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Generalized Kripke Frames

Abstract. Algebraic work [9] shows that the deep theory of possible world semantics is available in the more general setting of substructural logics, at least in an algebraic guise. The question is whether it is also available in a relational form. This article seeks to set the stage for answering this question. Guided by the algebraic theory, but purely relationally we introduce a new type of frames. These structures generalize Kripke structures but are two-sorted, containing both worlds and co-worlds. These latter points may be viewed as modelling irreducible increases in information where worlds model irreducible decreases in information. Based on these structures, a purely model theoretic and uniform account of completeness for the implication-fusion fragment of various substructural logics is given. Completeness is obtained via a generalization of the standard canonical model construction in combination with correspondence results.

1. Introduction

A plethora of models of computation and information are based on various types of non-classical propositional logics including modal logics and substructural logics. For modal logics, possible world semantics, or relational semantics, play an absolutely fundamental role in making these logics useful by providing a means of obtaining theoretical results as well as by providing computational viability and intuitive links to transition systems. Consequently, relational semantics for modal logics by now have an extensive, deep, and powerful theory. The immense success of relational semantics in the setting of modal logics has prompted a sprawling literature that attempts to apply the methods in the setting of substructural logics as well, but the level of unity and power of these methods in the setting of substructural logic is not the same.

Recent developments in the algebraic theory of canonical extensions have lead to results extending many aspects of the deep theory of Kripke semantics for modal logics, formulated algebraically, to a much broader setting including that of most substructural logics, e.g., see [11, 12, 9]. What is more is that the methods needed in this vastly generalized setting are easily

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seen to parallel the ones needed in the classical setting, and they are not more complicated. A case in point is the work in [9] which provides relational semantics for a hierarchy of substructural logics in a uniform and modular way. The methodology is completely parallel to the algebraic route to similar results in the classical modal setting.

The purpose of the current paper is to try to understand the semantics developed in [9] and the proof methods used there from a purely model theoretic perspective. Thus we treat the same semantics and the same results as in [9], but our entire approach is model theoretic. The result is relational semantics for a hierarchy of substructural logics obtained on the basis of a canonical model construction for the basic logic (i.e., non-associative Lambek calculus), augmented by a modular addition of first-order properties corresponding to various additional axioms via a Sahlqvist-like calculus. The modularity and uniformity of this approach reach beyond what has been achieved by other completeness proofs, see, e.g., [8].

One main feature of the models obtained is that they are two-sorted. Algebraically, this stems from the fact that the join-irredicibles (i.e., atoms in the Boolean case) are not enough to reconstruct the lattice when it is not distributive. The join-irreducibles, in terms of logic, model, in the Boolean case, maximal consistent theories, that is, maximal consistent pieces of information. The order-dual points that are part of these structures may be viewed as modelling irreducible jumps in information. The two-sorted approach is not new and has already been explored in the setting of relational semantics in [4]. There as well as here the structures used are essentially polarities in the sense of Birkhoff and the potential interpretants for logical formulas over these structures are Galois-closed subsets of these polarities. Not surprisingly, the strength of this approach over a one-sorted approach is that the toggle between the two sorts allows us to get around problems created by the lack of distributivity of the lattice operations. The two-sided approach to relational semantics is not restricted to the models themselves but is also carried over to the notion of interpretations. Thus an interpretation is encoded by two relations: a satisfaction relation, $\Vdash$, which specifies which formulas hold at which worlds, and a 'part of' relation, $\succ$, which specifies which co-worlds, or irreducible information jumps, are part of which formulas. The toggle between these two relations allows us to reproduce standard Sahlqvist-type correspondence arguments in a uniform way.

One problem with general polarities is that the lattice of its Galois-closed sets need not be discrete in any sense, e.g., in the Boolean setting, all complete Boolean algebras occur as the Galois-closed subsets of some polarity – not just the atomic ones. However, this problem is circumvented in our approach. The point is that, as dictated by the algebraic theory, we restrict our attention to polarities in which the elements of both sorts satisfy an irreducibility condition. This restriction is a very central and important feature of our approach and is what allows us to reproduce a kind of Sahlqvist correspondence without much complication. This feature separates our semantics from the ones in [4].

The paper is organized as follows: In Section 2 we introduce our two-sorted generalization of sets of worlds. We show how they generalize the sets of worlds used in the Boolean and general distributive settings and we show how they fit in with the special posets used in [9]. The fact that our two-sorted structures may actually be thought of as single posets with special properties is a great help for the intuition when working with these structures. In Section 3 we discuss morphisms of these structures and identify subobjects. We show that the morphisms as defined correspond exactly to the complete homomorphisms of the corresponding algebras and that they restrict correctly in the Boolean and distributive settings. In Section 4 we describe the frames needed for modelling the basic connectives of substructural logics. We then give the canonical model construction and give a detailed proof of the completeness of the basic logic.

Much prior work contains parts of the material presented here. The properties and results in Section 2 are essentially all part of the theory of polarities. A general reference where much of this work can be found is [10]. However, the algebraic perspective offered by [9] and from whence we arrived at these results has lead us to stress the order theoretic nature of the polarities, and we do believe that this makes arguments more transparent and intuitive. Some very interesting earlier work which seems to align very closely with our point of view is that of Crapo, see [7]. It treats only the finite case but has identified all the essential features we work with here. It also includes several ideas not pursued here which seem very interesting. The work in Section 4 on relational semantics is most closely related to the work of Bimbo and Dunn in [4].

2. Generalizing sets of worlds

Underlying a Kripke structure is a non-empty set W of worlds. A potential interpretant over a Kripke frame is an arbitrary subset of worlds, and the intuition is that these are the worlds at which the proposition being interpreted holds. There is already precedence for some logics requiring a richer notion of underlying structure than just a set. In the frames introduced by Jansana and Celani for positive modal logic, [6], the set of possible worlds is given an order and interpretants are required to be upward-closed subsets of worlds. Likewise in [13] the set underlying a frame is required to be ordered, but there interpretants are taken among downward closed sets. This difference is of course not essential.

Here, as in [4], we will take a particular kind of two-sorted structures as underlying structures for our frames. The idea is that interpretants are not only described 'from below' but also 'from above'. In order to understand this better, notice first that these notions of up and down easily can be reversed. In a Kripke structure worlds describe interpretants 'from below' in the sense that single worlds generally are smaller units and interpretants consist of sets with (typically) many worlds in them. On the other hand, in terms of information content, worlds are maximal: their theories are (in the Boolean case) the maximal consistent ones. One could, in this upside-down world of information, also ask to describe formulas from below. The units of such a description one may then call 'information quanta' or 'co-worlds'. These may be thought of, in the Boolean case, as the smallest units of information, that is, the pieces of information just below True (whereas worlds are maximal pieces of information, that is, the ones just above False). In the non-Boolean case, just like worlds need not be just above False, 'information quanta' need not be just below True but are rather irreducible jumps in information.

We begin with a general definition that goes back at least to Birkhoff [5] and that is at the heart of various approaches to logic including Formal Concept Analysis (FCA), Chu-spaces, and classifications [10, 17, 2].

Definition 2.1. A polarity is a triple $P = (X, Y, R)$ where X and Y are non-empty sets and $R \subseteq X \times Y$ is a binary relation from X to Y.

We will call the elements of X worlds, the elements of Y information quanta, and when xRy is the case, we say that y is a part of x.

One central idea of our models is that interpretants should be two-sorted, consisting both of a set of worlds and a set of information quanta, but that either completely determines the interpretant. Thus if a proposition p has interpretant consisting of the set A of worlds and the set B of information quanta, we will say that p holds exactly at those worlds that are in A and that p consists exactly of the information quanta that are in B. With these ideas in mind we see that the following are natural requirements: (1) If x is in A, and y is in B then y must be part of x as the information content of x includes that of p which includes that of y; (2) If x is a world containing all the information quanta that p consists of then x should be in A since p is completely determined by its information content; (3) If y is an information quantum that is part of each world in A, then, since p should be completely determined by the set of worlds in which it holds, y better be in B. These ideas, which are also at the heart of the concept-formation mechanism of FCA [10], exactly identify potential interpretants of a structure $P = (X, Y, R)$ as the Galois stable sets of the Galois connection associated with P. To fix notation, we define the Galois connection by:

$$\begin{array}{cccll} (\ \ )^R & : & \mathcal{P}(X) & \to & \mathcal{P}(Y) \\ & & A & \mapsto & \{y : \forall x\ (x \in A \text{ implies } xRy)\} \end{array}$$

$$\begin{array}{cccll} {}^R(\ \ ) & : & \mathcal{P}(Y) & \to & \mathcal{P}(X) \\ & & B & \mapsto & \{x : \forall y\ (y \in B \text{ implies } xRy)\} \end{array}$$

and then the complete lattice of Galois stable subsets of X is given by

$$\mathcal{G}(P) = \{A \subseteq X : A = {}^R(A^R)\}.$$

As we shall see, choosing polarities as the basic structures and the Galois stable subsets of such as the potential interpretants ensures that we do indeed have structures which simultaneously determine our potential interpretants from above and from below. However, polarities are too general for it to be reasonable to say that they capture the notion of sets of worlds in this two sorted setting. To see this, notice that for a set of worlds W, the corresponding set of potential interpretants is the power set Boolean algebra, $\mathcal{P}(W)$. Power set algebras are abstractly characterized as the complete and atomic Boolean algebras. By contrast, for a polarity P, the corresponding set of potential interpretants is the complete lattice of Galois stable sets $\mathcal{G}(P)$. These lattices are abstractly characterized simply as complete lattices. That is, within the Boolean setting they restrict to the class of all complete (and not necessarily atomic) Boolean algebras. Thus, in order to truly capture the notion of sets of worlds, we need to make further restrictions on the class of structures we will consider. The property that is missing is the irreducibility of worlds, and dually, the irreducibility of the information quanta. Before we make the necessary definitions, we derive some basic facts about polarities and their stable sets. Our first task is to restrict to those polarities for which X and Y are subsets of $\mathcal{G}(P)$ just like W is a subset of $\mathcal{P}(W)$.

Proposition 2.2. *Let $P = (X, Y, R)$ be a polarity and let*

$$\begin{array}{ll} \Xi\ :\ X \to \mathcal{G}(P) & \qquad \Upsilon\ :\ Y \to \mathcal{G}(P) \\ \quad\ \ x \mapsto {}^R(\{x\}^R) & \qquad\quad\ \ y \mapsto {}^R\{y\} = Ry. \end{array}$$

then the following hold:

1. *Ξ is one-to-one if and only if $\forall x_1, x_2 \in X(x_1 \neq x_2$ implies $x_1R \neq x_2R)$.*
2. *Υ is one-to-one if and only if $\forall y_1, y_2 \in Y(y_1 \neq y_2$ implies $Ry_1 \neq Ry_2)$.*

PROOF. Suppose Ξ is injective and let $x_1 \neq x_2$. Then

$$^R(\{x_1\}^R) = \Xi(x_1) \neq \Xi(x_2) =^R (\{x_2\}^R).$$

But then

$$x_1R = \{x_1\}^R = (^R(\{x_1\}^R)^R \neq (^R(\{x_2\}^R))^R = \{x_2\}^R = x_2R$$

since $(\quad)^R$ is an anti-automorphism from the stable sets in X to the stable sets in Y. For the converse, if $x_1R \neq x_2R$, then since $^R(\quad)$ is an anti-automorphism from the stable sets in Y to the stable sets in X and since $x_1R = \{x_1\}^R$ and $x_2R = \{x_2\}^R$ are stable sets in Y we conclude that $\Xi(x_1) =^R (\{x_1\}^R) \neq^R (\{x_2\}^R) = \Xi(x_2)$ as desired. Statement 2 is completely trivial since $\Upsilon(y) = Ry$ for any $y \in Y$. ■

DEFINITION 2.3. A polarity $P = (X, Y, R)$ is said to be a separating frame, or S frame, provided

$$\forall x_1, x_2 \in X(x_1 \neq x_2 \text{ implies } x_1R \neq x_2R)$$

and

$$\forall y_1, y_2 \in Y(y_1 \neq y_2 \text{ implies } Ry_1 \neq Ry_2)$$

Thus for an S frame $F = (X, Y, R)$ we may (and we will) think of X and Y as subsets of $\mathcal{G}(P)$.

REMARK 2.4. In Formal Concept Analysis this property of a context (i.e., polarity) is called clarified. The property on X alone is in Chu-spaces called separated (there the dual property is called spatial). Thus in the Chu-space terminology our separating frames are both separated and dually separated, or doubly separated.

DEFINITION 2.5. Given an S frame $F = (X, Y, R)$ we let $Z(F)$ (or simply Z) denote the partially ordered subset of $\mathcal{G}(P)$ given by $Im(\Xi) \cup Im(\Upsilon)$. We write $Z = X \cup Y$ identifying X and Y with their images under Ξ and Υ, respectively. Note that $X \cap Y = Im(\Xi) \cap Im(\Upsilon)$ need not be empty and that when we talk about S frames we tacitly assume that the intersection of X and Y corresponds exactly to $Im(\Xi) \cap Im(\Upsilon)$.

PROPOSITION 2.6. *Let $P = (X, Y, R)$ be a polarity and let $x \in X$ and $y \in Y$ then xRy if and only if $\Xi(x) \subseteq \Upsilon(y)$.*

PROOF. First note that for $x \in X$ and $y \in Y$ we have xRy if and only if $x \in Ry$ if and only if $x \in \Upsilon y$. Also, since Υy is Galois stable and $\Xi(x)$ is the Galois stable closure of $\{x\}$ we have $x \in \Upsilon y$ implies $\Xi(x) \subseteq \Upsilon y$. For the converse, we always have $x \in \Xi(x)$ so $\Xi(x) \subseteq \Upsilon y$ implies $x \in \Upsilon y$. Thus we have shown that xRy if and only if $x \in \Upsilon y$ if and only if $\Xi(x) \subseteq \Upsilon y$. ■

PROPOSITION 2.7. *Let $F = (X, Y, R)$ be an S frame and let $z_1, z_2 \in Z$ then the following hold:*

1. *If $z_1, z_2 \in X$ then $z_1 \leq z_2$ if and only if*

$$\forall y \in Y(z_2Ry \textit{ implies } z_1Ry);$$

2. *If $z_1, z_2 \in Y$ then $z_1 \leq z_2$ if and only if*

$$\forall x \in X(xRz_1 \textit{ implies } xRz_2);$$

3. *If $z_1 \in X$ and $z_2 \in Y$ then $z_1 \leq z_2$ if and only if*

$$z_1Rz_2$$

4. *If $z_1 \in Y$ and $z_2 \in X$ then $z_1 \leq z_2$ if and only if*

$$\forall x \in X \forall y \in Y(xRz_1 \textit{ and } z_2Ry \textit{ implies } xRy)$$

PROOF. This is a simple translation of the containment statements for the corresponding stable sets. We leave out the details. ■

REMARK 2.8. These descriptions of the order on the worlds and information quanta of an S frame are also intuitively natural: The order on worlds is given by reverse information content (the more information, the smaller the world); One information quantum is less than another provided it is part of less worlds (i.e., it is a bigger piece of information); The order from worlds to information quanta is simply the 'is a part of' relation (and thus again the smaller holds more information); A information quantum is below a world if the world contains less information than the information quantum or equivalently if the world is above every world that the information quantum is a part of (and here again, even though the formulation is more complicated, the order is the reverse order of information content).

Notation 2.9. In an S frame $F = (X, Y, R)$, the relation R is really the order from X to Y, so we will usually denote it by $\leq$ even though it is only the restriction of the order on $Z = X \cup Y$ to $X \times Y$. Accordingly, rather than calling the maps in the Galois connection associated with F for $(\quad)^R$ and ${}^R(\quad)$, we will call them $(\quad)^u$ and $(\quad)^l$ as is customary in order theory.

We now turn to the generation properties of the components of a polarity in its lattice of Galois stable sets. For a set W we have that $\mathcal{P}(W)$ is join generated by W. Here, as is well known, we have something similar:

Proposition 2.10. *Let $P = (X, Y, R)$ be any polarity then $Im(\Xi)$ join generates $\mathcal{G}(P)$, and $Im(\Upsilon)$ meet generates $\mathcal{G}(P)$.*

Proof. Let $S \subseteq X$, then

$$\begin{aligned} S^R &= \{y : \forall x (x \in S \text{ implies } xRy)\} \\ &= \bigcap \{xR : x \in S\}. \end{aligned}$$

Since S^R is stable and ${}^R(\quad)$ is an anti-automorphism from the Galois closed sets in Y to those in X we have

$$\begin{aligned} {}^R(S^R) &= \bigvee \{{}^R(xR) : x \in S\} \\ &= \bigvee \{\Xi(x) : x \in S\}. \end{aligned}$$

In particular if $S \in \mathcal{G}(P)$, then $S = ^R (S^R)$ and we get

$$S = \bigvee \{\Xi(x) : x \in S\}$$

and we see that $im(\Xi)$ does join generate $\mathcal{G}(P)$. Similarly for $S \in \mathcal{G}(P)$ we have

$$\begin{aligned} S = ^R (S^R) &= \{x : \forall y (y \in S^R \text{ implies } xRy)\} \\ &= \bigcap \{Ry : y \in S^R\} \\ &= \bigcap \{\Upsilon(y) : y \in S^R\} \end{aligned}$$

and thus $Im(\Upsilon)$ meet generates $\mathcal{G}(P)$. ■

Corollary 2.11. *Let $F = (X, Y, \leq)$ be an S frame and Z the associated poset. Then the following hold:*

1. *X join generates $\mathcal{G}(F)$ (and thus also Z);*

2. *Y meet generates $\mathcal{G}(F)$ (and thus also Z);*
3. *$\mathcal{G}(F) = \overline{Z}$ where $\overline{Z}$ is the Dedekind-MacNeille completion of Z.*

PROOF. For the proof of statement 3 one must know that $\overline{Z}$ is uniquely characterized as the complete lattice that is both join and meet generated by Z [1]. ■

As mentioned above, an essential property of $\mathcal{P}(W)$ among Boolean algebras is that it is atomic, that is, not only do the singletons $\{w\}$ for $w \in W$ join generate $\mathcal{P}(W)$, they are also completely join irreducible. In our two-sorted setting this irreducibility corresponds to the following property:

DEFINITION 2.12. Let $F = (X, Y, \leq)$ be an S frame. We say that F is reduced, and in this case we call F an RS frame, provided the following two properties hold:

1. $\forall x \in X \exists y \in Y$ with $x \not\leq y$ and $\forall x' \in X$, if $x' < x$, then $x' \leq y$.
2. $\forall y \in Y \exists x \in X$ with $y \not\geq x$ and $\forall y' \in Y$, if $y' > y$, then $y' \geq x$.

Note that, for an S frame, $F = (X, Y, \leq)$, being reduced exactly means that all the elements of X are completely join irreducible in X (or equivalently in Z; or equivalently in $\overline{Z} = \mathcal{G}(F)$), and, dually, that all the elements of Y are completely meet irreducible in Y (or equivalently in Z; or equivalently in $\overline{Z} = \mathcal{G}(F)$).

REMARK 2.13. In [9] we say a poset Z is perfect provided it is join generated by $J^\infty(Z)$, the set of its completely join irreducible elements, it is meet generated by $M^\infty(Z)$, its set of completely meet irreducible elements, and every element of the poset is completely join or completely meet irreducible. It is straight forward to check that if $F = (X, Y, \leq)$ is an RS frame then the associated poset is perfect, and conversely, if Z is a perfect poset, then $F = (J^\infty(Z), M^\infty(Z), \leq)$, where $\leq$ is the order on Z restricted to $J^\infty(Z) \times M^\infty(Z)$, is an RS frame. Thus RS frames are just a two-sorted way of describing the perfect posets of [9].

We now show that if one restricts to distributive and Boolean lattices, then one gets exactly the usual notion of interpretants.

THEOREM 2.14. *Let $F = (X, Y, \leq)$ be an RS frame. Then the following hold:*

1. *If $\mathcal{G}(F)$ is distributive, then $\mathcal{G}(F) \cong \mathcal{O}(X)$;*
2. *If $\mathcal{G}(F)$ is Boolean, then $\mathcal{G}(F) \cong \mathcal{P}(X)$.*

We prove this theorem by proving several lemmas. We will adhere, here and in the rest of the paper, to the convention that if a frame $F = (X, Y, \leq)$ is given then variables named by x (including x_i, x' etc.) range over the set X of the frame, variables named by y range over the set Y of the frame, variables named by z range over the poset Z corresponding to the frame, and variables named by u, v, and w range over the corresponding lattice $\mathcal{G}(F) = \overline{Z}$. While the point of view is different here, these results may be found in [10] where they are attributed to Erné.

Lemma 2.15. *Let $F = (X, Y, \leq)$ be an RS frame and suppose $\mathcal{G}(F)$ is distributive, then the following hold:*

1. $\forall x\, \exists! y\, [x \nleq y$ *and* $\forall x'(x' < x$ *implies* $x' \leq y)]$;
2. $\forall x\, \exists y\, [x \nleq y$ *and* $\forall u(x \leq u$ *or* $u \leq y)]$;
3. *The map* $\kappa : X \to Y$ *given by* $x \mapsto \bigvee\{x' : x \nleq x'\}$ *is an order isomorphism;*
4. $\forall x\, \forall u\, (x \nleq u$ *if and only if* $u \leq \kappa(x))$.

Proof. Let $F = (X, Y, \leq)$ be an RS frame, let $x \in X$ and suppose $\mathcal{G}(F)$ is distributive. The existence part of statement 1 is given by reducedness of F. Thus we just have to prove uniqueness. To this end suppose $y_1, y_2 \in Y$ with

$$x \nleq y_i \text{ and } \forall x'(x' < x \text{ implies } x' \leq y_i)$$

for both $i = 1$ and $i = 2$. Let $u = \bigvee\{x' : x' < x\}$, then from the assumptions on y_1 and y_2 it is clear that

$$x \wedge y_1 = u = x \wedge y_2,$$

and thus, by distributivity, we have $u = x \wedge (y_1 \vee y_2)$. Now either $y_1 \not< y_2$ or $y_2 \not< y_1$. We assume WLOG that $y_2 \not< y_1$. Let $v = x \vee y_1$, and $w = (y_1 \vee y_2) \wedge v$. Then we have

$$\begin{aligned} x \wedge w &= x \wedge (y_1 \vee y_2) \wedge v \\ &= u \wedge v \\ &= u. \end{aligned}$$

Now using the fact that $y_1 = y_1 \wedge v \leq (y_1 \vee y_2) \wedge v = w \leq v$ we also have

$$\begin{aligned} v &= x \vee y_1 \\ &\leq x \vee w \\ &\leq x \vee v = v \end{aligned}$$

and thus $x \vee w = v$. That is, $x \wedge w = u = x \wedge y_1$ and $x \vee w = v = x \vee y_1$, and thus, again by distributivity, it follows that $y_1 = w = (y_1 \vee y_2) \wedge v$. Also, recall that $y_1 \in Y$ is completely meet irreducible and $v = x \vee y_1 > y_1$ since $x \not\leq y_1$. Thus it follows that $y_1 = y_1 \vee y_2$. In combination with our assumption that $y_2 \not< y_1$ we get $y_1 = y_2$ as desired.

Now in order to prove statement 2, we show that the element $y \in Y$ stipulated to be unique in 1 does the job. So let $x \in X$, let $y \in Y$ with

$$x \not\leq y \text{ and } \forall x'(x' < x \text{ implies } x' \leq y)$$

and suppose $u \in \mathcal{G}(F)$ with $x \not\leq u$. Then by distributivity $x \not\leq y \vee u$, and thus, since Y is meet dense in $\mathcal{G}(F)$, there is $y' \in Y$ with $x \not\leq y'$ and $y \vee u \leq y'$. But $y \leq y \vee u \leq y'$ and $\forall x'(x' < x$ implies $x' \leq y)$ implies that $\forall x'(x' < x$ implies $x' \leq y')$. Now by the uniqueness, we must have $y = y'$ and thus $u \leq y$. This means that y is the largest element in $\mathcal{G}(F)$ that is not above x and thus statement 2 is true.

In order to prove statement 3, we first need to know that $\kappa(x) \in Y$, but this is clear as $\kappa(x)$ must be the element y stipulated in statement 2. Notice that this also means that for each x we have

$$x \not\leq \kappa(x) \text{ and } \forall u(x \leq u \text{ or } u \leq \kappa(x)).$$

Therefore if $x_1, x_2 \in X$ then $x_1 \not\leq \kappa(x_1)$ and if $x_2 \not\leq x_1$, then by the definition of κ we have $x_1 \leq \kappa(x_2)$. So $x_2 \not\leq x_1$ implies $\kappa(x_2) \not\leq \kappa(x_1)$. It is also clear from the definition of κ that it is an order preserving map. Thus κ is an oder embedding. Finally since all the assumptions are self-dual, it follows that for all $y \in Y$ there is $x \in X$ with

$$y \not\geq x \text{ and } \forall u(y \geq u \text{ or } u \geq x)$$

and thus $\kappa(x) = y$ and κ is also surjective.

Finally it is straight forward to see that statement 4 holds since $\kappa(x)$ is the element y stipulated in statement 2. ■

Lemma 2.16. *Let $F = (X, Y, \leq)$ be an RS frame and suppose $\mathcal{G}(F)$ is distributive, then $\mathcal{G}(F) \cong \mathcal{O}(X)$.*

Proof. For any RS frame F (in fact for any polarity), $\mathcal{G}(F)$ is a subposet of $\mathcal{O}(X)$, so we just need to prove that $\mathcal{G}(F)$ is not proper. Let $U \in \mathcal{O}(X)$ and let $x' \notin U$. Then for each $x \in U$ we have $x' \not\leq x$, and thus $x \leq \kappa(x')$. But then $\kappa(x') \in U^{\leq}$ and thus, for each $x'' \in^{\leq} (U^{\leq})$ we must have $x'' \leq \kappa(x')$. Since $x' \not\leq \kappa(x')$, we can conclude that $x' \notin^{\leq} (U^{\leq})$. That is, ${}^{\leq}(U^{\leq}) \subseteq U$ and thus ${}^{\leq}(U^{\leq}) = U$ and $u \in \mathcal{G}(F)$. ■

Lemma 2.17. *Let $F = (X, Y, \leq)$ be an RS frame and suppose $\mathcal{G}(F)$ is Boolean, then X is an antichain and thus $\mathcal{G}(F) \cong \mathcal{O}(X) \cong \mathcal{P}(X)$.*

Proof. Let $x \in X$ then for any $u \in \mathcal{G}(F)$ we have

$$x = (x \wedge u) \vee (x \wedge \neg u)$$

and since x is join irreducible, it follows that $x \leq u$ or $x \leq \neg u$, that is, $x \leq u$ or $u \leq \neg x$. Therefore we must have $\kappa(x) = \neg x$. So for each $u < x$ we have $u \leq \neg x$ and thus $u \leq x \wedge \neg x = 0$. That is, x is an atom. The rest now follows by the preceding lemma. ∎

These three lemmas have now proved the theorem. We also note the following correspondences obtained from this:

Corollary 2.18. *There is a one-to-one correspondence between RS frames whose Galois stable sets form a distributive lattice and posets. It is given by $F = (X, Y, \leq) \mapsto (X, \leq)$ where $\leq$ is the order on X induced by the frame F, and by $(X, \leq) \mapsto (X, X, R)$ where, for $x, y \in X$, we have xRy if and only if $y \not\leq x$. Furthermore, if $(X, \leq)$ and F correspond to each other then $\mathcal{G}(F) = \mathcal{O}(X)$.*

Corollary 2.19. *There is a one-to-one correspondence between RS frames whose Galois stable sets form a Boolean lattice and sets. It is given by $F = (X, Y, \leq) \mapsto X$ and by $X \mapsto (X, X, R)$ where, for $x, y \in X$, we have xRy if and only if $y \neq x$. Furthermore, if X and F correspond to each other then $\mathcal{G}(F) = \mathcal{P}(X)$.*

Remark 2.20. Notice that these corollaries tell us that in the distributive and Boolean setting information quanta are in one-to-one correspondence with worlds and simply consist of the information "does not hold at x" for some world x.

We close this section with the definition of an interpretation into an RS frame.

Definition 2.21. Let P be a set of variables and $F = (X, Y, \leq)$ an RS frame. Then an interpretation of P in F is a map $V : P \to \mathcal{G}(F)$. This yields a satisfaction relation defined for $x \in X$:

$$(F, V), x \Vdash p \iff x \in (V(p))^l \iff x \leq V(p),$$

and when $(F, V), x \Vdash p$ holds we say p holds at x in (F, V). We also obtain an information content relation defined for $y \in Y$:

$$(F, V), y \succ p \iff y \in (V(p))^u \iff y \geq V(p),$$

and when $(F, V), y \succ p$ holds we say y is part of p in (F, V).

This is of course the only possible definition of an interpretation as long as the interpretants have to be Galois closed subsets. Accordingly, it is the standard one considered for semantics involving Galois closed sets, e.g., see [4].

Having interpretations is not very interesting so far as we have not introduced any logic connectives. RS frames in themselves already encode a notion of conjunction, disjunction, True, and False. While we will not consider these logical connectives as such in the latter part of this paper, we spell out here how these are interpreted in an RS frame model. For this purpose, fix a set P of propositional variables and an RS frame $F = (X, Y, \leq)$. We then give the inductive conditions for extending the relations $\Vdash$ and $\succ$ to the set of all compound formulas in the connectives $\wedge, \vee, 0, 1$ over the set P. Suppose ϕ and ψ are formulas for which $(F, V), x \Vdash$ and $(F, V), y \succ$ have already been determined to hold or not to hold for each $x \in X$ and each $y \in Y$. Then we define for $x \in X$ and $y \in Y$:

$$
\begin{array}{lcl}
(F,V), x \ \Vdash\ \phi \wedge \psi & \iff & (F,V), x \ \Vdash\ \phi \ \text{ and } \ (F,V), x \ \Vdash\ \psi, \\
(F,V), y \ \succ\ \phi \wedge \psi & \iff & \forall x' \ ((F,V), x' \ \Vdash\ \phi \wedge \psi \ \Rightarrow\ x' \leq y), \\[6pt]
(F,V), y \ \succ\ \phi \vee \psi & \iff & (F,V), y \ \succ\ \phi \ \text{ and } \ (F,V), y \ \succ\ \psi, \\
(F,V), x \ \Vdash\ \phi \vee \psi & \iff & \forall y' \ ((F,V), y' \ \succ\ \phi \vee \psi \ \Rightarrow\ x \leq y'), \\[6pt]
(F,V), x \ \nVdash\ 0 & & (F,V), y \ \succ\ 0, \\
(F,V), x \ \Vdash\ 1 & & (F,V), y \ \not\succ 1.
\end{array}
$$

3. Morphisms for RS frames

In the previous section we provided an exposition of RS frames from a (more or less) purely relational structures point of view instead of starting from the dual concrete algebras point of view as was done in [9]. For the morphisms of these structures, we do not quite see how to do this yet, but since the structure preserving maps for any mathematical setting are crucial for the understanding of the structures, we do include a treatment of these here. Our treatment here, even though still rooted in algebra, hopefully gives a little more insight than what was already said in [9]. Nevertheless, it is clear that a lot of work still remains before we have an in depth relational understanding of RS frame homomorphisms.

Just like maps between sets are exactly duals of complete homomorphisms between the corresponding complex algebras, we want our morphisms to be dual to complete lattice homomorphisms between the corresponding perfect lattices. This requirement dictates the notion of morphism that we must choose. Because their origin is based in duality, it is not surprising that these morphisms are closely related to those defined in [14].

Consider RS frames $F_1 = (X_1, Y_1, \leq)$ and $F_2 = (X_2, Y_2, \leq)$ and $h : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$ a complete lattice homomorphism between their dual lattices. Since the lattices are complete, the fact that h is a complete homomorphism is equivalent to the fact that h is both residuated and dually residuated. That is, there are $f, g : \mathcal{G}(F_1) \to \mathcal{G}(F_2)$ with

$$\forall u_1 \in \mathcal{G}(F_1) \;\; \forall u_2 \in \mathcal{G}(F_2) \qquad \begin{aligned} f(u_1) \leq u_2 &\iff u_1 \leq h(u_2) \\ h(u_2) \leq u_1 &\iff u_2 \leq g(u_1) \end{aligned}$$

We list the following facts that are well known about residuated pairs of maps in general or that are simple consequences of the fact that here we have two pairs of residuated maps linked by h that is doubly residuated.

PROPOSITION 3.22. *Let $F_1 = (X_1, Y_1, \leq)$ and $F_2 = (X_2, Y_2, \leq)$ be RS frames, and $h : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$ a complete lattice homomorphism. Let $f, g : \mathcal{G}(F_1) \to \mathcal{G}(F_2)$ be the dual residual and the residual of h, respectively. Then the following statements are true:*

1. *For each $u_1 \in \mathcal{G}(F_1)$, we have that $g(u_1)$ is the greatest element of $\mathcal{G}(F_2)$ that is mapped below u_1 by h, and $f(u_1)$ is the least element of $\mathcal{G}(F_2)$ that is mapped above u_1 by h.*
2. *$Im(h)$ is a complete sublattice of $\mathcal{G}(F_1)$ that is isomorphic to both of the subposets $Im(f)$ and $Im(g)$ of $\mathcal{G}(F_2)$ and the isomorphisms are provided by the restriction of h in one direction and by the restrictions of f and g, respectively, in the other direction.*
3. *$Im(f)$ is a complete join semilattice of $\mathcal{G}(F_2)$ whereas $Im(g)$ is a complete meet semilattice of $\mathcal{G}(F_2)$.*
4. *h is surjective if and only if f is injective if and only if g is injective.*
5. *h is injective if and only if f is surjective if and only if g is surjective.*
6. *h is an isomorphism if and only if f is an isomorphism if and only if g is an isomorphism if and only if $f = g$.*

Given a homomorphism $h : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$, the idea is then essentially that the dual object is given by the maps f and g. However, these don't always restrict correctly to the dual structures. First we go through some cases where they do:

Proposition 3.23. *Let $F_1 = (X_1, Y_1, \leq)$ and $F_2 = (X_2, Y_2, \leq)$ be RS frames, and $h : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$ a complete lattice homomorphism. Let $f, g : \mathcal{G}(F_1) \to \mathcal{G}(F_2)$ be the dual residual and the residual of h, respectively. If $\mathcal{G}(F_1)$ is distributive, then $f \restriction X_1 : X_1 \to X_2$ and $g \restriction Y_1 : Y_1 \to Y_2$.*

Proof. We just show the first claim, the second being order dual. Let $x_1 \in X_1$ and suppose $f(x_1) = \bigvee \mathcal{C}$ where $\mathcal{C}$ is a subset of $\mathcal{G}(F_2)$. Then we have $x_1 \leq h(f(x_1)) = \bigvee\{h(u_2) : u_2 \in \mathcal{C}\}$ by the residuation property and because h preserves arbitrary joins. Now since $\mathcal{G}(F_1)$ is distributive we have $\mathcal{G}(F_1) = \mathcal{O}(X_1)$ and each completely join irreducible in $\mathcal{G}(F_1)$ is completely join prime. Therefore we conclude that there is $u_2 \in \mathcal{C}$ with $x_1 \leq h(u_2)$. And then $f(x_1) \leq f(h(u_2)) \leq u_2$. That is, $f(x_1)$ is completely join irreducible as required. ■

Remark 3.24. In the distributive and Boolean setting, i.e, when both $\mathcal{G}(F_1)$ and $\mathcal{G}(F_2)$ are at least distributive, then we can throw away half the dual structures and the dual of a complete homomorphism is just the order preserving map $f \restriction X_1 : X_1 \to X_2$. This is of course the usual dual map.

Essentially the same argument goes in general as long as h is surjective.

Proposition 3.25. *Let $F_1 = (X_1, Y_1, \leq)$ and $F_2 = (X_2, Y_2, \leq)$ be RS frames, and $h : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$ a complete lattice homomorphism. Let $f, g : \mathcal{G}(F_1) \to \mathcal{G}(F_2)$ be the dual residual and the residual of h, respectively. If h is surjective, then $f \restriction X_1 : X_1 \to X_2$ and $g \restriction Y_1 : Y_1 \to Y_2$.*

Proof. The proof is very similar to the one in the distributive case: Here also we just show the first claim, the second being order dual. Let $x_1 \in X_1$ and suppose $f(x_1) = \bigvee \mathcal{C}$ where $\mathcal{C}$ is a subset of $\mathcal{G}(F_2)$. Then, because h is surjective, we have $x_1 = h(f(x_1))$ rather than just $x_1 \leq h(f(x_1))$. Thus we get $x_1 = h(f(x_1)) = \bigvee\{h(u_2) : u_2 \in \mathcal{C}\}$ because h preserves arbitrary joins. Now since x_1 is completely join irreducible we conclude that there is $u_2 \in \mathcal{C}$ with $x_1 = h(u_2)$ and we get $f(x_1) \leq f(h(u_2)) \leq u_2$. That is, $f(x_1)$ is completely join irreducible. ■

The surjective homomorphisms on the lattice side of course correspond to subobjects on the RS frame side. First we show that RS subobjects are subobjects in the usual sense of relational structures.

Proposition 3.26. *Let $F_1 = (X_1, Y_1, \leq)$ and $F_2 = (X_2, Y_2, \leq)$ be RS frames, and $h : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$ a complete lattice homomorphism. Let $f, g : \mathcal{G}(F_1) \to \mathcal{G}(F_2)$ be the dual residual and the residual of h, respectively. If h is surjective, then the following hold:*

1. *$f \restriction X_1 : X_1 \to X_2$ is an order embedding;*
2. *$g \restriction Y_1 : Y_1 \to Y_2$ is an order embedding;*
3. *$x_1 \leq y_1$ if and only if $f(x_1) \leq g(y_1)$.*

Proof. Statements 1 and 2 follows as f and g are order isomorphisms between $\mathcal{G}(F_1)$ and $Im(f)$ and $Im(g)$, respectively, whenever h is surjective. If $x_1 \leq y_1$, then $x_1 \leq y_1 = h(g(y_1))$ and thus $f(x_1) \leq g(y_1)$ since f is the dual residual of h. Conversely, if $f(x_1) \leq g(y_1)$, then $x_1 = h(f(x_1)) \leq h(g(y_1)) = y_1$. ■

In [9] we described the structures studied here, not as two sorted structures, but as special posets, called perfect posets. As spelled out in Section 1, given an RS frame, $F = (X, Y, \leq)$, the corresponding perfect poset is the induced poset $Z = X \cup Y$. We still believe that there is great advantage to be gained by thinking of F and Z somewhat interchangeably, and to remember that an RS frame induces an (quasi-)order on $X \cup Y$ of which the relation that is part of the frame is exactly the order from X to Y. However, the danger of thinking of these frames as the special posets Z is that a subobject is NOT necessarily a subposet. On the two sorted level however, as we've just seen in the above proposition, subobjects are at least subobjects in the usual sense of relational structures.

Example 3.27. Consider the complete homomorphism $\pi_1 : 2^2 \to 2$ which projects the four element Boolean algebra onto its first coordinate. Then the two element chain $Z(2) = J(2) \cup M(2) = 2$ is not a subposet of the two element anti-chain $Z(2^2) = \{(1,0),(0,1)\}$ even though $f : 1 \mapsto (1,0)$ embeds $X(2)$ in $X(2^2)$, $g : 0 \mapsto (0,1)$ embeds $Y(2)$ in $Y(2^2)$, and the order from X_1 to Y_1 is empty just like the order from $Im(f)$ to $Im(g)$.

The following example shows that an RS frame which is a subobject in the usual sense of relational structures of another RS frame need not be a subobject of the second frame in the category of RS frames.

Example 3.28. Let D be the perfect lattice shown in Figure 1. The corresponding perfect poset, $Z(D)$, is obtained by removing 0 and 1. Furthermore, $X(D)$ consists of all the elements of $Z(D)$ except the element y, and $Y(D)$ consists of the maximal elements of $Z(D)$. Now the relational substructure of $(X(D), Y(D), \leq)$ obtained by removing y from $Y(D)$ yields again an RS frame. It is easy to verify that this RS frame corresponds to the perfect lattice C obtained from D by removing y. However, the injection $f : X(C) \to X(D)$ does not extend to a join preserving map from C to D since $x \vee_C x' = 1$ whereas $x \vee_D x' = y$.

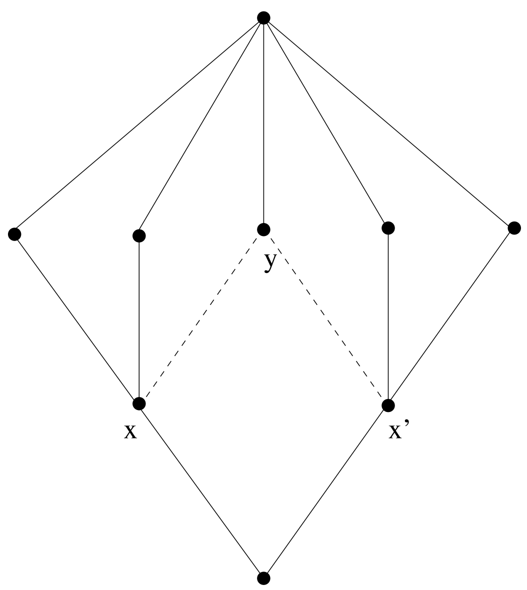

Figure 1. The lattice D.

PROPOSITION 3.29. *Let $F_1 = (X_1, Y_1, \leq)$ and $F_2 = (X_2, Y_2, \leq)$ be RS frames, and $h : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$ a surjective complete lattice homomorphism. Let $f, g : \mathcal{G}(F_1) \to \mathcal{G}(F_2)$ be the dual residual and the residual of h, respectively. We assume (WLOG) that $f(x_1) = x_1$ and $g(y_1) = y_1$ for all $x_1 \in X_1$ and $y_1 \in Y_1$. Then the following hold:*

1. *For all $x_2 \in X_2$, $(\downarrow x_2 \cap X_1)^u = \uparrow x_2 \cap Y_1$;*
2. *For all $y_2 \in Y_2$, $(\uparrow y_2 \cap Y_1)^l = \downarrow y_2 \cap X_1$.*

Here the arrows are taken over the frame F_2, whereas the (u, l)-connection is taken over the frame F_1.

PROOF. Throughout the proof, we will subscript various operations and relations with 1 and 2 to indicate which frame we are working over. Let $x_2 \in X_2$ and $y_1 \in (\downarrow_2 x_2 \cap X_1)^{u_1}$, then we have

$$\forall x_1\ (x_1 \leq_2 x_2 \Rightarrow x_1 \leq_1 y_1).$$

But $x_1 \leq_2 x_2$ means $f(x_1) \leq_2 x_2$ which is equivalent to $x_1 \leq_1 h(x_2)$. Thus we have

$$\forall x_1\ (x_1 \leq_1 h(x_2) \Rightarrow x_1 \leq_1 y_1).$$

That is, $h(x_2) \leq_1 y_1$ and thus $x_2 \leq_2 g(y_1) = y_1$ as desired. The other inclusion of statement 1 is obvious, and statement 2 is handled similarly. ■

DEFINITION 3.30. Let $F_1 = (X_1, Y_1, \leq_1)$ and $F_2 = (X_2, Y_2, \leq_2)$ be RS frames. We say that $F_1 = (X_1, Y_1, \leq_1)$ is an RS subframe of $F_2 = (X_2, Y_2, \leq_2)$ provided the following hold:

1. $X_1 \subseteq X_2$;
2. $Y_1 \subseteq Y_2$;
3. $\leq_1 = \leq_2 \cap (X_1 \times Y_1)$;
4. $\forall x_2 \in X_2 \quad (\downarrow x_2 \cap X_1)^u = \uparrow x_2 \cap Y_1$;
5. $\forall y_2 \in Y_2 \quad (\uparrow y_2 \cap Y_1)^l = \downarrow y_2 \cap X_1$;

THEOREM 3.31. *Let $F_1 = (X_1, Y_1, \leq)$ be an RS subframe of $F_2 = (X_2, Y_2, \leq)$, then*

$$\begin{aligned} h : \mathcal{G}(F_2) &\to \mathcal{G}(F_1) \\ u_2 &\mapsto \bigvee(\downarrow u_2 \cap X_1) = \bigwedge(\uparrow u_2 \cap Y_1) \end{aligned}$$

is a complete homomorphism. Furthermore, $f, g : \mathcal{G}(F_1) \to \mathcal{G}(F_2)$, the dual residual and the residual of h, are given by $f(x) = x$ and $g(y) = y$, respectively.

PROOF. Again, in the proof we will mark various constructs with subscripts of 1 or 2 to indicate which of the two frames it is referring to. Let $u_2 \in \mathcal{G}(F_2)$ then

$$\begin{aligned} (\downarrow_2 u_2 \cap X_1)^{u_1} &= (\bigcup\{\downarrow_2 x_2 \cap X_1 : x_2 \leq_2 u_2\})^{u_1} \\ &= \bigcap\{(\downarrow_2 x_2 \cap X_1)^{u_1} : x_2 \leq_2 u_2\} \\ &= \bigcap\{\uparrow_2 x_2 \cap Y_1 : x_2 \leq_2 u_2\} \\ &= \uparrow_2 u_2 \cap Y_1 \end{aligned}$$

and thus $\bigvee_1(\downarrow_2 u_2 \cap X_1) = \bigwedge_1(\uparrow_2 u_2 \cap Y_1)$, and h is well-defined.

Clearly h is order preserving. Let $U_2 \subseteq \mathcal{G}(F_2)$ with $\bigvee_2 U_2 = u_2$. Then $\bigvee_1 h(U_2) \leq_1 h(u_2)$. To show the reverse inequality, we show that any upper bound $y_1 \in Y_1$ in $\mathcal{G}(F_1)$ of $h(u_2)$ is a common upper bound in $\mathcal{G}(F_1)$ of all the elements of $h(U_2)$. Suppose $y_1 \geq_1 h(u_2) = \bigwedge_1(\uparrow_2 u_2 \cap Y_1)$. Recall that in order to show that h is well-defined, we computed that $(\downarrow_2 u_2 \cap X_1)^{u_1} = \uparrow_2 u_2 \cap Y_1$. Notice that this implies that $\uparrow_2 u_2 \cap Y_1$ is a stable set for F_1, thus $(\bigwedge(\uparrow_2 u_2 \cap Y_1))^{u_1} = \uparrow_2 u_2 \cap Y_1$. So $y_1 \in \uparrow_2 u_2 \cap Y_1$. But then for each $u_2' \in U_2$, since $u_2' \leq_2 u_2$, we must have $y_1 \in \uparrow_2 u_2' \cap Y_1 = (h(u_2'))^{u_1}$. We conclude that h is completely join preserving. The proof that h is completely meet preserving is of course order dual.

Finally we want to show that h is surjective. Let $u_1 \in \mathcal{G}(F_1)$ and let $u_2 = \bigvee_2\{x_1 : x_1 \leq_1 u_1\}$. Since h is completely join preserving, we have

$h(u_2) = \bigvee_1\{h(x_1) : x_1 \leq_1 u_1\}$. In order to show that $h(u_2) = u_1$, we show that $h(x_1) = x_1$ for each $x_1 \in X_1$. By definition $h(x_1) = \bigwedge_1(\uparrow_2 x_1 \cap Y_1)$, but for $y_1 \in Y_1$ we have $x_1 \leq_2 y_1$ if and only if $x_1 \leq_1 y_1$, and thus $h(x_1) = \bigwedge_1(\uparrow_1 x_1 \cap Y_1) = x_1$. Notice that this also shows that $f(x_1) = x_1$ for all $x_1 \in X_1$. The fact that $g(y_1) = y_1$ for all $y_1 \in Y_1$ is order dual. ∎

Corollary 3.32. *There is a one-to-one correspondence between surjective homomorphisms from the lattice of Galois stable subsets of an RS frame and RS subframes of that RS frame.*

When a complete homomorphism is not surjective, then $f \restriction X_1$ and $g \restriction Y_1$ do not necessarily map into X_2 and Y_2, respectively. Nevertheless, these two restrictions encode f and g and are therefore sufficient for describing the fact that h is a complete homomorphism. We sketch the development now. Some details are omitted, but they should be fairly straightforward to anyone familiar with residuation. Alternatively, the same content, presented differently, is proved in [9].

Proposition 3.33. *Let $F_1 = (X_1, Y_1, \leq)$ and $F_2 = (X_2, Y_2, \leq)$ be RS frames. Then the following statements hold:*

1. *If $h : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$ and $g : \mathcal{G}(F_1) \to \mathcal{G}(F_2)$ is a residuated pair, then both maps are uniquely determined by the relation $R_h \subseteq Y_1 \times X_2$ given by*
$$y_1 R_h x_2 \iff y_1 \geq h(x_2) \iff g(y_1) \geq x_2.$$
Furthermore, since $R_h[\_, x_2] = \uparrow h(x_2) \cap Y_1$ and $R_h[y_1, \_] = \downarrow g(y_1) \cap X_2$ for any $x_2 \in X_2$ and $y_1 \in Y_1$ it follows that these are stable sets.
2. *If $R \subseteq Y_1 \times X_2$ is a relation such that $R[\_, x_2]$ and $R[y_1, \_]$ are stable sets for all $x_2 \in X_2$ and $y_1 \in Y_1$, then the maps $h_R : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$ and $g_R : \mathcal{G}(F_1) \to \mathcal{G}(F_2)$ defined by*
$$\begin{aligned} h_R(x_2) &= \bigwedge R[\_, x_2] \quad \text{for } x_2 \in X_2, \\ h_R(u_2) &= \bigvee\{h_R(x_2) : u_2 \geq x_2 \in X_2\} \\ &= \bigwedge\{y_1 : \forall x_2(x_2 \leq u_2 \Rightarrow y_1 R x_2\} \quad \text{for } u_2 \in \mathcal{G}(F_2), \end{aligned}$$
$$\begin{aligned} g_R(y_1) &= \bigvee R[y_1, \_] \quad \text{for } y_1 \in Y_1, \\ g_R(u_1) &= \bigwedge\{g_R(y_1) : u_1 \leq y_1 \in Y_1\} \\ &= \bigvee\{x_2 : \forall y_1(y_1 \geq u_1 \Rightarrow y_1 R x_2\} \quad \text{for } u_1 \in \mathcal{G}(F_1) \end{aligned}$$
form a residuated pair.

3. *If $h : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$ and $g : \mathcal{G}(F_1) \to \mathcal{G}(F_2)$ is a residuated pair then $h_{R_h} = h$ and $g_{R_h} = g$, and if $R \subseteq Y_1 \times X_2$ is a relation such that $R[\_, x_2]$ and $R[y_1, \_]$ are stable sets for all $x_2 \in X_2$ and $y_1 \in Y_1$ then $R_{h_R} = R$.*

Now this allows us to define the dual of a complete lattice homomorphisms:

Definition 3.34. Let $F_1 = (X_1, Y_1, \leq)$ and $F_2 = (X_2, Y_2, \leq)$ be RS frames, and $h : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$ a complete lattice homomorphism. Then we define the dual of h to be the pair (R_h, S_h), where $R_h \subseteq Y_1 \times X_2$ is the relation described above, and which arises from the fact that h is residuated, and $S_h \subseteq X_1 \times Y_2$ is the corresponding relation arising from the fact that h is dually residuated. That is, R_h and S_h are given by

$$y_1 R_h x_2 \iff y_1 \geq h(x_2)$$

and

$$x_1 S_h y_2 \iff x_1 \leq h(y_2).$$

That is, we encode h by remembering the value of h both on the completely join irreducible elements and on the completely meet irreducible elements, instead of encoding only the value of h on the atoms as is done in the setting of sets and Boolean algebras.

We also make the following definition:

Definition 3.35. Let $F_i = (X_i, Y_i, \leq)$ be RS frames for $i = 1, \ldots, n$, and T an n-ary relation in $\clubsuit_1 \times \ldots \clubsuit_n$, where $\clubsuit_i$ is either X_i or Y_i for $i = 1, \ldots, n$. We say T is compatible (with the frames F_i) provided T yields stable sets whenever all but one coordinate is fixed.

We've already seen that one of the characterizing properties of R_h, and thus also of S_h, is that it is compatible with F_1 and F_2. The only additional information we need to encode to characterize the pairs of relations we get as dual morphisms is that the residuated map corresponding to R_h is equal to the dually residuated map corresponding to S_h. For this purpose, suppose $F_1 = (X_1, Y_1, \leq)$ and $F_2 = (X_2, Y_2, \leq)$ are RS frames, and that $R \subseteq Y_1 \times X_2$ and $S \subseteq X_1 \times Y_2$ are relations compatible with F_1 and F_2. Let $h_R : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$ and $g_R : \mathcal{G}(F_1) \to \mathcal{G}(F_2)$ be the residuated pair corresponding to R, $h_S = g_{(S^{-1})} : \mathcal{G}(F_2) \to \mathcal{G}(F_1)$ and $f_S = h_{(S^{-1})} : \mathcal{G}(F_1) \to \mathcal{G}(F_2)$ the dually residuated pair corresponding to S. Then we have:

$$\begin{aligned}
& \forall u_2 \in \mathcal{G}(F_2) \qquad h_R(u_2) \leq h_S(u_2) \\
\iff & \forall x_2\ \forall y_2 \qquad (x_2 \leq y_2 \ \Rightarrow h_R(x_2) \leq h_S(y_2)) \\
\iff & \forall x_2\ \forall y_2 \qquad (x_2 \leq y_2 \ \Rightarrow R[_, x_2]^l \subseteq S[_, y_2])
\end{aligned}$$

and for the reverse inequality we have to use the residuals:

$$\begin{aligned}
&\forall u_2 \in \mathcal{G}(F_2) \qquad h_S(u_2) \leq h_R(u_2)\\
\iff &\forall u_2\ \forall x_1\ \forall y_1 \qquad [(x_1 \leq h_S(u_2) \text{ and } h_R(u_2) \leq y_1) \Rightarrow x_1 \leq y_1]\\
\iff &\forall u_2\ \forall x_1\ \forall y_1 \qquad [(f_S(x_1) \leq u_2 \text{ and } u_2 \leq g_R(y_1)) \Rightarrow x_1 \leq y_1]\\
\iff &\forall x_1\ \forall y_1 \qquad (f_S(x_1) \leq g_R(y_1) \Rightarrow x_1 \leq y_1)\\
\iff &\forall x_1\ \forall y_1 \qquad (\bigwedge S[x_1, _] \leq \bigvee R[y_1, _] \Rightarrow x_1 \leq y_1)\\
\iff &\forall x_1\ \forall y_1 \qquad (S[x_1, _]^l \subseteq R[y_1, _] \Rightarrow x_1 \leq y_1)
\end{aligned}$$

These calculations yield the following definition of morphism for RS frames:

Definition 3.36. Let $F_1 = (X_1, Y_1, \leq)$ and $F_2 = (X_2, Y_2, \leq)$ be RS frames. A morphism from F_1 to F_2 is a pair (R, S) satisfying:

1. $R \subseteq Y_1 \times X_2$ is a compatible relation;
2. $S \subseteq X_1 \times Y_2$ is a compatible relation;
3. $\forall x_2\ \forall y_2 \qquad (x_2 \leq y_2 \Rightarrow R[\_, x_2]^l \subseteq S[\_, y_2])$;
4. $\forall x_1\ \forall y_1 \qquad (S[x_1, \_]^l \subseteq R[y_1, \_] \Rightarrow x_1 \leq y_1)$.

Corollary 3.37. *Let $F_1 = (X_1, Y_1, \leq)$ and $F_2 = (X_2, Y_2, \leq)$ be RS frames. There is a one-to-one correspondence between the complete lattice homomorphisms from $\mathcal{G}(F_2)$ to $\mathcal{G}(F_1)$ and the RS frame morphisms from F_1 to F_2.*

While the intuitive meaning of this definition is not so clear, the complexity is not too terrible. We believe that what is needed in order to be able to work efficiently with these structures is some kind of calculus that isolates and identifies the main manipulations corresponding to the toggling between the two functions in a residuated pair. The underlying RS frame has the (upper,lower) Galois connection (residuated pair with an order flip) at its heart. A morphism then consists of two relations each of which corresponds to a residuated pair. As we shall see, the logical implication and fusion then introduces another (binary this time) residuated map. Working with these structures then requires us to be able to manipulate first order statements involving various combinations of these. This might seem like a hard task, but the point is that, guided by the Sahlqvist correspondence ideas, we see a version of this calculus emerging in this setting as well. Working this out in detail is the subject of ongoing research with Palmigiano and Priestley. However, we already see the embryonic form in the two reductions right before Definition 3.36. The statement

$$\forall u_2 \in \mathcal{G}(F_2) \qquad h_R(u_2) \leq h_S(u_2)$$

is really of the form $\Diamond \leq \Box$, which is the simplest type of Sahlqvist inequality. We also see that the reduction is straightforward as it corresponds to a join less than a meet and thus only requires the individual parts to satisfy the corresponding inequality.

The statement

$$\forall u_2 \in \mathcal{G}(F_2) \qquad h_S(u_2) \leq h_R(u_2)$$

on the other hand, is of the form $\Box \leq \Diamond$, the simplest form of the more complex type of Sahlqvist inequality. Here indeed we see that a little more care is required, but that the essential content is that we have to translate the statement to one about the residual of $\Box$ and the dual residual of $\Diamond$ in order to be able to eliminate the second order quantifier.

4. Modeling implication and fusion

We now turn to the actual purpose of this paper, namely to spell out, in relational terms, the complete semantics given in [9] for the implication-fusion fragment of various substructural logics. Part of this material has been worked out in discussions with Manisha de Montgomery Nørgård. Her work can be found in her Master's Thesis [15].

We fix a set P of propositional variables and let $\mathcal{F}(P)$ be the set of all formulas in the binary connectives $\circ, \rightarrow, \leftarrow$. We will describe complete relational semantics based on RS frames for various logics of this connective type.

The basic logic we consider corresponds to requiring only that the implications $\rightarrow$ and $\leftarrow$ are the residuals of the fusion, $\circ$. To be specific, we consider the logic given by:

$$\frac{}{\phi \vdash \phi} \quad \text{and} \quad \frac{\Gamma \vdash \phi \qquad \Delta; \phi; \Sigma \vdash \psi}{\Delta; \Gamma; \Sigma \vdash \psi},$$

$$\frac{}{\phi; \phi \rightarrow \psi \vdash \psi} \quad \text{and} \quad \frac{\phi; \Gamma \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi},$$

and

$$\frac{}{\psi \leftarrow \phi; \phi \vdash \psi} \quad \text{and} \quad \frac{\Gamma; \phi \vdash \psi}{\Gamma \vdash \psi \leftarrow \phi}.$$

In order to capture this logic over RS frames we need, for an RS frame $(X, Y, \leq)$, to add structure that will capture the behavior of the logical connectives. For this purpose we will need a ternary relation $R \subseteq X \times X \times Y$ that is compatible with the frame. In order to keep the exposition simple, but at the expense of not differentiating our notion of frame from the plethora of others, we will just call the obtained relational structures frames.

DEFINITION 4.38. A frame is a structure $F = (X, Y, \leq, R)$ such that:

1. $(X, Y, \leq)$ is an RS frame;
2. $R \subseteq X \times X \times Y$ is a compatible relation.

Associated with the relation R, we have another compatible relation $R^{\downarrow} \subseteq X \times X \times X$ defined by:

$$\begin{array}{rcl} (x_1, x_2, x_3) \in R^{\downarrow} & \iff & \forall y\ ((x_1, x_2, y) \in R\ \Rightarrow\ y \geq x_3) \\ & \iff & x_3 \in R[x_1, x_2, \_]^l. \end{array}$$

For $x_1, x_2, x_3 \in X$ and $y \in Y$, if $(x_1, x_2, y) \in R$ then we say that y is part of the fusion of x_1 and x_2, and if $(x_1, x_2, x_3) \in R^{\downarrow}$ then we say that from x_3 the fusion of x_1 and x_2 is accessible. Note that there is no fusion operation defined on F or X but just an accessibility and/or part of relation describing fusion.

REMARK 4.39. Notice that we might as well take the accessibility relation $R^{\downarrow}$ as basic to the frame and then define the relation R from it. It is not hard to verify that compatibility of one is equivalent to compatibility of the other. Taking $R^{\downarrow}$ as basic would correspond more closely to what is usually done in Kripke frames, both in modal logic and substructural logic, see, e.g. [3]. However, with this choice the relation is not as natural for defining the interpretation of arbitrary formulas on frames as R is.

We are now ready to define models.

DEFINITION 4.40. A model is a pair $M = (F, V)$ where:

1. F is a frame;
2. V is an interpretation, that is, $V : P \to \mathcal{G}(X, Y, \leq)$ is a map.

Given a model $M = (F, V)$, we define relations $\Vdash\ \subseteq X \times \mathcal{F}(P)$ and $\succ\ \subseteq Y \times \mathcal{F}(P)$ inductively as follows:

1. For $x \in X$ and $p \in P$, $x \Vdash p$ if and only if $x \leq V(p)$, and for $y \in Y$ and $p \in P$, $y \succ p$ if and only if $y \geq V(p)$.
2. For $\phi, \psi \in \mathcal{F}(P)$ such that whether or not $x \Vdash \phi$, $x \Vdash \psi$, $y \succ \phi$, and $y \succ \psi$ have already been determined for each $x \in X$ and each $y \in Y$, and for $x \in X$ and $y \in Y$:

$$\begin{array}{rcl} y \succ \phi \circ \psi & \iff & \forall x_1, x_2\ [(x_1 \Vdash \phi \text{ and } x_2 \Vdash \psi) \Rightarrow R(x_1, x_2, y)], \\ x \Vdash \phi \circ \psi & \iff & \forall y\ (y \succ \phi \circ \psi \Rightarrow x \leq y), \end{array}$$

$$\begin{array}{rcl} x \Vdash \phi \to \psi & \iff & \forall x'\ \forall y\ [(x' \Vdash \phi \text{ and } y \succ \psi) \Rightarrow R(x', x, y)], \\ y \succ \phi \to \psi & \iff & \forall x\ (x \Vdash \phi \to \psi \Rightarrow x \leq y), \end{array}$$

$$\begin{array}{rcl} x \Vdash \psi \leftarrow \phi & \iff & \forall x'\ \forall y\ [(x' \Vdash \phi \text{ and } y \succ \psi) \Rightarrow R(x, x', y)], \\ y \succ \psi \leftarrow \phi & \iff & \forall x\ (x \Vdash \psi \leftarrow \phi \Rightarrow x \leq y). \end{array}$$

REMARK 4.41. The conditions given in the above inductive definition may look a little unfamiliar. For fusion, which is a $\Diamond$-like operation as it is join preserving in each coordinate, we'd expect a condition like:

$$x \Vdash \phi \circ \psi \quad \iff \quad \exists x_1, x_2\ (x_1 \Vdash \phi,\ x_2 \Vdash \psi \text{ and } R^{\downarrow}(x_1, x_2, x)) \quad (*)$$

We sketch the proof of the fact that this condition is indeed equivalent to the one given if we happen to be in a Boolean frame (something similar works in a distributive frame).

If the frame is Boolean, we know that there is a one-to-one correspondence between X and Y, say x corresponds to y_x, and

$$x \leq y' \qquad \iff \qquad y' \neq y_x.$$

In this case notice that

$$\begin{array}{rcl} R^{\downarrow}(x_1, x_2, x) & \iff & \forall y'\ (R(x_1, x_2, y') \Rightarrow x \leq y') \\ & \iff & \forall y'\ (R(x_1, x_2, y') \Rightarrow y' \neq y_x) \\ & \iff & \text{NOT } R(x_1, x_2, y_x), \end{array}$$

where NOT $R(x_1, x_2, y_x)$ means that it is not the case that $R(x_1, x_2, y_x)$.

Using these facts we get:

$$\begin{array}{rcl} x \Vdash \phi \circ \psi & \iff & \forall y'\ (y' \succ \phi \circ \psi \Rightarrow x \leq y') \\ & \iff & \forall y'\ (y' \succ \phi \circ \psi \Rightarrow y' \neq y_x) \\ & \iff & y_x \not\succ \phi \circ \psi \\ & \iff & \exists x_1, x_2\ (x_1 \Vdash \phi,\ x_2 \Vdash \psi \text{ and NOT } R(x_1, x_2, y_x)) \\ & \iff & \exists x_1, x_2\ (x_1 \Vdash \phi,\ x_2 \Vdash \psi \text{ and } R^{\downarrow}(x_1, x_2, x)). \end{array}$$

This equivalence does not hold in general, but one can still describe these semantics entirely on the basis of $R^{\downarrow}$ and $\Vdash$. However, it becomes very messy and impossible to understand. This can be done simply because

$$R(x_1, x_2, y) \qquad \iff \qquad \forall x\ (R^{\downarrow}(x_1, x_2, x) \Rightarrow x \leq y).$$

Thus one can substitute $\forall x\ (R^{\downarrow}(x_1, x_2, x) \Rightarrow x \leq y)$ in the place of $R(x_1, x_2, y)$ in the inductive condition defining $y \succ \phi \circ \psi$, and then one can substitute the resulting condition into the antecedent of the condition defining $x \Vdash \phi \circ \psi$ in the above inductive definition. The resulting very complicated condition does not simplify significantly in the general setting. Put in lattice theoretic terms, this is because join irreducibles in non-distributive (perfect) lattices aren't necessarily join prime.

Notice that the equivalence of our semantics with those defined by $(*)$ for distributive frames (and in particular Boolean ones), implies that if these special frames are enough to identify the given logic, then our more general structures are not needed. In algebraic terms, this is the case when the distributive members of the class have as general a theory (in the specified connectives) as the full class. Even when this is the case though, the approach may not be as uniform as with the broader class of structures.

It is straight forward to prove that $\Vdash$ and $\succ$ define one and the same extension of the interpretation V to the set of all formulas. In fact, it is the unique homomorphic extension of V once we put the appropriate residuated structure on the dual of the frame:

Definition 4.42. Let $F = (X, Y, \leq, R)$ be a frame, then the complex algebra of F is the residuated algebra $F^+ = (\mathcal{G}(X, Y, \leq), \leq, \circ_R, \rightarrow_R, \leftarrow_R)$ where

$$\begin{aligned} \circ : \mathcal{G}(X,Y,\leq) \times \mathcal{G}(X,Y,\leq) &\to \mathcal{G}(X,Y,\leq) \\ (u,v) &\mapsto \bigwedge\{y : \forall x_1, x_2\ [(x_1 \leq u \text{ and } x_2 \leq v) \Rightarrow\ R(x_1, x_2, y)]\} \end{aligned}$$

$$\begin{aligned} \rightarrow : \mathcal{G}(X,Y,\leq) \times \mathcal{G}(X,Y,\leq) &\to \mathcal{G}(X,Y,\leq) \\ (u,v) &\mapsto \bigvee\{x : \forall x'\ \forall y\ [(x' \leq u \text{ and }\ y \geq v)\ \Rightarrow\ R(x', x, y)]\} \end{aligned}$$

$$\begin{aligned} \leftarrow : \mathcal{G}(X,Y,\leq) \times \mathcal{G}(X,Y,\leq) &\to \mathcal{G}(X,Y,\leq) \\ (u,v) &\mapsto \bigvee\{x : \forall x'\ \forall y\ [(x' \leq v \text{ and }\ y \geq u)\ \Rightarrow\ R(x, x', y)]\}. \end{aligned}$$

The fact that F^+ is a residuated algebra whenever F is a frame and that this establishes a one-to-one correspondence (up to natural isomorphism) between frames and perfect residuated lattices was proved in [9]. In [9] this is stated as Proposition 6.6, and the pertinent proofs can be found in sections 4 and 5 (section 5 deals with the additional operations).

Proposition 4.43. *Let $M = (F, V)$ be a model. Then*

$$\begin{aligned} \overline{V} : \mathcal{F}(P) &\to F^+ \\ \phi &\mapsto \bigvee\{x : x \Vdash \phi\} = \bigwedge\{y : y \succ \phi\} \end{aligned}$$

is the unique homomorphic extension of the map V.

We leave the details to the reader.

DEFINITION 4.44. Let $M = (X, Y, \leq, R, V)$ be a model and ϕ and ψ formulas in $\mathcal{F}(P)$. We say the sequent $\phi \vdash \psi$ holds in M provided the following equivalent conditions hold:

1. $\forall x\ (x \Vdash \phi \ \Rightarrow\ x \Vdash \psi)$;
2. $\forall y\ (y \succ \psi \ \Rightarrow\ y \succ \phi)$;
3. $\overline{V}(\phi) \leq \overline{V}(\psi)$.

Let $F = (X, Y, \leq, R)$ be a frame and ϕ and ψ formulas in $\mathcal{F}(P)$. We say the sequent $\phi \vdash \psi$ is valid in F provided $\phi \vdash \psi$ holds in every model $M = (F, V)$ over F.

Let $\mathcal{K}$ be a class of frames and ϕ and ψ formulas in $\mathcal{F}(P)$. We say the sequent $\phi \vdash \psi$ is valid over $\mathcal{K}$ provided $\phi \vdash \psi$ is valid in each frame F in $\mathcal{K}$.

The following theorem was given an algebraic proof in [9]. Here we sketch the corresponding relational proof based on the canonical model.

THEOREM 4.45. *Let ϕ and ψ be formulas in $\mathcal{F}(P)$. The sequent $\phi \vdash \psi$ is valid over the class of all frames if and only if it can be deduced in the basic substructural logic.*

The main part of this proof consists of a standard construction of the canonical frame for the logic. We split the work up to make it more accessible.

LEMMA 4.46. *(Soundness) Let ϕ and ψ be formulas in $\mathcal{F}(P)$. If the sequent $\phi \vdash \psi$ can be deduced in the basic substructural logic then it is valid over the class of all frames.*

PROOF. The soundness of the rules and axioms of the basic logic with respect to the class of frames is straight forward to check. For illustration, we show the rule:

$$\frac{\phi \circ \psi \vdash \theta}{\psi \vdash \phi \to \theta}$$

is sound. For this purpose, let $F = (X, Y, \leq, R)$ be any frame and $M = (F, V)$ any model over F, and suppose $\phi \circ \psi \vdash \theta$ holds in M. Then

$$\forall y\ (y \succ \theta \ \Rightarrow\ y \succ \phi \circ \psi).$$

Furthermore, $y \succ \phi \circ \psi$ means

$$\forall x_1, x_2\ [(x_1 \Vdash \phi \ \text{ and } \ x_2 \Vdash \psi) \ \Rightarrow\ R(x_1, x_2, y)].$$

Now assume in addition that x is such that $x \Vdash \psi$. We want to show that $x \Vdash \phi \to \theta$. That is, we have to show

$$\forall x' \, \forall y' \, [(x' \Vdash \phi \text{ and } y' \succ \theta) \Rightarrow R(x', x, y')].$$

So let x' be such that $x' \Vdash \phi$ and let y' be such that $y' \succ \theta$. Then, as $y' \succ \theta$, it follows that $y' \succ \phi \circ \psi$. That is, for all x_1, x_2 such that $x_1 \Vdash \phi$ and $x_2 \Vdash \psi$ we have $R(x_1, x_2, y')$. Accordingly, since $x' \Vdash \phi$ and $x \Vdash \psi$, it follows that $R(x', x, y')$ as desired. ■

For the completeness, we have to show that any sequent $\phi \vdash \psi$ that is not deducible in the basic logic does not hold in some model. We construct the canonical model and show that any sequent not deducible in the logic does not hold in this model.

Definition 4.47. Let P be the set of propositional variables, $\mathcal{F}(P)$ the set of formulas in the given connective type over P, and L the given logic. We call a non-empty subset D of $\mathcal{F}(P)$ a filter for L provided D is upward closed and down-directed. That is, if $\phi \in D$ and $\phi \vdash \psi$ is deducible then $\psi \in D$, and anytime ϕ_1 and ϕ_2 are in D, there is ψ in D with $\psi \vdash \phi_i$ deducible for both $i = 1$ and $i = 2$. Dually, we call a subset U of $\mathcal{F}(P)$ an ideal for L provided it has the dual properties, that is, it is downward closed and up-directed. A pair (D, U) consisting of a filter and an ideal is a maximally disjoint pair (mdp) (for L) provided $D \cap U = \emptyset$ and for all pairs (D', U') consisting of a filter and an ideal that are disjoint from each other, if $D \subseteq D'$ and $U \subseteq U'$, then $D = D'$ and $U = U'$.

We are now ready to define the canonical frame $F(= F(L))$:

1. Let $X = \{D : \exists U \ (D, U) \text{ is a mdp}\}$;
2. Let $Y = \{U : \exists D \ (D, U) \text{ is a mdp}\}$;
3. For $D \in X$ and $U \in Y$, define $D \leq U$ if and only if $D \cap U \neq \emptyset$;
4. For $D_1, D_2 \in X$ and $U \in Y$, define $(D_1, D_2, U) \in R$ if and only if there are $\phi \in D_1$ and $\psi \in D_2$ so that $\phi \circ \psi \in U$, that is, the complex product $D_1 \circ D_2 = \{\phi \circ \psi : \phi \in D_1 \text{ and } \psi \in D_2\}$ intersects U.

We of course need to know that this is a frame.

Lemma 4.48. *Let P be the set of propositional variables, $\mathcal{F}(P)$ the set of formulas in the given connective type over P, and L the given logic. Then the polarity $(X, Y, \leq)$ underlying the canonical model $F(L)$ is an RS frame.*

Proof. Let $p \in P$, then it is easy to show that, e.g., there are models in which p is not implied by $p \to p$ (in a Boolean frame for example), so by soundness, it follows that $p \to p \vdash p$ is not deducible. Thus $D = \uparrow (p \to p) = \{\phi : p \to p \vdash \phi\}$ and $U = \downarrow p = \{\psi : \psi \vdash p\}$ form a disjoint pair. Because filters and ideals are closed under directed unions, it follows that any disjoint pair can be extended to a mdp by Zorn's Lemma. Thus X and Y are both non-empty. Now let $D_1, D_2 \in X$ with $D_1 \neq D_2$. Then there is a ϕ which is in one but not the other. Suppose WLOG $\phi \notin D_1$. Then $(D_1, \uparrow \phi)$ is a disjoint pair and it can be extended to a mdp (D, U). Then $\phi \in U \cap D_2$ and thus $D_2 \leq U$, but $D_1 \cap U \subseteq D \cap U = \emptyset$, so $D_1 \leq$ and $D_2 \leq$ are not equal. The fact that $U_1 \neq U_2$ implies $\leq U_1 \neq \leq U_2$ is proved dually. Thus the frame $(X, Y, \leq)$ is separated.

We show that F is reduced. It is easy to see that for $D' \in X$, $D' \leq D$ is equivalent to $D \subseteq D'$. Thus $D' < D$ is equivalent to $D \subsetneq D'$. Now let $D \in X$, and let U be any ideal such that (D, U) is a mdp. Then for each D' with $D \subsetneq D'$ we must have $D' \cap U \neq \emptyset$. Otherwise D is not maximal with respect to being disjoint from U. Of course the statement for ideals is proved dually, and we conclude that F is an RS frame. ∎

Lemma 4.49. *Let P be the set of propositional variables, $\mathcal{F}(P)$ the set of formulas in the given connective type over P, and L the given logic. Then the ternary relation R underlying the canonical model $F(L)$ is compatible.*

Proof. Fix $D_1, D_2 \in X$, then $R[D_1, D_2, \_] = \{U : \exists \phi \in D_1, \exists \psi \in D_2$ with $\phi \circ \psi \in U\}$. We show that $R[D_1, D_2, \_]^l = \{D : D_1 \circ D_2 \subseteq D\}$. Certainly these are all common lower bounds. On the other hand suppose there are $\phi \in D_1$ and $\psi \in D_2$ with $\phi \circ \psi \notin D$. Then $(D, \uparrow \phi \circ \psi)$ is a disjoint pair and can be extended to a mdp (D', U). Then $\phi \circ \psi \in U$ and thus $U \in R[D_1, D_2, \_]$. On the other hand $D \cap U \subseteq D' \cap U = \emptyset$ so that $D \not\leq U$ and thus $D \notin R[D_1, D_2, \_]^l$. Further we show that $\{D : D_1 \circ D_2 \subseteq D\}^u = \{U : D_1 \circ D_2 \cap U \neq \emptyset\} = R[D_1, D_2, \_]$. It is clear that the elements of $R[D_1, D_2, \_]$ are common upper bounds of the set $\{D : D_1 \circ D_2 \subseteq D\}$. Now suppose $(D_1 \circ D_2) \cap U = \emptyset$. Clearly then $\downarrow (D_1 \circ D_2) \cap U = \emptyset$. We claim that $\downarrow (D_1 \circ D_2)$ is a filter. Let $\phi_1, \phi_2 \in D_1$ and $\psi_1, \psi_2 \in D_2$, then there are $\phi \in D_1$ and $\psi \in D_2$ with $\phi \vdash \phi_i$ and $\psi \vdash \psi_i$ deducible for $i = 1, 2$. But then $\phi \circ \psi \vdash \phi_i \circ \psi_i$ for $i = 1, 2$ are also deducible. So $\downarrow (D_1 \circ D_2)$ is a filter. Now suppose $\theta \in U$ with $\theta \vdash \phi \circ \psi$ deducible where $\phi \in D_1$ and $\psi \in D_2$. Then $\phi \circ \psi \in U$ since U is an ideal. But $(D_1 \circ D_2) \cap U = \emptyset$ by assumption, so $(\downarrow (D_1 \circ D_2), U)$ is a disjoint pair and can be extended to a mdp (D, U'). But then $D \in \{D : D_1 \circ D_2 \subseteq D\}$ and $D \cap U \subseteq D \cap U' = \emptyset$ so that U is not above D.

Now let $D \in X$ and $U \in Y$ and consider $R[D, \_, U] = \{D' : (D \circ D') \cap U \neq \emptyset\}$. We claim that $R[D, \_, U]^u = \{U' : D \to U \subseteq U'\}$. Let U' be such that $D \to U \subseteq U'$. Now for D' with $(D \circ D') \cap U \neq \emptyset$ we have $\phi \in D$ and $\psi \in D'$ with $\phi \circ \psi \in U$. Then $\phi \to (\phi \circ \psi) \in D \to U$, and thus $\phi \to (\phi \circ \psi) \in U'$. Finally, since $\psi \vdash \phi \to (\phi \circ \psi)$ is deducible and since U' is an ideal we get $\psi \in U'$ and $D' \cap U' \neq \emptyset$. That is, $\{U' : D \to U \subseteq U'\} \subseteq R[D, \_, U]^u$. On the other hand, suppose $U' \in Y$ and there is $\phi \in D$ and $\theta \in U$ with $\phi \to \theta \notin U'$. Then $(\downarrow (\phi \to \psi), U')$ is a disjoint pair and may be extended to a mdp (D', U''). Then $D' \cap U' \subseteq D' \cap U'' = \emptyset$ so $D' \nsubseteq U'$. However, since $\phi \to \theta \in D'$ and $\phi \circ (\phi \to \theta) \vdash \theta$ is deducible in the logic, it follows that $\phi \circ (\phi \to \theta) \in U$. But $\phi \circ (\phi \to \theta) \in D \circ D'$, so $D' \in R[D, \_, U]$ and it follows that U' is not in $R[D, \_, U]^u$. Finally, to complete this part on proves that $\{U' : D \to U \subseteq U'\}^l = \{D' : (D \circ D') \cap U \neq \emptyset\}$. The last coordinate is handled identically. So R is compatible. ∎

Definition 4.50. Let P be the set of propositional variables, $\mathcal{F}(P)$ the set of formulas in the given connective type over P, and L the given logic. Let $F = F(L)$ be the canonical frame for L, then the canonical model for L is $M(L) = (F, V)$ where $V : P \to \mathcal{G}(X, Y, R)$ is given by by $V(p) = \bigvee\{D : p \in D\} = \bigwedge\{U : p \in U\}$.

It is straight forward to show that $\{D : p \in D\}^u = \{U : p \in U\}$ and $\{U : p \in U\}^l = \{D : p \in D\}$ and thus V is well-defined. In addition we show:

Lemma 4.51. *Let P be the set of propositional variables, $\mathcal{F}(P)$ the set of formulas in the given connective type over P, and L the given logic. Let $M = (F, V)$ be the canonical model for L. For an arbitrary formula $\phi \in \mathcal{F}(P)$, $D \Vdash \phi$ if and only if $\phi \in D$ and $U \succ \phi$ if and only if $\phi \in U$.*

Proof. We proceed by induction on the complexity of formulas. It is clear by the definition that it is true for $p \in P$. Now suppose that ϕ and ψ are formulas for which the claim is true, and Let $U \in Y$ with $U \succ \phi \circ \psi$. That is, $\forall D_1, D_2[(D_1 \Vdash \phi, D_2 \Vdash \psi) \Rightarrow R(D_1, D_2, U)]$. But $D_1 \Vdash \phi$ means $\phi \in D_1$, $D_2 \Vdash \psi$ means $\psi \in D_2$, and $R(D_1, D_2, U)$ means $(D_1 \circ D_2) \cap U \neq \emptyset$. So we have $U \succ \phi \circ \psi$ if and only if

$$\forall D_1, D_2[(\phi \in D_1, \psi \in D_2) \Rightarrow (D_1 \circ D_2) \cap U \neq \emptyset].$$

It is clear then that $\phi \circ \psi \in U$ implies $U \succ \phi \circ \psi$. We now prove the converse. Let $U \in Y$ with $\phi \circ \psi \notin U$. Let $U_1 = \{\phi' : \phi' \circ \psi \in U\}$. It is straight forward to check that U_1 is an ideal. Now since $\phi \circ \psi \notin U$, it follows that $\phi \notin U_1$. Thus $(\uparrow \phi, U_1)$ is a disjoint pair and it can be extended to a mdp (D_1, U_1'). Further, we let $U_2 = \{\psi' : \exists \phi'(\phi' \in D_1 \text{ and } \phi' \circ \psi' \in U\}$. We first show that

U_2 is an ideal. It is easy to see that U_2 is closed below. If ψ' and ψ'' are in U_2, then there are ϕ' and ϕ'' in D_1 with $\phi' \circ \psi' \in U$ and $\phi'' \circ \psi'' \in U$. Since D_1 is a filter, there is $\phi''' \in D_1$ with $\phi''' \vdash \phi'$ and $\phi''' \vdash \phi''$ both deducible. But then $\phi''' \circ \psi' \vdash \phi' \circ \psi'$ and $\phi''' \circ \psi'' \vdash \phi'' \circ \psi''$ are also deducible so that $\phi''' \circ \psi'$ and $\phi''' \circ \psi''$ are in U. Now since U is an ideal, there is $\theta \in U$ with $\phi''' \circ \psi' \vdash \theta$ and $\phi''' \circ \psi'' \vdash \theta$ both deducible. Now it is straight forward to show that $\phi''' \to \theta$ is in U_2 and that both $\psi' \vdash (\phi''' \to \theta)$ and $\psi'' \vdash (\phi''' \to \theta)$ are deducible. That is, U_2 is an ideal. Furthermore, notice that $\psi \notin U_2$, so $(\uparrow \psi, U_2)$ is a disjoint pair and can be extended to a mdp (D_2, U_2'). Now we see that $\phi \in D_1$, $\psi \in D_2$, and if $\phi' \in D_1$ and ψ' is such that $\phi' \circ \psi' \in U$, then $\psi' \in U_2$, and thus $\psi' \notin D_2$. That is, $U \not\succ \phi \circ \psi$ as desired.

Now for $D \in X$, we have by definition that $D \Vdash (\phi \circ \psi)$ if and only if

$$\forall U \in Y \ (U \succ \phi \circ \psi \ \Rightarrow \ D \leq U.$$

That is, $D \Vdash (\phi \circ \psi)$ if and only if D is a common lower bound of the set $\{U : U \succ \phi \circ \psi\} = \{U : \phi \circ \psi \in U\}$. But $\{U : \phi \circ \psi \in U\}^l$ is clearly $\{D : \phi \circ \psi \in D\}$. That is, $D \Vdash (\phi \circ \psi)$ if and only if $\phi \circ \psi \in D$.
The corresponding statements for the implications are proved similarly. ■

Lemma 4.52 (Completeness). *Let ϕ and ψ be formulas in $\mathcal{F}(P)$. If the sequent $\phi \vdash \psi$ is valid over the class of all frames then it can be deduced in the basic substructural logic.*

Proof. We show the contrapositive: If $\phi \vdash \psi$ cannot be deduced in the basic logic, then it is not valid over the class of all frames. In particular, we show that it does not hold in the canonical model. Indeed, if $\phi \vdash \psi$ is not deducible in the logic, then there is a mdp (D, U) so that $\phi \in D$ and $\psi \in U$, and thus $\phi \vdash \psi$ does not hold in the canonical model. ■

This completes the proof of the completeness theorem.

In the paper [9] this completeness result was proved algebraically, via the composition of a canonicity result and a discrete duality result. Then it was showed that various other rules/equations were canonical and had first order correspondence. In this manner we derived completeness theorems for the implication-fusion fragment of various substructural logics. These completeness theorems could of course also be proved purely relationally as we have done above with the completeness of the basic logic. However, the methods are not significantly different, so we just state the results here, referring the reader to the algebraic proofs in [9].

We consider the following rules:

$$\frac{\Gamma \ ; \ \Delta \ \vdash \ \psi}{\Gamma \ ; \ \phi \ ; \ \Delta \ \vdash \ \psi}, \tag{T}$$

$$\frac{\Gamma \ ; \ \phi \ ; \ \phi \ ; \ \Delta \ \vdash \ \psi}{\Gamma \ ; \ \phi \ ; \ \Delta \ \vdash \ \psi}, \tag{C}$$

$$\frac{\Gamma \ ; \ \phi \ ; \ \psi \ ; \ \Delta \ \vdash \ \theta}{\Gamma \ ; \ \psi \ ; \ \phi \ ; \ \Delta \ \vdash \ \theta}, \tag{P}$$

$$\frac{(\phi \ ; \ \psi) \ ; \ \theta \ \vdash \ \rho}{\phi \ ; \ (\psi \ ; \ \theta) \ \vdash \ \rho} \quad \text{and} \quad \frac{\phi \ ; \ (\psi \ ; \ \theta) \ \vdash \ \rho}{(\phi \ ; \ \psi) \ ; \ \theta \ \vdash \ \rho} \tag{A}$$

In [9] it was shown that adding each single one of these rules to the basic logic yields a logic complete with respect to the class of frames obtained by imposing each single one of the following first order properties, respectively:

$$\forall x_1, x_2 \forall y \qquad (x_1 \leq y \implies R(x_1, x_2, y)), \tag{T'}$$

$$\forall x \ \forall y \qquad (R(x, x, y) \implies x \leq y), \tag{C'}$$

$$\forall x_1, x_2 \ \forall y \ \ (R(x_1, x_2, y) \iff R(x_2, x_1, y)), \tag{P'}$$

$$\forall x_1, x_2, x_3 \ \forall y$$
$$[\forall x \ (R^{\downarrow}(x_2, x_3, x) \ \Rightarrow \ R(x_1, x, y)) \iff \forall x \ (R^{\downarrow}(x_1, x_2, x) \ \Rightarrow \ R(x, x_3, y))] \tag{A'}$$

In fact, the corresponding correspondence results were also shown. That is, e.g., the algebraic inequality $\alpha \leq \alpha \circ \alpha$ which corresponds to (T) holds in a 'complex algebra', $\mathcal{G}(F)$ if and only if (T') holds in F, and so on.

This of course yields completeness results, in a modular way, for the implication-fusion fragment of various substructural logics. In particular, the Lambek calculus is obtained from the basic logic by adding associativity, and thus it is complete with respect to the class of frames satisfying the first order condition (A'). The implication-fusion fragment of linear logic is obtained from the Lambek calculus by adding the permutation rule, and thus it is complete with respect to the class of frames satisfying both (A') and (P'). The implication-fusion fragment of relevance logic is obtained from linear logic by adding the contraction rule, and thus it is complete with respect to the class of frames satisfying (A'), (P'), and (C'). The implication-fusion fragment of BCK logic is obtained from linear logic by adding the thinning rule, and thus it is complete with respect to the class of frames satisfying (A'), (P'), and (T'). Finally, the implication-fusion fragment of intuitionistic propositional logic is obtained from linear logic by adding both the contraction and the thinning rule, and thus it is complete with respect to the class of frames satisfying (A'), (P'), (C') and (T'). However, all these are of course not needed as contraction and thinning already imply that fusion is infimum and thus both commutative and associative. We just have to have thinning both on the right and on the left. Call these (Tr) and (Tl), respectively. As an example we show:

PROPOSITION 4.53. *The logic obtained by adding (Tl), (Tr) and (C) to the basic logic is the implication-fusion fragment of intuitionistic propositional logic. In particular, bot (A) and (P) hold for this logic.*

PROOF. We show that if (Tl'), (Tr') and (C') hold in a frame $F = (X, Y, \leq, R)$, then $R(x_1, x_2, y)$ if and only if $x_1 \wedge x_2 \leq y$, where the meet is taken in $\mathcal{G}(F)$.

For this purpose, suppose $R(x_1, x_2, y)$ holds and let $x \leq x_1$ and $x \leq x_2$, then from the compatibility of R, we get that $R(x, x, y)$ holds and thus by (C) $x \leq y$. That is,

$$\forall x\ [(x \leq x_1, x \leq x_2) \quad \Rightarrow \quad x \leq y].$$

But this is exactly the meaning of $x_1 \wedge x_2 \leq y$. For the converse, suppose $x_1 \wedge x_2 \leq y$, and consider $R[x_1, x_2, \_]$. By (Tl') and (Tr') we have:

$$\forall y'\ (x_1 \leq y' \quad \Rightarrow \quad y' \in R[x_1, x_2, \_])$$

$$\forall y'\ (x_2 \leq y' \quad \Rightarrow \quad y' \in R[x_1, x_2, \_])$$

Now let $x' \in R[x_1, x_2, \_]^l$, then we have

$$\forall y'\ (x_1 \leq y' \quad \Rightarrow \quad x' \leq y')$$

$$\forall y'\ (x_2 \leq y' \quad \Rightarrow \quad x' \leq y').$$

That is, $x' \leq x_1$ and $x' \leq x_2$. But then $x' \leq x_1 \wedge x_2$ and thus $x' \leq y$. But then $y \in R[x_1, x_2, \_]^{lu} = R[x_1, x_2, \_]$ as desired. ∎

5. Other connectives

So far we have not treated any other connectives than fusion and its two associated implications. However, the methods do not restrict to these. In algebraic terms, a connective can be captured on RS frames if (once the order is dualized in any number of coordinates and/or in the codomain) its canonical extension is residuated.

One way this can happen is if the connective is already residuated in the logic. This is for example the case with $\circ, \rightarrow$, and $\leftarrow$. In [16], Mortgatt discussed adding other residuated binary families obtained by dualizing the order either in the domain or in the codomain thus getting four possible families. For example, the family $\oplus, \oslash, \obslash$ is given by the requirement:

$$\theta \oslash \psi \vdash \phi \quad \iff \quad \theta \vdash \phi \oplus \psi \quad \iff \quad \phi \obslash \theta \vdash \psi.$$

The canonical extension of a connective lives on a perfect lattice, that is in particular, a complete lattice. Thus being residuated (modulo some dualization in the domain) is equivalent to, in any one coordinate, either preserving arbitrary joins or turning arbitrary meets to joins, and being dually residuated (modulo some dualization in the domain) is equivalent to, in any one coordinate, either preserving arbitrary meets or turning arbitrary joins to meets. For unary operations this yields 4 different types of operations, and for binary operations it yields 8 different types.

If meet and join are part of the connective type and one of the 8 types of join/meet preservation/reversal is stipulated for finite joins/meets in the logic, then the infinitary versions hold in the canonical extensions. Thus this is another way that capturable connectives come about. This is for example the case with $\Box$ and $\Diamond$. These are not generally stipulated to be residuated but are rather given to be (finite) meet and join preserving, respectively. The point is though that in the canonical extension they become residuated/ dually residuated.

Finally there is a third entrance to the realm of connectives that can be captured on RS frames, and that is operations for which the corresponding variety of algebras is finitely generated. This has not been explored beyond the distributive setting yet though. We will not discuss this here.

The real problem of course with additional connectives is capturing their interactions. This is of course a much harder problem and already in the classical modal case, where we have only unary non-order determined connectives and the order is Boolean, there are axioms which do not behave well with respect to relational semantics. However, as was seen in [9] some Sahlqvist-like analysis is still available. Nevertheless, in substructural logic, with binary connectives and lots of order reversing connectives things can go wrong very quickly.

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