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Update to "A Survey of Abstract Algebraic Logic"

Abstract. A definition and some inaccurate cross-references in the paper *A Survey of Abstract Algebraic Logic*, which might confuse some readers, are clarified and corrected; a short discussion of the main one is included. We also update a dozen of bibliographic references.

Keywords: Algebraic logic, finitely algebraizable logic, abstract algebraic logic, quasivariety.

Our paper *A Survey of Abstract Algebraic Logic* [Su] presents, among other notions, the concept of ***algebraizability*** of a logical system. This concept was introduced by W. Blok and D. Pigozzi in [23]. In their monograph only finitary logics are considered, and their algebraic counterparts are restricted to quasivarieties; as a consequence, the interpretations involved in the original notion of an algebraizable logic are assumed to be finite. Later on all these finiteness restrictions have been removed in the works of other scholars such as Herrmann, Dellunde or Czelakowski. Accordingly the term "algebraizable" in the literature has come to be used for a more general notion, and some adjectives qualifying "algebraizable" are now in common usage to refer to several stricter versions, among them one incorporating the finitary original character.

Section 3.3 of [Su] introduces the notion of algebraizable logic by describing Blok and Pigozzi's original approach, but using the new terminology and without explicitly assuming that the logics are finitary, in a way that is not consistent with later usage in the same paper. Moreover, we have detected a linguistic confusion when cross-referencing two conditions that share the same label. Since the *Survey* intends to give compact and reliable references to notions and results that were scattered in the literature or were simply part of the *folklore* of the field at the time of publication, we believe it may be useful to publish some precisions.

The context where the algebraizability of logics is discussed in [Su] is the following. We reproduce here the main definitions and notations involved, as they appear in pages 38–39 of [Su]:

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Let $\mathcal{S}$ be a logic over the language $\mathcal{L}$, and let $\vdash_{\mathcal{S}}$ be its consequence relation. Let K be a class of algebras of type $\mathcal{L}$ and let $\models_{\mathsf{K}}$ be the equational consequence relative to the class K. This is the relation between a set of equations $\Gamma{\approx}\Delta = \{\gamma_i \approx \delta_i : i \in I\}$ and a single equation equation $\varphi \approx \psi$ of type $\mathcal{L}$ defined by:

$$\begin{aligned}\Gamma{\approx}\Delta \models_{\mathsf{K}} \varphi \approx \psi \quad \text{iff} \quad & \text{for every } \boldsymbol{A} \in \mathsf{K} \text{ and every } h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}),\\ & \text{if } \ h(\gamma_i) = h(\delta_i) \text{ for every } i \in I, \text{ then } h(\varphi) = h(\psi).\end{aligned}$$

If $K{\approx}\Lambda = \{\kappa_j \approx \lambda_j : j \in J\}$ is another set of equations of the same type, we write $\Gamma{\approx}\Delta \models_{\mathsf{K}} K{\approx}\Lambda$ for the conjunction of all the entailments $\Gamma{\approx}\Delta \models_{\mathsf{K}} \kappa_j \approx \lambda_j$ for all $j \in J$. A similar convention is used for the consequence relation $\vdash_{\mathcal{S}}$ of the logic $\mathcal{S}$.

A set of $\mathcal{L}$-equations $K(x){\approx}\Lambda(x) = \{\kappa_j(x) \approx \lambda_j(x) : j \in J\}$ in at most one variable is said to be a ***faithful interpretation of*** $\mathcal{S}$ ***in*** $\models_{\mathsf{K}}$ if for every $\Gamma \subseteq Fm$ and every $\varphi \in Fm$,

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \text{iff} \quad K(\Gamma){\approx}\Lambda(\Gamma) \models_{\mathsf{K}} K(\varphi){\approx}\Lambda(\varphi), \tag{3.5}$$

where $K(\Gamma){\approx}\Lambda(\Gamma) = \{\kappa_j(\psi) \approx \lambda_j(\psi) : j \in J, \psi \in \Gamma\}$ and $K(\varphi){\approx}\Lambda(\varphi) = \{\kappa_j(\varphi) \approx \lambda_j(\varphi) : j \in J\}$. When there is a faithful interpretation of a logic $\mathcal{S}$ in the equational consequence $\models_{\mathsf{K}}$ associated with a class of algebras K, the class is called an ***algebraic semantics*** for the logic $\mathcal{S}$.

A set of formulas $E(x,y) = \{\epsilon_i(x,y) : i \in I\}$ in at most two variables is said to be a ***faithful interpretation of*** $\models_{\mathsf{K}}$ ***in*** $\mathcal{S}$ if for every set of equations $\Gamma{\approx}\Delta$ and every equation $\varphi \approx \psi$ we have

$$\Gamma{\approx}\Delta \models_{\mathsf{K}} \varphi \approx \psi \quad \text{iff} \quad E(\Gamma, \Delta) \vdash_{\mathcal{S}} E(\varphi, \psi), \tag{3.6}$$

where $E(\Gamma, \Delta) = \{\epsilon_i(\gamma, \delta) : \gamma \approx \delta \in \Gamma{\approx}\Delta, i \in I\}$ and $E(\varphi, \psi) = \{\epsilon_i(\varphi, \psi) : i \in I\}$. The two interpretations are said to be ***mutually inverse*** if

$$x \dashv\vdash_{\mathcal{S}} E\big(K(x), \Lambda(x)\big) \ \text{ and } \ x \approx y \mathrel{=\!\!|\!\!\models_{\mathsf{K}}} K\big(E(x,y)\big){\approx}\Lambda\big(E(x,y)\big), \tag{3.7}$$

where $x \dashv\vdash_{\mathcal{S}} E\big(K(x), \Lambda(x)\big)$ is the conjunction of the entailments $x \vdash_{\mathcal{S}} E\big(K(x), \Lambda(x)\big)$ and $E\big(K(x), \Lambda(x)\big) \vdash_{\mathcal{S}} x$, and similarly for the other part of (3.7). A logic $\mathcal{S}$ over a language $\mathcal{L}$ is ***algebraizable*** if there is a class of algebras K of type $\mathcal{L}$ and a pair of faithful interpretations $E(x,y)$ and $K(x){\approx}\Lambda(x)$, respectively, of $\models_{\mathsf{K}}$ in $\mathcal{S}$ and conversely that are mutually inverse.

Page 39, lines 18–17 from the bottom of the page:

At this place in the paper (just after the definitions reproduced above), it is said that a logic is called ***finitely algebraizable*** when the two interpretations $E(x, y)$ and $K(x) \approx \Lambda(x)$ are finite sets. Although this choice of name may appear as fairly natural, it does not conform to previous nor to later usage in the literature, nor to its implicit usage in the statements of several results in the paper, where only the set $E(x, y)$ is assumed to be finite. Besides, this name does not match well with the notion of a *finitely equivalential* logic, and this is important for the theorems relating both notions. Strictly speaking, with the mentioned definition of finitely algebraizable logic part 6 of Theorem 3.13, part 3 of Corollary 3.14, part 6 of Theorem 3.15 and part 5 of Theorem 3.16 would not hold; the problem is apparent in the last mentioned result. Thus, *the paragraph comprising lines 18 to 11 (counting from the bottom) of page 39 should be replaced by the following one*:

> If the interpretation $E(x, y)$ is a finite set it is said that the logic is ***finitely algebraizable***. Independently of this, if an algebraizable logic is finitary, one can prove that the set $K(x) \approx \Lambda(x)$ is finite. Thus, for finitary logics the notion of a finitely algebraizable logic can be formulated by requiring that both interpretations $E(x, y)$ and $K(x) \approx \Lambda(x)$ are finite. Blok and Pigozzi in [23] considered exclusively finitary logics and finite interpretations, so the notion of algebraizable logic they introduced actually corresponds to what is now characterized as a "finitary and finitely algebraizable" logic. The extensions of the original notion to possibly nonfinitary consequences and with possibly infinite interpretations were considered by Herrmann [89, 90, 91] and by Czelakowski [45, 46]. Examples of algebraizable logics that are not finitely algebraizable can be found in [55, 91, 99]; notice that these examples are finitary logics.

As we remark in this new paragraph, if one considers only finitary logics then the "old" and the "new" definitions are in fact equivalent. However, the framework of [Su] is that of the more general theory, where finitarity is not assumed as a rule for the logics under study; this is what made us think that this correction was necessary.

Page 39, last sentence:

An oversight related to the just discusssed issue is also detected here; this sentence implicitly assumes that only finitary logics are considered, but as we said this is not the case in the context of the paper. Thus, *the last paragraph in page 39 should be replaced by the following one*:

> If $\mathcal{S}$ is an algebraizable logic, then $\mathsf{Alg}\mathcal{S}$ is the largest class K of algebras such that there exist mutually inverse faithful interpretations between $\mathcal{S}$ and $\models_{\mathsf{K}}$. It is called the ***equivalent algebraic semantics*** of $\mathcal{S}$. If $\mathcal{S}$ is finitary and finitely algebraizable then $\mathsf{Alg}\mathcal{S}$ is always a *quasivariety*, and in fact this property characterizes finitely algebraizable logics among the algebraizable finitary logics. More precisely, we can highlight two different facts: First, if $\mathcal{S}$ is an algebraizable logic such that $\mathsf{Alg}\mathcal{S}$ is a quasivariety then $\mathcal{S}$ will actually be finitely algebraizable; and second, the converse property can be proved for finitary logics but need not hold for infinitary logics.

Page 40, lines 4–8:

The first paragraph of this page contains three mixed-up cross-references to the two conditions collectively labelled as (3.7); moreover, the interderivability formula reproduced in line 4, corresponding to the left-hand half of (3.7), should be replaced by its right-hand half. Thus, *this paragraph should be replaced by the following one*:

> It turns out that to verify that $\mathcal{S}$ is algebraizable with equivalent algebraic semantics K it suffices only to show that there exists a faithful interpretation $K(x){\approx}\Lambda(x)$ of $\mathcal{S}$ in the equational consequence relation of K, and a set of formulas $E(x,y)$ in two variables such that $x \approx y \mathrel{=\!\!|\!\models_{\mathsf{K}}} K\big(E(x,y)\big){\approx}\Lambda\big(E(x,y)\big)$, i.e., it suffices to verify only (3.5) and the second (right-most) equivalence of (3.7), because (3.6) and the first (left-most) equivalence of (3.7) are easily shown to be a consequence of the other two conditions. Symmetrically, it also suffices to verify only (3.6) and the first (left-most) equivalence of (3.7).

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Updating references

Since the appearance of the paper in 2003, a number of sources quoted in the bibliography as "manuscripts" or as "to appear" have already been published, and some of them under a different title. We take this opportunity to update them. We have kept the same reference numbers used in the original paper. Below we list these updated references together with those explicitly mentioned in the present note. We do not intend here to add new references to those in [Su].

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