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A Few More Useful 8-valued Logics for Reasoning with Tetralattice $EIGHT_4$

Abstract. In their useful logic for a computer network Shramko and Wansing generalize initial values of Belnap's 4-valued logic to the set 16 to be the power-set of Belnap's 4. This generalization results in a very specific algebraic structure — the trilattice $SIXTEEN_3$ with three orderings: information, truth and falsity. In this paper, a slightly different way of generalization is presented. As a base for further generalization a set 3 is chosen, where initial values are **a** — incoming data is asserted, **d** — incoming data is denied, and **u** — incoming data is neither asserted nor denied, that corresponds to the answer "don't know". In so doing, the power-set of 3, that is the set 8 is considered. It turns out that there are not three but four orderings naturally defined on the set 8 that form the tetralattice $EIGHT_4$. Besides three ordering relations mentioned above it is an extra uncertainty ordering. Quite predictably, the logics generated by **a**–order (truth order) and **d**–order (falsity order) coincide with first-degree entailment. Finally logic with two kinds of operations (**a**–connectives and **d**–connectives) and consequence relation defined via **a**–ordering is considered. An adequate axiomatization for this logic is proposed.

Keywords: Generalized truth values, Dunn–Belnap logic, Shramko–Wansing logic, bilattice, trilattice, tetralattice, first-degree entailment

1. Preliminaries

Recently in a series of papers [17, 18, 19] Yaroslav Shramko and Heinrich Wansing highlighted the importance of generalizing the very notion of a truth value. This approach has been originated in an early work by J. Michael Dunn [8, 9] and developed further for a computer reasoning paradigm by Nuel D. Belnap [6, 7]. It rests essentially on a fundamental idea that any sentence can be treated as true, false, neither true nor false, as well as both true and false. The resulting (generalized) four values can be represented as the elements of power-set of the set of the classical two values.

It was Matthew Ginsberg [15] who introduced the concept of a bilattice, an algebraic structure combining two (complete) lattices and a negation operator serving as a lattice homomorphism. Ginsberg also indicated that the

Special Issue: Truth Values. Part II
Edited by **Yaroslav Shramko and Heinrich Wansing**; *Received* April 23, 2009

Studia Logica (2009) 92: 265–280
DOI: 10.1007/s11225-009-9198-x

four values mentioned above constitute the simplest bilattice (sometimes called $FOUR_2$) with distinct logical and information ordering relations. Bilattices have been intensively studied by Melvin Fitting, Arnon Avron and some others, see, e.g. [3, 4, 5, 11, 13, 14].

Shramko and Wansing [17] make use of what they call a *generalized truth value function* — a mapping from the sentences of our language into the subsets of some basic set of truth values. In this way the generalized truth values are modeled as the result of an application of the generalized truth value function. The first stage of generalization (on the basis of classical truth values) gives us exactly the Dunn-Belnap four valued logic. But Shramko and Wansing go further and put forward the collection of useful 16 values (suitable for reasoning within a computer network) to be a power-set of the Belnap's four values set. It turns out that the set 16 forms what can naturally be called a trilattice ($SIXTEEN_3$, cf. [16]) with three partial orderings: in information, truth and falsity.

In the present paper I employ a similar way of generalization but applied to a different set of initial values. These values result from a certain reconsideration of the Belnap's "told values" and "marked values". In fact, my point of departure is Kleene's "logic of uncertainty" rather than classical truth values. Consequently we arrive at the set of eight generalized truth values which, as I believe, may be regarded even more natural, as far as the intuitive perspective is concerned.

The motivation for this generalization will be clarified in detail in Section 2. In Section 3, I introduce on this ground a new algebraic structure — tetralattice $EIGHT_4$, which combines four different ordering relations defined independently: *information* ordering, *truth* ordering, *falsity* ordering, and *uncertainty* ordering. In Section 4 some useful logics generated on this algebraic basis are considered and adequately axiomatized.

2. Told values, marked values, and reported values

Consider Belnap's four valued computer receiving information from different independent sources. The sources may assert or deny something and the computer has to process the incoming information to use it as the base for correct reasoning. An important feature of this information exchange is that the computer must be able to operate properly even if it receives inconsistent information or no information at all. The founding father of "useful four-valued logic" clarifies this as follows: "We want to suggest a natural technique for employment in such cases: when an item comes in as asserted, mark it with a 'told True' sign, and when an item comes in denied, mark with a 'told False' sign" [2, p. 510]. Thus, there are two basic "told values" serving as special marks for incoming data. But there are also two other situations possible, which have to be taken into account and *marked* separately, the situations of incomplete and inconsistent information. This approach leads to the four well-known possibilities of how the computer can deal with the information at hand: mark it with **T** (just "told True"), mark it with **F** (just "told False"), mark it with **N** (no "told" values), or mark it with **B** (both "told True" and "told False"). Let us call the (generalized) truth values corresponding to these four possibilities "marked values".

In summary, there are two "told values" corresponding to the ways the data arrives in from the sources, and there are four "marked values", by which the computer, being in a cognitive situation different from that his sources find themselves in, marks the data. However, as far as the sources of information concerns, such a representation still appears not quite adequate. One may indeed wonder why the sources — human beings or artificial intelligence — have to keep silence when they do not know the correct answer. As rational agents, they definitely are to report consistent data, but why must they know answers to all questions, while even Belnap's computer may sometimes fail to provide relevant information (**N**)?

If we go further and permit the sources not to hide the lack of knowledge, but explicitly express it, it seems natural to extend the set of "told values" by an extra value — "told don't know". Such an enrichment of the told values set will highlight the ambiguity of value **N**, moreover, it will inevitably lead to the conclusion that the set of "marked values" should be re-defined as well. Here are two short arguments in favor of this.

First, the question arises — whether we should distinguish between the following two situations. In the first situation the computer deals with some statement A, which was not discussed with sources earlier and the information on which appears just missing in the database. In the other situation computer considers a statement A, on which the source — being aware of its content — has not been able to report anything certain, finding it problematic (has refrained from an answer or answered "I do not know"). Within standard four-valued interpretation this particular situation is simply impossible. If we do not extend the set of "told values", the latter situation remains absolutely indistinguishable from the first one. In both cases in reply to the inquiry the computer will answer **N**, which seems to be a short-sighted stance.

Next, one may ask whether the situation when two independent sources both tell the computer that a statement is true is the same as the situation when one source claims it is true but another says that he does not know?

For a standard Belnap's computer these situations again appear identical. However, one should distinguish, e.g., the case when the computer knew nothing about A and a source has "opened his eyes" from the case when we knew that A was true and another source has confirmed this. In other words, in the latter case, the honest computer must mark the statement as "definitely true", whereas in the former — with something like "probably/most likely true". Thus, in this way we get two values "True", however of a different "degree of truth".

These arguments support an insight that if we want to model the situation of information exchange more adequately, we have to consider an extra "told value", but also reconsider the set of "marked values". In order to avoid misleading associations and terminological confusion I introduce the notion of a "reported value". An essential difference of "reported values" from "told values" is that in former case an item may come in as

- **asserted** (report YES), mark it with **a** sign;
- **denied** (report NO), mark it with **d** sign;
- **uncertain** (report DON'T KNOW), mark it with **u** sign.

These "reported values" constitute the set of initial values $3 = \{\mathbf{a}, \mathbf{d}, \mathbf{u}\}$. It is natural to define truth ordering on this set as it is shown in Figure 2 to form a lattice $THREE_1$.

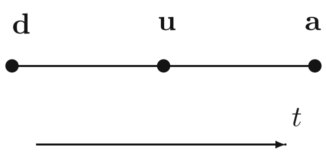

Figure 1. Lattice *THREE₁*

If we define an inversion operation $(-_3)$ as $-_3\mathbf{d} = \mathbf{a}$, $-_3\mathbf{a} = \mathbf{d}$, $-_3\mathbf{u} = \mathbf{u}$, we get exactly the base for Kleene's logic.

Now we can reconsider the set of "marked values" by way of generalization in the sense of Shramko and Wansing. Namely, by taking the power-set of the set 3 we naturally arrive at the set 8:

1. $\mathbf{N} = \varnothing$	5. $\mathbf{TU} = \{\mathbf{a}, \mathbf{u}\}$
2. $\mathbf{T} = \{\mathbf{a}\}$	6. $\mathbf{FU} = \{\mathbf{d}, \mathbf{u}\}$
3. $\mathbf{F} = \{\mathbf{d}\}$	7. $\mathbf{B} = \{\mathbf{a}, \mathbf{d}\}$
4. $\mathbf{U} = \{\mathbf{u}\}$	8. $\mathbf{BU} = \{\mathbf{a}, \mathbf{d}, \mathbf{u}\}$.

The difference between values **U** and **N** as well as between **B** and **BU** is quite remarkable. **U** means that an information source cannot report anything certain about the data, where **N** as before is the sign of no information at all. One may reckon that **N** means that the computer does not know, whereas **U** means that the source does not know, which turns to be more informative answer (in the later case the computer *knows* that the source does not know). Correspondingly, the information top is presented by **BU**, whereas **B** is just a contradiction.

3. Tetralattice $EIGHT_4$

Thus, we have eight values and it is the structure of generalized truth values that makes it possible to define four independent orders: **a**–order, **d**–order, **u**–order, and **i**–order to receive a 4-dimensional multilattice.

DEFINITION 3.1.
Let

$$x^{\mathbf{a}} = \{y \in x \mid \mathbf{a} = y\}, \quad x^{\mathbf{u}} = \{y \in x \mid \mathbf{u} = y\}, \quad x^{\mathbf{d}} = \{y \in x \mid \mathbf{d} = y\},$$
$$x^{-\mathbf{a}} = \{y \in x \mid \mathbf{a} \neq y\}, \quad x^{-\mathbf{u}} = \{y \in x \mid \mathbf{u} \neq y\}, \quad x^{-\mathbf{d}} = \{y \in x \mid \mathbf{d} \neq y\}.$$

Then

$$\begin{array}{lll} x \leqslant_{\mathbf{a}} y & \text{iff} & x^{\mathbf{a}} \subseteq y^{\mathbf{a}} \text{ and } y^{-\mathbf{a}} \subseteq x^{-\mathbf{a}}, \\ x \leqslant_{\mathbf{d}} y & \text{iff} & x^{\mathbf{d}} \subseteq y^{\mathbf{d}} \text{ and } y^{-\mathbf{d}} \subseteq x^{-\mathbf{d}}, \\ x \leqslant_{\mathbf{u}} y & \text{iff} & x^{\mathbf{u}} \subseteq y^{\mathbf{u}} \text{ and } y^{-\mathbf{u}} \subseteq x^{-\mathbf{u}}, \\ x \leqslant_{\mathbf{i}} y & \text{iff} & x \subseteq y. \end{array}$$

An algebraic structure that results as a combination of four complete lattices $\langle 8, \leqslant_{\mathbf{a}} \rangle$, $\langle 8, \leqslant_{\mathbf{d}} \rangle$, $\langle 8, \leqslant_{\mathbf{u}} \rangle$, $\langle 8, \leqslant_{\mathbf{i}} \rangle$ is the tetralattice $EIGHT_4$. The structure of tetralattice $EIGHT_4$ is presented by a Hasse diagram in Figure 2. Because of visual effects the axises are drawn rather tentatively and the best way to catch the idea of the 4-dimensional lattice is to observe its four projections as they are presented in next figure (Figure 3).

Notably, tetralattice $EIGHT_4$ preserves the Belnap's lattice $FOUR_2$ ordering relations.

There are straightforward and easy-to-test properties of meets and joints associated with $\leqslant_{\mathbf{a}}$ and $\leqslant_{\mathbf{d}}$:

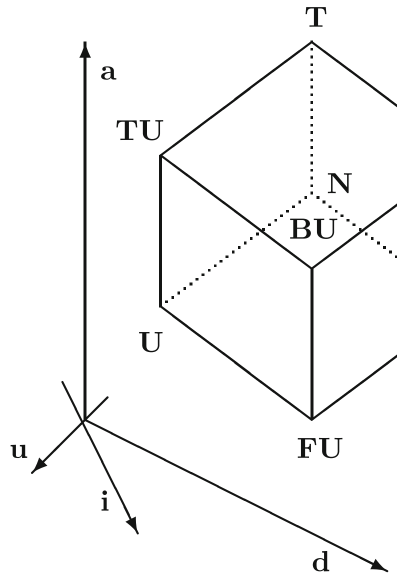

Figure 2. The tetralattice $EIGHT_4$

Proposition 3.1.

$$
\begin{aligned}
&(1) \quad \mathbf{a} \in x \cap_{\mathbf{a}} y \Leftrightarrow \mathbf{a} \in x \textit{ and } \mathbf{a} \in y\\
&\qquad \mathbf{d} \in x \cap_{\mathbf{a}} y \Leftrightarrow \mathbf{d} \in x \textit{ or } \mathbf{d} \in y\\
&\qquad \mathbf{u} \in x \cap_{\mathbf{a}} y \Leftrightarrow \mathbf{u} \in x \textit{ or } \mathbf{u} \in y\\
&(2) \quad \mathbf{a} \in x \cup_{\mathbf{a}} y \Leftrightarrow \mathbf{a} \in x \textit{ or } \mathbf{a} \in y\\
&\qquad \mathbf{d} \in x \cup_{\mathbf{a}} y \Leftrightarrow \mathbf{d} \in x \textit{ and } \mathbf{d} \in y\\
&\qquad \mathbf{u} \in x \cup_{\mathbf{a}} y \Leftrightarrow \mathbf{u} \in x \textit{ and } \mathbf{u} \in y\\
&(3) \quad \mathbf{a} \in x \cap_{\mathbf{d}} y \Leftrightarrow \mathbf{a} \in x \textit{ or } \mathbf{a} \in y\\
&\qquad \mathbf{d} \in x \cap_{\mathbf{d}} y \Leftrightarrow \mathbf{d} \in x \textit{ and } \mathbf{d} \in y\\
&\qquad \mathbf{u} \in x \cap_{\mathbf{d}} y \Leftrightarrow \mathbf{u} \in x \textit{ or } \mathbf{u} \in y\\
&(4) \quad \mathbf{a} \in x \cup_{\mathbf{d}} y \Leftrightarrow \mathbf{a} \in x \textit{ and } \mathbf{a} \in y\\
&\qquad \mathbf{d} \in x \cup_{\mathbf{d}} y \Leftrightarrow \mathbf{d} \in x \textit{ or } \mathbf{d} \in y\\
&\qquad \mathbf{u} \in x \cup_{\mathbf{d}} y \Leftrightarrow \mathbf{u} \in x \textit{ and } \mathbf{u} \in y
\end{aligned}
$$

Define several inversion operations ($-_\otimes$) with negation-like properties:

(1) $-_\otimes -_\otimes x = x$ (2) $x \leqslant_\otimes y \Rightarrow -_\otimes y \leqslant_\otimes -_\otimes x$, where $\otimes \in \{\mathbf{a}, \mathbf{d}, \mathbf{u}, \mathbf{i}\}$.

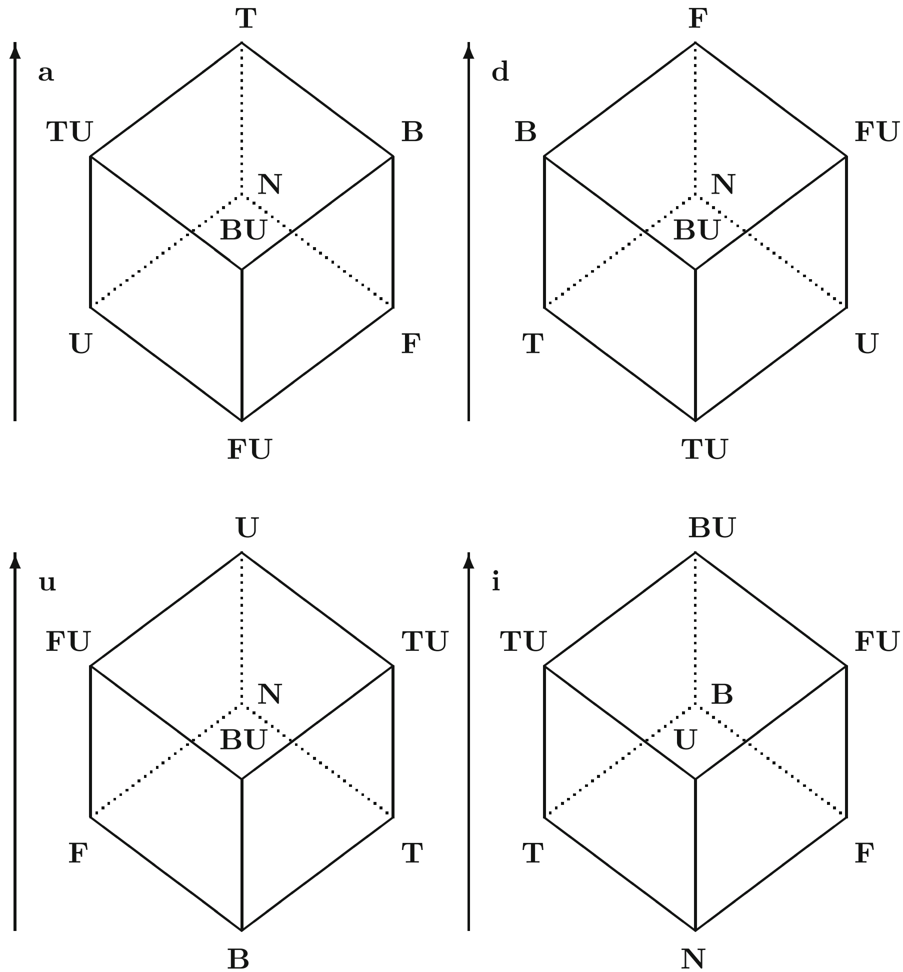

Figure 3. The tetralattice $EIGHT_4$: four projections

Definition 3.2.

c	$-_{\mathbf{a}}c$	$-_{\mathbf{d}}c$	$-_{\mathbf{u}}c$	$-_{i}c$
T	**FU**	**B**	**TU**	**TU**
TU	**U**	**F**	**T**	**T**
B	**F**	**T**	**U**	**U**
BU	**N**	**N**	**N**	**N**
N	**BU**	**BU**	**BU**	**BU**
U	**TU**	**FU**	**B**	**B**
F	**B**	**TU**	**FU**	**FU**
FU	**T**	**U**	**F**	**F**

It can be easily demonstrated that the following definition of $-_{\mathbf{u}}$ via $-_{\mathbf{a}}$ and $-_{\mathbf{d}}$ holds: $-_{\mathbf{a}} -_{\mathbf{d}} -_{\mathbf{a}}x = -_{\mathbf{d}} -_{\mathbf{a}} -_{\mathbf{d}}x = -_{\mathbf{u}}x$. Keeping in mind the coincidence of $-_{\mathbf{i}}$ with $-_{\mathbf{u}}$ the latter condition means that two inversion operations (and corresponding negations) are sufficient and the rest can be introduced by definition.

For inversions, the following proposition can be put forward.

Proposition 3.2.

(1)
$$\mathbf{a} \in -_{\mathbf{a}}x \Leftrightarrow \mathbf{a} \notin x$$
$$\mathbf{d} \in -_{\mathbf{a}}x \Leftrightarrow \mathbf{u} \notin x$$
$$\mathbf{u} \in -_{\mathbf{a}}x \Leftrightarrow \mathbf{d} \notin x$$

(2)
$$\mathbf{a} \in -_{\mathbf{d}}x \Leftrightarrow \mathbf{u} \notin x$$
$$\mathbf{d} \in -_{\mathbf{d}}x \Leftrightarrow \mathbf{d} \notin x$$
$$\mathbf{u} \in -_{\mathbf{d}}x \Leftrightarrow \mathbf{a} \notin x$$

This algebraic basis allows one to define syntactically and semantically an unified logical framework. In doing so first consider language $L_{\mathbf{ad}}$ with connectives $\wedge_{\mathbf{a}}, \vee_{\mathbf{a}}, \sim_{\mathbf{a}}, \wedge_{\mathbf{d}}, \vee_{\mathbf{d}}, \sim_{\mathbf{d}}$. A valuation function ν can be defined as the map from the set of propositional variables into 8, extended to compound formulas by the following conditions:

Definition 3.3.

$$\nu(A \wedge_{\mathbf{a}} B) = \nu(A) \cap_{\mathbf{a}} \nu(B); \quad \nu(A \wedge_{\mathbf{d}} B) = \nu(A) \cap_{\mathbf{d}} \nu(B);$$
$$\nu(A \vee_{\mathbf{a}} B) = \nu(A) \cup_{\mathbf{a}} \nu(B); \quad \nu(A \vee_{\mathbf{d}} B) = \nu(A) \cup_{\mathbf{d}} \nu(B);$$
$$\nu(\sim_{\mathbf{a}}A) = -_{\mathbf{a}}\nu(A); \quad \nu(\sim_{\mathbf{d}}A) = -_{\mathbf{d}}\nu(A).$$

Typically, truth (**a**) and falsity (**d**) orderings are interpreted as components of the so called *logical order*. That is why this paper centers mainly on both of these orderings. The next section presents some useful logics generated by them.

4. The logics of a–order and d–order

To establish logic over the tetralattice $EIGHT_4$ the latter should be equipped with an appropriate entailment relation.

The first in line is the logic generated by **a**-algebraic operations. Syntax of this logic is presented by language $L_{\mathbf{a}}$ with connectives $\wedge_{\mathbf{a}}, \vee_{\mathbf{a}}, \sim_{\mathbf{a}}$.

DEFINITION 4.1. For arbitrary formula of $L_{\mathbf{a}}$,

$$A \models_{\mathbf{a}} B \text{ iff } \forall \nu (\nu(A) \leqslant_{\mathbf{a}} \nu(B))$$

PROPOSITION 4.1. *The logic* $FDE^{\mathbf{a}}_{\mathbf{a}}$*, where the superscript indicates the type of language and the subscript indicates the type of consequence relation, is exactly* E_{fde}.

Hereinafter E_{fde} means the first-degree fragment of relevant logic R (or E). The proof is trivial just because $\langle 8, \leqslant_{\mathbf{a}} \rangle$ is famous M_0, that is a characteristic matrix for tautological entailment axiomatized by E_{fde} [1, §18.8].

If we take language $L_{\mathbf{d}}$ $(\wedge_{\mathbf{d}}, \vee_{\mathbf{d}}, \sim_{\mathbf{d}})$ with the following definition of **d**–variant of entailment relation

DEFINITION 4.2. For arbitrary formula of $L_{\mathbf{d}}$,

$$A \models_{\mathbf{d}} B \text{ iff } \forall \nu (\nu(B) \leqslant_{\mathbf{d}} \nu(A))$$

we again (remember — "first-degree everywhere" [18, p. 413–419]) get the same result:

PROPOSITION 4.2. *The logic* $FDE^{\mathbf{d}}_{\mathbf{d}}$ *is exactly* E_{fde}.

And $\langle 8, \leqslant_{\mathbf{d}} \rangle$ once again coincides with M_0.

My next task will be to construct a logical system which combines within a joint framework the logical connectives generated by *both* orderings $\leqslant_{\mathbf{a}}$ and $\leqslant_{\mathbf{d}}$. To that effect consider language $L^t_{\mathbf{ad}}$ with connectives $\wedge_{\mathbf{a}}$, $\vee_{\mathbf{a}}$, $\sim_{\mathbf{a}}$, $\wedge_{\mathbf{d}}$, $\vee_{\mathbf{d}}$, $\sim_{\mathbf{d}}$ and propositional constant t. I employ t for the sake of completeness proof, moreover, its introducing is justified by the idea that some generally invalid derivations can be quite legitimate under the assumption of reasoning within a complete and consistent theory. By means of this constant we can define a syntactical counterpart of a generator of principal prime filter (truth filter on M_0) as the set $\{A : t \vdash A\}$.

The valuation function ν assigns now **BU** to t and satisfies Definition 3.3. The entailment relation is defined by means of Definition 4.1 extended on the whole language $L^t_{\mathbf{ad}}$.

I formulate next a consequence system $FDE_{\mathbf{a}}^{\mathbf{ad}}$ in a natural for relevant logic way as a pair $(L_{\mathbf{ad}}^{t}, \vdash)$ where $\vdash$ is a consequence relation for which the following deductive postulates hold:

$A1.$ $A \wedge_{\mathbf{a}} B \vdash A$
$A2.$ $A \wedge_{\mathbf{a}} B \vdash B$
$A3.$ $A \vdash A \vee_{\mathbf{a}} B$
$A4.$ $B \vdash A \vee_{\mathbf{a}} B$
$A5.$ $A \wedge_{\mathbf{a}} (B \vee_{\mathbf{a}} C) \vdash (A \wedge_{\mathbf{a}} B) \vee_{\mathbf{a}} C$
$A6.$ $A \wedge_{\mathbf{d}} (B \vee_{\mathbf{d}} C) \vdash (A \wedge_{\mathbf{d}} B) \vee_{\mathbf{d}} C$
$A7.$ $A \wedge_{\mathbf{a}} B \vdash A \wedge_{\mathbf{d}} B$
$A8.$ $A \vee_{\mathbf{d}} B \vdash A \vee_{\mathbf{a}} B$
$A9.$ $\sim_{\mathbf{a}}\sim_{\mathbf{a}}A \vdash A$
$A10.$ $A \vdash \sim_{\mathbf{a}}\sim_{\mathbf{a}}A$
$A11.$ $\sim_{\mathbf{d}}\sim_{\mathbf{d}}A \vdash A$
$A12.$ $A \vdash \sim_{\mathbf{d}}\sim_{\mathbf{d}}A$
$A13.$ $t \vdash \sim_{\mathbf{a}}A \vee_{\mathbf{a}} A$
$A14.$ $\sim_{\mathbf{d}}\sim_{\mathbf{a}}\sim_{\mathbf{d}}A \vdash \sim_{\mathbf{a}}\sim_{\mathbf{d}}\sim_{\mathbf{a}}A$
$A15.$ $\sim_{\mathbf{a}}\sim_{\mathbf{d}}\sim_{\mathbf{a}}A \vdash \sim_{\mathbf{d}}\sim_{\mathbf{a}}\sim_{\mathbf{d}}A$
$A16.$ $\sim_{\mathbf{d}}(A \vee_{\mathbf{a}} B) \vdash \sim_{\mathbf{d}}A \vee_{\mathbf{a}} \sim_{\mathbf{d}}B$
$A17.$ $\sim_{\mathbf{d}}A \wedge_{\mathbf{a}} \sim_{\mathbf{d}}B \vdash \sim_{\mathbf{d}}(A \wedge_{\mathbf{a}} B)$

$R1.$ $A \vdash B, B \vdash C / A \vdash C$
$R2.$ $A \vdash B, A \vdash C / A \vdash B \wedge_{\mathbf{a}} C$
$R3.$ $A \vdash C, B \vdash C / A \vee_{\mathbf{a}} B \vdash C$
$R4.$ $A \vdash B / \sim_{\mathbf{a}}B \vdash \sim_{\mathbf{a}}A$
$R5.$ $A \vdash B, t \vdash \sim_{\mathbf{d}}A \ / t \vdash \sim_{\mathbf{d}}B$
$R6.$ $t \vdash A \wedge_{\mathbf{d}} B \ / t \vdash A \wedge_{\mathbf{a}} B$
$R7.$ $t \vdash A \vee_{\mathbf{a}} B \ / t \vdash A \vee_{\mathbf{d}} B$

Recalling Definition 3.2 one could define the negation corresponding to **u**–ordering ($\sim_{\mathbf{u}}A$) just as $\sim_{\mathbf{a}}\sim_{\mathbf{d}}\sim_{\mathbf{a}}A$. Note, however, that neither $\sim_{\mathbf{d}}\sim_{\mathbf{a}}A \vdash \sim_{\mathbf{a}}\sim_{\mathbf{d}}A$ nor $\sim_{\mathbf{a}}\sim_{\mathbf{d}}A \vdash \sim_{\mathbf{d}}\sim_{\mathbf{a}}A$ are theorems of $FDE_{\mathbf{a}}^{\mathbf{ad}}$.

Some extra theorems clarify the properties of connectives:

$t.1.$ $\sim_{\mathbf{a}}\sim_{\mathbf{d}}A \dashv\vdash \sim_{\mathbf{d}}\sim_{\mathbf{a}}\sim_{\mathbf{d}}\sim_{\mathbf{a}}A$
$t.2.$ $\sim_{\mathbf{d}}\sim_{\mathbf{a}}A \dashv\vdash \sim_{\mathbf{a}}\sim_{\mathbf{d}}\sim_{\mathbf{a}}\sim_{\mathbf{d}}A$
$t.4.$ $\sim_{\mathbf{d}}\sim_{\mathbf{a}}(A \wedge_{\mathbf{d}} B) \vdash \sim_{\mathbf{d}}\sim_{\mathbf{a}}A \vee_{\mathbf{a}} \sim_{\mathbf{d}}\sim_{\mathbf{a}}B$
$t.5.$ $\sim_{\mathbf{d}}\sim_{\mathbf{a}}A \vee_{\mathbf{a}} \sim_{\mathbf{d}}\sim_{\mathbf{a}}B \vdash \sim_{\mathbf{d}}\sim_{\mathbf{a}}(A \wedge_{\mathbf{d}} B)$
$t.6.$ $A \wedge_{\mathbf{a}} (B \vee_{\mathbf{d}} C) \vdash (A \wedge_{\mathbf{a}} B) \vee_{\mathbf{d}} C$
$t.7.$ $A \wedge_{\mathbf{d}} (B \vee_{\mathbf{a}} C) \vdash (A \wedge_{\mathbf{d}} B) \vee_{\mathbf{a}} C$
$t.8.$ $\sim_{\mathbf{d}} (A \wedge_{\mathbf{d}} B) \dashv\vdash \sim_{\mathbf{d}} A \vee_{\mathbf{d}} \sim_{\mathbf{d}} B.$

Now let us prove the semantic adequacy, starting with soundness.

THEOREM 4.3. *For any* $A, B \in L_{\mathbf{ad}}$*: If* $A \vdash B$*, then* $A \models_{\mathbf{a}} B$

To prove this theorem we should demonstrate that all deductive postulates are valid and inference rules preserve validity. It is mainly a routine check which is left to an interested reader as an exercise.

The core idea of the completeness proof is very close to that used by Shramko and Wansing in [17] and, in turn, is similar to the Henkin-style proof modified for first-degree relevant logic.

Define a canonical model in terms of theories. As usually, a *theory* is a set of formulas closed under consequence relation and **a**–adjunction. A theory α is **a**–*prime* iff the following condition holds: if $A \vee_{\mathbf{a}} B \in \alpha$, then $A \in \alpha$ or $B \in \alpha$. Hereinafter for the sake of simplicity **a**–*prime* theories will be referred just as *prime theories*. In what follows a special kind of theories will be used — t-theories which contain constant t. Note that every t-theory α is **a**–*complete* (that is either $A \in \alpha$ or $\sim_{\mathbf{a}} A \in \alpha$) and **a**–*consistent* (that is not both $A \in \alpha$ and $\sim_{\mathbf{a}} A \in \alpha$).

Now define for any t-theory α two sets of formulas:

$\alpha^{\star} = \{A \mid \sim_{\mathbf{d}} A \in \alpha\}$,
$\alpha^{\circledast} = \{A \mid \sim_{\mathbf{d}}\sim_{\mathbf{a}}\sim_{\mathbf{d}} A \in \alpha\}$.

LEMMA 4.1. *Let* α *be a t-theory and let* $\alpha^{\star}$ *and* $\alpha^{\circledast}$ *are defined as above. Then:*
(1) $\alpha^{\star}$ *is a theory and* $\alpha^{\circledast}$ *is a theory;*
(2) $\sim_{\mathbf{d}} A \in \alpha^{\star}$ *iff* $A \in \alpha$ *and* $\sim_{\mathbf{d}}\sim_{\mathbf{a}}\sim_{\mathbf{d}} A \in \alpha^{\circledast}$ *iff* $A \in \alpha$*;*
(3) $\alpha^{\star}$ *is prime iff* α *is prime and* $\alpha^{\circledast}$ *is prime iff* α *is prime.*

PROOF. Consider only $\star$-clauses, for $\circledast$ the proof is analogous.
(1) Let $A \vdash B$ and $A \in \alpha^{\star}$. Then, by definition of $\alpha^{\star}$, $\sim_{\mathbf{d}} A \in \alpha$ and hence, by R5, $\sim_{\mathbf{d}} B \in \alpha$, that is $B \in \alpha^{\star}$. Assume $A \in \alpha^{\star}$ and $B \in \alpha^{\star}$. Then, by definition of $\alpha^{\star}$, $\sim_{\mathbf{d}} A \in \alpha$ and $\sim_{\mathbf{d}} B \in \alpha$. Hence, $\sim_{\mathbf{d}} A \wedge_{\mathbf{a}} \sim_{\mathbf{d}} B \in \alpha$ and, by A17, $\sim_{\mathbf{d}}(A \wedge_{\mathbf{a}} B) \in \alpha$, that is $(A \wedge_{\mathbf{a}} B) \in \alpha^{\star}$. (2) $\sim_{\mathbf{d}} A \in \alpha^{\star}$ *iff* $\sim_{\mathbf{d}}\sim_{\mathbf{d}} A \in \alpha$, by definition of $\alpha^{\star}$, that is equivalent (by A11, A12) to $A \in \alpha$. (3) ($\Rightarrow$) Let α is not prime and show $\alpha^{\star}$ is not prime. That α is not prime means that there are A and B, $A \vee_{\mathbf{a}} B \in \alpha$ and $A \notin \alpha$ and $B \notin \alpha$. Then, by (2), $\sim_{\mathbf{d}}(A \vee_{\mathbf{a}} B) \in \alpha^{\star}$ and $\sim_{\mathbf{d}} A \notin \alpha^{\star}$ and $\sim_{\mathbf{d}} B \notin \alpha^{\star}$. Hence, applying A16, $\sim_{\mathbf{d}} A \vee_{\mathbf{a}} \sim_{\mathbf{d}} B \in \alpha^{\star}$, that is $\alpha^{\star}$ is not prime. ($\Leftarrow$) For the converse, by assuming that $\alpha^{\star}$ is not prime one obtains $\sim_{\mathbf{d}}(A \vee_{\mathbf{a}} B) \in \alpha$ and $A \notin \alpha$ and $B \notin \alpha$, what, by A16 again, provides α is not prime. ■

In what follows Lindenbaum's Lemma will be used, establishing the fact that *for any* $A, B \in L^t_{\mathbf{ad}}$*: if* $A \nvdash B$*, then there exists a prime theory* χ *such that* $A \in \chi$ *and* $B \notin \chi$.

Now, for α being a t-theory a canonical valuation ν_α can be defined as follows:

$$\begin{aligned}
\mathbf{a} \in \nu_\alpha(p) &\Leftrightarrow p \in \alpha \\
\mathbf{d} \in \nu_\alpha(p) &\Leftrightarrow\ \sim_{\mathbf{d}}\sim_{\mathbf{a}} p \in \alpha \\
\mathbf{u} \in \nu_\alpha(p) &\Leftrightarrow \sim_{\mathbf{d}} p \notin \alpha \\
\nu_\alpha(t) = \mathbf{BU} &\Leftrightarrow\ t \in \alpha.
\end{aligned}$$

Next, we have to show that the canonical valuation so defined can be extended to an arbitrary formula.

LEMMA 4.2. *Let* ν_α *be canonical valuation. Then for any* $A \in L^t_{\mathbf{ad}}$*:*

$$\begin{aligned}
\mathbf{a} \in \nu_\alpha(A) &\Leftrightarrow A \in \alpha \\
\mathbf{d} \in \nu_\alpha(A) &\Leftrightarrow\ \sim_{\mathbf{d}}\sim_{\mathbf{a}} A \in \alpha \\
\mathbf{u} \in \nu_\alpha(A) &\Leftrightarrow\ \sim_{\mathbf{d}} A \notin \alpha.
\end{aligned}$$

PROOF. To prove this lemma apply (simultaneous) induction on the length of a formula. I consider here only clauses with negations and **d**-conjunction, other cases being analogous.

Let $A = \sim_{\mathbf{a}} B$ and the lemma holds for B.

Ad **a**. $\mathbf{a} \in \nu_\alpha(\sim_{\mathbf{a}} B) \Leftrightarrow \mathbf{a} \notin \nu_\alpha(B)$ (proposition 3.2)
$\Leftrightarrow B \notin \alpha$ (inductive assumption)
$\Leftrightarrow\ \sim_{\mathbf{a}} B \in \alpha$ (**a**–completeness, **a**–consistency).

Ad **d**. $\mathbf{d} \in \nu_\alpha(\sim_{\mathbf{a}} B) \Leftrightarrow \mathbf{u} \notin \nu_\alpha(B)$ (proposition 3.2)
$\Leftrightarrow\ \sim_{\mathbf{d}} B \in \alpha$ (inductive assumption)
$\Leftrightarrow\ \sim_{\mathbf{d}}\sim_{\mathbf{a}}\sim_{\mathbf{a}} B \in \alpha$ ($A14$, $A16$, $t.1$).

Ad **u**. $\mathbf{u} \in \nu_\alpha(\sim_{\mathbf{a}} B) \Leftrightarrow \mathbf{d} \notin \nu_\alpha(B)$ (proposition 3.2)
$\Leftrightarrow\ \sim_{\mathbf{d}}\sim_{\mathbf{a}} B \notin \alpha$ (inductive assumption)

Let $A = \sim_{\mathbf{d}} B$ and the lemma holds for B.

Ad **a**. $\mathbf{a} \in \nu_\alpha(\sim_{\mathbf{d}} B) \Leftrightarrow \mathbf{u} \notin \nu_\alpha(B)$ (proposition 3.2)
$\Leftrightarrow\ \sim_{\mathbf{d}} B \notin \alpha$ (inductive assumption)

Ad **d**. $\mathbf{d} \in \nu_\alpha({\sim_\mathbf{d}}B) \Leftrightarrow \mathbf{d} \notin \nu_\alpha(B)$ (proposition 3.2)
$\Leftrightarrow\ {\sim_\mathbf{d}}{\sim_\mathbf{a}}B \notin \alpha$ (inductive assumption)
$\Leftrightarrow\ {\sim_\mathbf{a}}{\sim_\mathbf{d}}{\sim_\mathbf{a}}B \in \alpha$ (**a**-completeness, **a**-consistency)
$\Leftrightarrow\ {\sim_\mathbf{d}}{\sim_\mathbf{a}}{\sim_\mathbf{d}}B \in \alpha$ (t.1)

Ad **u**. $\mathbf{u} \in \nu_\alpha({\sim_\mathbf{d}}B) \Leftrightarrow \mathbf{a} \notin \nu_\alpha(B)$ (proposition 3.2)
$\Leftrightarrow B \notin \alpha$ (inductive assumption)
$\Leftrightarrow\ {\sim_\mathbf{d}}{\sim_\mathbf{d}}B \notin \alpha$ ($A11,\ A12$)

Let $A = B \wedge_\mathbf{d} C$ and the lemma holds for B and C.

Ad **a**. $\mathbf{a} \in \nu_\alpha(B \wedge_\mathbf{d} C) \Leftrightarrow \mathbf{a} \in \nu_\alpha(B)$ and $\mathbf{a} \in \nu_\alpha(C)$ (proposition 3.1)
$\Leftrightarrow\ B \in \alpha$ and $C \in \alpha$ (inductive assumption)
$\Leftrightarrow\ B \wedge_\mathbf{a} C \in \alpha$ (theory definition)
$\Leftrightarrow\ B \wedge_\mathbf{d} C \in \alpha$ ($\Rightarrow$ by A7, $\Leftarrow$ by R6)

Ad **d**. $\mathbf{d} \in \nu_\alpha(B \wedge_\mathbf{d} C) \Leftrightarrow \mathbf{d} \in \nu_\alpha(B)$ or $\mathbf{d} \in \nu_\alpha(C)$ (proposition 3.1)
$\Leftrightarrow\ {\sim_\mathbf{d}}{\sim_\mathbf{a}}B \in \alpha$ or ${\sim_\mathbf{d}}{\sim_\mathbf{a}}C \in \alpha$ (ind. assumption)
$\Leftrightarrow\ {\sim_\mathbf{d}}{\sim_\mathbf{a}}B \vee_\mathbf{a} {\sim_\mathbf{d}}{\sim_\mathbf{a}}C \in \alpha$ (prime theory def.)
$\Leftrightarrow\ {\sim_\mathbf{d}}{\sim_\mathbf{a}}(B \wedge_\mathbf{d} C) \in \alpha$ ($\Rightarrow$ by t.5, $\Leftarrow$ by t.4)

Ad **u**. $\mathbf{u} \in \nu_\alpha(B \wedge_\mathbf{d} C) \Leftrightarrow \mathbf{u} \in \nu_\alpha(B)$ and $\mathbf{u} \in \nu_\alpha(C)$ (proposition 3.1)
$\Leftrightarrow\ {\sim_\mathbf{d}}B \notin \alpha$ and ${\sim_\mathbf{d}}C \notin \alpha$ (inductive assumption)
$\Leftrightarrow\ {\sim_\mathbf{d}}B \vee_\mathbf{a} {\sim_\mathbf{d}}C \notin \alpha$ (prime theory definition)

And finally, ${\sim_\mathbf{d}}B \vee_\mathbf{a} {\sim_\mathbf{d}}C \notin \alpha \Rightarrow {\sim_\mathbf{d}}B \vee_\mathbf{d} {\sim_\mathbf{d}}C \notin \alpha$ (by A8) $\Rightarrow {\sim_\mathbf{d}}(B \wedge_\mathbf{d} C) \notin \alpha$ (by t.8); ${\sim_\mathbf{d}}(B \wedge_\mathbf{d} C) \notin \alpha \Rightarrow {\sim_\mathbf{d}}B \vee_\mathbf{d} {\sim_\mathbf{d}}C \notin \alpha$ (by t.8) $\Rightarrow {\sim_\mathbf{d}}B \vee_\mathbf{a} {\sim_\mathbf{d}}C \notin \alpha$ (by R7). ■

Moreover for a completeness proof we need another lemma. First recall some basic notions from a lattice theory. A non-empty set F of elements of a lattice $\mathbf{L}$ is a *filter* iff (1) if $x, y \in F$, then $x \cap y \in F$ and (2) if $x \in F$, then $x \cup y \in F$. A *prime* filter is a filter satisfying (3) if $x \cup y \in F$, then $x \in F$ or $y \in F$. A filter T on lattice $\mathbf{L}$ is called *truth filter* iff it is consistent (there is no $x \in \mathbf{L}$ such that both $x \in T$ and $-x \in T$) and complete (for all $x \in \mathbf{L}$, either $x \in T$ or $-x \in T$). A filter $[\theta)$ is a *principal filter (generated by the singleton θ)* iff $[\theta)$ is a set of all γ such that $\theta \leqslant \gamma$, where $\leqslant$ is the given lattice ordering on $\mathbf{L}$.

Lemma 4.3. *$A \in \chi$, where χ is a prime theory, iff $\nu_\alpha(A) \neq \boldsymbol{FU}$, for any prime t-theory α.*

Proof. ($\Rightarrow$) Let (1) $A \in \chi$, where χ is a prime theory, but (2) $\nu_\alpha(A) = \mathbf{FU}$, for some prime t-theory α. Consider a complete lattice $\langle 8, \leqslant_\mathbf{a}\rangle$. Let $[\theta)$ stands for principal filter generated by θ. Then obviously there are three distinct prime principal filters in $\langle 8, \leqslant_\mathbf{a}\rangle$ (distinct means here that not one of them is contained in another) - $[\mathbf{BU})$, $[\mathbf{F})$ and $[\mathbf{U})$. Filter $[\mathbf{FU})$, which coincides with $\langle 8, \leqslant_\mathbf{a}\rangle$, is not prime because it is not proper. Keeping in mind Lemma 4.1 and definition of canonical valuation, we may state the following, for any prime t-theory α, (3):

$$\mathbf{a} \in \nu_\alpha(A) \Leftrightarrow A \in \alpha \Leftrightarrow \nu_\alpha(A) \in [\mathbf{BU})$$
$$\mathbf{a} \in \nu_\alpha(\sim_\mathbf{a}\sim_\mathbf{d}\sim_\mathbf{a} A) \Leftrightarrow A \in \alpha^{\circledast} \Leftrightarrow \nu_\alpha(\sim_\mathbf{a}\sim_\mathbf{d}\sim_\mathbf{a} A) \in [\mathbf{F})$$
$$\mathbf{a} \in \nu_\alpha(\sim_\mathbf{d} A) \Leftrightarrow A \in \alpha^{\star} \Leftrightarrow \nu_\alpha(\sim_\mathbf{d} A) \in [\mathbf{U}).$$

(3) allows to define a map h from a set of prime filters into a set of prime theories as follows: $h([\theta)) = \chi \Leftrightarrow \forall A(\nu_\alpha(A) \in [\theta) \Rightarrow A \in \chi)$. Then
$h([\mathbf{BU})) = \alpha$,
$h([\mathbf{U})) = \alpha^{\star}$,
$h([\mathbf{F})) = \alpha^{\circledast}$.
Assumption (2) means that A$\notin \alpha$ and A$\notin \alpha^{\star}$ and A$\notin \alpha^{\circledast}$. Hence $h([\mathbf{BU}))\neq \chi \neq h([\mathbf{U})) \neq h([\mathbf{F}))$, and χ is not prime theory, that contradicts (1). Thus, $\nu_\alpha(A) \neq \mathbf{FU}$.

($\Leftarrow$) Can be proved precisely the same way. Assume $\nu_\alpha(A) \neq \mathbf{FU}$, by (3), applying h show $A \in \chi$, where χ is a prime theory. ■

Theorem 4.4. *For any $A, B \in L^t_{\mathbf{ad}}$: If $A \models_\mathbf{a} B$ then $A \vdash B$.*

Proof. Assume (1) $A \models_\mathbf{a} B$ and $A \nvdash B$. Then, by Lindenbaum lemma, (2) there is a prime theory χ such that $A \in \chi$ and $B \notin \chi$. Consider a canonical valuation ν_α on α to be some prime t-theory. Then $\nu_\alpha(A) \leq \nu_\alpha(B)$. From (2), by lemma 4.3, $\nu_\alpha(A) \neq \mathbf{FU}$ and $\nu_\alpha(B) = \mathbf{FU}$. Hence, $\nu_\alpha(A) \nleq \nu_\alpha(B)$ — a contradiction. ■

5. Conclusion: summary and future work

In this paper, I proposed a different basis for generalized truth values than that of Shramko and Wansing's. To my mind, the set 3 looks more natural and intuitively acceptable for such a basis because it corresponds better to a common-sense representation of an information exchange. It should be mentioned that Shramko and Wansing [18] discussed the possibility of generalizing Kleene's logic although with very different formal consideration.

The tetralattice $EIGHT_4$ is a new example of a multilattice with four complete and independent orderings. Three logics generated by this algebraic structure and considered in the paper are either the same first-degree entailment or (in the case of language $L_{\mathbf{ad}}$) its relevant generalization.

Surprisingly, the uncertainty ordering relation has fallen short of forming a converse of information ordering. It is typical to define the concept of information via uncertainty, or vagueness. However in this paper the notion of uncertainty seems to be used in a different sense. Definitely the interrelation between increasing of information and decreasing of uncertainty needs further examination and clarification.

Another problem is to axiomatize the logic with at least two consequence relations. My guess is that logic in the language $L_{\mathbf{ad}}$ with **d**–consequence relation will coincide with presented above logic $FDE^{\mathbf{ad}}_{\mathbf{a}}$ up to the connectives subscripts. Meanwhile, the question as to how logic $L^{\mathbf{ad}}_{\mathbf{ad}}$ should be treated remains unanswered.

Acknowledgements. I am grateful to Heinrich Wansing and Yaroslav Shramko for providing an opportunity to attend the 2008 International Workshop on Truth Values at Dresden University of Technology (May 29–June 1, 2008) and especially to Yaroslav Shramko whose friendly support and useful advice encouraged me to present results of my current research. I would also like to thank my colleague Oleg Grigoriev for his invaluable help in preparing this paper. Thanks are due to an anonymous referee whose helpful remarks allowed me to improve the presentation.

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