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Basic Hoops: an Algebraic Study of Continuous t-norms

Abstract. A *continuous t-norm* is a continuous map $*$ from $[0,1]^2$ into $[0,1]$ such that $\langle [0,1], *, 1\rangle$ is a commutative totally ordered monoid. Since the natural ordering on $[0,1]$ is a complete lattice ordering, each continuous t-norm induces naturally a residuation $\rightarrow$ and $\langle [0,1], *, \rightarrow, 1\rangle$ becomes a commutative naturally ordered residuated monoid, also called a hoop. The variety of basic hoops is precisely the variety generated by all algebras $\langle [0,1], *, \rightarrow, 1\rangle$, where $*$ is a continuous t-norm. In this paper we investigate the structure of the variety of basic hoops and some of its subvarieties. In particular we provide a complete description of the finite subdirectly irreducible basic hoops, and we show that the variety of basic hoops is generated as a quasivariety by its finite algebras. We extend these results to Hájek's *BL*-algebras, and we give an alternative proof of the fact that the variety of *BL*-algebras is generated by all algebras arising from continuous t-norms on $[0,1]$ and their residua. The last part of the paper is devoted to the investigation of the subreducts of *BL*-algebras, of Gödel algebras and of product algebras.

Keywords: basic hoops, continuous t-norms, subreducts of BL-algebras.

A *continuous t-norm* is a continuous map $*$ from $[0,1]^2$ into $[0,1]$ such that $\langle [0,1], *, 1\rangle$ is a commutative totally ordered monoid. There are three fundamental continuous t-norms: the Łukasiewicz t-norm defined by $x *_L y = \max(x+y-1, 0)$, the Gödel (or lattice) norm $x *_G y = x \wedge y$ and the product norm $x *_P y = xy$. Indeed it is known ([24, 35]) that, up to isomorphism, every continuous t-norm behaves locally as one of the above.

Since the natural ordering on $[0,1]$ is a complete lattice ordering, each t-norm induces naturally a residuation, or an implication in more logical terms, by $x \rightarrow y = \sup\{z : z * x \leq y\}$. The implications associated to the three fundamental norms are:

$$x \rightarrow_L y = \min(y - x + 1, 1)$$

$$x \rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

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$$\text{and} \quad x \to_P y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}.$$

It is clear that the residual $\to$ of a continuous t-norm on $[0,1]$ satisfies

$$\begin{aligned} & x \to x = 1 \\ & x \to 1 = 1 \\ & 1 \to x = x \\ & x \to y = 1 \quad \text{and} \quad y \to x = 1 \quad \text{imply} \quad x = y. \end{aligned}$$

Hence the variety $\mathcal{V}_{\mathcal{K}}$ generated by any class $\mathcal{K}$ of algebras of the form $\langle [0,1], *, \to, 0 \rangle$, where $*$ is a continuous t-norm and $\to$ is its residual, is *ideal-determined* [23] and therefore it is the *equivalent algebraic semantics* of its assertional logic (see [3], Section 3). In particular, there is a propositional calculus naturally associated to $\mathcal{V}_{\mathcal{K}}$ that is strongly complete with respect to $\mathcal{V}_{\mathcal{K}}$.

In his important monograph [24], Hájek considers in detail the three relevant cases and provides an axiomatization for the corresponding varieties of algebras and logics. Algebras in the variety $\mathcal{WA}$ generated by $\langle [0,1], \cdot_L, \to_L, 0, 1 \rangle$ are known as *Wajsberg algebras* [22] or *MV-algebras* [13] and the propositional calculus of which they constitute a complete semantics is the Łukasiewicz many-valued logic [13]. Algebras in the variety $\mathcal{GA}$ generated by $\langle [0,1], \cdot_G, \to_G, 0, 1 \rangle$ are called *Gödel algebras* and form an equivalent algebraic semantics for Dummett's Logic [17], also called Gödel Logic, the infinite-valued version of the infinitely many finitely-valued systems, which Gödel considered in his proof that Intuitionist Logic is not finitely-valued. The variety $\mathcal{PA}$ of *product algebras* is generated by $\langle [0,1], \cdot_P, \to_P, 0, 1 \rangle$, with *product logic* [25] as its associated propositional calculus. Last but not least, Hájek introduces the variety $\mathcal{BL}$ of *BL-algebras* and calls *basic logic* the associated propositional calculus; then he formulates the conjecture that this variety is in fact generated by all algebras of the form $\langle [0,1], *, \to, 0, 1 \rangle$, where $*$ is a continuous t-norm on $[0,1]$. This conjecture has been verified in [14]; we give a simpler proof of a stronger statement.

One of the relevant algebraic aspects of a continuous t-norm on $[0,1]$ is the fact that the associated monoid is residuated. Residuated (partially ordered) monoids have long been considered of interest by algebraists, starting from the classical example of the lattice-ordered monoid of the ideals of a ring with unit. In particular Bosbach [12] devoted several papers to the study of *left-complemented* monoids, i.e. the residuated monoids in which the underlying ordering is *natural*:

$$x \le y \qquad \text{if and only if} \qquad \exists\, z\, (x = zy).$$

Bosbach's work seems to have been the main source of inspiration for Büchi and Owens' research on commutative complemented monoids, which they called *hoops*. They prepared a manuscript entitled "Complemented Monoids and Hoops" in the middle seventies but, mainly because the manuscript was never published, their ideas caught on slowly. Blok and Pigozzi in [10] applied these ideas in the study of *hoops with dual normal operators*, which are a generalization of Boolean algebras with operators, but the first systematic study of the structural properties of hoops appeared in Ferreirim's thesis [19]. Some of the results obtained there can be found in two joint papers with Blok [5] and [6]; in particular the description of subdirectly irreducible hoops ([6, Theorem 2.9]) will play a crucial role in this paper.

Since the ordering induced by the residual of any continuous t-norm is the natural ordering on $[0, 1]$, any algebra of the form $\langle [0, 1], *, \rightarrow, 0, 1\rangle$ is a hoop that is also *bounded*, i.e., has a smallest element 0. Hence all the varieties we have considered so far are varieties of (bounded) hoops. This suggests the possibility that the structure theory of hoops can be used to achieve a better understanding of these varieties (and of the logics involved); conversely the class of varieties arising from these logics might shed more light on the behavior of other classes of hoops (and their implicative subreducts). The aim of this paper is to show that this enterprise can be successful. In Section 1 we investigate the variety of *basic hoops*, i.e., the variety of hoops naturally associated with basic logic; we clarify its relationship with the variety of basic BL-algebras and we characterize completely its finite subdirectly irreducible members in terms of ordinal sums of hoops. Then we do the same for its implicative subreducts. In Sections 2 and 3 we proceed to show that the variety of basic hoops is generated as a quasivariety by its finite algebras and the same holds for its implicative subreducts. As a by-product we are able to give a new proof and a slight improvement of the completeness result in [14]. In Section 4 we point out that the variety of G-hoops, naturally associated with Gödel logic, consists of well-known objects and we characterize its implicative subreducts. Sections 5 and 6 are devoted to *product hoops*, i.e., hoops coming from the product t-norm. In Section 5 we study the structure of product hoops and their implicative subreducts, while in Section 6 we describe completely the lattice of subvarieties of product hoops.

1. Basic hoops and BL-algebras

As mentioned in the introduction, a thorough algebraic study of the variety of hoops may be found in [6]. We start this section by recalling some known definitions and results. Then we study in detail the variety of basic hoops, which is generated by all totally ordered hoops.

A *hoop* is an algebra $\mathbf{A} = \langle A, \cdot, \to, 1\rangle$ such that $\langle A, \cdot, 1\rangle$ is a commutative monoid and for all $x, y, z \in A$

1. $x \to x = 1$
2. $x(x \to y) = y(y \to x)$
3. $x \to (y \to z) = xy \to z$.

If $\mathbf{A}$ is a hoop, define $a^0 = 1$, $a \overset{0}{\to} b = b$ and $a \overset{n+1}{\to} b = a \to (a \overset{n}{\to} b)$, for any natural number n. Then $a^n \to b = a \overset{n}{\to} b$ for all n.

If $\mathbf{A} = \langle A, \cdot, \to, 1\rangle$ is a hoop then the binary relation defined by $a \leq b$ if and only if $a \to b = 1$ is a partial order on A with respect to which $\langle A, \cdot, 1\rangle$ is a naturally ordered residuated commutative monoid, or naturally ordered *pocrim*. The residuation is given by $ab \leq c$ if and only if $a \leq b \to c$.

A *bounded* hoop is an algebra $\mathbf{A} = \langle A, \cdot, \to, 0, 1\rangle$ such that $\langle A, \cdot, \to, 1\rangle$ is a hoop and $0 \leq a$ for all $a \in A$.

A *Wajsberg hoop* is a hoop satisfying the identity

$$(x \to y) \to y \approx (y \to x) \to x. \qquad \text{(T)}$$

Bounded Wajsberg hoops are term-equivalent to *Wajsberg algebras* [22]—see [10]; they are also term-equivalent to Chang's *MV-algebras* [13] and Komori's *CN algebras* [33]. It follows from the fact that every Wajsberg hoop is a $\{\cdot, \to, 1\}$-subreduct of a Wajsberg algebra [6, Proposition 1.14] and the theory of algebraizable logics [9, Corollary 2.12] that the variety $\mathcal{WH}$ of Wajsberg hoops is the equivalent algebraic semantics of the positive fragment of Łukasiewicz's infinite-valued logic.

Finite Wajsberg hoops will play a crucial role in the sequel. For each natural number n, $\mathbf{C}_n$ denotes the finite totally ordered Wajsberg hoop whose universe is $C_n = \{1 = a^0, a, a^2, ..., a^n\}$ and $a^k a^m = a^{\min(k+m,n)}$, $a^k \to a^m = a^{\max(m-k,0)}$ for $0 \leq k, m \leq n$; similarly $\mathbf{Wa}_n$ denotes the finite totally ordered Wajsberg algebra $\langle C_n, \cdot, \to, a^n, 1\rangle$.

A *filter* of a hoop $\mathbf{A}$ is a subset F containing 1 and closed under *detachment*: if $a, a \to b \in F$, then $b \in F$. Given a subset X of a hoop $\mathbf{A}$, the filter generated by X in $\mathbf{A}$, denoted $\mathrm{Fg}_{\mathbf{A}}(X)$, is

$$\begin{aligned}\mathrm{Fg}_{\mathbf{A}}(X) &= \{b \in A \mid \exists\, a_1, \dots, a_n \in X \;\; a_1 \to (a_2 \to (\dots (a_n \to b) \dots)) = 1\} \\ &= \{b \in A \mid \exists\, a_1, \dots, a_n \in X \;\; a_1 a_2 \dots a_n \leq b\}.\end{aligned}$$

In particular, the principal filter generated by an element a is

$$\mathrm{Fg}_{\mathbf{A}}(a) = \{b \in A \mid \exists\, n \text{ with } a \overset{n}{\to} b = 1\} = \{b \in A \mid \exists\, n \text{ with } a^n \leq b\}.$$

If θ is a congruence on $\mathbf{A}$, then $1/\theta$ is a filter. Moreover, the map $\theta \longmapsto 1/\theta$ determines a lattice isomorphism between the congruence lattice and the filter lattice of a hoop, with inverse map $F \longmapsto \theta_F$, where $\theta_F = \{(a, b) : a \to b,\, b \to a \in F\}$ [12]. The variety of hoops is therefore *congruence regular at* 1, with witness term $(x \to y) \wedge (y \to x)$.

In the next proposition we collect two useful facts concerning the Wajsberg hoop $\langle [0,1], *_L, \to_L, 1 \rangle$ (resp. the Wajsberg algebra $\langle [0,1], *_L, \to_L, 0, 1 \rangle$).

PROPOSITION 1.1. *1. If $a < b \in \mathbb{R}$, then it is possible to define $\cdot$ and $\to$ on $[a, b]$ in such a way that $\langle [a, b], \cdot, \to, b \rangle$ is isomorphic to $\langle [0,1], *_L, \to_L, 1 \rangle$.*

2. Each $\mathbf{C}_n$ is embeddable in $\langle [0,1], *_L, \to_L, 1 \rangle$.

3. 1. and 2. hold for the Wajsberg algebras $\langle [0,1], *_L, \to_L, 0, 1 \rangle$ and $\mathbf{Wa}_n$.

PROOF. 1. Define for all $u, v \in [a, b]$

$$uv = \max(u + v - b, a) \qquad u \to v = \min(b - u + v, b).$$

Then the maps

$$f(x) = a + (b - a)x \quad \text{for } x \in [0,1] \qquad\qquad g(u) = \frac{u - a}{b - a} \quad \text{for } u \in [a, b]$$

are mutually inverse isomorphisms between $[0,1]$ and $[a, b]$.

2. The map defined by $a^k \longmapsto (n - k)/n$, for $k \leq n$, is an embedding of $\mathbf{C}_n$ into $\langle [0,1], *_L, \to_L, 1 \rangle$.

3. Clear. ■

We now turn our attention to implicative subreducts of hoops. It is well-known that an implicative subreduct of a hoop is always a BCK-algebra. BCK-algebras were introduced by Iséki [30] as algebraic models of C.A. Meredith's BCK-calculus and have been widely investigated since. They form a quasivariety $\mathcal{BCK}$ that is not a variety [42], [28]. However, if $\mathcal{V}$ is any variety of hoops then the class $\boldsymbol{S}^{\to}(\mathcal{V})$, consisting of all implicative subreducts of algebras in $\mathcal{V}$, is a *variety* of BCK-algebras. This fact was stated in [11, Section 4, Example III]; it also follows from [2, Theorem 2.14]. We give here a direct proof, based on the observation that, for every variety of hoops $\mathcal{V}$, the class $\boldsymbol{S}^{\rightarrow}(\mathcal{V})$ has the *filter extension property*, i.e., if $\mathbf{A}$, $\mathbf{B} \in \boldsymbol{S}^{\rightarrow}(\mathcal{V})$, $\mathbf{A} \leq \mathbf{B}$ and F is a filter of $\mathbf{A}$, then there exists a filter G of $\mathbf{B}$ such that $G \cap A = F$ (namely $G = \mathrm{Fg}_{\mathbf{B}}(F)$).

Proposition 1.2. *If $\mathcal{V}$ is any variety of hoops, then the class $\boldsymbol{S}^{\rightarrow}(\mathcal{V})$ is a variety of BCK-algebras.*

Proof. Recall that the variety of hoops satisfies the following identity, introduced by Cornish [16]:

$$(((x \to y) \to y) \to x) \to x \approx (((y \to x) \to x) \to y) \to y \tag{J}$$

[5, Corollary 4.8].

Let $\mathcal{V}$ be a variety of hoops; it is sufficient to show that $\boldsymbol{S}^{\rightarrow}(\mathcal{V})$ is closed under homomorphisms. Let $\mathbf{A}$ be a $\{\to, 1\}$-subreduct of $\mathbf{B} \in \mathcal{V}$; if $\mathbf{B}^{\rightarrow}$ denotes the $\{\to, 1\}$-reduct of $\mathbf{B}$, we may write $\mathbf{A} \leq \mathbf{B}^{\rightarrow}$. Let $\theta \in \mathrm{Con}(\mathbf{A})$ and let G be the filter of $\mathbf{B}$ generated by $1/\theta$; it is easy to see that $G = \mathrm{Fg}_{\mathbf{B}}(1/\theta) = \mathrm{Fg}_{\mathbf{B}^{\rightarrow}}(1/\theta)$. Clearly $\theta \subseteq \theta_G \cap A^2$. On the other hand, if $(a, b) \in \theta_G \cap A^2$, then $((a \to b) \to b) \to a \;\theta_G\; ((a \to a) \to a) \to a = 1$ and similarly $((b \to a) \to a) \to b \;\theta_G\; 1$. Thus both expressions belong to $1/\theta_G \cap A = G \cap A = 1/\theta$ (by the filter extension property) and hence, by (J),

$$\begin{aligned} a = 1 \to a \;\theta\; (((a \to b) \to b) \to a) \to a \\ = (((b \to a) \to a) \to b) \to b \;\theta\; 1 \to b = b. \end{aligned}$$

Thus $\theta = \theta_G \cap A^2$, and therefore $\mathbf{A}/\theta$ is a subreduct of $\mathbf{B}/\theta_G \in \mathcal{V}$. ■

Identity (J) determines a *variety* of BCK-algebras [16], which contains the variety $\mathcal{HBCK}$, consisting of all implicative subreducts of hoops. Indeed, $\mathcal{HBCK}$ is the variety of BCK-algebras satisfying identities (J) and

$$(x \to y) \to (x \to z) \approx (y \to x) \to (y \to z) \tag{H}$$

[5, 20]. A syntactic derivation of (J) from axioms for BCK-algebras together with (H) may be found in [34].

It is known that for BCK-algebras identity (T) implies (J). Therefore (T) defines a subvariety of $\mathcal{HBCK}$, the variety $\mathcal{LBCK}$ of Łukasiewicz BCK-algebras. Łukasiewicz BCK-algebras were introduced by Komori [32], under the name *C algebras*, to study the implicative fragment of Łukasiewicz many-valued logic. The variety $\mathcal{LBCK}$ coincides with the variety of implicative subreducts of Wajsberg hoops (algebras) [5] and it plays a relevant role in characterizing subdirectly irreducible members of $\mathcal{HBCK}$ [20, Theorem 4.9].

It is also known that if $\mathbf{A}$ is a BCK-algebra that belongs to a variety of BCK-algebras, then the filter lattice and the congruence lattice of $\mathbf{A}$ are isomorphic.

The underlying order of a hoop is always a $\wedge$-semilattice order (where $a \wedge b = a(a \to b)$), but not necessarily a lattice order.

When the join of two elements exists, it reflects in the lattice of filters as follows:

PROPOSITION 1.3. *Let $\mathbf{A}$ be a hoop and $a, b \in A$. If $a \vee b$ exists, then*

$$Fg_{\mathbf{A}}(a \vee b) = Fg_{\mathbf{A}}(a) \cap Fg_{\mathbf{A}}(b).$$

This property is shared by many ordered structures, for example Heyting algebras and Wajsberg algebras. In a more general context, it was first established for BCK-algebras whose underlying poset is a $\vee$-semilattice [39, Corollary 2], [41, Corollary 3]. However, its proof depends only on the existence of the required join; it appears in full generality in [40].

Next we investigate a class of hoops with a (term-definable) lattice order, closely related to the logical system introduced by Hájek in his monograph [24]. This propositional calculus, which Hájek called *basic (many-valued) logic*, is proposed as the "most general" many-valued logic with truth values in $[0, 1]$. The algebraic semantics for basic logic given in [24] consists of *BL-algebras.*

DEFINITION 1.4. A *BL-algebra* is an algebra $\langle A, \vee, \wedge, \cdot, \to, 0, 1\rangle$ such that

1. $\langle A, \vee, \wedge, \cdot, \to, 0, 1\rangle$ is a bounded residuated lattice;
2. $x \wedge y = x(x \to y)$;
3. $(x \to y) \vee (y \to x) = 1$.

The class of BL-algebras is denoted by $\mathcal{BL}$.

The binary connective $\vee$ defined in basic logic (BL) by

$$\phi \vee \psi := ((\phi \to \psi) \to \psi) \wedge ((\psi \to \phi) \to \phi)$$

corresponds, via the completeness theorem [24, Theorem 2.3.19], to the join operation on BL-algebras. Thus, join is term-definable using only meet and implication

$$x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x).$$

In [24] it was shown that BL-algebras satisfy all axioms of hoops. Thus, given a BL-algebra $\mathbf{A} = \langle A, \vee, \wedge, \cdot, \to, 0, 1\rangle$, it is appropriate to call $\langle A, \cdot, \to, 1\rangle$ the *hoop reduct* of $\mathbf{A}$.

Definition 1.5. A hoop is called a *basic hoop* if it is a hoop subreduct (i.e., a subhoop of the hoop reduct) of a BL-algebra. The class of basic hoops is denoted by $\mathcal{BH}$.

The algebraic translation of axiom (A6) in [24, Definition 2.2.4] is the identity

$$(x \to y) \to z \leq ((y \to x) \to z) \to z. \tag{B}$$

which holds in every BL-algebra. It follows that every basic hoop also satisfies (B). Next we show that identity (B) plays a crucial role in characterizing the class of basic hoops. Our main tools are the notion of ordinal sum, as well as the characterization of subdirectly irreducible hoops described in [6, Theorem 2.9(iii)]. Recall that, given two hoops $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A} \oplus \mathbf{B}$ denotes their ordinal sum. If $\mathbf{A}$ is a subdirectly irreducible hoop then it is of the form $\mathbf{F} \oplus \mathbf{S}$, where $\mathbf{F}$ and $\mathbf{S}$ are subhoops of $\mathbf{A}$, and $\mathbf{S}$ is a subdirectly irreducible Wajsberg hoop.

Theorem 1.6. *Let* $\mathbf{A}$ *be a hoop. The following are equivalent:*[1]

(i) $\mathbf{A}$ *is a basic hoop.*

(ii) $\mathbf{A}$ *satisfies the identity*

$$(x \to y) \to z \leq ((y \to x) \to z) \to z. \tag{B}$$

(iii) $\mathbf{A}$ *is isomorphic to a subdirect product of linearly ordered hoops.*

Proof. (i)⇒(ii). Clear from the remarks following the definition of basic hoop.

(ii)⇒ (iii). Without loss of generality, we may assume that $\mathbf{A}$ is subdirectly irreducible. Then, by [6, Theorem 2.9] $\mathbf{A}$ decomposes as $\mathbf{F} \oplus \mathbf{S}$, where $\mathbf{S}$ is a non-trivial subdirectly irreducible (hence linearly ordered) Wajsberg hoop. By way of contradiction, let $a, b \in A$ be such that $a \not\leq b$ and $b \not\leq a$. Then necessarily $a, b \in F$, since S is totally ordered and if $x \in S$ and $y \in F \setminus \{1\}$ then $y \leq x$. Let $c \in S \setminus \{1\}$ (such c exists since $|S| > 1$). Since $a \to b$ and $b \to a$ are in $F \setminus \{1\}$, one has $a \to b \leq c$ and $b \to a \leq c$, but $c \neq 1$, and (B) fails in $\mathbf{A}$, yielding a contradiction.

[1] The equivalence between (ii) and (iii) has been considered by Pałasinski in [38], with respect to BCK-algebras.

(iii)⇒(i). It is enough to prove that every linearly ordered hoop is basic. If $\mathbf{A}$ is a linearly ordered hoop, then $\mathbf{2} \oplus \mathbf{A}$ is a bounded linearly ordered hoop in which the lattice join is definable by $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)(= \max(x, y))$, and $(x \to y) \vee (y \to x) = 1$. Then, $\mathbf{2} \oplus \mathbf{A}$ is a BL-algebra, of which $\mathbf{A}$ is a hoop subreduct. ■

Note that if $\mathbf{A} = \langle A, \cdot, \to, 0, 1\rangle$ is a *bounded* hoop, its congruence lattice is the same as the congruence lattice of its hoop reduct $\langle A, \cdot, \to, 1\rangle$. Hence, a consequence of Theorem 1.6 is the following characterization of BL-algebras:

Theorem 1.7. *The variety $\mathcal{BL}$, of all BL-algebras, is term-equivalent to the variety of bounded hoops satisfying*

$$(x \to y) \to z \leq ((y \to x) \to z) \to z. \tag{B}$$

Proof. Given an arbitrary bounded hoop $\mathbf{A}$, satisfying (B), it is a subdirect product of subdirectly irreducible factors, each of which is linearly ordered, by Theorem 1.6. In $\mathbf{A}$, the operations $\wedge$ and $\vee$ are term definable by $x \wedge y = x(x \to y)$ and $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$ respectively. These are the lattice operations corresponding to the order on A, making $\langle A, \cdot, \to, \wedge, \vee, 0, 1\rangle$ a BL-algebra. ■

Recall that an algebra $\mathbf{A}$ is *finitely subdirectly irreducible* if any finite family of nontrivial congruences of $\mathbf{A}$ has a nontrivial intersection. Clearly, an algebra is finitely subdirectly irreducible if and only if any two nontrivial principal congruences have a nontrivial intersection.

In view of the lattice isomorphism between congruences and filters (under which principal congruences correspond to principal filters), one can say that a hoop $\mathbf{A}$ is finitely subdirectly irreducible if and only if any pair of nontrivial principal filters has a non-trivial intersection. In particular, if $\mathbf{A}$ is a totally ordered hoop, it is necessarily finitely subdirectly irreducible, since $a \leq b$ implies $Fg_{\mathbf{A}}(b) \subseteq Fg_{\mathbf{A}}(a)$. Moreover we have the following proposition.

Proposition 1.8. *A basic hoop is finitely subdirectly irreducible if and only if it is totally ordered.*

Proof. Let $\mathbf{A}$ be a basic hoop and $a, b \in A$. Then $\mathbf{A}$ is a hoop subreduct of a BL-algebra and therefore $(a \to b) \vee (b \to a) = 1$. It follows that $Fg_{\mathbf{A}}(a \to b) \cap Fg_{\mathbf{A}}(b \to a) = \{1\}$, by Proposition 1.3. If $\mathbf{A}$ is finitely subdirectly irreducible, this implies that either $Fg_{\mathbf{A}}(a \to b) = \{1\}$ or $Fg_{\mathbf{A}}(b \to a) = \{1\}$. Hence either $a \leq b$ or $b \leq a$ and $\mathbf{A}$ is totally ordered. ■

Ordinal sums can be defined for an arbitrary number of finitely many summands. Clearly, the class of totally ordered hoops is closed under ordinal sums.

Since, for finite algebras, being subdirectly irreducible is equivalent to being finitely subdirectly irreducible, and every totally ordered hoop is necessarily finitely subdirectly irreducible, then every finite totally ordered basic hoop is subdirectly irreducible. In particular, it follows from [32, Theorem 3.13] that each finite totally ordered Wajsberg hoop (algebra) is simple and isomorphic to some $\mathbf{C}_n$ ($\mathbf{Wa}_n$).

This yields an interesting classification of finite subdirectly irreducible basic hoops.

Corollary 1.9. $\mathbf{A}$ *is a finite subdirectly irreducible basic hoop if and only if there are* $k, n_1, \ldots, n_k \in \mathbb{N}$ *such that*

$$\mathbf{A} \cong \mathbf{C}_{n_1} \oplus \mathbf{C}_{n_2} \oplus \cdots \oplus \mathbf{C}_{n_k}.$$

Proof. First note that $\mathbf{C}_{n_1} \oplus \mathbf{C}_{n_2} \oplus \cdots \oplus \mathbf{C}_{n_k}$ is a (finite) totally ordered subdirectly irreducible basic hoop. Conversely, let $\mathbf{A}$ be a finite subdirectly irreducible basic hoop. We prove the claim by induction on the cardinality of A. If $|A| = 1$, then there is nothing to prove. Suppose that $|A| = r$; then by [6, Theorem 2.9 (iii)] there is a basic hoop $\mathbf{F}$ and a totally ordered Wajsberg hoop $\mathbf{S}$, with $\mathbf{A} \cong \mathbf{F} \oplus \mathbf{S}$ and $|S| > 1$. Since $\mathbf{A}$ is totally ordered, so is $\mathbf{F}$ and hence it is finitely subdirectly irreducible (by Proposition 1.8). Since $|F| < r$, by the induction hypothesis there exist $k-1, n_1, \ldots, n_{k-1}$ such that $\mathbf{F} \cong \mathbf{C}_{n_1} \oplus \cdots \oplus \mathbf{C}_{n_{k-1}}$. Moreover $\mathbf{S}$ is a finite totally ordered Wajsberg hoop, hence it is isomorphic to some $\mathbf{C}_{n_k}$. In conclusion we have

$$\mathbf{A} \cong \mathbf{C}_{n_1} \oplus \cdots \oplus \mathbf{C}_{n_k},$$

as claimed. ■

Since every BL-algebra is a bounded hoop satisfying (B), it is easy to see that the congruence lattice of a BL-algebra $\mathbf{A} = \langle A, \vee, \wedge, \cdot, \rightarrow, 0, 1\rangle$ is the same as the congruence lattice of its hoop reduct $\langle A, \cdot, \rightarrow, 1\rangle$ (and of its bounded hoop reduct $\langle A, \cdot, \rightarrow, 0, 1\rangle$). Therefore, congruences on BL-algebras correspond to filters and conversely. We can then extend the results above to BL-algebras.

It follows from Theorem 1.6 that the class $\mathcal{BH}$, of all basic hoops (i.e., hoop subreducts of BL-algebras), is a *variety* — the variety of hoops satisfying (B). This result may be generalized as follows. For any class $\mathcal{K}$ of BL-algebras, let $\boldsymbol{S}^h(\mathcal{K})$ denote the class of hoop subreducts of members of $\mathcal{K}$.

Proposition 1.10. *If $\mathcal{V}$ is a variety of BL-algebras, then the class $\boldsymbol{S}^h(\mathcal{V})$, of all hoop subreducts of members of $\mathcal{V}$, is a variety of basic hoops.*

Proof. If $\mathbf{A}$ is a basic hoop and $\mathbf{B}$ is a BL-algebra, let us write $\mathbf{A} \leq^h \mathbf{B}$ to mean that $\mathbf{A}$ is isomorphic to a hoop subreduct of $\mathbf{B}$. The class of hoop subreducts of $\mathcal{V}$ is clearly closed under subalgebras and direct products. To show that $\boldsymbol{S}^h(\mathcal{V})$ is closed under homomorphisms, let $\mathbf{A} \in \boldsymbol{S}^h(\mathcal{V})$ and let $\alpha \in \mathrm{Con}(\mathbf{A})$. Then $1/\alpha = F$ is a hoop filter and hence it has the detachment property (i.e., if $a,\ a \to b \in F$ then $b \in F$). Let $\mathbf{B}$ be a BL-algebra such that $\mathbf{A} \leq^h \mathbf{B}$, let G be the filter of $\mathbf{B}$ generated by F and let θ_G be the congruence associated to G in $\mathbf{B}$. Then $(u, v) \in \theta_G$ if and only if there are $a_1, \dots, a_n \in F$ such that

$$a_1 \to (a_2 \to \dots \to (a_n \to (u \to v)) \dots) = 1$$
$$a_1 \to (a_2 \to \dots \to (a_n \to (v \to u)) \dots) = 1$$

If $u, v \in A$, then $u \to v, v \to u \in F$ by the detachment property, hence $(u, v) \in \alpha$. Therefore $\alpha = \theta_G \cap A \times A$ and hence $\mathbf{A}/\alpha \leq^h \mathbf{B}/\theta_G$. ■

Corollary 1.11. *[24] The variety of BL-algebras consists entirely of subdirect products of totally ordered BL-algebras. Any subdirectly irreducible BL-algebra is totally ordered. A BL-algebra is finitely subdirectly irreducible if and only if it is totally ordered, hence any finite totally ordered BL-algebra is subdirectly irreducible.*

Corollary 1.12. *A finite BL-algebra $\mathbf{A}$ is subdirectly irreducible if and only if there are $k, n_1, \dots, n_k \in \mathbb{N}$ such that*

$$\mathbf{A} \cong \mathbf{Wa}_{n_1} \oplus \mathbf{Wa}_{n_2} \oplus \dots \oplus \mathbf{Wa}_{n_k}.$$

2. Generation by finite algebras

In this section we shall show that both $\mathcal{BH}$ and $\mathcal{BL}$ are generated as quasivarieties by their finite algebras. As a by-product we will obtain a new proof and a slight improvement of the completeness result [14] mentioned in the introduction. The results obtained in this section are a refinement for the varieties $\mathcal{BH}$ and $\mathcal{BL}$ of similar results obtained for a large class of varieties of hoops—see [6, section 3]. An algebra $\mathbf{A}$ in a class $\mathcal{K}$ has the *finite embeddability property* (FEP) with respect to $\mathcal{K}$ if for any finite partial subalgebra $\mathbf{A}'$ of $\mathbf{A}$ there exists a finite algebra $\mathbf{B} \in \mathcal{K}$ such that $\mathbf{A}'$ is embeddable in $\mathbf{B}$. A class $\mathcal{K}$ has the FEP if each of its members has the FEP with respect to $\mathcal{K}$. It is well known that if a variety $\mathcal{V}$ has the FEP, then $\mathcal{V} = \boldsymbol{HSP}(\mathcal{V}_{fin})$, where $\mathcal{V}_{fin}$ is the class of all finite members of $\mathcal{V}$ (see [18, Theorem 4]). This argument was extended to show that, if $\mathcal{V}$ has the FEP, then $\mathcal{V} = \boldsymbol{SPP}_u(\mathcal{V}_{fin})$, i.e., $\mathcal{V}$ is generated, as a quasivariety, by its finite members (see [6]).

In [6, Lemma 3.7] the authors proved that if the class of subdirectly irreducible members of $\mathcal{V}$ has the FEP, so does $\mathcal{V}$. Then they established that both the variety of hoops and the variety of Wajsberg hoops (hence Wajsberg algebras) have the FEP, hence $\mathcal{WH} = \boldsymbol{SPP}_u(\mathbf{C}_n : n \in \mathbb{N})$ and $\mathcal{WA} = \boldsymbol{SPP}_u(\mathbf{Wa}_n : n \in \mathbb{N})$. In particular they showed that the class of totally ordered Wajsberg hoops (algebras) has the FEP (see [6, proof of Theorem 3.9]). This fact is crucial to the following:

Theorem 2.1. *The class of totally ordered basic hoops has the FEP.*

Proof. Let $\mathbf{C}$ be a totally ordered basic hoop and let $\mathbf{C}'$ be a finite partial subhoop of $\mathbf{C}$. Let $C' = \{c_1, \ldots, c_n\}$, where $c_1 > c_2 > \cdots > c_n$. The proof is by induction on n. If $n = 1$, then trivially C' embeds into the trivial hoop. If $n > 1$, then the finite set $\{c_i \to c_j : 1 \leq i < j \leq n\}$ has a largest element, say $c_{i_0} \to c_{j_0}$. We claim that any maximal congruence φ separating c_{i_0} and c_{j_0} also separates any pair (c_i, c_j) with $i < j$. Suppose that $(c_i, c_j) \in \varphi$; then $(c_i \to c_j, 1) \in \varphi$ and, since $F = 1/\varphi$ is a filter and $c_i \to c_j \leq c_{i_0} \to c_{j_0}$ we get $(c_{i_0}, c_{j_0}) \in \varphi$, a contradiction. Therefore φ separates any pair (c_i, c_j) with $i < j$ and $\mathbf{C}'$ embeds into $\mathbf{C}/\varphi$, which is a subdirectly irreducible, totally ordered hoop. By [6, Theorem 2.9] $\mathbf{C}/\varphi = \mathbf{D} \oplus \mathbf{T}$, where $\mathbf{D}$ and $\mathbf{T}$ are totally ordered subhoops of $\mathbf{C}/\varphi$ and $\mathbf{T}$ is a Wajsberg hoop. Let $\overline{c}_i$ denote c_i/φ; by definition of φ, any nontrivial congruence of $\mathbf{C}/\varphi$ identifies $\overline{c}_{i_0}$ and $\overline{c}_{j_0}$, hence if $\overline{C}' = \{\overline{c}_1, \ldots, \overline{c}_n\}$, then $\overline{C}' \cap D$ has fewer elements than C'. Applying the induction hypothesis, there exists a finite totally ordered hoop $\mathbf{D}_1$ such that $\overline{\mathbf{C}}' \cap \mathbf{D}$ embeds into $\mathbf{D}_1$. On the other hand, $\overline{\mathbf{C}}' \cap \mathbf{T}$ is a totally ordered partial Wajsberg hoop, hence by the result in [6] quoted above, it can be embedded into a finite totally ordered Wajsberg hoop $\mathbf{T}_1$. Hence $\overline{\mathbf{C}}'$ embeds into $\mathbf{D}_1 \oplus \mathbf{T}_1$ and since it is isomorphic as a partial hoop to $\mathbf{C}'$, the result follows. ■

Since it is clear that the ordinal sum of a totally ordered BL-algebra and a totally ordered Wajsberg hoop is a totally ordered BL-algebra, the same proof goes through for totally ordered BL-algebras. Thus:

Corollary 2.2. *The class of totally ordered BL-algebras has the FEP.*

Combining the results above with the description of finite subdirectly irreducible basic hoops and BL-algebras in Corollary 1.9 and Corollary 1.12 we get:

Corollary 2.3. *The varieties* $\mathcal{BH}$ *and* $\mathcal{BL}$ *are generated as quasivarieties by their finite members. In fact*

$$\mathcal{BH} = \boldsymbol{SPP}_u(\mathbf{C}_{n_1} \oplus \cdots \oplus \mathbf{C}_{n_k} : k, n_1, \ldots, n_k \in \mathbb{N})$$
$$\mathcal{BL} = \boldsymbol{SPP}_u(\mathbf{Wa}_{n_1} \oplus \cdots \oplus \mathbf{Wa}_{n_k} : k, n_1, \ldots, n_k \in \mathbb{N})$$

A well-known result on logical systems due to Harrop [26], applied to quasivarieties [5, Lemma 3.13] and combined with FEP yields the following:

Corollary 2.4. *The quasi-equational theories of* $\mathcal{BH}$ *and* $\mathcal{BL}$ *are decidable.*

Since the variety of BL-algebras is the equivalent algebraic semantics of basic logic, the quasi-equational theory of $\mathcal{BL}$ is (equivalent to) propositional basic logic, which is therefore decidable. It follows that the consequence relation in basic logic is decidable. In [24] the author conjectured that basic logic is "the logic of continuous t-norms"; in algebraic terms this is equivalent to showing that $\mathcal{BL}$ is generated as a variety by all algebras of the form $\langle [0,1], *, \rightarrow, 1, 0\rangle$, where $*$ is a continuous t-norm on $[0,1]$ and $\rightarrow$ is the associated residual. This conjecture has been verified in [14]. Using a simple algebraic argument, we extend the result by showing that $\mathcal{BL}$ is generated as a *quasivariety* by all algebras of the form $\langle [0,1], *, \rightarrow, 1, 0\rangle$.

Theorem 2.5. *The variety of basic hoops is generated as a quasivariety by all algebras of the form* $\langle [0,1], *, \rightarrow, 1\rangle$*, where* $*$ *is a continuous t-norm on* $[0,1]$ *and* $\rightarrow$ *is its residual.*

Proof. By Corollary 2.3 it is sufficient to show that any finite subdirectly irreducible basic hoop $\mathbf{A}$ can be embedded in $\langle [0,1], *, \rightarrow, 1\rangle$ for some continuous t-norm $*$ on $[0,1]$. By Corollary 1.9 there are $k, n_1, \ldots, n_k \in \mathbb{N}$ with $\mathbf{A} \cong \mathbf{C}_{n_1} \oplus \cdots \oplus \mathbf{C}_{n_k}$. Moreover, by Proposition 1.1.2, each $\mathbf{C}_{n_k}$ is embeddable in $\langle [0,1], *_L, \rightarrow_L, 1\rangle$; let $\mathbf{B}_i$ be the copy of $\langle [0,1], *_L, \rightarrow_L, 1\rangle$ in which $\mathbf{C}_{n_i}$ is embedded. By Proposition 1.1.1, $\mathbf{B}_i$ is isomorphic to the algebra $\mathbf{D}_i = \langle [a_i, b_i], \cdot, \rightarrow, b_i\rangle$ where

$$a_i = \frac{i-1}{k} \qquad\qquad b_i = \frac{i}{k}$$

and the operations are defined accordingly. It follows that $\mathbf{C}_{n_1} \oplus \cdots \oplus \mathbf{C}_{n_k}$ is embeddable in $\mathbf{D} = \mathbf{D}_1 \oplus \cdots \oplus \mathbf{D}_k$. The universe of $\mathbf{D}$ is clearly $[0,1]$;

moreover, the binary operation $*$ defined on D, the ordinal sum of D_i, $i \leq k$, is clearly a t-norm. Continuity of $*$ is easy to verify: given $a, b \in [0,1]$, either $a, b \in D_i$ for some $i \neq k$; or $a \in D_i$, $b \in D_j$ for some i, j. In the first case, $a * b = a \cdot_i b$ in D_i and $\cdot_i$ is continuous. In the latter case, we may assume, with loss of generality, that $i > j$ and $b \neq 1$; hence $a * b = b$, which clearly preserves continuity. ■

Remark 2.6. By Theorem 2.5, the (finite) consequence relation defined by basic logic is complete with respect to continuous t-norms and their residuals, i.e. for every finite set Γ of basic logic formulas and for every basic logic formula ϕ, one has $\Gamma \vdash \phi$ if and only if for every BL-algebra $\mathbf{B} = \langle [0,1], *, \to, \vee, \wedge, 0, 1\rangle$, where $*$ is a continuous t-norm and $\to$ is its residual, and for every evaluation e in $\mathbf{B}$ such that $e(\psi) = 1$ for all $\psi \in \Gamma$, one has $e(\phi) = 1$. Hence, it follows from Corollary 2.4 that the consequence relation defined by basic logic is decidable.

3. The implicative reducts: basic BCK-algebras

In this section we shall characterize the class of implicative subreducts of basic hoops. By Proposition 1.2, the class of such subreducts is a variety of BCK-algebras, contained in $\mathcal{HBCK}$.

Definition 3.1. A BCK-algebra is called a *basic BCK-algebra* if it is a $\{\to, 1\}$-subreduct of a basic hoop. We denote the variety of basic BCK-algebras by $\mathcal{BBCK}$.

Note that every subdirectly irreducible basic BCK-algebra satisfies (B) and (H) and, hence, is a HBCK-algebra satisfying (B). By the proof of Theorem 1.6, (ii) $\Rightarrow$ (iii), every subdirectly irreducible HBCK-algebra satisfying (B) and therefore every subdirectly irreducible basic BCK-algebra, is totally ordered.

Remark 3.2. It is known that BCK-algebras that are subdirect products of totally ordered BCK-algebras coincide with BCK-algebras satisfying (B) [38, proof of Theorem 3]. However, the quasivariety of BCK-algebras satisfying (B) turns out to be strictly larger than $\mathcal{BBCK}$. Consider the algebra $\mathbf{A} = \langle [0,1], \to, 1\rangle$, where for $a, b \in A$

$$a \to b = \begin{cases} 1 & \text{if } a \leq b \\ \max\{1-a, b\} & \text{otherwise} \end{cases}.$$

Note that $\to$ is the residual of the *nilpotent minimum t-norm* [21], defined by

$$a \cdot b = \begin{cases} \min\{a, b\} & \text{if } a \geq 1 - b \\ 0 & \text{otherwise} \end{cases}.$$

Thus, $\mathbf{A}$ is the implicative reduct of a commutative integral residuated lattice, hence a BCK-algebra. However $\mathbf{A}$ does not satisfy (H). This can be seen by taking $x = 2/3$, $y = 1/3$ and $z = 0$. It follows that $\mathbf{A}$ is a (totally ordered) BCK-algebra that is not a subreduct of any hoop, hence does not belong to $\mathcal{BBCK}$.

Using [32, Theorem 3.13] and [5, Theorem 4.3] we can characterize the finite subdirectly irreducible HBCK-algebras satisfying (B). Here the building blocks are the implicative reducts of the hoops $\mathbf{C}_n$, which will be denoted henceforth by $\mathbf{L}_n$.

Proposition 3.3. *A finite HBCK-algebra* $\mathbf{A}$*, satisfying (B), is subdirectly irreducible if and only if there are* $k, n_1, \ldots, n_k \in \mathbb{N}$ *with*

$$\mathbf{A} \cong \mathbf{L}_{n_1} \oplus \cdots \oplus \mathbf{L}_{n_k}.$$

Our next goal is to show that the variety $\mathcal{BBCK}$ is generated by its finite algebras. Since every subdirectly irreducible basic BCK-algebra is a totally ordered HBCK-algebra, it suffices to show that the class of totally ordered HBCK-algebras has the FEP; using [5, Theorem 4.3], the proof of Theorem 2.1 will go through for totally ordered HBCK-algebras. It is only necessary to show that the class of totally ordered Łukasiewicz BCK-algebras has the FEP, so we proceed to prove it.

Lemma 3.4. *The class of totally ordered Łukasiewicz BCK-algebras has the FEP.*

Proof. Let $\mathbf{L}$ be a totally ordered Łukasiewicz BCK-algebra and let $\mathbf{P}$ be a partial subalgebra of $\mathbf{L}$. By [5, Lemma 4.1], it is known that $\mathbf{L}$ may be embedded in a totally ordered Wajsberg hoop $\mathbf{A}$; hence $\mathbf{P}$ may be regarded as a partial subalgebra of $\mathbf{A}$. It follows that there is a finite totally ordered Wajsberg algebra $\mathbf{A}'$ in which $\mathbf{P}$ is embeddable. But $\mathbf{P}$ is also embeddable in the $\{\to, 1\}$-reduct $\mathbf{L}'$ of $\mathbf{A}'$ and $\mathbf{L}'$ is a totally ordered Łukasiewicz BCK-algebra. ■

Proposition 3.5. *The class of totally ordered HBCK-algebras has the FEP.*

Theorem 3.6. *The variety* $\mathcal{BBCK}$ *is generated as a quasivariety by its finite members. In particular*

$$\mathcal{BBCK} = \boldsymbol{SPP}_u(\mathbf{L}_{n_1} \oplus \cdots \oplus \mathbf{L}_{n_k} : k, n_1, \ldots, n_k \in \mathbb{N})$$

Finally, we obtain an equational description of $\mathcal{BBCK}$.

Theorem 3.7. *The variety* $\mathcal{BBCK}$ *is the class of BCK-algebras which satisfy the identities (B) and (H).*

Proof. As observed before, every basic BCK-algebra satisfies (B) and (H).

Conversely, observe that the class of BCK-algebras which satisfy (B) and (H) is a variety, since it is the subvariety of HBCK-algebras defined by (B). Now, every algebra in a variety is a subalgebra of an ultraproduct of its finitely generated subalgebras, hence it suffices to show that every finitely generated BCK-algebra satisfying (B) and (H) is a $\{\to, 1\}$-subreduct of a basic hoop.

Using [5, Theorem 4.3] and the fact that subdirectly irreducible BCK-algebras satisfying (B) and (H) are totally ordered HBCK-algebras, argue by induction on the number of generators, as in [5, Theorem 4.4] (or [20, Theorem 5.1]) to prove that every totally ordered finitely generated HBCK-algebra is a $\{\to, 1\}$-subreduct of a totally ordered (hence basic) hoop. ■

4. G-algebras and their reducts

In [24] a *G-algebra* is defined to be a BL-algebra satisfying the equation $x^2 = x$; the variety $\mathcal{GA}$ of G-algebras is proposed as an equivalent algebraic semantics for *Gödel logic*, i.e. the propositional calculus naturally associated with the Gödel t-norm. It is clear that the variety of hoop-subreducts of G-algebras is the variety $\mathcal{GH}$ of basic hoops that satisfy $x^2 = x$, i.e. idempotent basic hoops. It is almost immediate to see that in an idempotent hoop the meet and the product coincide: an idempotent hoop is always a subreduct of a Heyting algebra [29] and it is better known as a *Brouwerian semilattice* [31], [8] and [7]. It follows that the variety $\mathcal{GH}$ coincides with the variety of Brouwerian semilattices that are subdirect products of chains.

The algebras in $\mathcal{GH}$ are term-equivalent to *relative Stone algebras* [27] or *semi-Boolean lattices* [37]; their properties have been thoroughly investigated. In particular, it is well-known that this variety and the variety of Heyting algebras generated by chains are locally finite and so the quasi-equational theories of $\mathcal{GH}$ and $\mathcal{GA}$ are decidable. Thus, Gödel logic is decidable, a fact already known, since Gödel logic is coNP-complete [24].

We consider now the variety $\mathcal{GBCK}$ consisting of implicative subreducts of algebras in $\mathcal{GH}$. Since algebras in $\mathcal{GH}$ are Brouwerian semilattices, the variety $\mathcal{GBCK}$ consists entirely of *Hilbert algebras*. It is well-known that the variety of Hilbert algebras is the variety of BCK-algebras satisfying

$$x \to (x \to y) = x \to y \tag{G}$$

hence the algebras in $\mathcal{GBCK}$ are basic BCK-algebras satisfying (G). We shall see that the converse also holds.

THEOREM 4.1. *$\mathcal{GBCK}$ is the variety of basic BCK-algebras satisfying (G).*

PROOF. It is enough to show that every subdirectly irreducible basic BCK-algebra satisfying (G) is a reduct of basic hoop. If **A** is a subdirectly irreducible basic BCK-algebra satisfying (G) then it is totally ordered and, by (G), a Hilbert algebra. But a linearly ordered Hilbert algebra is the reduct of a (linearly ordered) Brouwerian semilattice, hence of a basic hoop. ■

5. Product algebras and their reducts

Product algebras were introduced in [24] as an algebraic semantics of the logical system naturally associated with the product t-norm. In any BL-algebra it is possible to define a unary operation of negation ($\neg$) by $\neg x = x \to 0$; a *product algebra* is a BL-algebra satisfying the equations

$$x \wedge \neg x = 0 \tag{PA1}$$

$$\neg\neg z \to ((xz \to yz) \to (x \to y)) = 1. \tag{PA2}$$

The variety $\mathcal{PA}$ of product algebras seems to be the most interesting object associated with t-norms and has attracted a lot of attention recently. In particular, in [1] it is shown that product algebras are term equivalent to a class of bounded hoops and that $\mathcal{PA}$ is the equivalent algebraic semantics of product logic. In [15] it is shown that $\mathcal{PA}$ is generated by any infinite totally ordered product algebra and the finitely generated free product algebras are characterized. Moreover it is observed that for any product algebra **A** the set $A \setminus \{0\}$ is a filter, hence the only simple product algebra is the two element bounded hoop that is term-equivalent to the two element Boolean algebra. Since any product algebra has the two element Boolean algebra as a subalgebra, there are exactly three varieties of product algebras: the trivial variety, the variety of Boolean algebras and the entire variety. Recall the following lemma, necessary in the sequel.

Lemma 5.1. *[24] Let* $\mathbf{A}$ *be a totally ordered product algebra and let* $a, b, c \in A$. *Then:*

1. *if* $a \neq 0$, *then* $\neg a = 0.$*;*
2. *if* $a \neq 0$, *then* $ba = ca$ *implies* $b = c$*;*
3. *if* $a \neq 0$, *then* $ba < ca$ *implies* $b < c$.

A hoop $\mathbf{A}$ is *cancellative* if its underlying monoid is cancellative. This is equivalent to saying that for all $a, b \in A$ $a \to ba = b$. Cancellative hoops form a variety $\mathcal{C}$ of Wajsberg hoops [6]; hence they satisfy (T).

Proposition 5.2. $\mathbf{A}$ *is a subdirectly irreducible product algebra if and only if either* $\mathbf{A} = \mathbf{2}$ *or* $\mathbf{A} = \mathbf{2} \oplus \mathbf{C}$, *where* $\mathbf{2}$ *is the two element Boolean algebra and* $\mathbf{C}$ *is a subdirectly irreducible cancellative hoop.*

Proof. If $\mathbf{A}$ is subdirectly irreducible and different from $\mathbf{2}$, then it follows from Lemma 5.1.2 and [6, Theorem 2.9] that $\mathbf{A} = \mathbf{2} \oplus \mathbf{C}$ for some subdirectly irreducible hoop $\mathbf{C}$. Hence, by Lemma 5.1.2, $\mathbf{C}$ is a totally ordered hoop and its underlying monoid is cancellative. Conversely, let $\mathbf{C}$ be any cancellative subdirectly irreducible hoop. It is easy to check that $\mathbf{2} \oplus \mathbf{C}$ is a subdirectly irreducible product algebra. ■

Now, we want to describe the variety $\mathcal{PH}$ of *product hoops*, i.e. the variety of hoop subreducts of product algebras. This is not nearly as straightforward as the previous cases, since the constant 0, disguised as negation, appears in the defining axioms. Hence we adopt indirect reasoning.

It is well-known [4] that the lattice of subvarieties of hoops has exactly two atoms: the variety $\mathcal{C}$ of cancellative hoops and the variety $\mathcal{G}$ of *generalized Boolean algebras*, i.e. the variety generated by the two element hoop $\mathbf{2}$. Both varieties consist of basic hoops, so they are the only atoms in the lattice of subvarieties of $\mathcal{BH}$. The interesting fact is that the hoops belonging to these varieties are both Wajsberg hoops and product hoops; this follows from well-known facts about Wajsberg hoops and Proposition 5.2. Hence the varieties $\mathcal{WH}$ and $\mathcal{PH}$ both lie above the join of $\mathcal{G}$ and $\mathcal{C}$. However it is clear that no finite Wajsberg chain other than $\mathbf{2}$ is a product hoop.

A hoop is *semi-cancellative* if it satisfies the following first order formula

$$\forall xy (\exists z\ z < xy) \implies x = y \to xy;$$

a hoop is *quasi-cancellative* if it satisfies

$$\forall xyz\, (\exists w < z) \implies (xz \to yz) \to (x \to y) = 1.$$

Any cancellative hoop is quasi-cancellative and any quasi-cancellative hoop is semi-cancellative; moreover an unbounded quasi-cancellative hoop is cancellative and if $\mathbf{A}$ is a bounded (by 0) quasi-cancellative hoop, then $\mathbf{A} \setminus \{0\}$ is cancellative.

Proposition 5.3. *Any totally ordered Wajsberg hoop is semi-cancellative. No finite Wajsberg chain with more than two elements is quasi-cancellative.*

Proof. The second claim is easily proved by inspection, using the above description of finite Wajsberg chains. Let $\mathbf{A}$ be a totally ordered Wajsberg hoop and $a, b, c \in A$ with $c < ab$. Then $ab \not\leq c$ and by residuation $a \not\leq b \to c$. Since $\mathbf{A}$ is totally ordered, $b \to c \leq a$. By a repeated use of equation (T) we get

$$\begin{aligned} a &= a \vee (b \to c) \\ &= (a \to (b \to c)) \to (b \to c) \\ &= (ab \to c) \to (b \to c) \\ &= b \to ((ab \to c) \to c) \\ &= b \to (ab \vee c) \\ &= b \to ab. \end{aligned} \qquad \blacksquare$$

Corollary 5.4. *Let* $\mathbf{A}$ *be a nontrivial totally ordered hoop. Then* $\mathbf{A}$ *is cancellative if and only if it is an unbounded Wajsberg hoop.*

Proof. If $\mathbf{A}$ is cancellative, then no element different from 1 can be idempotent. However in a bounded hoop the lowermost element is always idempotent. It follows that a cancellative hoop must be unbounded.

Conversely, let $\mathbf{A}$ be an unbounded totally ordered Wajsberg hoop. Then for all $a, b \in A$ there is a $c \in A$ with $c < ab$. Thus, by Proposition 5.3, $a = b \to ab$ and $\mathbf{A}$ is cancellative. $\blacksquare$

We are now ready to characterize the variety of product hoops. We start with a lemma.

Lemma 5.5. *Let* $\mathbf{A}$ *be a subdirectly irreducible basic hoop satisfying the equation*

$$(x \to y) \to y \leq ((y \to z) \to ((y \to x) \to x)) \to ((y \to x) \to x). \qquad \text{(PB)}$$

Then either $\mathbf{A}$ *is a Wajsberg hoop or* $\mathbf{A} = \mathbf{2} \oplus \mathbf{B}$ *for some Wajsberg hoop* $\mathbf{B}$*.*

Proof. Let $\mathbf{A}$ be a subdirectly irreducible basic hoop satisfying (PB). Then $\mathbf{A} = \mathbf{F} \oplus \mathbf{B}$, where $\mathbf{B}$ is a subdirectly irreducible Wajsberg hoop. The thesis will be proved if we show that either **F** is trivial or it is equal to **2**. Suppose that $b \in F$, $b < 1$ and b is not the minimum element of **A**. Then there are $a, c \in A$ with $c < b < a < 1$ and $a \in B$. However, **A** is a basic hoop, in which $b = a \to b = a(a \to b) = ab$. Since $b \to c \in F \setminus \{1\}$ and $a \in B$ we have $b \to c < a$.

Hence

$$\begin{aligned} a &= 1 \to a = ((b \to c) \to a) \to a \qquad \text{since } b \to c \leq a \\ &= ((b \to c) \to ((b \to a) \to a)) \to ((b \to a) \to a) \qquad \text{since } b < a \\ &\geq (a \to b) \to b = b \to b = 1 \qquad \text{by (PB)} \end{aligned}$$

and, since $a \to b = b$, that is a contradiction. This concludes the proof. ■

THEOREM 5.6.[2] *The variety $\mathcal{PH}$ consists exactly of basic hoops satisfying*

$$(y \to z) \vee ((y \to xy) \to x) = 1. \tag{PH}$$

PROOF. We prove first that (PH) holds in all product hoops. It suffices to verify it in any totally ordered product hoop **H**. Now, if y is the minimum of **H** then $y \to z = 1$; otherwise $(y \to xy) \to x = 1$. Thus (PH) holds.

Conversely, we show that if **H** is a subdirectly irreducible (hence totally ordered) basic hoop satisfying (PH), then **H** is a product hoop. Assume that y is not the minimum of **H**. Take $z < y$. Then $y \to z < 1$, therefore $(y \to xy) \to x = 1$ for every x. Thus if **H** has no minimum, then it is a cancellative hoop, hence a product hoop. If **H** has a minimum 0, then by the above argument, $\mathbf{H} \setminus \{0\}$ is the domain of a cancellative hoop **C**, and $\mathbf{H} = \mathbf{2} \oplus \mathbf{C}$ is (the hoop reduct of) a product algebra, hence a product hoop. ■

The description of finite subdirectly irreducible basic hoops in Theorem 1.9 is hardly useful in case of product hoops, since the only finite subdirectly irreducible product hoop is **2**. However subdirectly irreducible product hoops are easily described.

COROLLARY 5.6. *An algebra in $\mathcal{PH}$ is subdirectly irreducible if and only if it is the two element generalized Boolean algebra* **2** *or is a subdirectly irreducible cancellative hoop or else is of the form* $\mathbf{2} \oplus \mathbf{C}$*, where* **C** *is a subdirectly irreducible cancellative hoop.*

Moreover subdirectly irreducible product hoops turn out to coincide with subdirectly irreducible quasi-cancellative hoops.

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[2]This axiom basis for $\mathcal{PH}$ was suggested by P. Jipsen.

PROPOSITION 5.7. *A subdirectly irreducible hoop is quasi-cancellative if and only if it belongs to $\mathcal{PH}$.*

PROOF. It is easily seen that any subdirectly irreducible hoop in $\mathcal{PH}$ is quasi-cancellative. Let now $\mathbf{A}$ be a quasi-cancellative subdirectly irreducible hoop. Then $\mathbf{A} = \mathbf{B} \oplus \mathbf{C}$, where $\mathbf{C}$ is a subdirectly irreducible Wajsberg hoop. If $\mathbf{A}$ is unbounded, then it is a cancellative hoop, therefore is in $\mathcal{PH}$, being a hoop subreduct of the product algebra $\mathbf{2} \oplus \mathbf{A}$. If $\mathbf{A}$ is bounded by 0, then $0 \in B$. If there is a $b \in B$ with $0 < b < 1$, then for $c \in C \setminus \{1\}$ we have $cb = b = 1b$. Since $b > 0$ and $\mathbf{A}$ is quasi-cancellative we must have $1 \to c = 1$, i.e. $c = 1$, a contradiction. Hence $\mathbf{B} \cong \mathbf{2}$ and $\mathbf{A} \cong \mathbf{2} \oplus \mathbf{C}$. Moreover, since $\mathbf{A}$ is quasi-cancellative and $c > 0$ for all $c \in C$, then $\mathbf{C}$ must be cancellative. ■

We now turn to implicative subreducts of product hoops (algebras). Recall that the variety $\mathcal{LBCK}$ of Łukasiewicz BCK-algebras coincides with implicative subreducts of Wajsberg hoops. Since the variety $\mathcal{C}$ of cancellative hoops is contained in the variety $\mathcal{WH}$ of Wajsberg hoops it is clear that $\boldsymbol{S}^{\to}(\mathcal{C}) \subseteq \mathcal{LBCK}$. The fact that the converse also holds will be crucial in the sequel.

Let $\mathbf{G}$ be an abelian ℓ-group; the *positive cone* of $\mathbf{G}$ is the hoop $\mathbb{P}(\mathbf{G}) = \langle \{x \in G : x \geq 0\}, \cdot, \to, 1_{\mathbb{P}(\mathbf{G})} \rangle$ where

$$
\begin{aligned}
xy &= x + y \\
x \to y &= 0 \vee (y - x) \\
1_{\mathbb{P}(\mathbf{G})} &= 0_{\mathbf{G}}.
\end{aligned}
$$

It is easily seen that $\mathbb{P}(\mathbf{G})$ is cancellative. The converse is true as well [6, Theorem 1.17]: if $\mathbf{A}$ is a cancellative hoop then there is an abelian ℓ-group $\mathbf{G}$ such that $\mathbf{A} \cong \mathbb{P}(\mathbf{G})$.

LEMMA 5.8. *Any Łukasiewicz BCK-algebra is isomorphic to a subreduct of a cancellative hoop; hence $\boldsymbol{S}^{\to}(\mathcal{C}) = \mathcal{LBCK}$.*

PROOF. Let $\mathbf{A} \in \mathcal{LBCK}$. We may assume that $\mathbf{A}$ is subdirectly irreducible, hence totally ordered. Therefore there exists a bounded Wajsberg hoop $\mathbf{B}$ of which $\mathbf{A}$ is a subreduct [20, Theorem 3.3]. Being bounded, $\mathbf{B}$ is (equivalent to) a Wajsberg algebra. By [36] there is an abelian ℓ-group $\mathbf{G} = \langle G, +, -, \vee, \wedge, 0_{\mathbf{G}} \rangle$ and $u \in G$ such that $\mathbf{B}$ is isomorphic to the Wajsberg algebra

$$\langle \{x \in G : 0 \leq x \leq u\}, \to, \cdot, 0_{\mathbf{B}}, 1_{\mathbf{B}} \rangle$$

where:

$$\begin{aligned} xy &= u \wedge (x+y) \\ x \to y &= 0 \vee (y-x) \\ 1_{\mathbf{A}} &= 0_{\mathbf{G}} \\ 0_{\mathbf{A}} &= u. \end{aligned}$$

Now it is easily checked that **A** is an implicative subreduct of the cancellative hoop $\mathbb{P}(\mathbf{G})$. ■

An implicative subreduct of a product hoop will be called a *product BCK-algebra*; we denote by $\mathcal{PBCK}$ the variety of product BCK-algebras.

THEOREM 5.9. *The variety $\mathcal{PBCK}$ consists exactly of basic BCK-algebras satisfying the equation (PB).*

PROOF. Since (PB) holds in any subdirectly irreducible product algebra and involves only implication, *a fortiori* it holds in any product BCK-algebra.

Conversely suppose that **A** is subdirectly irreducible basic BCK-algebra satisfying (PB). As in Lemma 5.5 we can prove that either **A** is a Łukasiewicz BCK-algebra or else $\mathbf{A} = \mathbf{2} \oplus \mathbf{B}$, where **B** is a Łukasiewicz BCK-algebra. In the first case, by Lemma 5.8, **A** is isomorphic to a subreduct of a cancellative hoop **C**. Then $\mathbf{2}\oplus\mathbf{C}$ is a (subdirectly irreducible) product hoop (by Corollary 5.6) of which **A** is a subreduct. In the other case let **C** be the cancellative hoop of which **B** is a subreduct. Then $\mathbf{A} = \mathbf{2} \oplus \mathbf{B}$ is a subreduct of $\mathbf{2} \oplus \mathbf{C}$ that is again a product hoop. ■

COROLLARY 5.10. *An algebra in $\mathcal{PBCK}$ is subdirectly irreducible if and only if it is **2**, or it is a subdirectly irreducible Łukasiewicz BCK-algebra or it is equal to $\mathbf{2} \oplus \mathbf{C}$ for some subdirectly irreducible Łukasiewicz BCK-algebra **C**.*

6. The lattice of subvarieties of product hoops

The link between abelian ℓ-groups, cancellative hoops and product algebras has been explored in [6] and [15]. A similar result holds for product hoops. It is well-known [4] that the variety $\mathcal{C}$ of cancellative hoops is generated as a quasivariety by the cancellative hoop $\mathbf{C}_\omega = \langle C_\omega, \to, \cdot, 1\rangle$ (i.e. $\mathcal{C} = \boldsymbol{SPP}_u(\mathbf{C}_\omega)$), where $C_\omega = \{a^n : n \in \omega\}$ is the free monogenerated monoid and

$$a^n a^m = a^{n+m} \qquad\qquad a^n \to a^m = a^{\max(m-n,0)}.$$

It is also clear that $\mathbf{C}_\omega \cong \mathbb{P}(\mathbb{Z})$. Recall that if **A** is a cancellative hoop, then $\mathbf{2} \oplus \mathbf{A}$ is a product algebra. So there is an abelian ℓ-group **G** such that

$\mathbf{A} \cong \mathbb{P}(\mathbf{G})$. Since ℓ-groups are torsion-free, then $\mathbb{Z}$ (regarded as a ℓ-group) is embeddable in $\mathbf{G}$ and since $\mathbb{P}(\mathbb{Z}) \cong \mathbf{C}_\omega$ we have:

LEMMA 6.1. *The product algebra $\mathbf{2} \oplus \mathbf{C}_\omega$ is embeddable in any product algebra of the form $\mathbf{2} \oplus \mathbf{A}$, where $\mathbf{A}$ is a cancellative hoop.*

THEOREM 6.2. *The variety $\mathcal{PH}$ is generated as a quasivariety by $\mathbf{2} \oplus \mathbf{C}_\omega$, i.e. $\mathcal{PH} = \boldsymbol{SPP}_u(\mathbf{2} \oplus \mathbf{C}_\omega)$.*

PROOF. First we show that any subdirectly irreducible algebra $\mathbf{A}$ in $\mathcal{PH}$ belongs to $\boldsymbol{SPP}_u(\mathbf{2} \oplus \mathbf{C}_\omega)$. This is obvious if $\mathbf{A} = \mathbf{2}$; if $\mathbf{A}$ is a cancellative hoop, then $\mathbf{A} \in \boldsymbol{SPP}_u(\mathbf{C}_\omega) \subseteq \boldsymbol{SPP}_u(\mathbf{2} \oplus \mathbf{C}_\omega)$. Let now $\mathbf{A} = \mathbf{2} \oplus \mathbf{C}$, where $\mathbf{C}$ is cancellative, and let $\mathbf{C}_\omega^I/U$ the ultrapower in which $\mathbf{C}$ embeds. It is easily seen that $\mathbf{A}$ embeds into $(\mathbf{2} \oplus \mathbf{C}_\omega)^I/U$. By Corollary 5.6 there is nothing else to check.

Since any algebra in $\mathcal{PH}$ is a subdirect product of subdirectly irreducible product hoops, from the above we get that $\mathcal{PH} \subseteq \boldsymbol{SPSPP}_u(\mathbf{2} \oplus \mathbf{C}_\omega) \subseteq \boldsymbol{SPP}_u(\mathbf{2} \oplus \mathbf{C}_\omega)$. This is enough to prove the result. ■

We can now describe completely the lattice of subvarieties of product hoops.

THEOREM 6.3. *The lattice of subvarieties of product hoops (depicted in Figure 1) consists exactly of five subvarieties: the trivial variety $\mathcal{T}$, the variety $\mathcal{PH}$, the variety $\mathcal{C}$ of cancellative hoops, the variety $\mathcal{G}$ of generalized Boolean algebras and the variety $\mathcal{G} \vee \mathcal{C}$.*

PROOF. We know that $\mathcal{G} = \boldsymbol{V}(\mathbf{2})$, $\mathcal{C} = \boldsymbol{V}(\mathbf{C}_\omega)$ and $\mathcal{PH} = \boldsymbol{V}(\mathbf{2} \oplus \mathbf{C}_\omega)$. Moreover $\mathcal{G} \vee \mathcal{C} < \mathcal{PH}$, since it consists entirely of Wajsberg hoops. Hence, by Corollary 5.6, it is enough to show that $\mathcal{PH}$ is generated by any subdirectly irreducible product algebra $\mathbf{2} \oplus \mathbf{A}$, where $\mathbf{A}$ is a subdirectly irreducible cancellative hoop. But by Lemma 6.1 and Theorem 6.2

$$\mathcal{PH} = \boldsymbol{V}(\mathbf{2} \oplus \mathbf{C}_\omega) \subseteq \boldsymbol{V}(\mathbf{2} \oplus \mathbf{A}) \subseteq \mathcal{PH},$$

hence equality holds. ■

It is easy to check that the variety $\mathcal{G}$ and $\mathcal{C}$ are axiomatized (relative to $\mathcal{PH}$) by $x \to x^2 \approx 1$ and $x \to x^2 \approx x$ respectively. Moreover:

PROPOSITION 6.4. *The variety $\mathcal{G} \vee \mathcal{C}$ is axiomatized, relative to $\mathcal{PH}$ by the equation (T).*

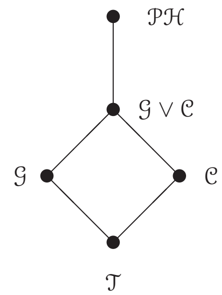

Figure 1. The lattice of subvarieties of $\mathcal{PH}$

Proof. It is clear that the only subdirectly irreducible product hoops in $\mathcal{G} \vee \mathcal{C}$ are **2** and the subdirectly irreducible cancellative hoops. Since they are all Wajsberg hoops, equation (T) holds in them.

Conversely suppose that $\mathcal{V}$ is a variety of product hoops satisfying (T) and suppose, by way of contradiction, that $\mathbf{2} \oplus \mathbf{A} \in \mathcal{V}$ for some subdirectly irreducible cancellative hoop **A**. If $a \in A$ with $a < 1$ from (T) we obtain

$$1 = (a \to 0) \to 0 = (0 \to a) \to a = 1 \to a = a$$

a contradiction. It follows that no subdirectly irreducible product hoop of the form $\mathbf{2} \oplus \mathbf{A}$ belongs to $\mathcal{V}$ and hence $\mathcal{V} = \mathcal{G} \vee \mathcal{C}$. ■

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