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Dialogue Games for Many-Valued Logics — an Overview

Abstract. An overview of different versions and applications of Lorenzen's dialogue game approach to the foundations of logic, here largely restricted to the realm of many-valued logics, is presented. Among the reviewed concepts and results are Giles's characterization of Łukasiewicz logic and some of its generalizations to other fuzzy logics, including interval based logics, a parallel version of Lorenzen's game for intuitionistic logic that is adequate for finite- and infinite-valued Gödel logics, and a truth comparison game for infinite-valued Gödel logic.

Keywords: dialogue games, fuzzy logic, many-valued logic, hypersequents.

Introduction

> According to Lorenzen, valid arguments are those patterns from premises to conclusions in which the proponent of the conclusion has a winning strategy against any opponent granting the premises. Thus, there is a third independent pragmatic intuition of logical validity, based on viewing argumentation as a game. I have been converted to that view ever since, even though most of my professional life has been under camouflage as a model theorist, or occasionally a proof theorist. Johan van Benthem, [34], p. 10

The above quotation by an eminent logician nicely summarizes the gist of the dialogue game theoretic approach to logic. But, indirectly, it acknowledges that this foundational paradigm cannot compete in popularity with the mainstream view on modern logic, which refers to the duality of semantics, largely identified with model theory, and proof theory, either Gentzen-style or based on Frege-Hilbert-type calculi. There are many reasons for this; not least among them is the fact that Paul Lorenzen, who introduced logical dialogue games in the late 1950s [38], insisted on the centrality of constructive logic (his term for intuitionistic logic) to the exclusion of classical logic and other logics. Consequently the emphasis of Lorenzen and his immediate followers has been on the justification of particular procedural rules of the original game that guarantee that the characterized set of formulas does not

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'collapse' into classical logic. In contrast, we want to show that, once freed from these somewhat narrow ideological concerns, dialogue games provide a flexible and versatile tool for the characterization of many different logics, in particular also many-valued logics.

We emphasize that this overview on dialogue games for many-valued logics is not intended to cover all types of logical games that can be found in the literature. While there are connections to, e.g., Hintikka-style evaluation games [35] and to Ehrenfeucht-Fraïssé games [15], but also to Mundici's (and others') analysis of Ulam-Renyí games [39, 9], we focus on Lorenzen style dialogue games and variations of it. In particular we will not engage with so-called game semantics [7, 1, 2, 36] that, although closely related, is based on the more abstract view of connectives as game operators and is successfully applied to sub-structural logics and functional programming languages. The stream of research reviewed here directly refers to the Lorenzen/Giles proposal to consider properly regulated dialogues as semantic and pragmatic foundation of logics (cf. [38, 28, 5, 42]). As we will see, this limited perspective is still quite diverse in scope. We readily admit that the resulting selection is biased towards our own work. In fact it can be seen as an introduction to an ongoing research programme, currently pursued in Vienna, that explores various applications of Lorenzen/Giles-style dialogue games in the context of fuzzy logics and related logics.

The paper is organized as follows. In Section 1 we briefly review Lorenzen's original dialogue game for intuitionistic logic. Section 2 is devoted to Giles's game for Łukasiewicz logic. More exactly, we concentrate on the implication rules first and look at appropriate rules for other connectives in Section 3. In Section 4 we explore a connection between Łukasiewicz logic and a supervaluationistic account of vagueness. Generalizations of Giles's game to other t-norm based fuzzy logics are reviewed in Section 5. In Section 6 Giles-style games are employed to settle a puzzle about the interpretation of truth functions in interval based fuzzy logics. Section 7 reviews parallel versions of Lorenzen's game that characterize Gödel logics. In Section 8 we look at a different type of dialogue games for Gödel logic that is based on the idea of truth comparison. We conclude in Section 9 with a summary and hints at further related topics.

To achieve a concise presentation we focus solely on propositional logics.

1. Lorenzen's dialogue game for intuitionistic logic

Different versions and formats of logical dialogue games can be found in the literature. Here, we refer directly to Paul Lorenzen's original idea, dating back to the late 1950s (see e.g., [38]), to identify logical validity of a formula F with the existence of a winning strategy for a *proponent* **P** in an idealized confrontational dialogue, in which **P** tries to uphold F against systematic doubts (attacks) by an *opponent* **O**.

Although the claim that this leads to an adequate characterization of Brouwer's *intuitionistic logic* was implicit already in Lorenzen's early essays, it took more than twenty years until the first rigorous, complete and error free proof of this central claim was published in [18]. Many variants of Lorenzen's original dialogue games have appeared in the literature since. (Already Lorenzen and his collaborators defined different versions of the game. See, eg., [19, 37] for further references.) We review a version of Lorenzen's game for propositional intuitionistic logic, introduced in [21], that makes the connection to Gentzen's sequent calculus **LI** explicit and moreover allows to define parallel versions of the game characterizing some intermediate logics, including many-valued logics (see Section 7).

Notation. An *atomic formula (atom)* is either a propositional variable or the constant $\bot$ (*falsum*). As usual, *compound formulas* are built up from atoms using the connectives $\to$, $\wedge$, $\vee$; $\neg A$ abbreviates $A \to \bot$. In addition to formulas, the special signs ?, l?, r? can be stated in a dialogue by the players **P** and **O**, as specified below.

Dialogue games are characterized by two sorts of rules: logical ones and structural ones. The *logical rules* define how to attack a compound formula and how to defend against such an attack. They are summarized in the following table. (If **X** is the proponent **P** then **Y** refers to the opponent **O**, and vice versa.[1])

Logical dialogue rules:

X:	attack by **Y**	defense by **X**
$A \wedge B$	l? or r? (**Y** chooses)	A or B, accordingly
$A \vee B$	?	A or B (**X** chooses)
$A \to B$	A	B

We will see below that **O**, but not **P**, may also attack *atoms*, including $\bot$.

A *dialogue* is a sequence of *moves*, which are either attacking or defending, in accordance with the presented logical rules. Each dialogue refers to a finite multiset of formulas that are *initially granted* by **O**, and to an *initial formula* to be defended by **P**.

[1] Note that *both* players may launch attacks as well as defending moves during the course of a dialogue. For motivation and detailed exposition of these rules we refer to [18, 19].

Moves can be viewed as state transitions. In any state of the dialogue the (multiset of) formulas, that have been either initially granted or stated by **O** so far, are called the *granted formulas* (at this state). The last formula that has been stated by **P** and that either already has been attacked or must be attacked in **O**'s next move is called *active formula.* (Note that the active formula, in general, is *not* the last formula stated by **P**; since **P** may have stated formulas *after* the active formula, that are not attacked by **O**.) With each state of a dialogue we thus associate a *dialogue sequent* $\Pi \vdash A$, where Π denotes the granted formulas and A the active formula.

We stipulate that each move carries the information (pointers) necessary to reconstruct which formula is attacked or defended in which way in that move. However, we do not care about the exact way in which this information is coded.

Structural rules (*Rahmenregeln* in the diction of Lorenzen and his school) regulate the succession of moves. Quite a number of different systems of structural rules have been proposed in the literature (See e.g., [41, 19, 37]; in particular, [37] compares and discusses different systems.). The following rules, together with the winning conditions stated below, amount to a version of dialogues traditionally called *Ei*-dialogues (i.e., Felscher's *E*-dialogues combined with the so-called *ipse dixisti* rule; see, e.g., [37]).

Structural dialogue rules:

Start: The first move of the dialogue is carried out by **O** and consists in an attack on the initial formula.

Alternate: Moves strictly alternate between players **O** and **P**.

Atom: Atomic formulas, including $\bot$, may be stated by both players, but can neither be attacked nor defended by **P**.

E: Each (but the first) move of **O** reacts directly to the immediately preceding move by **P**. I.e., if **P** attacks a granted formula then **O**'s next move either defends this formula or attacks the formula used by **P** to launch this attack. If, on the other hand, **P**'s last move was a defending one then **O** has to attack immediately the formula stated by **P** in that defense move.

Winning conditions (for P):

W: The game ends with **P** winning if **O** has attacked a formula that has already been granted (either initially or in a later move) by **O**.

W⊥: The game ends with **P** winning if **O** has granted $\bot$.

A *dialogue tree* τ *for* $\Pi \vdash C$ is a rooted directed tree with nodes labelled by dialogue sequents and edges corresponding to moves, such that each branch of τ is a dialogue with initially granted formulas Π and initial formula C. We thus identify the nodes of a dialogue tree with states of a dialogue. We distinguish **P**-nodes and **O**-nodes, according to whether it is **P**'s or **O**'s turn to move at the corresponding state.

A finite dialogue tree is a *winning strategy* (for **P**) if the following conditions hold:

1. Every **P**-node has at most one successor node.
2. All leaf nodes are **P**-nodes at which the winning conditions for **P** are satisfied.
3. Every **O**-node has a successor node for each move by **O** that is a permissible continuation of the dialogue (according to the rules) at this stage.

Winning strategies for a player **X** in an extensive-form two-person game with perfect information are commonly described as *functions* assigning a move of **X** to every state of the game at which it is **X**'s turn to move. Observe that the tree form of a winning strategy specifies the corresponding function in a manner that makes the step-wise evolution of permissible dialogues more explicit.

As already mentioned, a dialogue game may be viewed as a state transition system, where moves in a dialogue correspond to transitions between **P**-nodes and **O**-nodes. A dialogue then is a possible trace in the system; winning strategies can be obtained by a systematic 'unraveling' of all possible traces. In any case, the following statement summarizes the adequateness of the game for intuitionistic logic.

THEOREM 1.1. *F is intuitionistically valid if and only if* **P** *has a winning strategy for initial formula F in the game presented above.*

2. Giles's game for Łukasiewicz logic

Giles's analysis [28, 30] of approximate reasoning, that builds on to Lorenzen's dialogue game approach, originally referred to the phenomenon of 'dispersive' experiments in the context of physics. Later Giles [29] explicitly applied the same concept to the problem of providing 'tangible meanings' to fuzzy propositions. For this purpose he introduces a game that consists of two, largely independent components:

(1) Betting for positive results of experiments.

Two players — say: *me* and *you* — agree to pay 1€ to the opponent player for every false statement they assert. By $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ we denote an *elementary state* of the game, where I assert each q_i in the multiset $\{q_1, \dots, q_n\}$ of atomic statements (represented by propositional variables) and you assert each atomic statement $p_i \in \{p_1, \dots, p_m\}$.

Every propositional variable q refers to an experiment E_q with binary (yes/no) result. The statement q can be read as 'E_q yields a positive result'. Things get interesting as the experiments may show dispersion; i.e., the same experiment may yield different results when repeated. However, the results are not completely arbitrary: for every run of the game, a fixed *risk value* $\langle q \rangle^r \in [0, 1]$ is associated with q, denoting the probability that E_q yields a negative result. For the special atomic formula $\bot$ (*falsum*) we define $\langle \bot \rangle^r = 1$. The risk associated with a multiset $\{p_1, \dots, p_m\}$ of atomic formulas is defined as $\langle p_1, \dots, p_m \rangle^r = \sum_{i=1}^{m} \langle p_i \rangle^r$. The risk $\langle \rangle^r$ associated with the empty multiset is defined as 0. The risk associated with an elementary state $[p_1, \dots, p_m \| q_1, \dots, q_n]$ is calculated from my point of view. Therefore the condition $\langle p_1, \dots, p_m \rangle^r \geq \langle q_1, \dots, q_n \rangle^r$ expresses that I do not expect any loss of money (but possibly some gain) when betting on the truth of atomic statements according to the scheme explained above.

(2) A dialogue game for the reduction of compound formulas.

Giles refers to Lorenzen's game, reviewed in Section 1, and specifies the meaning of logical connectives by reference to rules of a dialogue game that proceeds by systematically reducing arguments about compound formulas to arguments about their subformulas.

To achieve a concise presentation, first assume that formulas are built up from propositional variables, the falsity constant $\bot$, and the connective $\to$ only.[2] The central dialogue rule can then be stated as follows:

($R_\to$) If I assert $A \to B$ then, whenever you choose to attack this statement by asserting A, I have to assert also B. (And vice versa, i.e., for the roles of me and you switched.)

This rule reflects the idea that the meaning of implication is specified by the principle that an assertion of 'if A, then B' ($A \to B$) obliges one to assert B, if A is granted.

In contrast to dialogue games for intuitionistic logic (cf. Section 1) no special regulations (structural rules) on the succession between moves of me and you — roughly corresponding to Lorenzen's players **P** and **O** — are

[2]Remember that in Ł all other connectives can be defined from $\to$ and $\bot$ alone. (See, e.g., [32].)

required here. However, we stipulate that each assertion is attacked at most once. This is reflected in the removal of $A \rightarrow B$ from the multiset of all formulas asserted by a player during a run of the game, as soon as the other player has either attacked by asserting A, or has indicated that she will not attack $A \rightarrow B$ at all. Note that every run of the dialogue game ends in an elementary state $[p_1, \dots, p_m \| q_1, \dots, q_n]$. Given an assignment $\langle \cdot \rangle^r$ of risk values to all p_i and q_i we say that I *win*[3]

the corresponding run of the game if I do not expect any loss, i.e., if $\langle p_1, \dots, p_m \rangle^r \geq \langle q_1, \dots, q_n \rangle^r$.

As a trivial example consider the game where I initially assert $p \rightarrow q$ for some atomic formulas p and q; i.e., the initial state is $[\|p \rightarrow q]$. In response you can either assert p in order to force me to assert q, or you explicitly refuse to attack $p \rightarrow q$. In the first case, the game ends in the elementary state $[p\|q]$. In the second case it ends in state $[\|]$. If an assignment $\langle \cdot \rangle^r$ of risk values gives $\langle p \rangle^r \geq \langle q \rangle^r$, then I win, whatever move you choose to make. In other words: I have a winning strategy for $p \rightarrow q$ in all assignments of risk values where $\langle p \rangle^r \geq \langle q \rangle^r$.

Recall (e.g. from [32]) that a *valuation* v for Łukasiewicz logic Ł is a function assigning values $\in [0, 1]$ to the propositional variables and 0 to $\bot$, extended to compound formulas using the truth function $x \Rightarrow_{Ł} y = \inf\{1, 1 - x + y\}$ for implication.

THEOREM 2.1 (R. Giles [28]). *Every assignment $\langle \cdot \rangle^r$ of risk values to atomic formulas occurring in a formula F induces a valuation $v_{\langle \cdot \rangle^r}$ for Ł such that $v_{\langle \cdot \rangle^r}(F) = 1$ if and only if I have a winning strategy for F in the game presented above.*

COROLLARY 2.2. *F is valid in Ł if and only if for all assignments of risk values to atomic formulas occurring in F I have a winning strategy for F.*

3. Other connectives

Although all other connectives can be defined in Łukasiewicz logic from $\rightarrow$ and $\bot$ alone, let us illustrate the idea that the meaning of *all* relevant

[3]Note that 'winning' here refers to *expected* gain. Although, by definition, I 'win' in state $[p\|p]$, I may still have to pay 1€ to you: namely in those concrete runs of the game, where the instance of the experiment E_p referring to *my* assertion of p results in 'no', while the instance of E_p referring to *your* assertion of p yields 'yes', which is possible by dispersion, unless $\langle p \rangle^r$ is 0 or 1. In other words: 'winning' here does not refer to concrete instances of pay-offs, but rather means that I don't loose any money *in average* in a corresponding state.

connectives can be specified directly by intuitively plausible dialogue rules. Interestingly, for conjunction two different rules seem to be plausible candidates at a first glance:

($R_\wedge$) If I assert $A_1 \wedge A_2$ then I have to assert also A_i for any $i \in \{1,2\}$ that you may choose.

($R_{\wedge'}$) If I assert $A_1 \wedge' A_2$ then I have to assert also A_1 as well as A_2.

Of course, both rules turn into rules referring to *your* claims of a conjunctive formula by simply switching the roles of the players ('I' and 'you').

Rule ($R_\wedge$) is dual to the following natural candidate for a disjunction rule:

($R_\vee$) If I assert $A_1 \vee A_2$ then I have to assert also A_i for some $i \in \{1,2\}$ that I myself may choose.

It follows already from results in [28] that rules ($R_\wedge$) and ($R_\vee$) are adequate for 'weak' conjunction and disjunction in Ł, respectively. $\wedge$ and $\vee$ are also called 'lattice connectives' in the context of fuzzy logics, since their truth functions are given by $v(A \wedge B) = \inf\{v(A), v(B)\}$ and $v(A \vee B) = \sup\{v(A), v(B)\}$, respectively.

The question arises, whether one can use the remaining rule ($R_{\wedge'}$) to characterize strong conjunction (&) which corresponds to the t-norm $x *_{\text{Ł}} y = \sup\{0, x + y - 1\}$. However, rule ($R_{\wedge'}$) is inadequate in the context of our betting scheme for random evaluation in a precisification space. The reason for this is that we have to ensure that for each (not necessarily atomic) assertion that we make, we risk a *maximal* loss of 1€ only. It is easy to see that rules ($R_\rightarrow$), ($R_\wedge$), and ($R_\vee$) comply with this *principle of limited liability*. However, if I assert $p \wedge' q$ and we proceed according to ($R_{\wedge'}$), then I end up with a loss of 2€, in case both experiments E_p and E_q fail. There is a simply way to redress this situation to obtain a rule that is adequate for &: Allow any player who asserts $A_1 \& A_2$ to hedge her possible loss by asserting $\bot$ instead of A_1 and A_2, if wished. Asserting $\bot$, of course, corresponds to the obligation to pay 1€ (but not more) for this assertion in the resulting final state. We obtain the following rule for strong conjunction:

($R_\&$) If I assert $A_1 \& A_2$ then I either have to assert A_1 as well as A_2, or else I have to assert $\bot$.

In a similar way, also dialogue rules for negation, strong disjunction, and equivalence can be formulated directly, instead of just derived from ($R_\rightarrow$).

4. Connections to supervaluation

Supervaluation is a widely discussed concept in philosophical logic. Kit Fine has pioneered its application to formal languages that accommodate vague propositions in [27], a paper that remains an important reference point for philosophers of language and logic. The main idea is to evaluate propositions not simply with respect to classical interpretations — i.e., assignments of the truth values 0 ('false') and 1 ('true') to atomic statements — but rather with respect to a whole *space* Π of (possibly) partial interpretations. For every partial interpretation I in Π, Π is required to contain also a classical interpretation I' that extends I. I' is called an *admissible (complete) precisification* of I. A proposition is called *supertrue* in Π if it evaluates to 1 in all admissible precisifications, i.e., in all classical interpretations contained in Π.

Supervaluation and fuzzy logics can be viewed as capturing contrasting, but individually coherent intuitions about the role of logical connectives in vague statements. Consider a sentence like

(*) The sun is orange and is not orange.

When formalized as $s \,\&\, \neg s$, (*) is superfalse in all precisification spaces, since either s or $\neg s$ is evaluated to 0 in each precisification. This fits Kit Fine's motivation in [27] to capture 'penumbral connections' that, in this case, prevent any mono-colored object from having two colors at the same time. According to Fine's intuition the statement 'The sun is orange' absolutely contradicts the statement 'The sun is not orange', even if neither statement is definitely true or definitely false. Consequently (*) is judged as definitely false, although admittedly composed of vague sub-statements. On the other hand, by asserting (*) one may intend to convey the information that both component statements are true *only to some degree*, different from 1 but also from 0. The statement that the sun is orange is not deemed completely incompatible with the opposite statement. In one and the same interpretation, both statements might be deemed partially true and partially false. With this reading and under certain 'natural' choices of truth functions for $\&$ and $\neg$ the statement $s \,\&\, \neg s$ is *not* definitely false, but receives some intermediary truth value.

In [25], we have worked out a dialogue game based attempt to reconcile supervaluation and t-norm based ('fuzzy') evaluation within a common formal framework. To this aim we interpret 'supertruth' as a modal operator and define a logic $\mathsf{S Ł}$ that extends both, Łukasiewicz logic $\mathsf{Ł}$, as well as the classical modal logic $\mathsf{S5}$.

Formulas of $\mathsf{S}Ł$ are built up from the propositional variables $p \in V = \{p_1, p_2, \ldots\}$ and the constant $\bot$ using the connectives & and $\rightarrow$. (Remember that the additional connectives $\neg$, $\wedge$, and $\vee$ can be defined from these.) In accordance with our informal semantic considerations, above, a precisification space is formalized as a triple $\langle W, e, \mu \rangle$, where $W = \{\pi_1, \pi_2, \ldots\}$ is a non-empty countable set, whose elements π_i are called *precisification points*, e is a mapping $W \times V \mapsto \{0, 1\}$, and μ is a probability measure on the σ-algebra formed by all subsets of W. Given a precisification space $\Pi = \langle W, e, \mu \rangle$ a *local truth value* $\|A\|_\pi$ is defined for every formula A and every precisification point $\pi \in W$ inductively by

$$\begin{array}{rcl} \|p\|_\pi & = & e(\pi, p), \text{ for } p \in V \\ \|\bot\|_\pi & = & 0 \\ \|A \,\&\, B\|_\pi & = & \begin{cases} 1 & \text{if } \|A\|_\pi = 1 \text{ and } \|B\|_\pi = 1 \\ 0 & \text{otherwise} \end{cases} \\ \|A \rightarrow B\|_\pi & = & \begin{cases} 1 & \text{if } \|A\|_\pi = 1 \text{ and } \|B\|_\pi = 0 \\ 0 & \text{otherwise} \end{cases} \\ \|\mathsf{S}A\|_\pi & = & \begin{cases} 1 & \text{if } \forall \sigma \in W : \|A\|_\sigma = 1 \\ 0 & \text{otherwise} \end{cases} . \end{array}$$

Local truth values are classical and do not depend on the underlying t-norm $*_Ł$. In contrast, the *global truth value* $\|A\|_\Pi$ of a formula A is defined by

$$\begin{array}{rcl} \|p\|_\Pi & = & \mu(\{\pi \in W | e(\pi, p) = 1\}), \text{ for } p \in V \\ \|\bot\|_\Pi & = & 0 \\ \|A \,\&\, B\|_\Pi & = & \|A\|_\Pi *_Ł \|B\|_\Pi \\ \|A \rightarrow B\|_\Pi & = & \|A\|_\Pi \Rightarrow_Ł \|B\|_\Pi \\ \|\mathsf{S}A\|_\Pi & = & \|\mathsf{S}A\|_\pi \text{ for any } \pi \in W. \end{array}$$

Note that $\|\mathsf{S}A\|_\pi$ is the same value (either 0 or 1) for all $\pi \in W$. In other words: 'local' supertruth is in fact already global, which justifies the above clause for $\|\mathsf{S}A\|_\Pi$. Further observe that we could have used the global conditions, referring to $*_Ł$ and $\Rightarrow_Ł$, also to define $\|A \,\&\, B\|_\pi$ and $\|A \rightarrow B\|_\pi$, since the t-norm based truth functions coincide with the (local) classical ones, when restricted to $\{0, 1\}$.

Most importantly for our current purpose, it is demonstrated in [25] that the evaluation of formulas of $\mathsf{S}Ł$ can be characterized by a dialogue game extending Giles's game for Ł, where 'dispersive elementary experiments' (see Section 2) are replaced by 'indeterministic evaluations' over precisification spaces. The dialogue rule for the supertruth modality involves a relativization to specific precisification points:

(R_{S}) If I assert $\mathsf{S}A$ then I also have to assert that A holds at any precisification point π that you may choose. (And *vice versa*, i.e., for the roles of me and you switched.)

The resulting game is adequate for $\mathsf{S}Ł$:

THEOREM 4.1 ([25]). *A formula F is valid in $\mathsf{S}Ł$ if and only if for every precisification space Π I have a winning strategy for the game starting with my assertion of F.*

5. Generalizations of Giles's game

There is an interesting ambiguity in the phrase 'betting for positive results of (a multiset of) experiments' that describes the evaluation of elementary states of the dialogue game. As explained in Section 2, Giles identifies the combined risk for such a bet with the *sum* of risks associated with the single experiments. Other ways of interpreting the combined risk are worth exploring, too. In [8, 23] a second version of the game is considered, where an elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ corresponds to my single bet that *all* experiments associated with $q_1, \ldots, q_n$ show a positive result, against your single bet that *all* experiments associated with $p_1, \ldots, p_m$ show a positive result. A third form of the game arises if one decides to perform only a *single* experiment for each of the two players, where the relevant experiment is chosen by the opponent (again, see [8, 23]).

These three betting schemes constitute three versions of Giles's game that turn out to be adequate for the three fundamental logics $Ł$ (Łukasiewicz logic), P (Product logic), and G (Gödel logic), respectively. To understand this result it is convenient to invert risk values into probabilities of *positive* results (yes-answers) of the associated experiments. More formally, the *value* of an atomic formula q is defined as $\langle q \rangle = 1 - \langle q \rangle^r$; in particular, $\langle \bot \rangle = 0$.

My expected gain in an elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ in Giles's game for $Ł$ is the sum of money that I expect you to have to pay me minus the sum that I expect to have to pay you. This amounts to $\sum_{i=1}^{m}(1 - \langle p_i \rangle) - \sum_{i=1}^{n}(1 - \langle q_i \rangle)$ €. Therefore, my expected gain is greater or equal to zero if and only if $1 + \sum_{i=1}^{m}(\langle p_i \rangle - 1) \leq 1 + \sum_{i=1}^{n}(\langle q_i \rangle - 1)$ holds. The latter condition is called winning condition $W_{\sum}$.[4]

[4] Remember from Section 2 that 'winning' here refers to *expected* gain and thus abstracts from concrete pay-offs triggered by particular results of associated experiments.

In the second version of the game, you have to pay me 1€ unless every experiment associated with your assertions (p_i) tests positively, and I have to pay you 1€ unless every experiment associated with my assertions (q_i) tests positively. My expected gain therefore is $1 - \prod_{i=1}^{m} \langle p_i \rangle - (1 - \prod_{i=1}^{n} \langle q_i \rangle)$ €; the corresponding winning condition $W_{\prod}$ is $\prod_{i=1}^{m} \langle p_i \rangle \leq \prod_{i=1}^{n} \langle q_i \rangle$.

To maximize the expected gain in the third version of the game I will choose a $p_i \in \{p_1, \ldots, p_m\}$ where the probability of a positive result of the associated experiment is least; and you will do the same for my assertions. Therefore, my expected gain is $(1 - \min_{1 \leq i \leq m} \langle p_i \rangle) - (1 - \min_{1 \leq i \leq n} \langle q_i \rangle)$ €, and consequently the corresponding winning condition $W_{\min}$ is $\min_{1 \leq i \leq m} \langle p_i \rangle \leq \min_{1 \leq i \leq n} \langle q_i \rangle$.

In contrast to Ł, the dialogue game rule $(R_{\rightarrow})$ does not suffice to characterize P and G. To see this, consider the state $[p \rightarrow \bot \| q]$. According to rule $(R_{\rightarrow})$ I may assert p in order to force you to assert $\bot$. Since $\langle \bot \rangle = 0$, the resulting elementary state $[\bot \| p, q]$ fulfills the winning conditions $\langle \bot \rangle \leq \langle p \rangle \cdot \langle q \rangle$ and $\langle \bot \rangle \leq \min\{\langle p \rangle, \langle q \rangle\}$, that correspond to P and G, respectively. However, this is at variance with the fact that for assignments where $\langle p \rangle = 0$ and $\langle q \rangle < 1$ you have asserted a statement $(p \rightarrow \bot)$ that is definitely true $(v(p \rightarrow \bot) = 1)$, whereas my statement q is not definitely true $(v(q) < 1)$.[5]

There are different ways to address the indicated problem. They all seem to imply a break of the symmetry between the roles of the two players. We have to distinguish between elementary states in which my expected gain is non-negative and those in which my expected is strictly positive. Accordingly, we introduce a (binary) signal or *flag* ¶ into the game that, when raised, announces that I will be declared the winner of the current run of the game, only if the evaluation of the final elementary state yields a *strictly positive* (and not just non-negative) expected gain for me. This allows us to come up with a version of the dialogue rules for implication that can be shown to lead to adequate games for all three logics considered here (Ł, P, G):

$(R^{I*}_{\rightarrow})$ If I assert $A \rightarrow B$ then, whenever you choose to attack this statement by asserting A, I have the following choice: either I assert B in reply or I challenge your attack on $A \rightarrow B$ by replacing the current game with a new one in which you assert A and I assert B.

[5]The problem does not arise in logic Ł, since there the expected gain for state $[\bot \| p, q]$ is $\langle p, q \rangle_{Ł} - \langle \bot \rangle_{Ł} = 1 - (\langle p \rangle - 1) - (\langle q \rangle - 1) - (1 - 1) = \langle p \rangle + \langle q \rangle - 1$ and therefore, indeed, negative, as expected, if $\langle p \rangle = 0$ and $\langle q \rangle < 1$.

In formulating an adequate rule for my attacks on your assertions of an implicative formulas we have to use the flag signalling the strict case of the winning condition:

($R^{Y*}_{\rightarrow}$) If you assert $A \rightarrow B$ then, whenever I choose to attack this statement by asserting A, you have the following choice: either you assert B in reply or you challenge my attack on $A \rightarrow B$ by replacing the current game with a new one in which the flag ¶ is raised and I assert A while you assert B.

In contrast to Ł, in G and P the other connectives cannot be defined from $\rightarrow$ and $\bot$ alone. However, the rules for $\wedge$, $\&$, and $\vee$, presented in Section 3 turn out to be adequate for G and P, too.

THEOREM 5.1 ([23]). *Consider the version of Giles's game with rules* ($R^{I*}_{\rightarrow}$), ($R^{Y*}_{\rightarrow}$), ($R_\wedge$), ($R_\vee$), *and* ($R_\&$). *A formula F is valid in* Ł, P, *or* G *if and only if for every valuation $\langle\cdot\rangle$ I have a winning strategy for the game starting with my assertion of F, where 'winning' refers to condition $W_\sum$, $W_\prod$, or $W_{\min}$, respectively.*

In the case of Gödel logic (G), the two versions of conjunction ('strong' and 'weak') coincide. This fact, that is well known from the algebraic view of t-norm based logic (see, e.g., [32]) can also be obtained by comparing optimal strategies involving the rules ($R_\wedge$) and ($R_\&$), respectively. It is also interesting to note that already the original rules ($R_\rightarrow$), ($R_\wedge$), ($R_\vee$), and ($R_\&$) (i.e., without flag) combined with winning condition $W_\prod$ suffice to characterize cancellative hoop logic CHL [16].

6. Interval based logics

To demonstrate the usefulness of the dialogue game approach beyond the familiar logics considered so far, we address a conceptual puzzle about truth functions in the context of interval based fuzzy logics.[6]

A number of researchers have pointed out that, while modelling *degrees of truth* by values in $[0,1]$ might be a justifiable choice in principle, it is problematic to assume that we can always assign adequate values to all concrete, interpreted atomic propositions in a coherent and principled manner. While this problem may be ignored as long as one is only interested in an abstract characterization of logical consequence in contexts of graded truth,

[6]This section draws on as yet unpublished work. Talks on the topic have been presented at *LOGICA 2008*, Hejnice, and at *LOFT 2008*, Amsterdam.

it is sometimes deemed desirable to refine the model by incorporating 'imprecision due to possible incompleteness of the available information' [17] about truth values. Accordingly, it seems reasonable to replace single values $x \in [0,1]$ by whole *intervals* $[a,b] \subseteq [0,1]$ of truth values as the basic semantic unit assigned to propositions. The 'natural truth ordering' $\leq$ can be generalized to intervals in different ways. Following [17] we arrive at these definitions:

Weak truth ordering: $[a_1, b_1] \leq^* [a_2, b_2]$ iff $a_1 \leq a_2$ and $b_1 \leq b_2$.

Strong truth ordering: $[a_1, b_1] \prec [a_2, b_2]$ iff $b_1 \leq a_2$ or $[a_1, b_1] = [a_2, b_2]$.

On the other hand, set inclusion ($\subseteq$) is called *imprecision ordering* in this context. The set of closed subintervals $Int_{[0,1]}$ of $[0,1]$ is augmented by the empty interval $\emptyset$ to yield so-called *enriched bilattice* structures $\langle Int_{[0,1]}, \leq^*, 0, 1, \emptyset, L, N^* \rangle$ as well as $\langle Int_{[0,1]}, \prec, 0, 1, \emptyset, L, N^* \rangle$, where L is the standard lattice on $[0,1]$, with minimum and maximum as operators, and N^* is the extension of the negation operator N to intervals; in our particular case $N^*([a,b]) = [1-b, 1-a]$ and $N^*(\emptyset) = \emptyset$.

Quite a number of papers have been devoted to the study of logics based on such interval generated bilattices. Let us just mention that the Ghent school of Kerre, Deschrijver, Cornelis, and colleagues has produced an impressive amount of work on interval bilattice based logics (see, e.g., [10, 11, 12] and further references therein).

While it is straightforward to generalize both types of conjunction (t-norm and minimum) as well as disjunction (maximum) from $[0,1]$ to $Int_{[0,1]}$ by applying the operators point-wise, it seems less clear how the 'right' generalization of the corresponding truth function $\Rightarrow$ for implication should look like. In [11, 12] $[a,b] \Rightarrow^*_C [c,d] =_{df.} [\min(a \Rightarrow c, b \Rightarrow d), b \Rightarrow d]$ is studied, but in [17] the authors suggest $[a,b] \Rightarrow^*_E [c,d] =_{df.} [b \Rightarrow c, a \Rightarrow d]$. As has been pointed out in [33] there seems to be a kind of trade-off involved here. While $\Rightarrow^*_C$ preserves a lot of algebraic structure — in particular it yields a *residuated* lattice which contains the underlying lattice over $[0,1]$ as a substructure — the function $\Rightarrow^*_E$ is not a residuum, but leads to the following desirable preservation property that $\Rightarrow^*_C$ is lacking. If $\mathcal{M}_2$ is a precisiation of $\mathcal{M}_1$ — meaning: for each propositional variable p, $\mathcal{M}_2$ assigns a subinterval of the interval assigned to p by $\mathcal{M}_1$ — then any formula satisfied by $\mathcal{M}_1$ is also satisfied by $\mathcal{M}_2$.[7]

It is interesting to note that both, Esteva et al. [17] and Cornelis et al. [11, 12], refer to Ginsberg [31], who explicitly introduced bilattices for logics

[7] A formula is defined to be satisfied if it evaluates to the degenerate interval $[1,1]$.

following ideas of Belnap [6]. Most prominently[8] Ginsberg considers $\mathcal{B} = \langle\{0, \top, \bot, 1\}, \leq_t, \leq_k, \neg\rangle$ as endowed with the following intended meaning:

- 0 and 1 represent (classical) falsity and truth, respectively, $\top$ represents 'inconsistent information' and $\bot$ represents 'no information'. The idea here is that truth values are assigned after receiving relevant information from different sources. Accordingly $\top$ is identified with $\{0, 1\}$, $\bot$ with $\emptyset$ and the classical truth values with their singleton sets.
- $\leq_t$, defined by $0 \leq_t \top/\bot \leq 1$, is the '*truth ordering*'.
- $\leq_k$, defined by $\bot \leq_t 0/1 \leq 1$, is the '*knowledge ordering*'.
- Negation is defined by $\neg(0) = 1$, $\neg(1) = 0$, $\neg(\top) = \top$, $\neg(\bot) = \bot$.

While the four 'truth values' of $\mathcal{B}$ may justifiably be understood to represent *states of knowledge* about propositions, it is very questionable to try to define corresponding 'truth functions' for connectives other than negation. Indeed, it is surprising to see how many authors followed Belnap [6] in defending a four valued, truth functional logic based on $\mathcal{B}$. It should be clear that, in the underlying classical setting that is taken for granted by Belnap, the formula $A \wedge \neg A$ can only be false (0), independently of what kind of information, if any, we have about the truth of A. On the other hand, if we have no information about both, A and B, then also $B \wedge \neg A$ could be true as well as false, and therefore $\bot$ should be assigned not only to A, B, and $\neg A$, but also to $B \wedge \neg A$ (in contrast to $A \wedge \neg A$). This well known fact, namely that knowledge does not propagate compositionally, seems to have been ignored repeatedly. (For a recent, forceful reminder on the incoherency of the intended semantics for Belnap's logic we refer to [13].)

In our context this warning about truth functionality is relevant at two separate levels. First, one should realize that 'degrees of truth' cannot be interpreted *epistemically* while upholding truth functionality. Indeed, most fuzzy logicians correctly emphasize that the concept of degrees of truth is *orthogonal* to the concept of degrees of belief. While truth functions for degrees of truth can be motivated and justified in various ways — in particular also by a dialogue game based approach, as we have seen in previous sections — degrees of belief simply don't propagate compositionally and call for other types of logical models (e.g., 'possible world' semantics). Second, concerning the concept of *intervals* of degrees of truth, one should recognize that it is incoherent to insist on both at the same time:

(1) *truth functions* for all connectives, lifted from $[0, 1]$ to $Int_{[0,1]}$, and

[8] Dozens of papers have been published about Belnap's corresponding 4-valued logic.

(2) the interpretation of an interval $[a, b] \subseteq [0, 1]$ assigned to a (possibly logically complex) proposition F as representing the fact our *best knowledge* about the (definite) degree of truth d of F is that $d \in [a, b]$.

Given the mathematical elegance of (1), resulting, among other desirable properties, in a low computational complexity of the involved logics[9], one should look for alternatives to (2). Godo and Esteva[10] have pointed out that, if we insist on (2) just for *atomic* propositions, then at least we can assert that the corresponding 'real', but unknown truth degree of a proposition F cannot lie outside the interval assigned to F according to the truth functions considered in [17] (see above). However, a stronger and more satisfying assertion emerges by taking clues from Giles's game.

Recall the interpretation of elementary states in Giles's game (Section 2): success probabilities are assigned to binary experiments that determine the expected amount of money that the players, *you* and *me*, are expected to have to pay to the opponent. Let v^* be an assignment of intervals $\subseteq [0, 1]$ to the propositional variables. In reference to the partial information represented by v^*, we assign two different success probabilities to each experiment E_q corresponding to a propositional variable q, reflecting whether q is asserted by me or by you. More exactly, my expected loss for the final state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ when evaluated v^*-*cautiously* is given by $\sum_{i=1}^{n} \langle q_i \rangle_h^r - \sum_{i=1}^{m} \langle p_i \rangle_l^r$, but when evaluated v^*-*boldly* it is given by $\sum_{i=1}^{n} \langle q_i \rangle_l^r - \sum_{i=1}^{m} \langle p_i \rangle_h^r$, where the risk values $\langle q \rangle_h^r$ and $\langle q \rangle_l^r$ are determined by the limits of the interval $v^*(q) = [a, b]$ as follows: $\langle q \rangle_h^r = 1 - a$ and $\langle q \rangle_l^r = 1 - b$. The different expectations clearly reflect upper and lower (subjective) success probabilities of the experiments associated with atomic assertions.

THEOREM 6.1. *The following are equivalent:*

(i) Formula F evaluates to $v_{\text{Ł}}^*(F) = [a, b]$.

(ii) For the dialogue game presented in Sections 2 and 3, starting with my assertion of F: if elementary states are evaluated v^*-cautiously then the minimal expected loss I can achieve by an optimal strategy is $(1 - b)$€; if elementary states are evaluated v^*-boldly then my optimal expected loss is $(1 - a)$€.

Note that this result justifies the truth functions suggested in [17], but at the same time dismisses the intended interpretation of $v_{\text{Ł}}^*(F) = [a, b]$ as

[9] It is easy to see that the coNP-completeness of testing validity for Ł (and many other t-norm based logics) carries over to the interval based logics described above.

[10] Private communication.

representing the best available knowledge about the 'real truth degree' of F. It rather suggests an alternative interpretation of truth value intervals, where no 'real degree of truth' exists. That interpretation refers to pessimistic and optimistic expectations about dispersively evaluated atomic assertions that result from implicit commitments made by asserting complex statements.

7. Parallel Lorenzen dialogues

In [21] we have shown that parallel versions of Lorenzen's original dialogue game lead to characterizations of some intermediate logics, i.e., logics that are stronger than intuitionistic logic, but weaker than classical logics. All finite-valued Gödel logics, as well as infinite-valued Gödel-logic (also called Dummett's LC in reference to [14]) belong to this important family of logics. We quickly review the basic concept of 'parallelizing' Lorenzen's dialogue game and the corresponding results for Gödel logics, but refer to [21] for details.

We build on Lorenzen's dialogue game for intuitionistic logic as presented in Section 1. To make the reference to that version of the game precise, we will speak of I-*dialogues* for the corresponding runs of that game. Our parallel versions of the I-dialogue game share the following features:

1. The logical and structural rules of I-games remain unchanged. Indeed, ordinary I-game dialogues appear as sub-case of the more general parallel framework.
2. The proponent **P** may initiate additional I-dialogues by 'cloning' the dialogue sequent of one of the parallel I-dialogues, in which it is her turn to move.
3. To win a set of parallel dialogues the proponent **P** has to win at least one of the component dialogues.

Note that the level of individual dialogue moves is strictly separated from the initiation of new dialogues and the interaction between dialogues. Moreover, we like to consider **P** as the sole 'scheduler' of parallel dialogues. (These features should be contrasted with alternative concepts of abstract dialogue games, like the ones in [1, 7].)

Notation. A *parallel* I-*dialogue (P-*I*-dialogue)* is a sequence of nodes connected by moves. Each node ν is labelled by a *global state* $\Sigma(\nu)$. A global state is a non-empty finite set $\{\Pi_1 \vdash_{\iota 1} C_1, \ldots, \Pi_n \vdash_{\iota n} C_n\}$ of *indexed* I-*dialogue sequents*. Each index ιk uniquely names one of the n elements, called *component dialogue sequents* or simply *components*, of the global state.

In each of the components it is either **P**'s or **O**'s turn to move. We speak of a **P**-component or an **O**-component, accordingly. We distinguish *internal* and *external* moves.

Internal moves combine single I-dialogue moves for some (possibly also none or all) of the components of the current global state. An internal move from global state $\{\Pi_1 \vdash_{\iota 1} C_1, \ldots, \Pi_n \vdash_{\iota n} C_n\}$ to global state $\{\Pi'_1 \vdash_{\iota 1} C'_1, \ldots, \Pi'_n \vdash_{\iota n} C'_n\}$ consists in a set of indexed I-dialogue moves $\{\iota i_1 : \mathsf{move}_1, \ldots, \iota i_m : \mathsf{move}_m\}$, such that the indices ιi_j, $1 \leq j \leq m$, are pairwise distinct elements of $\{\iota 1, \ldots, \iota n\}$. $\Pi'_k \vdash_{\iota k} C'_k$ denotes the component corresponding to the result of move_k applied to the component indexed by ιk if $k \in \{i_1, \ldots, i_m\}$; otherwise $\Pi_k = \Pi'_k$ and $C_k = C'_k$.

External moves, in contrast to internal moves, may add or remove components of a global state, but do not change the local status (**P** or **O**) of existing components.

The basic external move, that enables the 'parallelization' of I-dialogues is the following:

fork is a move by **P** and consists in duplicating one of the **P**-components of the current global state.

Clearly, fork corresponds to item 2 in the above list of basic features of our parallel dialogue games. We call the new index generated by fork a child of the original index of the duplicated component.

The central condition in the definition of a *P*-I-dialogue is the following:

- each sequence of I-dialogue moves that carry the same index in consecutive internal moves forms a (part of an) ordinary I-dialogue.

The *initial global state* $\Sigma(\nu)$, i.e. the state labelling the root node ν of a *P*-I-dialogue, consists only of **O**-components.

The parallel version of the dialogue game may be viewed as a finite collection of state transition systems that are coordinated by referring to a global, discrete flow of time. At each time step some (possibly also none or all) of the component dialogues advance by one move. In a fork-move the component dialogues remain in their individual current states but a new dialogue, that copies the state of one of the old ones, is created.

Observe that our definition of a *P*-I-dialogue game allows for considerable flexibility in 'implementing' the involved parallelism. We may, for example, require that *all* component dialogues have to advance at each time step; or, alternatively, that at most k dialogues may advance simultaneously (even if there are more than k components). If $k = 1$ his corresponds to a 'sequentialization' by dove-tailing moves that might otherwise be made in parallel.

As presented so far, the 'winning powers' of the players remain unchanged in generalizing from I-dialogues to P-I-dialogues and thus the resulting logic remains intuitionistic. However, one may go beyond intuitionistic logic by considering special *synchronization rules*. Synchronization between I-dialogues consists in *merging* two or more I-dialogues into one according to the following general principle: **P** selects certain **P**-components from the global state. The selected components are then merged into a single new dialogue in some specific way. For some synchronization rules, there are different possible ways to merge the components selected by **P**. In those cases **O** may choose one of these possibilities.

The synchronization rule for infinite-valued Gödel logic G is:

g_∞ – **P**-*part*: **P** picks two (indices of) **P**-components $\Pi_1 \vdash_{\iota 1} C_1$ and $\Pi_2 \vdash_{\iota 2} C_2$ from the current global state and thus indicates that $\Pi_1 \cup \Pi_2$ will be the granted formulas of the resulting merged dialogue sequent.

g_∞ – **O**-*part*: In response to this external **P**-move, **O** chooses either C_1 or C_2 as the active formula of the merged component, which is indexed by $\iota 1$ or $\iota 2$, correspondingly.

The appropriate synchronization rule for n-valued Gödel logics G_n ($n \geq 2$) is as follows:

g_n – **P**-*part*: **P** picks $n-1$ **P**-components $\Pi_1 \vdash_{\iota 1} C_1, \ldots \Pi_{n-1} \vdash_{\iota[n-1]} C_{n-1}$, and a **P**-component of form $\Pi_n \vdash_{\iota n} \bot$ from the current global state for merging.

g_n – **O**-*part*: **O** chooses one of the components $\Pi_1 \cup \Pi_2 \vdash_{\iota 1} C_1, \Pi_2 \cup \Pi_3 \vdash_{\iota 2} C_2, \ldots$ or $\Pi_{n-1} \cup \Pi_n \vdash_{\iota[n-1]} C_{n-1}$ as the merged component, that replaces the components picked by **P**.

Remember that **P** wins a P-I-dialogue for a formula F if there is a sequence of (internal and external) moves, starting with initial state $\{\vdash_{\iota 1} F\}$ and resulting in a global state, where at least one component satisfies the winning conditions for I-dialogues. This means that, from the proponents point of view, P-I-dialogues can be seen as (parallel) explorations of different ways to conduct a dialogue. It suffices to find one way to win a dialogue: the component dialogues are joined disjunctively.

THEOREM 7.1 ([26]). *F is valid in G (G_n) if and only if* **P** *has a winning strategy for initial formula F in the P-I-dialogue game, where synchronization rule g_∞ (g_n) may be employed.*

8. Truth comparison games

In [26] yet another dialogue game based approach to reasoning in Gödel logic G is presented. It relies on the fact that G is the single t-norm based logic, where the validity of formulas depends only on the *relative order* of the values of the involved propositional variables, but not on their absolute values, except for the values 0 and 1. This observation arguably is of philosophical interest, given frequently expressed scepticism concerning the meaning of particular real numbers $\in [0, 1]$ as 'truth values'. To emphasize that only the *comparison* of degrees of truth — using the standard order relations $<$ and $\leq$ — is needed to evaluate formulas in G we refer to a dialogue game founded on the idea that any logical connective $\circ$ of G can be characterized via an adequate response by a player $\mathbf{X}$ to player $\mathbf{Y}$'s attack on $\mathbf{X}$'s claim that a statement of form $(A \circ B) \lhd C$ or $C \lhd (A \circ B)$ holds, where $\lhd$ is either $<$ or $\leq$.

We need the following definitions. An assertion $F \lhd G$ is *atomic* if F and G are both atomic; otherwise it is a *compound assertion*. Atomic assertions of form $p < p$, $p < \bot$, $\top < p$, or $\top \leq \bot$ are called *elementary contradictions*. An *elementary order claim* is a set of two atomic assertions of the form $\{E \lhd_1 F, F \lhd_2 G\}$, where $\lhd_1, \lhd_2 \in \{<, \leq\}$.

Following Lorenzen (cf. Section 1), we call the player that initially claims the validity of a chosen formula the *Proponent* $\mathbf{P}$, and the player that tries to refute this claim the *Opponent* $\mathbf{O}$. The dialogue game proceeds in rounds as follows:

1. A dialogue starts with $\mathbf{P}$'s claim that a formula F is valid. $\mathbf{O}$ answers to this move by contradicting this claim with the assertion $F < \top$. (We say that the game is 'on F'.)
2. Each following round consists in two steps:
   (i) $\mathbf{P}$ either attacks a compound assertion or an elementary order claim, contained in the set of assertions that have been made by $\mathbf{O}$ up to this state of the dialogue, but that have not yet been attacked by $\mathbf{P}$.
   (ii) $\mathbf{O}$ answers to the attack by adding a set of assertions according to the rules of Table 1 (for compound assertions) and Table 2 (for elementary order claims).
3. The dialogue ends with $\mathbf{P}$ as winner if $\mathbf{O}$ has asserted an elementary contradiction. Otherwise, $\mathbf{O}$ wins if there is no further possible attack for $\mathbf{P}$, i.e., if all compound assertions and order claims of $\mathbf{O}$ have already been attacked by $\mathbf{P}$.

Table 1. Rules for connectives

P attacks:	**O** asserts as answer:
$A \wedge B \lhd C$	$\{A \lhd C\}$ or $\{B \lhd C\}$
$C \lhd A \wedge B$	$\{C \lhd A,\ C \lhd B\}$
$A \vee B \lhd C$	$\{A \lhd C,\ B \lhd C\}$
$C \lhd A \vee B$	$\{C \lhd A\}$ or $\{C \lhd B\}$
$A \to B < C$	$\{B < A,\ B < C\}$
$C < A \to B$	$\{C < B\}$ or $\{A \leq B,\ C < \top\}$
$A \to B \leq C$	$\{\top \leq C\}$ or $\{B < A, B \leq C\}$
$C \leq A \to B$	$\{A \leq B\}$ or $\{C \leq B\}$

$\lhd$ is either $<$ or $\leq$, consistently within each line.

Table 2. Rules for elementary order claims

P attacks:	**O** asserts as answer:
$\{p \leq q,\ q \leq r\}$	$\{p \leq r\}$
$\{p < q,\ q \lhd r\}$	$\{p < r\}$
$\{p \lhd q,\ q < r\}$	$\{p < r\}$

$\lhd$ is either $<$ or $\leq$.

Instead of considering the rules of Table 1 and 2 as derived from the truth functions for G, one may argue that the dialogue rules are derived from fundamental principles about reasoning in a many-valued logic, where only the relative order of truth values matters.

Consider the example of conjunction. We contend that anyone who claims '$A \wedge B$ is at least as true as C' (for any concrete statements A, B, and C) has to be prepared to defend the claim that 'A is at least as true as C' and the claim that 'B is at least as true as C'. In a similar manner, the claim that 'C is at least as true as $A \wedge B$', obliges one to assert either 'C is at least as true as A' or 'C is at least as true as B'. (Likewise, if we replace 'at least as true' by 'truer than'.) One may then go on to argue that this reading of the rules for $\wedge$ in Table 1 completely determines correct reasoning about assertions of this form. Form this assumption, one can *derive* that $v(A \wedge B) = \min(v(A), v(B))$ is the only adequate definition of the semantics of conjunction in this setting.

The case for disjunction is very similar. Implication, as usual, is more controversial. However, it is easy to see that there are no reasonable alternatives to our rules if the truth value (i.e., degree of truth) of any assertion involving a formula $A \to B$ should only depend on the relative degrees of truth of A and B. (In particular the resulting value should not refer to any arithmetical operation that had to be performed on the absolute values of A and B, respectively).

Clearly the mentioned dialogue rules for logical connectives guide a stepwise, systematic reduction of any claim involving propositions of arbitrary logical complexity to claims about the relative degree of truth of atomic propositions. Sets of claims of the latter form are further reduced as specified in Table 2. The intuitive justification of these latter dialogue rules is obvious: if player **O** asserts, e.g., both $p < q$ and $q \leq r$, then **P** is entitled to force **O** to assert $p < r$, too. Clearly, a claim of the form $p < p$ is indefensible. (Likewise for $p < \bot$, $\top < p$, and $\top \leq \bot$.) Therefore **P** is declared winner of a run of the dialogue game if she succeeds in forcing **O** to assert an elementary contradiction.

Formally we may summarize this analysis of Gödel logic as follows:

THEOREM 8.1 ([26]). *A formula F is valid in G if and only if there exists a winning strategy for* **P** *on F in the presented comparison game.*

Without going into details we mention that winning strategies for **P** directly correspond to proofs in a so-called 'sequent of relation' system for G, as defined in [4].

Note that, in the parlor of game theoreticians, the presented dialogue game is an extensive-form zero-sum game with perfect information. Moreover all runs of the game are finite, if we stipulate that compound assertions and elementary order claims (by **O**) can be attacked at most once (by **P**). Therefore our truth comparison dialogue game is determinate: for any initial state either **P** or **O** has a winning strategy. While winning strategies for **P** witness validity, more specific semantic information can be extracted from winning strategies for **O**. Call a Gödel logic valuation v *compatible* with an elementary order claim $p \lhd q$ if and only if $v(p) \lhd v(q)$, where $\lhd \in \{<, \leq\}$. Then the following holds:

THEOREM 8.2 ([26]). *For all formulas F and Gödel logic valuations v: $v(F) < 1$ if and only if there exists a winning strategy for* **O** *on F, where v is compatible with all elementary order claims made by* **O** *in the corresponding runs of the game.*

Since all sets of elementary order claims either contain an elementary contradiction or else are compatible with some valuation, Theorem 8.2 implies that winning strategies for **O** implicitly specify counter-models and, vice versa, counter-models induce winning strategies for **O**.

9. Conclusions

We have provided a short tour through the varied landscape of logical dialogue games. The tour took off from its natural starting place at Lorenzen's original game for intuitionistic logic, but then remained within the terrain of many-valued logics, in particular of fuzzy logics. Even in this limited realm we had to confine to just a few stops, glimpsing at Giles's characterization of Łukasiewicz logic and some variants for other important fuzzy logics, but also to parallel versions of Lorenzen's game that are adequate for infinite- and finite-valued Gödel logics. A further type of dialogue games, visited in Section 8, made explicit that Gödel logic can be viewed as a logic of truth comparison. In Section 6 we hinted at a recent result that illustrates the intended application of dialogue games for the analysis of conceptual problems relating to fuzzy logics: An appropriate variant of Giles's game for Łukasiewicz logic may be used to justify certain truth functions for an interval based logic [17] by providing an alternative interpretation that avoids the incoherence of claiming that intervals of truth values optimally represent incomplete knowledge of unknown truth values of complex formulas.

A number of related topics had to be omitted from this brief overview. Although Lorenzen's game [38, 18, 19] was originally conceived for first order (intuitionistic) logic, and also Giles [28, 30] makes interesting remarks about the treatment of quantifiers, we confined attention to propositional logics here. In some cases the results straightforwardly generalize to first order logics. However, for many of the mentioned logics quantification is more delicate and calls for further investigations.

Among other neglected topics at least one also deserves to be mentioned: Winning strategies for the proponent ('me' in Giles's terminology) of a dialogue game are closely connected to analytic, i.e., cut-free proofs in appropriate Gentzen-style calculi. In fact winning strategies for the parallel games reviewed in Section 7 directly correspond to analytic proofs in hypersequent systems [3, 40]. Our original (rather sketchy) presentation of the variants of Giles's game for all three major fuzzy logics (G, $\mathsf{Ł}$, P) in [8] was in fact motivated by the observation that rules of a uniform hypersequent calculus for the mentioned logics correspond to rules for constructing optimal strategies for 'me' in adequate versions of Giles's game. Even if one is not interested in hypersequent systems *per se*, one should note that dialogue strategies, as well as corresponding hypersequent derivations, can be viewed as representations of certain proof search strategies in more traditional proof systems, like sequent or tableau calculi (as is explained, e.g., in [22]).

In any case, we may conclude with the statement that the dialogue game approach to many-valued logics offers not only fresh perspectives and insights on foundational issues, but also provides useful tools for a better understanding of analytic proof systems and corresponding proof search.

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