensional transitive interpretation* $(\mathcal{M}^E, \beta^E)$ *w.r.t.* T *such that for all equation- and reference-free expressions* φ*:* $(\mathcal{M}, \beta) \vDash \varphi \Leftrightarrow (\mathcal{M}^E, \beta^E) \vDash \varphi$*.*

Proof. Let $(\mathcal{M}, \beta)$ be any interpretation w.r.t. some $\mathcal{L}$-theory T. Of course, $(\mathcal{M}, \beta)$ satisfies the empty set E of identifications. Let $(\mathcal{M}^E, \beta^E)$ be the intensional transitive interpretation obtained in the proof of Theorem 5.4. Claim 1 in the proof of the Theorem then yields the assertion. ■

Is there a model that satisfies *exactly* the set E of equations between logically equivalent expressions? Note that every two-element model satisfies all equations of E — but also many more. $c \equiv c$ and $c \equiv (c : true)$ are in E. By the Substitution Principle (Theorem 3.7), a model of E must also satisfy $K_i c \equiv K_i(c : true)$ and $(c \equiv c) \equiv (c \equiv (c : true))$. However, $K_i c$ and $K_i(c : true)$ are not logically equivalent, neither are $c \equiv c$ and $c \equiv (c : true)$ ($c \equiv c$ is valid but $c \equiv (c : true)$ is not: there are models which contain truth-tellers and others which do not (see [11])). Hence, such a model cannot exist. In a two-element model all expressions of the same truth-value are identified. However, this model does not satisfy all satisfiable equations. For instance, $\top \equiv \neg(c_1 \equiv c_2) \wedge \neg(c_1 \equiv c_3) \wedge \neg(c_2 \equiv c_3)$ is satisfied in some model that contains at least three propositions but it is false in the two-element model. Of course, the reason for this is that the constant $\top$ must denote a true proposition. A different argument was already given by Sträter [15]: consider the equation $c \equiv (c \rightarrow \varphi)$ which expresses Löb's paradox (see [1]). Note that a model of this equation necessarily satisfies φ (the equation is paradoxical if φ denotes a false proposition). Both equations $c \equiv (c \rightarrow x)$ and $c \equiv (c \rightarrow \neg x)$ are satisfiable but they cannot be satisfied in the same model.

In the following example we apply Theorem 5.4 to get an infinite standard model that contains two self-referential propositions asserted by the following equations.

$$c_1 \equiv \neg(K_1 c_1 \vee \cdots \vee K_n c_1),$$
$$c_2 \equiv K_1(c_2 : false) \vee \cdots \vee K_n(c_2 : false).$$

The proposition denoted by c_1 says: "This proposition is not known by anyone". This proposition stands for a paradox known in the literature as the *knower paradox*. In contrast to the truth-teller which may assume any truth value one easily recognizes that the knower proposition can only have the truth value *true*. Note that the so-called knower paradox is paradoxical only if one of the agents has enough reasoning power. For instance, if agent i knows all true propositions of the given model, i.e. $TRUE_i = TRUE$, then the equation expressing the knower paradox is false. If we reduce the number of agents to one and $TRUE_1 = TRUE$, then the knower paradox turns out to be the (strengthened) liar paradox expressed by $c_1 \equiv \neg(c_1 : true)$. This equation is unsatisfiable. The equation expressing the knower paradox is satisfiable in models where the reasoning power of all agents is weak enough, i.e. where $TRUE_i$ is a proper subset of $TRUE$.

The proposition denoted by c_2 says "Someone knows that this proposition is false". This equation is satisfiable only if c_2 denotes a false proposition. For similar reasons as above, the equation $c_2 \equiv K_1(c_2 : false) \vee \ldots \vee K_n(c_2 : false)$ can be satisfied only if the reasoning power of agents is weak enough. If one agent knows all true propositions of the given model, then the statement expressed by this equation turns out to be paradoxical. Indeed, if we reduce the number of agents to one and this agent knows all true propositions, then the equation is equivalent with the equation expressing the following form of the liar paradox: $c_2 \equiv (c_2 : false)$. In this way, we may consider these two epistemic "paradoxes" as relaxations of the semantic liar paradox (in its weak and its strong form).[5]

Note that both equations are satisfiable in the finite models constructed at the beginning of this section if $f_\top \neq f_{\top_i}$ for $i = 1, \ldots, n$. It is clear that the set $E = \{(c_1, \neg(K_1 c_1 \vee \ldots \vee K_n c_1)), (c_2, K_1(c_2 : false) \vee \ldots \vee K_n(c_2 : false))\}$ of identifications is simple. Theorem 5.4 yields a standard model that satisfies exactly the equations $\varphi \equiv \psi$ with $(\varphi, \psi) \in E^*$. From (RP) it follows that, in particular, $c_1 < c_1$ and $c_2 < c_2$ are true in the model, that is, c_1 and c_2 denote self-referential propositions.

We leave it as an exercise to construct a finite standard model (modify the given construction of the finite model) with two new true propositions f_{c_3} and f_{c_4} such that additionally the equation $c_3 \equiv K_1 c_3$ and the system of n equations $c_4 \equiv K_1 c_4, \ldots, c_4 \equiv K_n c_4$ are satisfied. Then Theorem 5.4 yields an infinite standard model with four self-referential propositions denoted by c_1, c_2, c_3, c_4, respectively. c_3 says "Agent 1 knows this proposition". By Theorem 3.7, $c_3 \equiv K_1^k c_3$ for all $k \geq 0$. If in such a model $TRUE_1 = TRUE$, then c_3 denotes a truth-teller. The proposition denoted by c_4 says "This proposition is known by everyone". If c_4 denotes a true proposition, then $K_{i_1} \ldots K_{i_m} c_4$ holds for all $m \geq 0$ and all sequences $i_1, \ldots, i_m \in \{1, \ldots, n\}$. This follows again from Theorem 3.7. Since the model is a standard model, from Lemma 3.3 it follows that c_4 is common knowledge in $G = \{1, \ldots, n\}$. This shows that common knowledge can be achieved by such a self-referential proposition.

[5] The liar proposition asserted by $c \equiv (c : false)$ may exist if one admits "paraconsistent" models where $TRUE \cap FALSE \neq \varnothing$ is possible. Such a logic is studied in [10].

5.3. Equi-expressiveness with respect to the syntactic approach to explicit knowledge

Among the most prominent approaches to explicit knowledge are the syntactic approach, the awareness approach and the impossible-worlds approach. It is well-known that these approaches are in a certain sense equi-expressive (see, for instance, [19, 5, 7]). There are recent approaches to explicit knowledge, e.g. epistemic versions of *justification logics* (see, e.g. [6]), that use a kind of resource measurement in order to tackle logical omniscience. We consider here the syntactic approach and prove a result concerning equi-expressiveness. Let $true, false$ be truth values of formulas. We define a syntactic structure as a pair (S, δ, T) consisting of a set S of states, a complete $\mathcal{L}$-theory T, and a standard syntactic assignment δ which is defined here as a function $\delta : S \times Expr(\mathcal{C}, \mathcal{L}, n) \to \{true, false\}$ having the following properties, for all $s \in S$ and for all formulas φ, ψ:

(i) $\delta(s, a) = true$ iff $a \in T$
(ii) $\delta(s, \top) = \delta(s, K_1\top_1) = \cdots = \delta(s, K_n\top_n) = true$ and $\delta(s, \bot) = false$
(iii) $\delta(s, \varphi : true) = true$ iff $\delta(s, \varphi) = true$
(iv) $\delta(s, \varphi : false) = true$ iff $\delta(s, \varphi) = false$
(v) $\delta(s, \neg\varphi) = true$ iff $\delta(s, \varphi) = false$
(vi) $\delta(s, \varphi \to \psi) = true$ iff $\delta(s, \varphi) = false$ or $\delta(s, \psi) = true$
(vii) $\delta(s, \varphi \equiv \psi) = true$ iff $\varphi = \psi$
(viii) $\delta(s, \varphi < \psi) = true$ iff $\varphi \prec \psi$
(ix) $\delta(s, \varphi) = true$, if $\delta(s, K_i\varphi) = true$
(x) $\delta(s, K_i\varphi) = true$ and $\delta(s, C_G K_i\varphi) = true$ for all $i \in G$, if $\delta(s, C_G\varphi) = true$
(xi) $\delta(s, C_{G'}\varphi) = true$, if $\delta(s, C_G\varphi) = true$ and $G' \subseteq G$.

Truth or satisfaction at state $s \in S$ is defined as follows: $((S, \delta, T), s) \vDash \varphi$ iff $\delta(s, \varphi) = true$. If we abandon the truth conditions concerning knowledge and common knowledge and restrict the language to usual propositional logic, then our notion of syntactic structure coincides with the usual one (see, e.g., [5]).

THEOREM 5.8. *If $(\mathcal{M}, \beta)$ is an intensional $\in_K$-interpretation w.r.t. T, then there is a syntactic structure (S, δ, T) such that for any $\varphi \in Expr(\mathcal{C}, \mathcal{L}, n)$ and $s \in S$, $(\mathcal{M}, \beta) \vDash \varphi$ iff $((S, \delta, T), s) \vDash \varphi$. If (S, δ, T) is a syntactic structure and $s \in S$, then there is an intensional transitive $\in_K$-interpretation $(\mathcal{M}, \beta)$ such that for all formulas $\varphi \in Expr(\mathcal{C}, \mathcal{L}, n)$, $((S, \delta, T), s) \vDash \varphi$ iff $(\mathcal{M}, \beta) \vDash \varphi$.*

Proof. Let $(\mathcal{M}, \beta)$ be an intensional $\in_K$-interpretation w.r.t. T. Put $S = \{s\}$, where s is a new symbol. Define $\delta(s, \varphi) := true$ iff $\Gamma(\varphi, \beta) \in TRUE$, for any φ. Then (S, δ, T) is a syntactic structure that satisfies the same expressions as $(\mathcal{M}, \beta)$. Now suppose that (S, δ, T) is any syntactic structure and $s \in S$. We define an $\in_K$-model. Put $TRUE := \{\varphi \mid \delta(s, \varphi) = true\}$, $TRUE_i := \{\varphi \mid \delta(s, K_i\varphi) = true\}$, $FALSE := \{\varphi \mid \delta(s, \varphi) = false\}$, $M := TRUE \cup FALSE$, $<^{\mathcal{M}} := \prec$, and $\Gamma(\varphi, \gamma) := \varphi[\gamma]$ for all φ and all γ. (EP) follows immediately; (RP) and (CP) follow by induction on expressions. We show (SP). Suppose $\sigma : V \to M$ is a substitution, γ is an assignment and φ is an expression. Let $x \in var(\varphi)$. By the definitions, $\sigma \circ \gamma(x) = \sigma(x)[\gamma] = \Gamma(\sigma(x), \gamma) = \gamma\sigma(x)$. By Lemma 2.3 and (CP): $\Gamma(\varphi[\sigma], \gamma) = \varphi[\sigma][\gamma] = \varphi[\sigma \circ \gamma] = \Gamma(\varphi, \sigma \circ \gamma) = \Gamma(\varphi, \gamma\sigma)$. Thus, (SP) holds. The truth conditions follow easily from the construction. Hence, $\mathcal{M} = (M, TRUE, FALSE, (TRUE_i)_{1 \leq i \leq n}, <^{\mathcal{M}}, \Gamma, T)$ is a model. For the assignment β, $x \mapsto x$, we get $\Gamma(\varphi, \beta) = \varphi$, for every φ. $(\mathcal{M}, \beta)$ is clearly intensional and transitive and satisfies the same formulas as $((S, \delta, T), s)$. ■

If we restrict the language to equation- and reference-free formulas, then we get full equi-expressiveness.

Corollary 5.9. *If $(\mathcal{M}, \beta)$ is any $\in_K$-interpretation w.r.t. T, then there is a syntactic structure (S, δ, T) such that for any equation- and reference-free expression φ and state $s \in S$, $(\mathcal{M}, \beta) \vDash \varphi$ iff $((S, \delta, T), s) \vDash \varphi$. If (S, δ, T) is a syntactic structure and $s \in S$, then there is a transitive $\in_K$-interpretation $(\mathcal{M}, \beta)$ such that for all expressions φ, $((S, \delta, T), s) \vDash \varphi$ iff $(\mathcal{M}, \beta) \vDash \varphi$.*

Proof. The first assertion follows from Corollary 5.7 and Theorem 5.8, the second assertion follows from Theorem 5.8. ■

5.4. Constructing intensional standard models with specific epistemic properties

Shortly, we will present a construction method that models epistemic properties in a very flexible way. But now let us briefly discuss some well-known 3-valued logics within the framework of classical abstract logics. These logics will be useful for our model construction. As generator sets of these abstract logics we identify certain "complete theories". The set of complete theories in an abstract logic is, roughly speaking, the greatest set of theories which is in a sense stable under the considered connectives (see [12]). For instance, in classical logics the set of complete theories is given by the set of maximal theories whereas in intuitionistic logics it is the collection of all prime theories.

Definition 5.10. Suppose $\mathcal{L}$ is a classical abstract logic and $T_0 = \bigcap Th_{\mathcal{L}}$ is its smallest theory, i.e. the set of all valid formulas. Three further abstract logics $X(\mathcal{L})$, where $X \in \{K_3, LP, Ł_3^-\}$, are defined in terms of pairs $(A, \overline{A})$ of sets of expressions of the underlying classical abstract logic $\mathcal{L}$. Let $A, \overline{A} \subseteq Expr_{\mathcal{L}}$ such that for all $a, b \in Expr_{\mathcal{L}}$ the following hold:
(i) $\sim a \in A \Leftrightarrow a \in \overline{A}$; $\sim a \in \overline{A} \Leftrightarrow a \in A$;
(ii) $a \rightarrowtail b \in A \Leftrightarrow a \in \overline{A}$ or $b \in A$; $a \rightarrowtail b \in \overline{A} \Leftrightarrow a \in A$ and $b \in \overline{A}$.
If A is $\mathcal{L}$-consistent, then A is called a complete $K_3(\mathcal{L})$-theory and $\overline{A}$ is the $K_3(\mathcal{L})$-complement of A. If A contains all valid formulas, i.e. $T_0 \subseteq A$ (and A is not necessarily $\mathcal{L}$-consistent), then A is a complete $LP(\mathcal{L})$-theory and $\overline{A}$ is its $LP(\mathcal{L})$-complement. Now we consider the following condition
(ii)' $a \rightarrowtail b \in A \Leftrightarrow a \in \overline{A}$ or $b \in A$ or ($a \in A \Leftrightarrow b \in A$ and $a \in \overline{A} \Leftrightarrow b \in \overline{A}$); and $a \rightarrowtail b \in \overline{A} \Leftrightarrow a \in A$ and $b \in \overline{A}$.
If $A, \overline{A}$ satisfy (i) and (ii)' (instead of (i) and (ii)) for all expressions a, b, and A is $\mathcal{L}$-consistent, then A is a complete $Ł_3^-(\mathcal{L})$-theory, and $\overline{A}$ is its $Ł_3^-(\mathcal{L})$-complement. We denote the abstract logic generated by the set of all complete $K_3(\mathcal{L})$-theories by $K_3(\mathcal{L}) = (Expr_{\mathcal{L}}, Th_{K_3(\mathcal{L})}, \{\sim, \rightarrowtail\})$, where $Th_{K_3(\mathcal{L})}$ is the set of $K_3(\mathcal{L})$-theories, i.e., the set of all intersections of non-empty sets of complete $K_3(\mathcal{L})$-theories. Analogously, we define the abstract logics $LP(\mathcal{L})$ and $Ł_3^-(\mathcal{L})$. If the context $\mathcal{L}$ is clear, then we may write K_3, LP, $Ł_3^-$ for these abstract logics, respectively.

Of course, K_3 stands for Kleene's 3-valued logic and LP stands for Priest's 3-valued Logic of Paradox. The minus-superscript in $Ł_3^-$ indicates a deviation from the principles of Łukasiewicz 3-valued logic $Ł_3$: recall that we have defined $a \curlyvee b := \sim a \rightarrowtail b$, which would be inadequate in logic $Ł_3$. Our aim is to model some suitable properties of knowledge such as "if a is known or b is known, then $a \curlyvee b$ is known" working with the base $\{\sim, \rightarrowtail\}$ of connectives.

Remark 5.11. *If $\mathcal{L}$ is a classical abstract logic and A is a complete $K_3(\mathcal{L})$- or a complete $LP(\mathcal{L})$-theory, then*
$a \curlywedge b \in A \Longleftrightarrow a \in A$ *and* $b \in A$;
$a \curlyvee b \in A \Longleftrightarrow a \in A$ *or* $b \in A$;
$a \rightarrowtail b \in A \Longrightarrow (a \in A \Rightarrow b \in A)$.
If A is a complete $Ł_3^-(\mathcal{L})$-theory, then the condition concerning disjunction turns out to be weaker: $a \in A$ or $b \in A \Longrightarrow a \curlyvee b \in A$.

Lemma 5.12. *Let $\mathcal{L}$ be a classical abstract logic. Then $A \subseteq Expr_{\mathcal{L}}$ is a complete $K_3(\mathcal{L})$-theory iff A is $\mathcal{L}$-consistent and the following hold for all expressions a, b:*

(i) $a \in A \Leftrightarrow \sim\sim a \in A$;
(ii) $a \rightarrowtail b \in A \Leftrightarrow \sim a \in A$ *or* $b \in A$;
(iii) $\sim (a \rightarrowtail b) \in A \Leftrightarrow a \in A$ *and* $\sim b \in A$.
Similarly, A is a complete $LP(\mathcal{L})$-theory iff A contains the smallest $\mathcal{L}$-theory (i.e., A contains all tautologies of $\mathcal{L}$) and satisfies the above conditions (i)-(iii). On the other hand, A is a complete $Ł_3^-(\mathcal{L})$-theory iff A is $\mathcal{L}$-consistent and (i), (iii) and the following condition are satisfied:
(ii') $a \rightarrowtail b \in A \Leftrightarrow \sim a \in A$ *or* $b \in A$ *or* $(a \in A \Leftrightarrow b \in A$ *and* $\sim a \in A \Leftrightarrow \sim b \in A)$

Corollary 5.13. *Let $\mathcal{L}$ be a classical abstract logic. Then $K_3(\mathcal{L})$, $LP(\mathcal{L})$ and $Ł_3^-(\mathcal{L})$ are minimally generated.*

Proof. Note that the set of all complete $K_3(\mathcal{L})$-theories is by definition a generator set for $K_3(\mathcal{L})$. From Lemma 5.12 it follows that the set of complete K_3-theories is closed under union of chains. That is, if $\alpha > 0$ is any ordinal and $(A_i \mid i < \alpha)$ is a chain of complete $K_3(\mathcal{L})$-theories, then $\bigcup_{i<\alpha} A_i$ is a complete $K_3(\mathcal{L})$-theory. In [12] (Theorem 2.11) we have shown that this chain condition is sufficient for the existence of a minimal generator set. Analogously, one shows the assertions for $LP(\mathcal{L})$ and $Ł_3^-(\mathcal{L})$. ■

Let now $\mathcal{L}$ be a classical abstract logic, T a complete (maximal) $\mathcal{L}$-theory, and $A \subseteq T$ a complete $K_3(\mathcal{L})$ or a complete $Ł_3^-(\mathcal{L})$-theory. We read "$a \in T$" as "a is true (w.r.t. T)", and "$\sim a \in T$" as "a is false (w.r.t. T)". Remark 5.11 shows that interpreting "$a \in A$" as "a is known" yields some useful epistemic properties with respect to the connectives.

Definition 5.14. Let $\mathbf{Ax}_n^+$ be the system $\mathbf{Ax}_n$ extended by the following:
$K_i\varphi \leftrightarrow K_i\neg\neg\varphi$
$K_i(\varphi \rightarrow \psi) \rightarrow (K_i\varphi \rightarrow K_i\psi)$ (the **K**-axiom)
$K_i(\varphi \wedge \psi) \leftrightarrow K_i\varphi \wedge K_i\psi$
$K_i\varphi \vee K_i\psi \rightarrow K_i(\varphi \vee \psi)$
$K_i \sim a \leftrightarrow K_i\neg a$, where $a \in Expr_{\mathcal{L}}$
$K_i(a \rightarrowtail b) \leftrightarrow K_i(a \rightarrow b)$, where $a, b \in Expr_{\mathcal{L}}$.
Let $J \subseteq \{1, \ldots, 10\}$. Then $\mathbf{Ax}_n^+ + J$ is the system $\mathbf{Ax}_n^+$ together with the axioms (Aj), $j \in J$, of the following list.
(A1) $K_i\varphi \rightarrow K_iK_i\varphi$ (positive introspection)
(A2) $C_G\varphi \rightarrow K_iC_G\varphi$, whenever $i \in G$
(A3) $\neg K_i\varphi \rightarrow K_i\neg K_i\varphi$ (negative introspection)
(A4) $\neg C_G\varphi \rightarrow K_i\neg C_G\varphi$, whenever $i \in G$
(A5) $C_G\varphi \rightarrow C_GC_G\varphi$

(A6) $\neg C_G\varphi \rightarrow C_G\neg C_G\varphi$
(A7) $C_G\varphi \leftrightarrow C_G\neg\neg\varphi$
(A8) $C_G(\varphi \rightarrow \psi) \rightarrow (C_G\varphi \rightarrow C_G\psi)$
(A9) $C_G(\varphi \wedge \psi) \leftrightarrow C_G\varphi \wedge C_G\psi$
(A10) $C_G\varphi \vee C_G\psi \rightarrow C_G(\varphi \vee \psi)$

First we construct models of $\mathbf{Ax}^+ + J$ for $J = \{1, \dots, 10\}$. Then we will see that abandoning specific conditions in the construction one gets models of $\mathbf{Ax}^+ + J'$, for any given $J' \subseteq \{1, \dots, 10\}$. Of course, if axiom (A5) holds, then we must also ensure that (A2) is true; similarly for the axioms (A6) and (A4). We mention here that our models also satisfy the axioms $K_i(\varphi : true) \leftrightarrow K_i\varphi$, $K_i\varphi \leftrightarrow K_i(\varphi : false : false)$, $K_i(\varphi : false) \leftrightarrow K_i\neg\varphi$, $K_i(\varphi \equiv \varphi)$, $K_i(\varphi < \psi)$ whenever $\varphi \prec \psi$. This, however, is not essential and can be avoided. Let $T \in MTh_{\mathcal{L}}$. The propositional universe is $M := Sent(\mathcal{C}, \mathcal{L}, n)$, and the reference relation $<^{\mathcal{M}}$ is given by $\prec$ restricted to sentences. We define $TRUE$, $TRUE_i$ and $FALSE$ inductively. For this we choose a partition $\mathcal{C} = \mathcal{C}_t \cup \mathcal{C}_f$ such that $\bot \in \mathcal{C}_f$ and $\top, \top_1, \dots, \top_n \in \mathcal{C}_t$. For each $i = 1, \dots, n$ let $A_i \subseteq T$ be a complete $K_3(\mathcal{L})$-theory and let $\overline{A_i}$ its $K_3(\mathcal{L})$-complement. We will extend each A_i to a set $TRUE_i \supseteq A_i$ of sentences according to the rules of K_3. Any set $\{\varphi \in Expr(\mathcal{C}, \mathcal{L}, n) \mid \Gamma(\varphi, \gamma) \in TRUE_i\}$ then will be a complete $K_3(\mathcal{L}^n_{\mathcal{C}})$-theory. In order to represent the corresponding $K_3(\mathcal{L}^n_{\mathcal{C}})$-complements we define sets $FALSE_i$. We will carry out the construction by induction on the rank $R : Expr(\mathcal{C}, \mathcal{L}, n) \rightarrow \omega$ which is defined as follows:

(i) $R(u) = R(\varphi \equiv \psi) = R(\varphi < \psi) = 0$, where u is an atomic expression and φ, ψ are any expressions.
(ii) $R(\varphi : true) = R(\varphi : false) = R(\neg\varphi) = R(K_i\varphi) = R(C_G\varphi) = R(\varphi) + 1$.
(iii) $R(\varphi \rightarrow \psi) = max\{R(\varphi), R(\psi)\} + 1$.

For every $i = 1, \dots, n$ let $TRUE^0_i = \{\top_i\} \cup A_i \cup \{\varphi \equiv \varphi \mid \varphi \text{ is a sentence}\} \cup \{\varphi < \psi \mid \varphi \prec \psi \text{ are sentences}\}$ and $FALSE^0_i = \overline{A_i}$. For every group G we choose a set $COMMON^0_G \subseteq \bigcap_{i \in G} TRUE^0_i$ such that $G \subseteq G'$ implies $COMMON^0_{G'} \subseteq COMMON^0_G$.

Now suppose for all agents i and all groups G the sets $TRUE^m_i$, $FALSE^m_i$ and $COMMON^m_G$ are already defined for some $m \geq 0$. Then let $TRUE^{m+1}_i$ be the union of the set $TRUE^m_i$ with $\{K_j\varphi \mid G \text{ is any group with } i, j \in G \text{ and } \varphi \in COMMON^m_G\}$ and with the following sets:

(i) $\{\varphi : true \mid \varphi \in TRUE^m_i\}$
(ii) $\{\varphi : false \mid \varphi \in FALSE^m_i\}$
(iii) $\{\neg\varphi \mid \varphi \in FALSE^m_i\}$

(iv) $\{\varphi \rightarrow \psi \mid \varphi \in FALSE_i^m$ or $\psi \in TRUE_i^m\}$
(v) $\{C_G\varphi \mid G$ is any group with $i \in G$ and $\varphi \in COMMON_G^m\}$.

$FALSE_i^{m+1}$ is given as the union of $FALSE_i^m$ with the following sets:
(i) $\{\varphi : true \mid \varphi \in FALSE_i^m\}$
(ii) $\{\varphi : false \mid \varphi \in TRUE_i^m\}$
(iii) $\{\neg\varphi \mid \varphi \in TRUE_i^m\}$
(iv) $\{\varphi \rightarrow \psi \mid \varphi \in TRUE_i^m$ and $\psi \in FALSE_i^m\}$
(v) $\{K_i\varphi \mid R(\varphi) \leq m$ and $\varphi \notin TRUE_i^m\}$
(vi) $\{C_G\varphi \mid G$ is any group with $i \in G, R(\varphi) \leq m$ and $\varphi \notin COMMON_G^m\}$.

For each G we choose a $X_G^{m+1} \subseteq \bigcap_{j\in G} TRUE_j^{m+1} \smallsetminus \{\varphi \mid R(\varphi) \leq m\}$ such that $G \subseteq G'$ implies $X_{G'}^{m+1} \subseteq X_G^{m+1}$. Then let $COMMON_G^{m+1}$ be the union of $COMMON_G^m$ with $\{K_j\varphi \mid G' \supseteq G$ is any group with $j \in G'$ and $\varphi \in COMMON_{G'}^m\}$ and with the following sets:
(i) X_G^{m+1}
(ii) $\{C_{G'}\varphi \mid G' \supseteq G$ and $\varphi \in COMMON_{G'}^m\}$
(iii) $\{\neg C_{G'}\varphi \mid G' \supseteq G, R(\varphi) \leq m-1$ and $\varphi \notin COMMON_{G'}^{m-1}\}$

Clearly, $TRUE_i^m$, $FALSE_i^m$, $COMMON_G^m$ contain only sentences of R-rank $\leq m$. It is not hard to check that $G \subseteq G'$ implies $COMMON_{G'}^m \subseteq COMMON_G^m$. For every i and every G let $TRUE_i := \bigcup_{m<\omega} TRUE_i^m$ and $COMMON_G := \bigcup_{m<\omega} COMMON_G^m$.

Claim1: For any G, $COMMON_G \subseteq \bigcap_{i\in G} TRUE_i$.

Proof of Claim1. Let $\varphi \in COMMON_G^{m+1}$. If $\varphi = K_j\varphi'$ for some $\varphi' \in COMMON_{G'}^m$ and $j \in G' \supseteq G$, then for any $i \in G'$ we have $K_j\varphi' \in TRUE_i^{m+1}$. Thus, $\varphi \in \bigcap_{i\in G'} TRUE_i \subseteq \bigcap_{i\in G} TRUE_i$. The remaining cases $\varphi \in X_G^{m+1}$, $\varphi = C_{G'}\varphi'$, $\varphi = \neg C_{G'}\varphi'$ follow similarly.

Claim2: If $COMMON_G^0 = \bigcap_{j\in G} TRUE_j^0$ and $X_G^{m+1} = \bigcap_{j\in G} TRUE_j^{m+1} \smallsetminus \{\varphi \mid R(\varphi) \leq m\}$ for all $m \geq 0$, then $COMMON_G = \bigcap_{j\in G} TRUE_j$.
Claim2 follows by induction on m.

Claim3: $COMMON_G$ is the greatest set $Y \subseteq \bigcap_{i\in G} TRUE_i$ such that for any sentence φ: $\varphi \in Y \Rightarrow$ for all $i \in G$, $K_i\varphi \in Y$.

Proof of Claim3. $COMMON_G$ is closed under G. Suppose $Y \subseteq \bigcap_{i\in G} TRUE_i$ has this property. Let $\varphi \in Y$ with $R(\varphi) = m$. For every $j \in G$, $K_j\varphi \in Y \subseteq \bigcap_{i\in G} TRUE_i$. Since $R(K_j\varphi) = m+1$, $K_j\varphi \in TRUE_i^{m+1}$, for any $i \in G$. The definition of $TRUE_i^{m+1}$ implies $\varphi \in COMMON_G^m \subseteq COMMON_G$.

The sets $TRUE$ and $FALSE$ are inductively defined as follows:

- $u \in TRUE$, if $u \in \mathcal{C}_t \cup T$; $u \in FALSE$, if $u \in \mathcal{C}_f \cup (Expr_{\mathcal{L}} \smallsetminus T)$;
- $\varphi \equiv \psi \in TRUE$, if $\varphi = \psi$; $\varphi \equiv \psi \in FALSE$, if $\varphi \neq \psi$;
- $\varphi < \psi \in TRUE$, if $\varphi \prec \psi$; $\varphi < \psi \in FALSE$, if $\varphi \not\prec \psi$;
- $\varphi : true \in TRUE$, if $\varphi \in TRUE$; $\varphi : true \in FALSE$, if $\varphi \in FALSE$;
- $\varphi : false \in TRUE$, if $\varphi \in FALSE$; $\varphi : false \in FALSE$, if $\varphi \in TRUE$;
- $\neg\varphi \in TRUE$, if $\varphi \in FALSE$; $\neg\varphi \in FALSE$, if $\varphi \in TRUE$;
- $\varphi \rightarrow \psi \in TRUE$, if $\varphi \in FALSE$ or $\psi \in TRUE$; $\varphi \rightarrow \psi \in FALSE$, if $\varphi \in TRUE$ and $\psi \in FALSE$;
- $K_i\varphi \in TRUE$, if $\varphi \in TRUE_i$; $K_i\varphi \in FALSE$, if $\varphi \notin TRUE_i$;
- $C_G\varphi \in TRUE$, if $\varphi \in COMMON_G$;
  $C_G\varphi \in FALSE$, if $\varphi \notin COMMON_G$.

Then $TRUE_i \subseteq TRUE$, $TRUE \cap FALSE = \varnothing$ and $M = TRUE \cup FALSE$. We define $\Gamma(\varphi, \gamma) := \varphi[\gamma]$. The structure conditions follow in the same way as in the proof of Theorem 5.8. The truth conditions follow directly from the construction. Thus, $\mathcal{M} = (M, TRUE, (TRUE_i)_{1 \leq i \leq n}, FALSE, \prec, \Gamma, T)$ is a model. In fact, $\mathcal{M} \vDash TRUE$. We have $\mathcal{M} \vDash \varphi \equiv \psi \Leftrightarrow \Gamma(\varphi) = \Gamma(\psi) \Leftrightarrow \varphi = \psi$; and $\mathcal{M} \vDash \varphi < \psi \Leftrightarrow \Gamma(\varphi) \prec \Gamma(\psi) \Leftrightarrow \varphi \prec \psi$, for any sentences φ, ψ. By Claim3, $COMMON_G = GREATEST_G$ for any G. Trivially, every proposition is the denotation of a sentence. Thus, $\mathcal{M}$ is an intensional standard model.

By (iii), (iv) of $TRUE_i^{m+1}$ and $FALSE_i^{m+1}$, $(\mathcal{M}, \gamma)$ satisfies the axioms of $\mathbf{Ax}_n^+$, for any assignment γ. Let us look at the axioms (A1)-(A10). If $G = \{i\}$ and we choose in the construction $COMMON_G^0 = TRUE_i^0$ and $X_G^{m+1} = TRUE_i^{m+1} \smallsetminus \{\varphi \mid R(\varphi) \leq m\}$, for all $m \geq 0$, then by Claim2 we have $COMMON_G = TRUE_i$ (knowledge of i is common knowledge in the trivial group G). It follows that $\Gamma(\varphi, \gamma) \in TRUE_i$ implies $\Gamma(K_i\varphi, \gamma) \in TRUE_i$, for any φ and any γ. That is, i has positive introspection. So if we do this for every agent i and group $G = \{i\}$, then axiom (A1) holds. If we choose $COMMON_G \subsetneq TRUE_i$ for $G = \{i\}$, then (A1) does not hold. This follows from Claim3. The validity of axiom (A2) depends on item (v) of the definition of $TRUE_i^{m+1}$. Negative introspection of i, axiom (A3), is guaranteed by item (v) of $FALSE_i^{m+1}$ and by item (iii) of $TRUE_i^{m+1}$. So if we abandon (v) of $FALSE_i^{m+1}$, then i has no longer negative introspection. Axiom (A4) depends on item (vi) of $FALSE_i^{m+1}$ and item (iii) of $TRUE_i^{m+1}$. Hence, if we abandon item (vi) of $FALSE_i^{m+1}$, then (A4) no longer holds. Axiom (A5) depends on item (ii) of the definition of $COMMON_G^{m+1}$ and axiom (A6) depends on both (ii) and (iii) of $COMMON_G^{m+1}$. One possibility to guarantee the axioms (A7)-(A10) is to choose $COMMON_G^0 = \bigcap_{i\in G} TRUE_i^0$ and $X_G^{m+1} = \bigcap_{i\in G} TRUE_i^{m+1} \smallsetminus \{\varphi \mid R(\varphi) \leq m\}$, for all $m \geq 0$. Then by Claim2, $COMMON_G = \bigcap_{i\in G} TRUE_i$, and (A7)-(A10) hold. Another possibility is to choose $COMMON_G^0 \subseteq \bigcap_{i\in G} TRUE_i^0$ as a complete $K_3(\mathcal{L})$-theory. This is always possible since the empty set is a complete $K_3(\mathcal{L})$-theory. Now one defines $COMMON_G^{m+1}$ and its $K_3(\mathcal{L})$-complement according to the K_3-rules. One can do this in such a way that also (A5) and (A6) remain true.

Instead of $K_3(\mathcal{L}_\mathcal{C}^n)$ we may work with the logic $Ł_3^-(\mathcal{L}_\mathcal{C}^n)$. We leave it to the reader to consider further modifications in order to model specific properties of knowledge. The logic LP is possibly a canditate for modeling belief instead of knowledge. The new axioms of $\mathbf{Ax}_n^+ + J$ correspond to the following truth conditions.

Definition 5.15. Let $\mathcal{S}(\mathbf{Ax}_n^+)$ be the class of all transitive $\in_K$-structures that satisfy the following additional truth conditions:

- $\Gamma(\varphi,\gamma) \in TRUE_i \Leftrightarrow \Gamma(\neg\neg\varphi,\gamma) \in TRUE_i$
- $\Gamma(\varphi \to \psi,\gamma) \in TRUE_i \Rightarrow$ if $\Gamma(\varphi,\gamma) \in TRUE_i$, then $\Gamma(\psi,\gamma) \in TRUE_i$
- $\Gamma(\varphi \wedge \psi,\gamma) \in TRUE_i \Leftrightarrow \Gamma(\varphi,\gamma) \in TRUE_i$ and $\Gamma(\psi,\gamma) \in TRUE_i$
- $\Gamma(\varphi,\gamma) \in TRUE_i$ or $\Gamma(\psi,\gamma) \in TRUE_i \Rightarrow \Gamma(\varphi \vee \psi,\gamma) \in TRUE_i$
- $\Gamma(\sim a) \in TRUE_i \Leftrightarrow \Gamma(\neg a) \in TRUE_i$, where $a \in Expr_\mathcal{L}$
- $\Gamma(a \rightarrowtail b) \in TRUE_i \Leftrightarrow \Gamma(a \to b) \in TRUE_i$, where $a, b \in Expr_\mathcal{L}$

Let $J \subseteq \{1,\dots,10\}$. Then $\mathcal{S}(\mathbf{Ax}_n^+ + J)$ is the class of all transitive $\in_K$-structures that satisfy the six truth conditions above and additionally the truth conditions (Tj), $j \in J$, of the following list.

- (T1) $\Gamma(\varphi,\gamma) \in TRUE_i \Rightarrow \Gamma(K_i\varphi,\gamma) \in TRUE_i$
- (T2) $\Gamma(C_G\varphi,\gamma) \in TRUE \Rightarrow \Gamma(C_G\varphi,\gamma) \in TRUE_i$, whenever $i \in G$
- (T3) $\Gamma(\varphi,\gamma) \notin TRUE_i \Rightarrow \Gamma(\neg K_i\varphi,\gamma) \in TRUE_i$
- (T4) $\Gamma(C_G\varphi,\gamma) \notin TRUE \Rightarrow \Gamma(\neg C_G\varphi,\gamma) \in TRUE_i$, whenever $i \in G$
- (T5) $\Gamma(C_G\varphi,\gamma) \in TRUE \Rightarrow \Gamma(C_GC_G\varphi,\gamma) \in TRUE$
- (T6) $\Gamma(C_G\varphi,\gamma) \notin TRUE \Rightarrow \Gamma(C_G\neg C_G\varphi,\gamma) \in TRUE$
- (T7) $\Gamma(C_G\varphi,\gamma) \in TRUE \Leftrightarrow \Gamma(C_G\neg\neg\varphi,\gamma) \in TRUE$
- (T8) $\Gamma(C_G(\varphi \to \psi),\gamma) \in TRUE \Rightarrow \Gamma(C_G\varphi \to C_G\psi,\gamma) \in TRUE$

- (T9) $\Gamma(C_G(\varphi \wedge \psi), \gamma) \in TRUE \Leftrightarrow \Gamma(C_G\varphi \wedge C_G\psi, \gamma) \in TRUE$
- (T10) $\Gamma(C_G\varphi \vee C_G\psi, \gamma) \in TRUE \Rightarrow \Gamma(C_G(\varphi \vee \psi), \gamma) \in TRUE$

The system $\mathbf{Ax}_n^+ + J$ together with modus ponens gives rise to a relation of derivability $\vdash_J^+$. Any non-empty class $\mathcal{I}$ of transitive $\in_K$-interpretations generates a sublogic of $\mathcal{L}_\mathcal{C}^n$ with consequence relation $\Vdash_\mathcal{I}$. We say that $\mathbf{Ax}_n^+ + J$ is sound and complete with respect to the sublogic generated by $\mathcal{I}$ if for all $\Phi \cup \{\varphi\} \subseteq Expr(\mathcal{C}, \mathcal{L}, n)$: $\Phi \vdash_J^+ \varphi \Leftrightarrow \Phi \Vdash_\mathcal{I} \varphi$. We have seen that $\mathcal{S}(\mathbf{Ax}_n^+ + J)$ is non-empty. It is clear that the axioms of $\mathbf{Ax}_n^+ + J$ are valid in the $\in_K$-sublogic generated by all interpretations $(\mathcal{M}, \gamma)$ with $\mathcal{M} \in \mathcal{S}(\mathbf{Ax}_n^+ + J)$. Following the same strategy as above one gets the following completeness result.

THEOREM 5.16. *For every $J \subseteq \{1, \ldots, 10\}$ the system $\boldsymbol{Ax}_n^+ + J$ is sound and complete with respect to the $\in_K$-sublogic generated by the class of all transitive interpretations $(\mathcal{M}, \gamma)$, where $\mathcal{M} \in \mathcal{S}(\boldsymbol{Ax}_n^+ + J)$ and γ is any assignment.*

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