Dimiter Vakarelov

Nelson's Negation on the Base of Weaker Versions of Intuitionistic Negation

Abstract. Constructive logic with Nelson negation is an extension of the intuitionistic logic with a special type of negation expressing some features of constructive falsity and refutation by counterexample. In this paper we generalize this logic weakening maximally the underlying intuitionistic negation. The resulting system, called *subminimal logic with Nelson negation*, is studied by means of a kind of algebras called *generalized N-lattices*. We show that generalized N-lattices admit representation formalizing the intuitive idea of refutation by means of counterexamples giving in this way a counterexample semantics of the logic in question and some of its natural extensions. Among the extensions which are near to the intuitionistic logic are the *minimal logic with Nelson negation* which is an extension of the Johansson's minimal logic with Nelson negation and its in a sense dual version — the *co-minimal logic with Nelson negation.* Among the extensions near to the classical logic are the well known 3-valued logic of Łukasiewicz, two 12-valued logics and one 48-valued logic. Standard questions for all these logics — decidability, Kripke-style semantics, complete axiomatizability, conservativeness are studied. At the end of the paper extensions based on a new connective of *self-dual conjunction* and an analog of the Łukasiewicz *middle value* **1/2** have also been considered.

Keywords: Nelson negation, subminimal logic, counterexample semantics, many-valued logics.

Introduction

The aim of this paper is to study some generalizations of the constructive logic with strong negation. This logic was suggested by Nelson [13] and independently by Markov [12] and formalized by Vorob'ev [30, 31, 32]. The Vorob'ev's axiomatization is an extension of the intuitionistic propositional logic by a new connective $\sim$, called *strong negation* or *Nelson negation*, satisfying the following axioms:

$(*)\ \sim A \Rightarrow (A \Rightarrow B)$,

$(\sim\Rightarrow)\ \sim (A \Rightarrow B) \Leftrightarrow A \wedge \sim B$,

$(\sim \wedge)\ \sim (A \wedge B) \Leftrightarrow \sim A \vee \sim B$,

Special Issue Negation in Constructive Logic
Edited by **Heinrich Wansing, Sergei Odintsov, and Yaroslav Shramko**

$(\sim \vee)$ $\sim (A \vee B) \Leftrightarrow \sim A \wedge \sim B$,

$(\sim \neg)$ $\sim \neg A \Leftrightarrow A$,

$(\sim\sim)$ $\sim\sim A \Leftrightarrow A$.

Following some tradition in the literature, constructive logic with strong negation will be called in this paper Nelson logic, or simply N-logic.

The name "strong negation" comes from the fact that the formula (S) $\sim A \Rightarrow \neg A$ is a theorem of the N-logic. The adjective "constructive" has more deep reasons contained in Nelson's and Markov's criticism of some non-constructive features of the intuitionistic negation. Namely for the intuitionistic negation $\neg$ the derivability of $\neg(A \wedge B)$ in intuitionistic logic does not imply that at least one of the formulas $\neg A$, $\neg B$ is derivable, while for the constructive negation this is true.

Since in this paper we will consider generalizations of the N-logic in which the formula $\sim A \Rightarrow \neg A$ need not be a theorem, the name "strong negation" is not suitable. The name "constructive negation" also is not suitable, because, as it was pointed out by Kracht [8], the above mentioned feature of constructivity of $\sim A$ need not be true for some extensions of the N-logic. So we prefer to use for $\sim$ the more neutral name of *Nelson negation.*

In this paper we will concentrate our attention on another feature of the Nelson negation, pointed out by Markov that we have in general two different ways to refute a sentence A. One way is by *reductio ad absurdum*, namely refuting A is replaced by proving that A implies absurdum. This role of negation is played both by the intuitionistic negation and by the classical negation. Another way to refute A is to construct a counterexample of A. Obviously one sentence A may have many counterexamples and each of them have to contradict A. For instance, a counterexample of the sentence *"This apple is red"* is for instance *"This apple is green"*, or, *"This apple is yellow"*, etc. It is quite strange that neither in the intuitionistic logic, nor in the classical logic there is a formal means for a refutation by a counter-example. Namely the Nelson negation in the N-logic may play such a role. A similar role for an extension of the classical logic can play a negation, satisfying the Vorob'ev's axioms, added to the axioms of the classical logic. Strangely enough, the obtained in this way logic is equivalent to the well known three-valued logic of Łukasiewicz [9](see for this fact [27, 28]). This new view on the Nelson negation leads to a reading of $\sim A$ as *a counterexample of A*. Having in mind this reading, then the Vorob'ev's axioms have a quite clear meaning: the axiom $(*)$ describes a relation between a given sentence and its counterexample. Namely it says in some sense that $\sim A$ together with A imply everything. This means that the counterexample $\sim A$ of

A contradicts A. The other axioms can be treated as constructive ways (algorithms) of how to construct a counterexample of a compound sentence if we have already constructed counterexamples of its components. That is why we call them *algorithmic axioms*. For instance the axiom ($\sim\Rightarrow$) says that a counterexample of the implication $A \Rightarrow B$ is just a conjunction of A and a counterexample $\sim B$ of B. The axiom ($\sim \vee$) says that a counterexample $\sim (A \vee B)$ of $A \vee B$ can be constructed by a conjunction $\sim A \wedge \sim B$ of counterexamples of A and B. ($\sim\sim$) says that if we want to construct a counterexample of $\sim A$ then simply we have to go back to A. And similarly for the other axioms.

All this leads to consider a formal treating of the above idea as a *counterexample semantics* for the N-logic. This was done in [24, 26, 27]. The counterexample semantics is more stable and can be preserved in one way or another trough some extensions or generalizations of the N-logic. Example of such an extension is for instance the three-valued logic of Łukasiewicz. Another example is an extension of the minimal logic of Johansson with Nelson negation given in [25] and presenting a counterexample semantics for it. A generalization of the counterexample semantics to the so called paraconsistent Nelson's logic has been given by Odintsov [14, 15].

The first formal semantics for the N-logic is algebraic one and has been given by Rasiowa [19] and Bialinicki-Birula and Rasiowa [2] by means of *N-lattices* (N-lattices are studied also in [20] under the name of *quasi-pseudo-Boolean algebras*; other authors [21, 22, 8] called them simply *Nelson algebras*). Let us note that the semantics of the N-logic by means of N-lattices is not intuitive one, even after the topological representation theorem given in [2]. In [24, 27] the author was able to find a construction of a special kind of N-lattices, formalizing the intuitive idea of counterexample. I will call this construction a *counter-example construction*. The completeness theorem for the N-logic with respect to the counterexample semantics was proved in [27] by a representation theorem, stating that every N-lattice can be isomorphically embedded into an N-lattice obtained by the counterexample construction. The counterexample construction had been independently invented also by Fidel [4]. It then has been successfully used as a tool for studying N-logic and its extensions by several authors: Goranko [7], Sendlewski [21, 22], Kraht [8]. Applications to logic programming have been given by Pearce and Wagner [16, 17].

In this paper we will present a maximally possible generalization of the N-logic, for which the counterexample semantics is possible. The generalization will try to satisfy the following requirements:

(I) To preserve the language of the N-logic and to weaker maximally the intuitionistic negation.

(II) To preserve all algorithmic axioms of the Nelson negation, relaxing only the axiom $(*)$.

(III) I and II to be chosen in such a way as to be possible to make a reasonable generalization of the counterexample semantics.

For (I) we decide to drop all axioms of the intuitionistic negation and to add only the (provable in intuitionistic logic) rule of extensionality: $\frac{A \Leftrightarrow B}{\neg A \Leftrightarrow \neg B}$.

For (II) we preserve all axioms for the Nelson negation replacing the axiom $(*)$ with a new equivalent one. First note that the axiom $(*) \sim A \Rightarrow (A \Rightarrow B)$, is equivalent on the base of the intuitionistic logic to each of the following two formulas:

$(\mathsf{S}) \sim A \Rightarrow \neg A$,

$(\sharp) \sim A \Rightarrow (\neg B \Rightarrow \neg A)$.

However the equivalence of the above three formulas is no longer true if the requirement (I) is made. Obviously (S) implies $(\sharp)$ on the base of positive logic. Axiom $(*)$ is too strong, because it identify the Nelson contradiction $A \wedge \sim A$ with the intuitionistic (and classical) contradiction, while we want to relax this. So to realize (II) we decide to replace the axiom $(*)$ with the axiom $(\sharp)$ and to preserve all algorithmic axioms for the Nelson negation.

Fixing in this way (I) and (II) we obtain an axiomatic system, containing two negations: the Nelson negation $\sim$ and the weaker version $\neg$ of the intuitionistic negation. We call this new negation *subminimal negation*. We chose this name because adding only one very simple axiom for $\neg$, $\neg\neg\top$, we obtain the *minimal* negation of Johansson, and so the name "subminimal". Making the above assumption the intuitionistic negation is lost and in order to have it at hand and to make the things more simple, we will assume that the axiomatic system contains the sign $\bot$ of absurdity with the axiom $\bot \Rightarrow A$. For symmetry we also assume that we have the sign $\top$ for the logical truth with the axiom $A \Rightarrow \top$. Having $\bot$ we may define the intuitionistic negation (using the sign $-$ for it) with the standard definition: $-A =_{def} A \Rightarrow \bot$. Preserving in this way $\bot$ and $\top$ in the language, a question arises how to treat in this weaker logic $\sim \bot$ and $\sim \top$? There are many choices: any sentence is a counterexample of $\bot$ and also many sentences can be counterexamples of $\top$, for instance $\bot$. In the Nelson's logic we have $\sim \top \equiv \bot$ and $\neg\bot \equiv \top$. So in order not to make the things complicate we also assume the following two axioms for $\bot$ and $\top$:

$(\sim \top) \qquad \sim \top \Rightarrow \bot \quad$ and $(\sim \bot) \qquad \sim \bot$.

The above two axioms are in a sense algorithmic, because they fix some counterexamples of $\bot$ and $\top$, do not concern the negation $\neg$, so their choice is in accordance with the assumptions (I) and (II).

We call the new logic *subminimal logic with Nelson negation* and denote it by $\mathbf{SUBMIN^N}$. Note that now (S) is no more a theorem and hence $\sim A$ is no more a "strong negation".

The main aim of the paper is to show that on the base of the assumptions we have just made for (I) and (II) the requirement (III) can be realized. We study also some extensions of the subminimal logic with Nelson negation: the minimal logic with Nelson negation — $\mathbf{MIN^N}$, the co-minimal logic with Nelson negation — $\mathbf{CO-MIN^N}$, the join of these two, which is just the Nelson's logic — $\mathbf{INT^N}$, and also the classical versions of all these logics, having classical implication — $\mathbf{Class.SUBMIN^N}$, $\mathbf{Class.MIN^N}$, $\mathbf{Class.INT^N} = \mathbf{Class^N}$. It is notable that all these classical versions of the logics with Nelson negation are finitely valued logics: the classical version of the Nelson's logic, $\mathbf{Class^N}$, is the well known 3-valued logic of Łukasiewicz, $\mathbf{Class.MIN^N}$ and $\mathbf{Class.CO-MIN^N}$ are 12-valued logics, and the logic $\mathbf{Class.SUBMIN^N}$ is a 48-valued logic. For all introduced logical systems an algebraic semantics is given, based on generalized N-lattices and a generalization of counterexample construction for these lattices is given. It is shown also how this construction can be rephrased to obtain Kripke style semantics for the logics in consideration. The main result of the paper is a class of completeness theorems with respect to all of the introduced semantics and an application of these theorems to obtain the decidability of the introduced logics. In order to realize this program a special attention is given to the sublogics of the introduced systems in a language without the sign of the Nelson negation. In the final section we show that the counterexample semantics is also meaningful for some extensions of the language with some new connectives, which are interesting only in the presence of the Nelson negation. One of them is the self-dual conjunction $A \bullet B$ which satisfies the axioms $A \bullet B \Leftrightarrow A \wedge B$ and $\sim (A \bullet B) \Leftrightarrow \sim A \wedge \sim B$ from which self-duality with respect to $\sim$ easily follows. The first axiom says that with respect to the equivalence $\Leftrightarrow$ $A \bullet B$ is just the ordinary conjunction while the second axiom can be understand as a new way of forming a counterexample of conjunction.

Acknowledgements. The cooperation for this work was partially supported by EU COST action 274 "Theory and Applications of Relational Structures as Knowledge Instruments" (TARSKI). The author was also supported by the RILA project sponsored by the Bulgarian Ministry of Science and Education. Thanks are due to Heinrich Wansing and to the organizers of the workshop "Negation in constructive logic", in which the results of this paper were presented. The author thanks both referees for their valuable and detailed remarks improving the quality of the text.

1. Nelson logic, N-lattices and counterexample construction

In this section we remind some definitions and facts concerning Nelson logic. We will denote this logic by $\mathbf{INT^N}$. The upper index $\mathbf{N}$ in $\mathbf{INT}$ indicates that the intuitionistic logic is extended with the Nelson negation.

The Nelson logic.

As we have already mentioned, the language of Nelson logic is an extension of the language $L(\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow)$ of the intuitionistic propositional logic with the sign of Nelson negation $\sim$. The axiomatization of this logic can be obtained from an axiomatization of the intuitionistic logic by adding the Vorob'ev's axioms $(*)$, $(\sim\vee)$, $(\sim\wedge)$, $(\sim\Rightarrow)$, $(\sim\neg)$, $(\sim\sim)$.

Note that for the N-logic the rule of replacement of equivalents is not true just because the rule $\frac{A\Leftrightarrow B}{\sim A\Leftrightarrow\sim B}$ is not true. The failure of this rule can be expected if we interpret $\sim A$ as a counterexample of A: one sentence may have in general many non-equivalent counterexamples.

Now we will introduce an algebraic semantics for the N-logic.

N-lattices.

Definition 1.1. [19] An algebraic system $N = (N, \leq, 0, 1, \wedge, \vee, \Rightarrow, \neg, \sim)$, is said to be an N-lattice if the following conditions are satisfied:

R1 Let $a \vdash b$ iff $a \Rightarrow b = 1$ and $a \equiv b$ iff $a \vdash b$ and $b \vdash a$. Then $\vdash$ is a quasi-ordering in N and consequently $\equiv$ is an equivalence relation in N.

R2 $a \leq b$ iff $a \vdash b$ and $\sim b \vdash \sim a$,

R3 $x \wedge a \vdash b$ iff $x \vdash a \Rightarrow b$,

R4 $a \vdash c$ and $b \vdash c$ iff $a \vee b \vdash c$,

R5 $c \vdash a$ and $c \vdash b$ iff $c \vdash a \wedge b$,

R6 $\neg a = a \Rightarrow 0$,

R7 $(N, \leq, 0, 1, \vee, \wedge, \sim)$ is a quasi-Boolean algebra, i.e.:

$\sim (a \vee b) = \sim a \wedge \sim b$,

$\sim (a \wedge b) = \sim a \wedge \sim b$,

$\sim\sim a = a$,

R8 $\sim (a \Rightarrow b) \equiv a \wedge \sim b$,

R9 $\sim \neg a \equiv a$,

R10 $\sim a \vdash \neg a$.

DEFINITION 1.2 ("Counterexample" construction [24, 27]).

Let $(B, 0, 1, \leq, \wedge, \vee, \Rightarrow)$ be a Heyting algebra. Let $N(B) = \{(a, b) \in B^2 : a \wedge b = 0\} = \{(a, b) \in B^2 : \neg(a \wedge b) = 1\}$. Define the following operations in $N(B)$:

$0 = (0, 1)$, $1 = (1, 0)$,
$(a_1, a_2) \wedge (b_1, b_2) = (a_1 \wedge b_1, a_2 \vee b_2)$,
$(a_1, a_2) \vee (b_1, b_2) = (a_1 \vee b_1, a_2 \wedge b_2)$,
$(a_1, a_2) \Rightarrow (b_1, b_2) = (a_1 \Rightarrow b_1, a_1 \wedge b_2)$,
$\neg(a_1, a_2) = (\neg a_1, a_1)$, $\sim (a_1, a_2) = (a_2, a_1)$.

The "counterexample construction" has the following intuitive explanation. The elements of the pseudo-Boolean algebra can be considered as sentences and their operations as logical operations on sentences. The condition $a \wedge b = 0$, or equivalently $\neg(a \wedge b) = 1$, is interpreted as the counterexample relation: "b is a counterexample of a". Since one sentence may have many (in general non-equivalent) counterexamples, then each pair (a, b) with $a \wedge b = 0$ carries both the sentence a with a given counterexample b. Then the operations on the pairs can be considered as algorithms for constructing counterexamples for complex sentences by means of given counterexamples of the arguments.

THEOREM 1.3 ([27] Representation theorem for N-lattices).

(i) The set $N(B)$ with the above defined operations is an N-lattice, called **special N-lattice**.
(ii) Each N-lattice can be isomorphically embedded into a special N-lattice.

THEOREM 1.4 ([27] Completeness theorem for the Nelson logic).

The following conditions are equivalent for very formula A of $\mathbf{INT^N}$:

(i) A is a theorem of $\mathbf{INT^N}$,

(ii) A is true in all N-lattices,

(iii) A is true in all special N-lattices.

2. Subminimal logic with Nelson negation

The logic $\mathbf{SUBMIN^N}$.

In this section we will introduce with details the subminimal logic with Nelson negation, discussed briefly in the introduction. We denote it by $\mathbf{SUBMIN^N}$.

The language of the subminimal logic with Nelson negation is an extension of the language of the Nelson logic with the symbols $\bot$ and $\top$.

We adopt the following axiomatics for $\mathbf{SUBMIN^N}$.

(I) Axioms for positive logic + $\bot$, $\top$:

P1 $A \Rightarrow (B \Rightarrow A)$,

P2 $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$,

P3 $A \wedge B \Rightarrow A$,

P4 $A \wedge B \Rightarrow B$,

P5 $(C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow A \wedge B))$,

P6 $A \Rightarrow A \vee B$,

P7 $B \Rightarrow A \vee B$,

P8 $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \vee B \Rightarrow C))$,

P9 $A \Rightarrow \top$,

P10 $\bot \Rightarrow A$.

(II) Axioms for the Nelson negation.

$(\sharp)$ $\sim A \Rightarrow (\neg B \Rightarrow \neg A)$,

$(\sim\Rightarrow)$ $\sim (A \Rightarrow B) \Leftrightarrow A \wedge \sim B$,

$(\sim \wedge)$ $\sim (A \wedge B) \Leftrightarrow \sim A \vee \sim B$,

$(\sim \vee)$ $\sim (A \vee B) \Leftrightarrow \sim A \wedge \sim B$,

$(\sim \neg)$ $\sim \neg A \Leftrightarrow A$,

$(\sim\sim)$ $\sim\sim A \Leftrightarrow A$,

$(\sim \top)$ $\sim \top \Rightarrow \bot$,

$(\sim \bot)$ $\sim \bot$.

(III) Rules of inference.

Modus Ponens: $\frac{A, A \Rightarrow B}{B}$,

The **rule of extensionality** for the subminimal negation: **(Ext-$\neg$)** $\frac{A \Leftrightarrow B}{\neg A \Leftrightarrow \neg B}$.

The above axiomatic system is identified with the logic $\mathbf{SUBMIN^N}$. Note that the rule $\frac{A \Leftrightarrow B}{\sim A \Leftrightarrow \sim B}$ is not true for this logic and consequently the rule of replacement of equivalents does not hold in general. It however holds if the replacements of equivalents is not in the scope of the sign $\sim$.

The presented axiom system for $\mathbf{SUBMIN^N}$ is not very suitable, because the axiomatization is not separated in a sense that we have no special group of axioms for the subminimal negation from which one can derive all theorems for it not containing the Nelson negation. Moreover it contains one more rule (Ext-$\neg$), which for the Nelson logic is derivable. The purpose of the given axiomatization is just to show that all information about the subminimal negation is contained in the group of axioms of the Nelson negation. We will give separated axiomatization of $\mathbf{SUBMIN^N}$ in which the rule (Ext-$\neg$) is replaced by some axioms. For that purpose we will derive some theorems for $\mathbf{SUBMIN^N}$. In order to simplify the proofs we will use the notations:

- $A \vdash B$ iff $A \Rightarrow B$ is a theorem of $\mathbf{SUBMIN^N}$.
- $A \equiv B$ iff $A \vdash B$ and $B \vdash A$.

Obviously $\vdash$ is a reflexive and transitive relation between formulas and $\equiv$ is an equivalence relation. Later on we will use these notations for different logics. Also in the proofs we will use some obvious intuitionistic calculations and replacement of equivalents if the replacement is not in the scope of the Nelson negation.

Proposition 2.1. *The following conditions are provable in* $\mathbf{SUBMIN^N}$*:*

(i) $A \wedge \neg B \vdash \neg(A \Rightarrow B)$,

(ii) $A \wedge \neg A \vdash \neg\top$, *and* $\neg A \vdash A \Rightarrow \neg\top$,

(iii) $\neg B \wedge A \vdash \neg \sim A$,

(iv) $\neg \sim (A \Rightarrow \neg B) \equiv \neg(A \wedge B)$,

(v) $\neg C \wedge (A \Rightarrow \neg B) \vdash \neg(A \wedge B)$,

(vi) $\neg A \vdash \neg\neg\top$,

(vii) $\neg\neg\top \wedge (A \Rightarrow \neg\top) \vdash \neg A$,

(viii) $\neg A \equiv (A \Rightarrow \neg\top) \wedge \neg\neg\top$,

(ix) $\neg\bot \equiv \neg\neg\top$,

(x) $\neg A \equiv (A \Rightarrow \neg\top) \wedge \neg\bot$,

(xi) $A \Rightarrow B \vdash \neg B \Rightarrow \neg A$,

(xii) $\neg C \vdash \neg(A \wedge \sim A)$.

PROOF. **(i)** $A \wedge \neg B \equiv$ by $(\sim\sim)$ $\neg B \wedge (A \wedge \sim\sim \neg B) \equiv$ by $(\sim\Rightarrow)$ $\neg B \wedge \sim (A \Rightarrow\sim \neg B) \vdash$(by $(\sharp)$) $\neg(A \Rightarrow\sim \neg B) \equiv$ by $(\sim \neg)$ $\neg(A \Rightarrow B)$.

(ii) By (i): $A \wedge \neg A \vdash \neg(A \Rightarrow A) \equiv \neg\top$. From here: $\neg A \vdash A \Rightarrow \neg\top$.

(iii) By $(\sim\sim)$ and $(\sharp)$: $\neg B \wedge A \equiv \neg B \wedge \sim\sim A \vdash \neg \sim A$.

(iv) By $(\sim\Rightarrow)$ and $(\sim \neg)$: $\sim (A \Rightarrow\sim \neg B) \equiv A \wedge \sim \neg B \equiv A \wedge B$. Then by the rule of extensionality for $\neg$ we obtain: $\neg \sim (A \Rightarrow \neg B) \equiv \neg(A \wedge B)$.

(v) By (iii) and (iv): $\neg C \wedge (A \Rightarrow \neg B) \vdash \neg \sim (A \Rightarrow \neg B) \equiv \neg(A \wedge B)$.

(vi) By (v) we have: $\neg A \wedge (\neg\top \Rightarrow \neg\top) \vdash \neg(\neg\top \wedge \top)$. After simplification we get $\neg A \vdash \neg\neg\top$.

(vii) By (v): $\neg\neg\top \wedge (A \Rightarrow \neg\top) \vdash \neg(A \wedge \top) \equiv \neg A$.

(viii) By (ii) and (vi) we get $\neg A \vdash (A \Rightarrow \neg\top) \wedge \neg\neg\top$. Then by (vii) we obtain (viii).

(ix) By (viii) $\neg\bot \equiv (\bot \Rightarrow \neg\top) \wedge \neg\neg\top$. After simplification we obtain (ix)

(x) By direct application of (viii) and (ix).

(xi) By the positive logic and (ii) we obtain:

$(A \Rightarrow B) \wedge A \wedge \neg B \vdash B \wedge \neg B \vdash \neg\top$. From here we get

(1) $(A \Rightarrow B) \wedge \neg B \vdash A \Rightarrow \neg\top$.
By (vi) we obtain
(2) $(A \Rightarrow B) \wedge \neg B \vdash \neg\neg\top$. Then from (1) and (2) we get
(3) $(A \Rightarrow B) \wedge \neg B \vdash (A \Rightarrow \neg\top) \wedge \neg\neg\top$. Then (3) and (viii) imply
(4) $(A \Rightarrow B) \wedge \neg B \vdash \neg A$. From (4) we obtain (xi).

(xii) By $(\sharp)$ and (ii): $\neg C \wedge \sim A \vdash \neg A \vdash A \Rightarrow \neg\top$. From here we have:

$\neg C \vdash \sim A \Rightarrow (A \Rightarrow \neg\top) \equiv \sim A \wedge A \Rightarrow \neg\top$ and by (vi) we obtain

$\neg C \vdash (\sim A \wedge A \Rightarrow \neg\top) \wedge \neg\neg\top$. Then applying (viii) we obtain (xii).

■

Separated axiomatization for SUBMIN$^{\mathbf{N}}$.

Now we can simplify the axiomatization of **SUBMIN$^{\mathbf{N}}$** replacing the rule of extensionality for the subminimal negation: **(Ext-$\neg$)** $\frac{A \Leftrightarrow B}{\neg A \Leftrightarrow \neg B}$ by the following two axioms ((xi) and (vi) from proposition 2.1):

P11 $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$,

P12 $\neg A \Rightarrow \neg\neg\top$, Axiom P11 is the well known law of contraposition. It implies the rule of contraposition $\frac{A \Rightarrow B}{\neg B \Rightarrow \neg A}$, which in turn implies the rule of extensionality for $\neg$. Axiom P12 is superfluous, because it follows from the axioms for the Nelson negation, but it is added in order to obtain a separated set of axioms for the subminimal negation in a sense that all theorems for $\neg$ not containing $\sim$ can be proved without using the axioms for $\sim$. This fact will be proved later on. It is possible to make P12 independent if we replace the axiom ($\sharp$) by the following formula:

($\heartsuit$) $A \wedge \sim A \Rightarrow (\neg\neg\top \Rightarrow \neg\top)$

Then ($\heartsuit$) is still deductively equivalent to ($\sharp$) but now P12 is independent.

Extensions of SUBMIN$^{\mathbf{N}}$.

We will consider extensions of **SUBMIN$^{\mathbf{N}}$** with some of the following formulas as candidates for additional axioms:

(Nor) $\neg\neg\top$ (or $\neg\bot$) — **normality axiom**,

(Nor^*) $\neg\top \Rightarrow A$, — **co-normality axiom**,

(Class) $(A \Rightarrow B) \vee A$ — **classical implication**.

We will use notations for the introduced extensions in the form $\mathbf{L}^{\mathbf{N}}$ where **L** is a logic in the language of intuitionistic logic (in the form of positive logic + the signs of and $\top$ and $\bot$) and the superscript N will denote that **L** is extended with the axioms of Nelson negation. Names for the different **L** will be given in the next section.

- **MIN$^{\mathbf{N}}$** = **SUBMIN$^{\mathbf{N}}$** + **Nor**,
- **Class**.**SUBMIN$^{\mathbf{N}}$** = **SUBMIN$^{\mathbf{N}}$** + **Class**,
- **Class**.**MIN$^{\mathbf{N}}$** = **MIN$^{\mathbf{N}}$** + **Class**,
- **Class**.**CO** – **MIN$^{\mathbf{N}}$** = **SUBMIN$^{\mathbf{N}}$** + **Class**,

In order to obtain a semantics for the logic **SUBMIN$^{\mathbf{N}}$** and some of its natural extensions we have to study in more details the sublogics **L** of the introduced above logics,

3. Intuitionistic logic with subminimal negation and its extensions

The logic SUBMIN.

In this section we will study the subsystem of **SUBMIN$^{\mathbf{N}}$** which does not contain the sign of Nelson negation. The new system is based on the axioms P1-P12 and Modus Ponens. It will be called intuitionistic logic with subminimal negation, and will be denoted by **SUBMIN**.

We consider the following extensions of **SUBMIN**:

- Intuitionistic logic with minimal negation: $\mathbf{MIN} = \mathbf{SUBMIN} + \mathbf{Nor}$.
- The co-minimal logic: $\mathbf{CO-MIN} = \mathbf{SUBMIN} + \mathbf{Nor}^*$.

  It can be shown that we have the following equality:

  $\mathbf{INT} = \mathbf{SUBMIN} + \mathbf{Nor} + \mathbf{Nor}^*$.

  When we add the axiom **Class** to some logic we always have that the underlying logic is the classical logic with definable classical negation $-A =_{def} A \Rightarrow \bot$.

- The classical logic with subminimal negation:

  $\mathbf{Class}.\mathbf{SUBMIN} = \mathbf{SUBMIN} + \mathbf{Class}$.

- The classical logic with minimal negation:

  $\mathbf{Class}.\mathbf{MIN} = \mathbf{MIN} + \mathbf{Class}$.

- The classical logic with co-minimal negation:

  $\mathbf{Class}.\mathbf{CO-MIN} = \mathbf{CO-MIN} + \mathbf{Class}$.

REMARKS 3.1. **(i)** *The logic* **SUBMIN** *and some of its important extensions was introduced by the author for the first time in [26] (see also [29]). Let us note, as one of the referees pointed out, that the name "subminimal logic" was used also in [3] for a similar system.*

(ii) *Most of the results of this section, sometimes in a modified form or based on a different terminology, were obtained by the author in [26, 29]. So proofs will be given only for the new things.*

(iii) *If we drop the sign of $\bot$ from the logic* **MIN** *we obtain exactly the minimal logic of Johansson.*

Algebraic semantics for the logic SUBMIN and some of its extensions.

Now we will introduce an algebraic semantics for the logic **SUBMIN**.

DEFINITION 3.2. A system $(A, \leq, 0, 1, \wedge, \vee, \Rightarrow, \neg)$ is called a subminimal algebra if it satisfies the following axioms:

(Int) The system $(A, \leq, 0, 1, \wedge, \vee, \Rightarrow)$ is a Heyting algebra.

(Submin1) $(a \Rightarrow b) \wedge \neg b \leq \neg a$,

(Submin2) $\neg 0 = \neg\neg 1$.

A subminimal algebra is called:

- a **minimal algebra** if it satisfies the axiom (Min) $\neg 0 = 1$,
- a **co-minimal algebra** if it satisfies the axiom (Co-min) $\neg 1 = 0$.
- a **classical subminimal algebra** if it satisfies the axiom (Class) $a \vee (a \Rightarrow b)$.

Proposition 3.3 (Characterization theorem for subminimal, minimal and co-minimal algebras).

Let $(A, \leq, 1, \wedge, \vee, \Rightarrow)$ be a positive algebra, and $\neg$ be an unary operation in A. Then:

(i) *A is a subminimal algebra iff there exist two fixed elements $\mathbf{p}, \mathbf{q}$ in A with the properties:*

(i1) $\mathbf{p} \leq \mathbf{q}$,

(i2) $\neg a = (a \Rightarrow \mathbf{p}) \wedge \mathbf{q}$, $\neg 1 = \mathbf{p}$, $\neg 0 = \mathbf{q}$.

(ii) *A is a minimal algebra iff for the element $\mathbf{q}$ from (i) we have $\mathbf{q} = 1$,*

(iii) *A is a co-minimal algebra iff for the element $\mathbf{p}$ from (i) we have $\mathbf{p} = 0$,*

(iv) *A is both minimal and co-minimal algebra iff A is a Heyting algebra (in a sense that $\neg a = a \Rightarrow 0$).*

Proof. (i) ($\leftarrow$) — by a straightforward verification.

($\rightarrow$) Put $\mathbf{p} = \neg 1$ and $\mathbf{q} = \neg 0$. Then by the axiom (Submin1) we have: $(a \Rightarrow \neg 1) \wedge \neg\neg 1 \leq \neg a$ and hence $(a \Rightarrow \neg 1) \wedge \neg 0 \leq \neg a$. For the converse inclusion we have: $a \wedge \neg a = (1 \Rightarrow a) \wedge \neg a \leq \neg 1$. Then from here we get $\neg a \leq (a \Rightarrow \neg 1)$ and by axiom (Submin1) we obtain $\neg a \leq (a \Rightarrow \neg 1) \wedge \neg 0$. Thus we have $\neg a = (a \Rightarrow \neg 1) \wedge \neg 0$. From this equality we obtain that $\neg 1 \leq \neg\neg 1$.

The proofs for (ii) and (iii) are similar to the proof of (i).

For (iv) suppose that A is both a minimal and a co-minimal algebra. Then it satisfies $\mathbf{q} = 1$ and $\mathbf{p} = 0$ and consequently $\neg a = a \Rightarrow 0$ which shows that A is a Heyting algebra. The converse is obvious. ■

Kripke semantics for the logic SUBMIN and some of its extensions.

Definition 3.4. By a **subminimal frame** we will consider any relational structure $W = (W, \leq, N, N^*)$ where W is a non-empty set of possible worlds, and the subsets $N, N^* \subseteq W$, with the following names: N — the set of normal worlds, and N^* — the set of co-normal worlds, or "bad normal worlds". We assume that W satisfies the following conditions:

(S1) $\leq$ is a reflexive and transitive relation (quasi-order) in W,

(S2) $N^* \subseteq N$,

(S3) N and N^* are upwards monotone with respect to $\leq$ (a subset $A \subseteq W$ is upwards monotone iff $(\forall x, y \in W)(x \in A$ and $x \leq y \rightarrow y \in A))$.

The frame $(W, \leq, N, N^*)$ is called:

- a **minimal** frame if $N = W$,
- a **co-minimal** frame if $N^* = \varnothing$,
- a **classical** subminimal frame if "$\leq$" is the identity relation "$=$".

It is easy to see that if we take $N = W$ and $N^* = \varnothing$ then we obtain just the frames for the intuitionistic logic.

The Kripke semantics of the language of **SUBMIN** in the class of subminimal frames is like the semantics of intuitionistic logic in frames of the form $(W, \leq)$, i.e. we consider all upwards monotone valuations v $((\forall x, y \in W)(x \in v(p), x \leq y \rightarrow y \in v(p)))$ and define the satisfaction relation $x \Vdash_v A$ inductively for all formulas in the standard way:

- $x \Vdash_v p$ iff $x \in v(p)$ if p is a propositional letter,
- $x \nVdash_v \bot$, $x \Vdash \top$,
- $x \Vdash_v A \wedge B$ iff $x \Vdash_v A$ and $x \Vdash_v B$,
- $x \Vdash_v A \vee B$ iff $x \Vdash_v A$ or $x \Vdash_v B$,
- $x \Vdash_v A \Rightarrow B$ iff $(\forall y \in W)(x \leq y$ and $y \Vdash_v A \rightarrow y \Vdash_v B)$,
- $x \Vdash_v \neg A$ iff $(\forall y \in W)(x \leq y \rightarrow y \in N^*)$ and $x \in N$.

As we have seen above, the subsets N and N^* are used just for the semantics of the subminimal negation. The worlds from N are called normal by the analogy with the semantics of non-normal modal logics. It is easy to see that:

$x \Vdash_v \neg\top$ iff $x \in N^*$, $x \Vdash \neg\bot$ iff $x \in N$.

From here we can see that the co-normal worlds, i.e. the worlds from the set N^*, are just the worlds in which $\neg\top$ is true, something which is not desirable. That is why we call these worlds "bad normal worlds".

Obtaining subminimal algebras from subminimal Kripke frames.

Definition 3.5. Let $W = (W, \leq, N, N^*)$ be a subminimal Kripke frame. Let $A(W)$ be the set of all upwards monotone subsets of W. Define a lattice in $A(W)$ taking the set theoretical operations of intersection and union as lattice operations, $1 = W$, $0 = \varnothing$ and define implication and negation as in the semantics: $A \Rightarrow B = \{x \in W : (\forall y \in W)(x \leq y \text{ and } y \in A \rightarrow y \in B)\}$, $\neg A = \{x \in N : (\forall y \in W)(x \leq y \rightarrow y \in N^*)\}$.

The following lemma can easily be proved.

LEMMA 3.6. **(i)** *The algebra $A(W)$ with the operations defined in Definition 3.5 is a subminimal algebra, called* **subminimal set-algebra** *over the frame W.*

(ii) *If the frame is minimal ($N = W$) then $A(W)$ is a minimal algebra.*

(iii) *If the frame is co-minimal ($N^* = \varnothing$) then $A(W)$ is a co-minimal algebra.*

(iv) *If the frame is classical ($\leq$ is "$=$") then the algebra $A(W)$ is a classical algebra.*

As in the case of Heyting algebras one can prove the following theorem.

THEOREM 3.7 (Representation theorem for subminimal, minimal and classical algebras). *Each subminimal algebra A can be isomorphically embedded into a subminimal set-algebra over a subminimal frame $W(A)$. If A is a minimal (co-minimal, classical) then the frame $W(A)$ can be chosen to be minimal (co-minimal, classical) and hence the algebra $A(W(A))$ to be minimal (co-minimal, classical).*

PROOF. (Idea) Define $W(A)$ to be the set of all prime filters of A with $\Gamma \leq \Delta$ iff $\Gamma \subseteq \Delta$. Define $N(A) = \{\Gamma \in W(A) : \neg 0 \in \Gamma\}$ and $N^*(A) = \{\Gamma \in W(A) : \neg 1 \in \Gamma\}$. Then proceed as in the well known representation theorem for Heyting algebras over quasi-ordered sets. ■

THEOREM 3.8 (Completeness Theorem for the logics **SUBMIN**, **MIN**, **CO − MIN** and their classical extensions). *The logics* **SUBMIN**, **MIN**, **CO − MIN** *and their classical extensions are sound and strongly complete in the corresponding algebraic and Kripke semantics.*

PROOF. Soundness and the completeness with respect to the algebraic semantics is easy. The completeness with respect to Kripke semantics in a slightly different language is given in [29] via the canonical construction, which is an adaptation of the corresponding construction for the intuitionistic logic. Strong completeness can be obtained by the same method. Alternatively, the representation theorem 3.7 can also be used for the Kripke completeness. ■

A more strong completeness theorem for the logics **Class.SUBMIN**, **Class.MIN**, **Class.CO – MIN** with respect to single finite matrices will be given at the end of this section.

The following proposition can be proved either directly using the axiomatics or just by using the completeness theorem.

PROPOSITION 3.9. *The subminimal negation has the following properties:*

(i) $\neg A \equiv (A \Rightarrow \neg\top) \wedge \neg\neg\top$,

(ii) $\neg\bot \equiv \neg\neg\top$, $\neg\neg\neg A \equiv \neg A$, $\neg\neg\bot \equiv \neg\top$,

(iii) $\neg A \equiv (A \Rightarrow \neg\top) \wedge \neg\bot$,

(iv) $\neg C \vdash A \Rightarrow \neg\neg A$,

(v) $\neg C \vdash \neg(A \wedge \neg A)$,

(vi) $A \wedge \neg(A \wedge B) \vdash \neg B$,

(vii) $\neg(A \vee B) \equiv \neg A \wedge \neg B$,

(viii) $\bot \vdash \neg\top \vdash \neg\neg\top \vdash \top$,

(ix) $A \wedge \neg A \Rightarrow \neg B$.

Now we will establish a special property of the logic **SUBMIN**. Let us look at the chain $\bot \vdash \neg\top \vdash \neg\neg\top \vdash \top$ from Proposition 3.9. It is easy to see by using the completeness theorem for **SUBMIN** that the formulas $\bot, \neg\top, \neg\neg\top, \top$ are not equivalent. So we may assume that they represent in the syntax some truth-value constants for which we will use the following names:

- $\bot$ — logical falsity (absurdity).
- $\neg\top$ — weak logical falsity,
- $\neg\bot$ — strong logical truth,
- $\top$ — logical truth.

Adopting this terminology we may say that:

- A is logically false if $A \equiv \bot$,
- A is weakly logically false if $A \equiv \neg\top$,
- A is strongly logically true if A$\equiv \neg\neg\top \equiv \neg\bot$,
- A is logically true if $A \equiv \top$.

According to this terminology the following simple lemma for **SUBMIN** can be stated.

LEMMA 3.10. *There is no formula A in* **SUBMIN** *such that $\neg A$ is logically true.*

PROOF. Suppose that such a formula exists. Then $\neg A$ is a theorem of **SUBMIN**. Then by Modus Ponens and axiom P11 we obtain that $\neg\neg\top$ is a theorem for **SUBMIN**, which by the completeness theorem is not true — just take a subminimal frame in which $N \neq W$. ■

Decidability of the logics SUBMIN, MIN and CO – MIN.

The decidability of the logics **SUBMIN**, **MIN** and **CO – MIN** can be obtained by proving that they have finite model property by adapting the filtration of the intuitionistic case. Another way for proving decidability is by interpetting them in **INT**. For **SUBMIN** the idea is the following. It follows by proposition 3.9 that each formula of **SUBMIN** is equivalent to a formula in which the negation $\neg$ appears only in subformulas of the form $\neg\top$ and $\neg\bot$ called a negation normal form. Then define a translation of the formulas from the language of **SUBMIN** into the language of the intuitionistic logic in the following way. Let p, q be two propositional variables not contained in the formula A. Then replace all occurrences of $\neg\top$ by $p \wedge q$ and all occurrences of $\neg\bot$ by p and denote the obtained formula by $\tau(A)$.

The following theorem is true:

THEOREM 3.11 (Translation theorem for **SUBMIN**). *For any formula A of* **SUBMIN***: A is a theorem of* **SUBMIN** *iff $\tau(A)$ is a theorem in the intuitionistic logic.*

PROOF. ($\leftarrow$). Suppose $\tau(A)$ is a theorem of the intuitionistic logic. Since it is a part of **SUBMIN** then $\tau(A)$ is a theorem of **SUBMIN**. Then substituting p with $\neg\top$ and q with $\neg\bot$ and using the fact that $\neg\top \wedge \neg\bot$ is equivalent in **SUBMIN** to $\neg\top$ we obtain that the obtained after this substitution formula is equivalent to A and hence A is a theorem of **SUBMIN**.

($\rightarrow$). We will reason by contraposition. Suppose that $\tau(A)$ is not a theorem of intuitionistic logic. Then by the Kripke semantics of the intuitionistic logic there is a a model $(W, \leq, v)$ falsifying A. Let $N = \{x \in W : x \Vdash_v p\}$ and $N^* = \{x \in W : x \Vdash_v p \wedge q\}$. Then obviously $(W, \leq, N, N^*, v)$ is a subminimal model which falsifies the negation normal form of A and hence A is not a theorem of **SUBMIN**. ■

COROLLARY 3.12. *The logic* **SUBMIN** *is decidable.*

Similar translations can be defined for the logics **MIN** and **CO − MIN**. Since $\neg\neg\top$ (or equivalently $\neg\bot$) is a theorem for **MIN** the negation normal form for **MIN** will contain only subformulas of the form $\neg\top$. Then $\tau(A)$ is obtained from the negational normal form of A by replacing in it all subformulas of the form $\neg\top$ by a fresh variable p. For **CO − MIN** the negation normal form contains only subformulas of the form $\neg\bot$ and we replace such subformulas by a fresh variable p. Then for both logics an analog of the translation theorem holds, which implies the decidability of the both logics. Similar decidability result, based on the algebraic semantics for the minimal logic of Johansson is contained in [20], which inspired the above translations.

Let us note that the described translations can be used to obtain the decidability of the classical extensions of the above logics: **Class**.**SUBMIN**, **Class**.**MIN** and **Class**.**COMIN**. Then the translation is in the classical logic.

Another way to obtain decidability of **Class**.**SUBMIN**, **Class**.**MIN** and **Class**.**CO − MIN** is to show that all these logics have finite characteristic matrices. This fact will be used later to obtain finite characteristic matrices for the extensions of these logics with Nelson negation.

Existence of finite characteristic matrices for the logics **Class**.**SUBMIN**, **Class**.**MIN** and **Class**.**COMIN** and the corresponding completeness theorem for them follow from the following result from [23].

Let $\mathcal{L}$ be a logic based on an extension of the language of classical logic with some n-place logical connective F. $\mathcal{L}$ is called strongly extensional if the following formula is a theorem of $\mathcal{L}$:

(SExt) $(A_1 \Leftrightarrow B_1) \wedge \ldots \wedge (A_n \Leftrightarrow B_n) \Rightarrow (F(A_1, \ldots, A_n) \Leftrightarrow F(B_1, \ldots, B_n))$.

It is proved in [23] that every strongly extensional logic $\mathcal{L}$ has a finite Boolean logical matrix with distinguished element 1 in which $\mathcal{L}$ is sound and weakly complete. Obviously the logics in question are strongly extensional. Here we shall give another completeness proof, giving strong completeness. This proof then will be extended for completeness proofs for some logics with Nelson negation. First let us identify the logical matrices of the corresponding logics.

Finite matrix for the logic Class.SUBMIN.

Let $(B, 0, 1, \wedge, \vee, \Rightarrow, \neg)$ be a classical subminimal algebra in which $0 < \neg 1 < \neg 0 < 1$ and let $\mathbf{p} = \neg 1$ and $\mathbf{q} = \neg 0$. Note that $(B, 0, 1, \wedge, \vee, -)$ with $-a = a \Rightarrow 0$ is a Boolean algebra. It is clear that the set $\{\mathbf{p}, \mathbf{q}\}$ generates an **8**-element Boolean subalgebra of B, denoted by **B8**, which is closed under the operation $\neg$ and consequently is a subminimal subalgebra of B. Note that **B8** is a subalgebra of every classical subminimal algebra with $0 < \neg 1 < \neg 0 < 1$. Namely the subminimal algebra **B8** is the required characteristic logical matrix for the logic **Class.SUBMIN**. In order to give an explicit description of this matrix let $\mathbf{r} = -\mathbf{p} \wedge \mathbf{q}$. Then $-\mathbf{r} = \mathbf{p} \vee -\mathbf{q}$. We code and identify these elements and their complements by binary 3-dimensional vectors as follows:

$\mathbf{0} = (000), \mathbf{p} = (001), -\mathbf{q} = (010), -\mathbf{r} = (011), -\mathbf{p} = (100), \mathbf{q} = (101), \mathbf{r} = (110), \mathbf{1} = (111)$.

Now the logical matrix **B8** in this coding can be obtained as follows. Boolean operations on codes are just coordinatewise. For the negation $\neg$ we obtain the following formula: $\neg(xyz) = (\overline{x}01)$. The proof is as follows: $\neg(xyz) = ((xyz) \Rightarrow \mathbf{p}) \wedge \mathbf{q} = ((xyz) \Rightarrow (001)) \wedge (101) = (\overline{x}01)$.

This semantics determines the following universal 3-point Kripke frame K3 for **Class.SUBMIN**: $W = \{x_1, x_2, x_3\}$, $N = \{x_1, x_3\}$ and $N^* = \{x_3\}$. Then the semantics for $\neg A$ is the following: $x_1 \Vdash_v \neg A$ iff $x_1 \nVdash_v A$, $x_2 \nVdash_v \neg A$, $x_3 \Vdash_v \neg A$. For the Boolean connectives the semantics is the standard one.

Another equivalent form of the **B8** semantics is to use an ordered triple (v_1, v_2, v_3) of two-valued Boolean valuations, extended to arbitrary formulas as follows: for $i = 1, 2, 3$, $v_i(-A) = 1$ iff $v_i(A) = 0$, $v_i(A \wedge B) = 1$ iff $v_i(A) = 1$ and $v_i(B) = 1$, $v_1(\neg A) = 1$ iff $v_1(A) = 0$, $v_2(\neg A) = 0$ and $v_3(\neg A) = 1$. A formula A is a **Class.SUBMIN**-tautology iff for all triples of valuations (v_1, v_2, v_3) and for all $i = 1, 2, 3$ we have $v_i(A) = 1$. A formula A is satisfiable if there is a triple of valuations (v_1, v_2, v_3) such that $v_i(A) = 1$ for some $i = 1, 2, 3$. A set of formulas Σ is jointly satisfiable in **B8** if there is a triple of valuations (v_1, v_2, v_3) and there is $i = 1, 2, 3$ such that $v_i(A) = 1$ for all $A \in \Sigma$. We will use this last version of the semantics to formulate the strong completeness theorem for **Class.SUBMIN**. We say that a set of formulas Σ is classically consistent if $\Sigma \nvdash \bot$. A different notion of consistency can be defined by the negation $\neg$, that is why we use the adjective "classical".

The following lemma can easily be proved.

LEMMA 3.13. *If a set of formulas Σ in* **Class.SUBMIN** *is jointly satisfiable in* **B8** *then it is classically consistent.*

Now we shall prove the converse.

THEOREM 3.14 (Strong completeness theorem for **Class.SUBMIN** with respect to **B8** semantics).

If Σ is a classically consistent set of formulas in **Class.SUBMIN** *then Σ is jointly satisfiable in* **B8***.*

Proof. First we need the following lemma.

Lemma 3.15 (Truth lemma for the canonical valuations).
Let $\gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$, be a triple of maximal consistent sets in **Class.SUBMIN** *satisfying the following conditions:*

(1) $\neg\top \notin \Gamma_1$ *and* $\neg\bot \in \Gamma_1$,

(2) $\neg\bot \notin \Gamma_2$ *and*

(3) $\neg\top \in \Gamma_3$.

Let for each variable p define a triple (v_1, v_2, v_3) of canonical valuation as follows:

- $v_i(p) = 1$ *iff* $p \in \Gamma_i$, $i = 1, 2, 3$.

Then for every formula A we have the following:

- $v_i(A) = 1$ *iff* $A \in \Gamma_i$, $i = 1, 2, 3$.

Proof. The proof is by induction on the construction of A. When $A = p$ is a variable the assertion is by the definition of the canonical valuations. The case of Boolean combinations of the formulas uses the fact that Γ_i, $i = 1, 2, 3$ are maximal consistent sets. Consider the case of $\neg A$ and suppose that for A the statement is true (induction hypothesis (i.h.)). We shall use the following facts:

$i = 1$. $v_1(\neg A) = 1$ iff (by the semantics)$v_1(A) = 0$ iff (by the i.h.) $A \notin \Gamma_1$ iff (by (1)) $(A \Rightarrow \neg\top) \wedge \neg\bot \in \Gamma_1$ iff $\neg A \in \Gamma_1$.

$i = 2$. By the semantics we have for every A: $v_2(\neg A) = 0$. So we have to show that for every A, $\neg A \notin \Gamma_2$. Suppose that for some A we have that $\neg A \in \Gamma_2$. Then $\neg\bot \in \Gamma_2$ which contradicts (2).

$i = 3$. By the semantics we have for every A that $v_2(\neg A) = 1$. So we have to show that for every A, $\neg A \in \Gamma_3$. By (3) $\neg\top \in \Gamma_3$ so $(A \Rightarrow \neg\top) \in \Gamma_3$. Also $\neg\top \vdash \neg\bot$, so again by (3) we have that $\neg\bot \in \Gamma_3$. Hence $(A \Rightarrow \neg\top) \wedge \neg\bot \in \Gamma_3$ and consequently $\neg A \in \Gamma_3$. ■

Now we turn to the proof of the theorem 3.14. Suppose that Σ is a consistent set of formulas. Define the following three sets of formulas.

(4) $\Sigma_1 = \Sigma \cup \{-\neg\top \wedge \neg 0\}$,

(5) $\Sigma_2 = \Sigma \cup \{-\neg 0\}$,

(6) $\Sigma_3 = \Sigma \cup \{\neg\top\}$.

We claim that at least one of this sets is consistent. Suppose that all are inconsistent. This implies the following

(7) $\Sigma \vdash -(-\neg\top \wedge \neg 0)$ (by (1)),

(8) $\Sigma \vdash \neg\neg\top$ by (2),

(9) $\Sigma \vdash -\neg\top$ by (3),

(10) By Boolean calculations we see that $-(-\neg\top \wedge \neg 0) \wedge \neg 0 \wedge -\neg\top \equiv \bot$.

So from (7), (8), (9) and (10) we get $\Sigma \vdash \bot$ which shows that Σ is not consistent — a contradiction. So there exists $i = 1, 2, 3$ such that Σ_i is consistent. Extend Σ_i into a maximal consistent set Γ_i. Define Γ_j, $j = 1, 2, 3$, for $j \neq i$ as to satisfy the corresponding condition (j) from lemma 3.15. This is possible because the formulas mentioned in (j) are consistent (this follows by the semantics). Now we may define the canonical valuations (v_1, v_2, v_3) from the just defined triple $\gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$ as in lemma 3.15. Since $\Sigma \subseteq \Gamma_i$ then by the lemma we have for every formula $A \in \Sigma$ that $v_i(A) = 1$, which has to be proved. ■

Finite matrices for the logics Class.MIN and Class.CO – MIN.

We may define finite logical matrices for the logics **Class.MIN** and **Class.CO – MIM** in a similar way as for the logic **Class.SUBMIN**.

Let $B = (B, 0, 1, \wedge, \vee, -, \neg)$ be a classical minimal algebra such that there exist an element $\mathbf{p} \in B$ such that $0 < \mathbf{p} < 1$ and $\neg a = a \Rightarrow \mathbf{p}$. Then the element $\mathbf{p}$ determines a 4-element Boolean subalgbra of B which is also a classical minimal subalgebra of B. We denote this algebra by **MinB4**. It is a subalgebra of every classical minimal algebra having an element $\mathbf{p}$ with $0 < \mathbf{p} < 1$. This namely is the characteristic matrix for the logic **Class.MIN**. The elements of **MinB4** can be coded by binary two-dimensional vectors as follows: $0 = (00)$, $\mathbf{p} = (01)$, $-\mathbf{p} = (10)$, $1 = (11)$. Boolean operations between the vectors are defined coordinatewise and for $\neg$ we have the formula: $\neg(xy) = (\overline{x}1)$. The same semantics can be reformulated also by pairs (v_1, v_2) of two-valued valuations as in the case of **B8**. Characteristic two-element Kripke structure for **Class.SUB** is the following: $W = \{x_1, x_2\}$ and $N = \{x_1\}$. As in the completeness theorem for **Class.SUBMIN** we may prove the following theorem.

Theorem 3.16 (Strong completeness theorem for **Class.MIN** with respect to **MinB4**). *Every consistent set of formulas of the logic* **Class.MIN** *is jointly satisfiable in the semantics with* **MinB4***.*

In an analogous way we define a finite 4-valued matrix for the logic **Class.CO – MIN**.

Let $(B, 0, 1, \wedge, \vee, -, \neg)$ be a classical co-minimal algebra, such that there exists an element $\mathbf{q}$ such that $0 < \mathbf{q} < 1$ and $\neg a = (a \Rightarrow \bot) \wedge \neg 0$. Then $\mathbf{q}$ generates a 4-element classical subalgebra **Comin.B4**. The binary coding of this matrix is similar to that of **MinB4** the difference is only for the $\neg$: $\neg(xy) = (0\overline{y})$. Also we have:

THEOREM 3.17 (Strong completeness theorem for **Class.CO** − **MIN** with respect to **CominB4**). *Every consistent set of formulas of the logic* **CO**− −**MIN** *is jointly satisfiable in the semantics by* **CominB4**.

REMARK 3.18. *The logics* **Class.MIN** *and* **Class.CO** − **MIN** *are equivalent to some modal systems introduced by Łukasiewicz in [10, 11] with semantics given by two-valued binary vectors. Matrix similar to* **B8** *was also mentioned in [11]. A general study of logics with matrices of similar kinds is given by the author in [23].*

4. Generalized N-lattices

In this section we will give generalizations of N-lattices in order to obtain an algebraic semantics for subminimal logic with Nelson negation and the introduced so far extensions.

DEFINITION 4.1. An algebraic system $N = (N, \leq, D, 0, 1, \wedge, \vee, \Rightarrow, \neg, \sim)$ is called a **generalized N-lattice** if it satisfies the following axioms:

- **(N1)** The reduct $(N, \leq, D, \wedge, \vee)$ is a distributive lattice and D is a filter in N,
- **(N2)** Define for every $a, b \in N$ the relations: $a \vdash b$ iff $a \Rightarrow b \in D$ and $a \equiv b$ iff $a \vdash b$ and $b \vdash a$. Then $\vdash$ is a quasi-ordering in N and consequently $\equiv$ is an equivalence relation in N,
- **(N3)** $0 \vdash a$, $a \vdash 1$,
- **(N4)** $a \leq b$ iff $a \vdash b$ and $\sim b \vdash \sim a$,
- **(N5)** If $a \in D$, and $a \Rightarrow b \in D$ then $b \in D$,
- **(N6)** $a \in D$ iff $(\forall b \in N)(b \vdash a)$,
- **(N7)** $x \wedge a \vdash b$ iff $x \vdash a \Rightarrow b$,
- **(N8)** $c \vdash a$ and $c \vdash b$ iff $c \vdash a \wedge b$,
- **(N9)** $a \vdash c$ and $b \vdash c$ iff $a \vee b \vdash c$,
- **(N10)** If $(a \equiv b)$ then $\neg a \equiv \neg b$,

(N11) $\sim\sim a = a$,

(N12) $\sim (a \vee b) = \sim a \wedge \sim b$,

(N13) $\sim (a \wedge b) = \sim a \vee \sim b$,

(N14) $\sim (a \Rightarrow b) \equiv a \wedge \sim b$,

(N15) $\sim \neg a \equiv a$,

(N16) $\neg b \wedge \sim a \vdash \neg a$,

(N17) $\sim 0 \equiv 1$,

(N18) $\sim 1 \equiv 0$.

REMARKS 4.2. *Note that by N11, N12 and N13 the reduct* $(N, \leq, \wedge, \vee, \sim)$ *is a De Morgan lattice. Calculations with the axioms of generalized N-lattice are similar with these in the subminimal logic with Nelson negation. Here the elements of the filter* D *can be considered as "theorems". Note also that the element* 1 *is not in general an unit element of the lattice — it can be considered just as an analog of* $\top$, *i.e. as a fixed element of* D. *Also* 0 *is not a zero element of the lattice — it is just an analog of* $\bot$. *Having in mind these analogies one may copy the proofs of the propositions 2.1 and 3.9 and to obtain the following statement for generalized N-lattices.*

PROPOSITION 4.3. *The following conditions are true for every generalized N-lattice:*

(i) $a \wedge \neg b \vdash \neg(a \Rightarrow b)$,

(ii) $a \wedge \neg a \vdash \neg 1$, *and* $\neg a \vdash a \Rightarrow \neg 1$,

(iii) $\neg b \wedge a \vdash \neg \sim a$,

(iv) $\neg \sim (a \Rightarrow \neg b) \equiv \neg(a \wedge b)$,

(v) $\neg c \wedge (a \Rightarrow \neg b) \vdash \neg(a \wedge b)$,

(vi) $\neg a \vdash \neg\neg 1$, $\neg 0 \equiv \neg\neg 1$,

(vii) $\neg\neg 1 \wedge (a \Rightarrow \neg 1) \vdash \neg a$,

(viii) $\neg a \equiv (a \Rightarrow \neg\top) \wedge \neg\neg 1 \equiv (a \Rightarrow \neg\top) \wedge \neg 0$,

(ix) $a \Rightarrow b \vdash \neg b \Rightarrow \neg a$,

(x) $\neg c \vdash \neg(a \wedge \sim a)$,

(xi) $\neg 0 \equiv \neg(a \wedge \sim a)$,

(xii) *If* $a \in D$ *then* $\neg \sim a \equiv \neg 0$,

(xiii) $\neg c \vdash a \Rightarrow \neg\neg a$,

(xiv) $\neg c \vdash \neg(a \wedge \neg a)$,

(xv) $a \wedge \neg(a \wedge b) \vdash \neg b$,

(xvi) $\neg(a \vee b) \equiv \neg a \wedge \neg b$,

(xvii) $a \wedge \sim a \vdash (\neg 0 \Leftrightarrow \neg 1)$.

Proof. As an example we shall prove (xii) and (xvii).
(xii) Suppose $a \in D$. By (iii) we have $a \Rightarrow (\neg 0 \Rightarrow \neg \sim a) \in D$ and by axiom (N5) we obtain $\neg 0 \Rightarrow \neg \sim a \in D$, so $\neg 0 \vdash \neg \sim a$. By (vi) we have $\neg \sim a \vdash \neg 0$ and consequently — $\neg \sim a \equiv \neg 0$.

(xvii) From axiom N6 we derive $\neg 0 \wedge \sim a \vdash \neg a$. Since $\neg a \vdash (a \Rightarrow \neg 1)$ we obtain $\neg 0 \wedge \sim a \vdash (a \Rightarrow \neg 1)$. From here we get $a \wedge \sim a \vdash (\neg 0 \Rightarrow \neg 1)$. Applying (vi) we may replace $\Rightarrow$ by $\Leftrightarrow$: $a \wedge \sim a \vdash (\neg 0 \Leftrightarrow \neg 1)$ which is what we need. ∎

We will consider generalized N-lattices satisfying some additional conditions.

Definition 4.4. Let $N = (N, \leq, D, 0, 1, \wedge, \vee, \Rightarrow, \neg, \sim)$ be a generalized N-lattice. Then we say that:

- N is a **minimal N-lattice** if it satisfies the condition

(N-min) $\neg 0 \equiv 1$,

- N is a **co-minimal N-lattice** if it satisfies the condition

(N-co-min) $\neg 1 \equiv 0$,

- N is a **classical** generalized N-lattice if it satisfies the condition

(N-class) $a \vee (a \Rightarrow b) \in D$.

Remarks 4.5. *(i) Minimal N-lattices (without 0) under the name of "Generalized Nelson lattices" were introduced in [25].*

(ii) It can be easily proved that a generalized N-lattice is an N-lattice iff it is both a minimal and a co-minimal N-lattice. This shows that generalized N-lattices indeed are generalizations of N-lattices.

Definition 4.6 (Counterexample relation). Let $(A, 0, 1, \leq, \wedge, \vee, \Rightarrow, \neg)$ be a subminimal algebra and let $a, b \in A$. We say that "b is a counterexample of a", in symbols aCb, iff $\neg(a \wedge b) = \neg 0$.

Lemma 4.7. *The counterexample relation C satisfies the following conditions:*

(i) *The following are some equivalent definitions of* C*:*

aCb *iff* $\neg 0 \leq \neg(a \wedge b)$ *iff* $a \wedge b \wedge \neg 0 \leq \neg 1$ *iff* $a \wedge b \leq (\neg 0 \Rightarrow \neg 1)$ *iff* $a \wedge b \leq (\neg 0 \Leftrightarrow \neg 1)$,

(ii) *If* aCb *then* bCa,

(iii) $aC\neg a$,

(iv) *If* $a_1 C a_2$ *and* $b_1 C b_2$ *then:*

- **(1)** $(a_1 \wedge b_1) C (a_2 \vee b_2)$,
- **(2)** $(a_1 \vee b_1) C (a_2 \wedge b_2)$,
- **(3)** $(a_1 \Rightarrow b_1) C (a_1 \wedge b_2)$.

PROOF. — by a routine calculations with the axioms of subminimal algebra. For the proof of (i) use the equality $\neg a = (a \Rightarrow \neg 1) \wedge \neg 0$. ■

REMARK 4.8. *Formally the above definition of the counterexample relation* C *differs from the corresponding definition in Heyting algebras (see Definition 1.2). In the former definition we have* aCb *iff* $a \wedge b = 0$*. Having in mind that in Heyting algebras* $0 = (1 \Leftrightarrow 0)$*,* $1 = \neg 0$ *and* $0 = \neg 1$*, the* C*-relation can be rewritten in an equivalent form as follows:* aCb *iff* $a \wedge b \leq 0 = (1 \Leftrightarrow 0) = (\neg 0 \Leftrightarrow \neg 1)$*, i.e.* aCb *iff* $a \wedge b \leq (\neg 0 \Leftrightarrow \neg 1)$*. As can be seen from lemma 4.7 (i) the relation* C *in subminimal algebras is equivalent just to this form. Intuitively this means the following: "a is a counterexample of b iff a and b together imply the equivalence of "weak falsity"* $(\neg 1)$ *with "strong truth"* $(\neg 0)$*. So we see again the role of these new logical constants.*

DEFINITION 4.9 (Counterexample construction). Let $(A, 0, 1, \leq, \wedge, \vee, \Rightarrow, \neg)$ be a subminimal algebra and let $N(A) = \{(a, b) \in A^2 : aCb\}$. Define the following relations and operations in $N(A)$:

- $(a_1, a_2) \leq (b_1, b_2)$ iff $a_1 \leq b_1$ and $b_2 \leq a_2$,
- $1 = (1, 0)$, $0 = (0, 1)$,
- $D = \{(1, a) : \neg a = \neg 0\}$,
- $(a_1, a_2) \wedge (b_1, b_2) = (a_1 \wedge b_1, a_2 \vee b_2)$,
- $(a_1, a_2) \vee (b_1, b_2) = (a_1 \vee b_1, a_2 \wedge b_2)$,
- $(a_1, a_2) \Rightarrow (b_1, b_2) = (a_1 \Rightarrow b_1, a_1 \wedge b_2)$,
- $\neg(a_1, a_2) = (\neg a_1, a_1)$,
- $\sim (a_1, a_2) = (a_2, a_1)$.

Obtaining generalized N-lattices from subminimal algebras by counterexample construction.

Lemma 4.10. **(i)** *The set $N(A)$ with the relations and operations defined above is a generalized N-lattice, called special generalized N-lattice over the subminimal algebra A.*

(ii) *If A is a minimal algebra then $N(A)$ is a minimal N-lattice.*

(iii) *If A is a co-minimal algebra then $N(A)$ is a co-minimal N-lattice.*

(iv) *If A is a classical subminimal algebra then $N(A)$ is a classical generalized N-lattice.*

(v) *Let $(a, b) \in N(A)$ and let $h((a, b)) = a$, then h is a homomorphism from the N-lattice $N(A)$ onto the lattice A, namely h preserves all operations in $N(A)$ which are from the signature of A.*

Proof. **(i)** The proof that $N(A)$ is closed under the introduced operations follows from lemma 4.7. The verification of the axioms of generalized N-lattice is a routine exercise.

(ii) Let A be minimal algebra. Then $\neg 0 = 1$. In $N(A)$ we have: $\neg(01) = (\neg 00) = (10)$.

(iii) The proof is similar to that of (ii).

(iv) Suppose A is classical. Then $a \vee (a \Rightarrow b) = 1$. Now in $N(A)$ we have: $(a_1, a_2) \vee ((a_1, a_2) \Rightarrow (b_1, b_2)) = (a_1 \vee (a_1 \Rightarrow b_1), a_1 \wedge a_2 \wedge b_2) = (1, a_1 \wedge a_2 \wedge b_2)$. It follows from $\neg 0 = \neg(a_1 \wedge a_2)$ that $\neg(a_1 \wedge a_2 \wedge b_2) = \neg 0$ which shows that $(1, a_1 \wedge a_2 \wedge b_2) \in D$. This proves that $N(A)$ is a classical generalized N-lattice.

(v) The proof is obvious. ∎

Obtaining set-theoretical generalized N-lattices from subminimal Kripke frames.

Definition 4.11. Let $W = (W, \leq, N, N^*)$ be a subminimal Kripke frame and let A(W) be the subminimal algebra over W. Then the N-lattice $N(A(W))$ over the subminimal algebra $A(W)$ is called **set-theoretical generalized N-lattice over the frame** W.

Corollary 4.12. *Let W be a subminimal frame. Then:*

(i) *If W is a minimal frame then $N(A(W))$ is a minimal N-lattice.*

(ii) *If W is a co-minimal frame then $N(A(W))$ is a co-minimal N-lattice.*

(iii) *If W is both a minimal and a co-minimal frame then $N(A(W))$ is an N-lattice.*

(iv) *If W is a classical subminimal frame then $N(A(N))$ is a classical generalized N-lattice.*

Obtaining subminimal algebras from generalized N-lattices.

DEFINITION 4.13. Let $N = (N, \leq, D, 1, \wedge, \vee, \Rightarrow, \neg, \sim)$ be a generalized N-lattice. For each $a \in N$ let $| a |= \{b \in N : a \equiv b\}$ and let $A(N) = \{| a |: a \in N\}$. Define the following relations and operations in the set $A(N)$:

$1 = |1| = D$, $0 = |0|$, $|a| \leq |b|$ iff $a \vdash b$, $|a| \wedge |b| = |a \wedge b|$, $|a| \vee |b| = |a \vee b|$, $|a| \Rightarrow |b| = |a \Rightarrow b|$, $\neg|a| = |\neg a|$.

LEMMA 4.14. **(i)** *The set $A(N)$ with the above defined relations and operations is a subminimal algebra.*

(ii) *If N is a minimal N-lattice, then $A(N)$ is a minimal algebra.*

(iii) *If N is a co-minimal N-lattice, then $A(N)$ is a co-minimal algebra.*

(iv) *If N is a classical generalized N-lattice then $A(N)$ is a classical subminimal algebra.*

PROOF. **(i)** First note that the relation $\equiv$ is a congruence with respect to the operations $\wedge, \vee, \Rightarrow, \neg$ which shows that the definitions of the operations are correct. Then the verifications of the axioms of subminimal algebra is a routine check.

(ii) The proof is immediate: let $\neg 0 = 1$. Them $\neg|0| = |\neg 0| = |1|$. The proof of (iii) is similar. ■

COROLLARY 4.15. *Let N be a generalized N-lattice. Let $A(N)$ be the subminimal algebra obtained from N by lemma 4.14 and let $N(A(N))$ be the special generalized N-lattice obtained from $A(N)$ by lemma 4.10. Then:*

(i) *If N is a minimal N-lattice then $N(A(N))$ is a minimal N-lattice too.*

(ii) *If N is a co-minimal N-lattice then $N(A(N))$ is a co-minimal N-lattice too.*

(iii) *If N is both a minimal and a co-minimal N-lattice then $N(A(N))$ is an N-lattice.*

(iv) *If N is classical generalized N-lattice then $N(A(N))$ is a classical generalized N-lattice too.*

Lemma 4.16. *Let* $N = (N, \leq, D, 1, \wedge, \vee, \Rightarrow, \neg, \sim)$ *be a generalized N-lattice and let* $N(A(N))$ *be the generalized N-lattice from corollary 4.15. Let for* $a \in N$ $h(a) =_{def} (|a|, |\sim a|)$. *Then* h *is an isomorphic embedding of* N *into* $N(A(N))$.

Proof. First, let us show that $h(a) \in N(A(N))$. By lemma 4.3 we have $\neg(a \wedge \sim a) \equiv \neg 0$ and hence $\neg(|a| \wedge |\sim a|) = \neg|0|$, which shows that $h(a) \in N(A(N))$.

Second, we have to show that h is a strong homomorphism in a sense that h preserves D biconditionally and that h is an ordinary homomorphism with respect to the operations.

First we show that h preserves D biconditionally: $a \in D$ iff $h(a) \in D'$ where D' is the filter in $N(A(N))$.

($\rightarrow$) Let $a \in D$. Then by lemma 4.3 (xii) we have $\neg 0 \equiv \neg \sim a$, so $\neg|0| = \neg|\sim a|$ which shows that $(|1|, |\sim a|) \in D'$. Since $a \in D$ then $|a| = |1|$ and $f(a) = (|a|, |\sim a|) = (|1|, |\sim a|) \in D'$.

($\leftarrow$) Now suppose that $f(a) \in D'$ i.e. $(|a|, |\sim a|) \in D'$. Then $|a| = |1|$ and consequently $a \in D$.

This property shows that $1 \in D$ iff $f(1) \in D'$.

Second we have to show that h preserves the operations. For $\wedge$ we have:

$h(a \wedge b) = (|a \wedge b|, |\sim (a \wedge b)|) = (|a| \wedge |b|, |\sim a| \vee |\sim b|) = (|a|, |\sim a|) \wedge (|b|, |\sim b|) = h(a) \wedge h(b)$. Here we have used axiom (N13). The verification for the remaining operations is analogous.

Finally we have to show that h is an injective mapping. Suppose that $h(a) = h(b)$ and proceed to show that $a = b$. From the assumption we have that $(|a|, |\sim a|) = (|b|, |\sim b|)$, and consequently $|a| = |b|$ and $|\sim a| = |\sim b|$. This implies $a \equiv b$ and $\sim a \equiv \sim b$. Then by axiom (N4) we obtain $a = b$. ∎

Corollary 4.17 (Representation theorem for generalized N-lattices). *For each generalized N-lattice* N *there exists a special N-lattice* N' *and an isomorphic embedding* h *from* N *into* N', *and if* N *satisfies some of the axioms (Min), (Co-min) and (Class), then* N' *satisfies the same axioms.*

Proof. Put $N' = N(A(N))$. By lemma 4.16 there exist an embedding h from N into the special generalized N-lattice N'. If N satisfies some of the axioms (Min), (Co-min) and (Class) then by corollary 4.15 N' satisfies the same axioms. ∎

The above representation theorem contains as special cases the representation theorems from [25, 27].

A representation of generalized N-lattices into set-theoretical generalized N-lattices can be obtained by the construction in the next theorem.

THEOREM 4.18 (Set-theoretical representatin theorem for generalized N-lattices). *Let N be a generalized N-lattice and let $A(N)$ be the subminimal algebra obtained from N by definition 4.13. By theorem 3.7 there exists a subminimal frame $W(A(N))$ and an isomorphic embedding f from $A(N)$ into the subminimal algebra $A(W(A(N))$ over the frame $W(A(N))$. Let $N' = N(A(W(A(N)))$ be the generalized N-lattice over the subminimal algebra $A(W(A(N))$. Define for any $a \in N$, $h(a) = (f(|a|), f(|\sim a|))$. Then f is an isomorphic embedding from N into N'. If N satisfies some of the axioms (Min), (Co-min) and (Class), then N' satisfies the same axioms.*

PROOF. — The proof is almost the same as the proof of theorem 4.17. ■

5. Semantics, completeness theorems and decidability for logics with Nelson negation

In this section we will use generalized N-lattices as semantics for the considered in this paper logics with Nelson negation.

Algebraic semantics for Nelson negation.

Let $N = (N, \leq, D, 1, \wedge, \vee, \Rightarrow, \neg, \sim)$ be a generalized N-lattice and v be a mapping from the sets of propositional variables into N. Then in a standard way we extend v from the set of all formulas into N. We say that v is a model for a formula A, or that A is true at the valuation v in N if $v(A) \in D$. A formula A is true in N if it is true at every valuation v in N. If Σ is a class of generalized N-lattices then A is true in Σ if A is true in every algebra from Σ.

Let $\mathbf{L}$ be any logic from the list $\mathbf{SUBMIN^N}$, $\mathbf{MIN^N}$, $\mathbf{CO-MIN^N}$, $\mathbf{Class.SUBMIN^N}$, $\mathbf{Class.MIN^N}$, $\mathbf{Class.CO-MIN^N}$, $\mathbf{Class.INT^N}$. A lattice N is called $\mathbf{L}$-lattice if N is the lattice from the corresponding class of: generalized-, minimal-, co-minimal-, classical generalized-, classical minimal-, classical co-minimal- N-lattices. We will use similar correspondence between the above list of logics and the corresponding classes of submminimal frames, called shortly $\mathbf{L}$-frames.

The following is one of the main theorems in this paper.

THEOREM 5.1 (Completeness theorem for logics with Nelson negation).

Let $\mathbf{L}$ be any logic from the list $\mathbf{SUBMIN^N}$, $\mathbf{MIN^N}$, $\mathbf{CO-MIN^N}$, $\mathbf{Class.SUBMIN^N}$, $\mathbf{Class.MIN^N}$, $\mathbf{Class.CO-MIN^N}$ and $\mathbf{Class.INT^N}$. Then the following conditions are equivalent for any formula A of $\mathbf{L}$:

(i) *A is a theorem of* **L**,

(ii) *A is true in the class of all* **L**-*lattices,*

(iii) *A is true in all special* **L**-*lattices,*

(iv) *A is true in all set-theoretical* **L**-*lattices.*

PROOF. The implications $(i) \to (ii) \to (iii) \to (iv)$, forming the soundness part of the theorem, are straightforward.

$(ii) \to (i)$. This is a standard proof using the construction of the Lindenbaum algebra of $\mathbf{SUBMIN^N}$ (see for a similar proof [20]). Note that the equivalence $\equiv$ determined by $\Leftrightarrow$ is not a congruence with respect to the Nelson negation and as in [20] one have to use the "strong" equivalence: $A \cong B$ iff $A \equiv B$ and $\sim A \equiv \sim B$.

The implication $(iii) \to (ii)$ follows from the representation theorem 4.17 for Generalized N-lattices.

The implication $(iv) \to (ii)$ follows from the set-theoretical representation theorem 4.18 for generalized N-lattices . ■

For the logics **L** with classical implication a more strong completeness theorem with respect to finite **L**-matrix will be proved.

Kripke semantics for Nelson negation.

The semantics of the logics **L** with respect to the set-theoretical **L**-lattices can be rephrased in an equivalent way as a Kripke semantics over the class of **L**-frames (see Kracht [8] for a similar reformulation for the case of Nelson's logic).

Let $W = (W, \leq, N, N^*)$ be a subminimal frame. Let $v = (v^+, v^-)$ of upwards monotone valuations in (W) satisfying the following condition: for any variable p, $N \cap v^+(p) \cap v^-(p) \subseteq N^*$. Define a pair of satisfaction relations $\Vdash^+$ and $\Vdash^-$ inductively as follows:

- $x \Vdash^+_v p$ iff $x \in v^+(p)$,
- $x \Vdash^-_v (p)$ iff $x \in v^-(p)$,
- $x \Vdash^+_v A \wedge B$ iff $x \Vdash^+_v A$ and $x \Vdash^+_v B$,
- $x \Vdash^-_v A \wedge B$ iff $x \Vdash^-_v A$ or $x \Vdash^-_v B$,
- $x \Vdash^+_v A \vee B$ iff $x \Vdash^+_v A$ or $x \Vdash^+_v B$,
- $x \Vdash^-_v A \vee B$ iff $x \Vdash^-_v A$ and $x \Vdash^-_v B$,
- $x \Vdash^+_v A \Rightarrow B$ iff $(\forall y \in W)(x \leq y$ and $y \Vdash^+_v A \to y \Vdash^+_v B)$,
- $x \Vdash^-_v A \Rightarrow B$ iff $x \Vdash^+_v A$ and $x \Vdash^-_v B$,

- $x \Vdash_v^+ \neg A$ iff $(\forall y \in W)(x \leq y \rightarrow y \in N^*)$ and $x \in N$,
- $x \Vdash_v^- \neg A$ iff $x \Vdash_v^+ A$,
- $x \Vdash_v^+ \sim A$ iff $x \Vdash_v^- A$,
- $x \Vdash_v^- \sim A$ iff $x \Vdash_v^+ A$,

The intuitive readings: $x \Vdash_v^+ A$: "A is accepted at x", $x \Vdash_v^- A$: "A is rejected at x". The following theorem is another version of the completeness theorem with respect to set-theoretical generalized N-lattices.

THEOREM 5.2. **Kripke completeness for logics with Nelson negation.** *Let* **L** *be any logic from the list* $\mathbf{SUBMIN^N}$, $\mathbf{MIN^N}$, $\mathbf{CO - MIN^N}$, $\mathbf{Class.SUBMIN^N}$, $\mathbf{Class.MIN^N}$, $\mathbf{Class.CO - MIN^N}$ *and* $\mathbf{Class.INT^N}$. *Then the following conditions are equivalent for any formula A of* **L***:*

(i) *A is a theorem of* **L***,*

(i) *A is true in all* **L***-frames.*

COROLLARY 5.3. *The following are true:*

(i) $\mathbf{SUBMIN^N}$ *is a conservative extension of* **SUBMIN***,*

(ii) $\mathbf{MIN^N}$ *is a conservative extension of* **MIN***,*

(i) $\mathbf{CO - MIN^N}$ *is a conservative extension of* $\mathbf{CO - MIN}$.

PROOF. Use the completeness theorem 5.2 for the above logics with respect to their Kripke semantics. ■

Finite matrix semantics for classical logics with Nelson negation.

Finite matrix for $\mathbf{Class.SUBMIN^N}$.

First we will introduce a finite matrix for $\mathbf{Class.SUBMIN^N}$. This is just $N(\mathbf{B8})$. Applying the counterexample construction to the subminimal algebra **B8** coded by 3-dimensional binary vectors we obtain the following finite classical generalized N-lattice which we will denote by **N48**, because it has exactly 48 different elements, which we will represent by 6-dimensional binary vectors.

($N48$) $N48 = \{(x_1x_2x_3; y_1y_2y_3) \in \{0,1\}^6 : x_1 \wedge y_1 = 0\}$. It follows from this representation of the set $N48$ that it indeed has exactly 48 elements.

(D) $D = \{(111; 0yz)\}$, $1 = (111; 000)$, $0 = (000; 111)$,

($\sim$) $\sim (x_1x_2x_3; y_1y_2y_3) = (y_1y_2y_3; x_1x_2x_3)$,

(¬) $\neg(x_1x_2x_3; y_1y_2y_3) = (\overline{x_1}01; x_1x_2x_3)$,

(∧) $(x_1x_2x_3; y_1y_2y_3) \wedge (u_1u_2u_3; v_1v_2v_3) =$
$((x_1 \wedge u_1)(x_2 \wedge u_2)(x_3 \wedge u_3); (y_1 \vee v_1)(y_2 \vee v_2)(y_3 \vee v_3))$,

(∨) $(x_1x_2x_3; y_1y_2y_3) \vee (u_1u_2u_3; v_1v_2v_3) =$
$((x_1 \vee u_1)(x_2 \vee u_2)(x_3 \vee u_3); (y_1 \wedge v_1)(y_2 \wedge v_2)(y_3 \wedge v_3))$,

(⇒) $(x_1x_2x_3; y_1y_2y_3) \Rightarrow (u_1u_2u_3; v_1v_2v_3) = ((x_1 \Rightarrow u_1)(x_2 \Rightarrow u_2)(x_3 \Rightarrow u_3); (x_1 \wedge v_1)(x_2 \wedge v_2)(x_3 \wedge v_3))$.

A valuation v in **N48** in the above binary representation can be represented by ordered 6-tuple $v = (v_1v_2v_3; v'_1v'_2v'_3)$ of binary valuations satisfying the condition that for every propositional variable p we have $v_1(p) \wedge v'_1(p) = 0$. So the above semantics can be rephrased in terms of these binary valuations. We say that a set of formulas Σ in **Class.SUBMIN$^{\mathbf{N}}$** is jointly satisfiable if there exist a valuation $v = (v_1v_2v_3; v'_1v'_2v'_3)$ and i=1,2,3 such that for any formula $A \in \Sigma$ we have $v_i(A) = 1$. Σ is said to be classically consistent in **Class.SUBMIN$^{\mathbf{N}}$** if $\Sigma \nvdash \bot$. The following lemma is straightforward.

LEMMA 5.4. *If* **Class.SUBMIN$^{\mathbf{N}}$** *is jointly satisfiable then* Σ *is classically consistent.*

THEOREM 5.5 (Strong completeness of **Class.SUBMIN$^{\mathbf{N}}$** in **N48**).
If Σ *is classically consistent set of formulas in* **Class.SUBMIN$^{\mathbf{N}}$** *then it is jointly satisfiable.*

PROOF. First we need the following lemma.

LEMMA 5.6 (Truth lemma for the canonical valuations).
Let $\gamma = (\Gamma_1, \Gamma_2, \Gamma_3)$*, be a triple of maximal consistent sets in the logic* **Class.SUBMIN$^{\mathbf{N}}$** *satisfying the following conditions:*

(1) $\neg\top \notin \Gamma_1$ *and* $\neg\bot \in \Gamma_1$,

(2) $\neg\bot \notin \Gamma_2$ *and*

(3) $\neg\top \in \Gamma_3$.

Let for each variable p define a 6-tuple $v = (v_1, v_2, v_3; v'_1, v'_2, v'_3)$ *of canonical valuation as follows:*

- $v_i(p) = 1$ *iff* $p \in \Gamma_i$ $v'_i(p) = 1$ *iff* $\sim p \in \Gamma_i$, $i = 1, 2, 3$.

Then for every formula A we have the following:

- $v_i(A) = 1$ *iff* $A \in \Gamma_i$, $v'_i(A) = 1$ *iff* $\sim A \in \Gamma_i$ $i = 1, 2, 3$.

PROOF OF THE LEMMA. The lemma is similar to lemma 3.15 and we will make a use of it. The proof goes by induction on the complexity of the formula A. For a propositional variable p the assertion is true by the definition of the canonical valuations. Induction hypothesis (i.h.): let for A, B the assertion is true.

- **Case for** $A = \top, \bot$**.** By the semantics we have $v(\top) = 1 = (111; 000)$, so for all $i = 1, 2, 3$ we have $v_i(\top) = 1$ and $v'_i(\top) = 0$. The first requires $\top \in \Gamma_i$, which is always true. The second requires $\sim \top \notin \Gamma_i$. By axiom $(\sim \top)$ we have that $\sim \top \equiv \bot$ and since we have always $\bot \notin \Gamma_i$ — the requirement is also fulfilled. The case for $\bot$ is similar.
- **Case for** $A = B \wedge C$**.** We have: $v_i(B \wedge C) = 1$ iff (by the semantics) $v_i(A) = 1$ and $v_i(B) = 1$ iff (by i.h.) $B \in \Gamma_i$ and $C \in \Gamma_i$ iff (by maximality of Γ_i) $A \wedge B \in \Gamma_i$.

  For the valuations v'_i we have: $v'_i(B \wedge C) = 1$ iff (by the semantics) $v'_i(B) = 1$ or $v'_i(C) = 1$ iff (by the i.h.) $\sim A \in \Gamma_i$ or $\sim B \in \Gamma_i$ iff (by maximality) $\sim A \vee \sim B \in \Gamma_i$ iff (by axiom $(\sim \wedge)$) $\sim (A \wedge B) \in \Gamma_i$.

  The cases for $A = (B \vee C)$ and $A = (B \Rightarrow C)$ are similar and make use of the axioms $(\sim \vee)$ and $(\sim\Rightarrow)$.
- **Case for** $A =\sim B$**.** We calculate: $v_i(\sim B) = 1$ iff (by the semantics) $v'_i(B) = 1$ iff (by the i.h.) $\sim B \in \Gamma_i$.

  For the valuations v' we have: $v'(\sim B) = 1$ iff (by the semantics) $v_i(B) = 1$ iff (by the i.h.) $B \in \Gamma_i$ iff (by the axiom $(\sim\sim)$ $\sim\sim B \in \Gamma_i$.
- **Case** $A = \neg B$**.** For the valuations v_i, $i = 1, 2, 3$ the proof is the same as in lemma 3.15. For the valuations v'_i we have: $v'_i(\neg B) = 1$ iff (by the semantics) $v_i(B) = 1$ iff (by the i.h.) $B \in \Gamma_i$ iff (by the axiom $(\sim \neg)$ $\sim \neg B \in \Gamma_i$. ■

Now the proof of theorem 5.5 is exactly the same as the proof of theorem 3.14. Instead of lemma 3.15, used in the proof of theorem 3.14 now we use the just proved lemma 5.6. ■

Finite matrices for Class.MIN$^{\mathbf{N}}$ and Class.CO – MIN$^{\mathbf{N}}$.

The finite matrix for the logic **Class.MIN$^{\mathbf{N}}$** is **N(MIN – B4)** and for the logic **Class.CO – MIN$^{\mathbf{N}}$** is **N(Co – min – B4)**. Both matrices have 12 elements, so the corresponding logics can be considered as 12-valued logics. We left to the reader to represent these matrices in a binary coding. In the same way as theorem 5.5 one can prove the strong completeness theorem for these logics with respect to their matrix semantics.

Decidability of $\mathbf{SUBMIN^N}$, $\mathbf{MIN^N}$ and $\mathbf{CO-MIN^N}$.

The decidability of the logics $\mathbf{SUBMIN^N}$, $\mathbf{MIN^N}$ and $\mathbf{CO-MIN^N}$ can be obtained via proving finite model property similar for the case for Nelson logic in [27] or [20]. We will use the translation method, similar to those of [31, 20].

Let A be a formula of $\mathbf{SUBMIN^N}$. The axioms for Nelson negation make possible to move all occurrences of the Nelson negation inside the formula and to obtain a negation normal form $Neg(A)$ equivalent to A, in which the Nelson negation occurs only in subformulas of the form $\sim p$, where p is a propositional variable. Then $Neg(A) \equiv B(p_1, \dots, p_n; \sim p_1, \dots, \sim p_n)$, where $p_1, \dots p_n$ is the list of all variables of B. We define a translation $\tau(B)$ only for formulas in a negation normal form as follows. Let $q_1, \dots, q_n$ be a list of different propositional variables not occurring in B. Then $\tau(B) =_{def}$ $((p_1 \wedge q_1 \wedge \neg\bot \Rightarrow \neg\top) \wedge \dots \wedge (\neg(p_n \wedge q_n) \Rightarrow \neg\bot)) \Rightarrow B(p_1, \dots, p_n; q_1, \dots, q_n))$. Then $\tau(B)$ does not contain $\sim$ and is a formula of **SUBMIN**. The following theorem holds for this translation.

THEOREM 5.7 (Translation theorem for the logics $\mathbf{SUBMIN^N}$, $\mathbf{MIN^N}$, $\mathbf{CO-MIN^N}$). *The following is true for every formula B in negational normal form:*

B is a theorem of $\mathbf{SUBMIN^N}$ *(* $\mathbf{MIN^N}$*,* $\mathbf{CO-MIN^N}$*) iff $\tau(B)$ is a theorem of* **SUBMIN** *(* $\mathbf{MIN^N}$*,* $\mathbf{CO-MIN^N}$*).*

PROOF. We will consider only the case of $\mathbf{SUBMIN^N}$. The other cases can be proved in a similar way.

• $(\leftarrow)$. Let $\tau(B)$ be a theorem of **SUBMIN**. Replacing every q_i with $\sim p_i$ the prefix of $\tau(B)$ becomes a theorem and the rest is just the formula B. So, by Modus Ponens B is a theorem of $\mathbf{SUBMIN^N}$.

• $(\rightarrow)$. For this case we will reason by contraposition. Suppose $\tau(B)$ is not a theorem of **SUBMIN**. Then by the Kripke completeness theorem 5.2 for **SUBMIN** there exists a subminimal model $M = (W, \leq, N, N^*, v)$ and an element $x_0 \in W$ such that

(1) $x_0 \Vdash_v (p_i \wedge q_i \wedge \neg\bot \Rightarrow \neg\top)$, $i = 1, \dots, n$, and

(2) $x_0 \nVdash_v B(p_1, \dots, p_n; q_1, \dots, q_n)$.

Let $M_0 = (W_0, \leq, N_0, N_0^*, v_0)$ be the submodel of M generated by x_0. Then by the well known lemma for generated submodels we obtain from (1) and (2) the following:

(3) $x_0 \Vdash_{v_0} (p_i \wedge q_i \wedge \neg\bot \Rightarrow \neg\top)$, $i = 1, \dots, n$, and

(4) $x_0 \not\Vdash_{v_0} B(p_1, \ldots, p_n; q_1, \ldots, q_n)$.

It is easy to see that from (3) we obtain

(5) $v_0(p_i) \cap v_0(q_i) \cap N_0 \subseteq N_0^*$.

Now we will define a pair of valuations v^+ and v^- on W_0 as follows:

(6) $v^+(p_i) = v_0(p_i)$ and $v^-(p_i) = v_0(q_i)$.

By (5) the model $M' = (W_0, \leq, N_0, N_0^*, v^+, v^-)$ is an **SUBMIN^N**-model. Obviously by (4) we have:

(7) $x_0 \not\Vdash_{v^+} B(p_1, \ldots, p_n; \sim p_1, \ldots, \sim p_n)$.

which shows that B is not a theorem of $\mathbf{SUBMIN^N}$. ■

COROLLARY 5.8. *The logics* $\mathbf{SUBMIN^N}$, $\mathbf{MIN^N}$, $\mathbf{CO-MIN^N}$ *are decidable.*

6. Extensions of Nelson logic with new connectives

In this concluding section we will discus informally some connections between logics with Nelson negation based on the intuitionistic logic and logics with Nelson negation based on the classical logic.

The fact that adding the characteristic axiom for the classical implication to the Nelson's logic $\mathbf{INT^N}$ turns it to be equivalent to the 3-valued logic **L3** of Łukasiewicz, makes possible to consider the $\mathbf{INT^N}$ as an **intuitionistic version** of **L3**. It is a well known fact, however, that **L3** is not functionally complete. In order to obtain an intuitionistic version of the functionally complete 3-valued logic **P3** of Post we consider the following new connective $\bullet$ with the name **auto-dual conjunction**. We consider the following axioms for $\bullet$:

$(\bullet 1) \quad (A \bullet B) \Leftrightarrow (A \wedge B), \quad (\bullet 2) \quad \sim (A \bullet B) \Leftrightarrow (\sim A \wedge \sim B)$.

In the definition of the generalised N-lattice with $\bullet$ we add analogs of the above axioms:

$(\bullet 1) \quad (A \bullet B) \equiv (A \wedge B), \quad (\bullet 2) \quad \sim (A \bullet B) \equiv (\sim A \wedge \sim B)$.

The above axioms easily imply the auto-duality of the new conjunction: $a \bullet b = \sim (\sim A \bullet \sim B)$.

By the new conjunction we may define a new logical constant analogous to the Łukasiewicz **middle value 1/2** $1/2 =_{def} \bot \bullet \top$.

For the counterexample construction we add the definitions: $(a_1, a_2) \bullet (b_1, b_2) = (a_1 \wedge b_1, a_2 \wedge b_2)$, $1/2 = (0, 0)$.

If we add the axiom (Class) for the classical implication to the extension of Nelson's logic with $\bullet$ we obtain the functionally complete 3-valued logic P3 of Post (see for this fact [28]). This shows that the new conjunction is not definable by means of the remaining operations and that the new system is indeed an intuitionistic version of the 3-valued logic of Post.

We may axiomatize separately the "midle value" 1/2 by the following axioms:

$(1/2, 1) \quad 1/2 \Leftrightarrow \bot$, and $\quad (1/2, 2) \quad \sim (1/2) \Leftrightarrow \bot$,

From these axioms we see that $\sim (1/2) \equiv \bot$, a feature which is characteristic for the 3-valued tables of Łukasiewicz. This fact shows that the roots of the Nelson negation go back to the old Łukasiewicz negation in **L3**.

We may add the self-dual conjunction $\bullet$ or the constant 1/2 with the corresponding axioms to the logics $\mathbf{Class.SUBMIN^N}$, $\mathbf{Class.MIN^N}$ and $\mathbf{Class.CO - MIN^N}$. Obviously the obtained new logics are more expressible but probably not functionally complete.

Note that the logical connective $A \bullet B$ was introduced for the first time in [28] on the base of the 3-valued logic of Łukasiewicz. After that similar connectives in the context of bilattice logics have also been considered (see for instance [1, 5, 6, 18]).

References

[1] Arieli, O., and A. Avron, 'Reasoning with logical bilattices', *Journal of Logic Language and Information*, 5:25–63, 1996.

[2] Białynicki-Birula, A., and H. Rasiowa, 'On constructible falsity in constructive logic with strong negation', *Colloquium Mathematicum*, 6:287–310, 1958.

[3] Dunn, J. M., 'Generalized ortho negation', in H. Wansing, Walter de Gruyer, (eds.), *Negation, A notion in Focus*, pp. 3-26, Berlin, New York, 1996.

[4] Fidel, M. M., 'An algebraic study of a propositional system of Nelson', in *Mathematical Logic, Proc. of the First Brasilian Conference, Campinas 1977*, Lecture Notes in pure Appl. Math., 39, pp. 99-117, 1978.

[5] Fitting, M., 'Bilattices and the semantics of logic programming', *Journal of logic programming*, 11(2):91–116, 1991.

[6] Gargov, G., 'Knowledge, uncertainty and ignorance in logic: bilattices and beyond', *Journal of Applied Non-Classical Logics*, 9:195–283, 1999.

[7] Goranko, V., 'The Craig Interpolation Theorem for Propositional Logics with Strong Negation', *Studia Logica*, 44:291–317, 1985.

[8] Kracht, M., 'On extensions of intermediate logics by strong negation', *Journal of Philosophical Logic*, 27:49–73, 1998.

[9] Łukasiewicz, J., 'O logice trójwartościowej', *Ruch Filozoficzny*, 5:170–171, 1920.

[10] Łukasiewicz, J., 'A system of Modal Logic', *The Journal of Computing Systems*, 1:111-149 1953.

[11] Łukasiewicz, J., Aristotle's syllogistic from the standpoint of modern formal logic, Oxford, Clarendon Press, 1957.

[12] Markov, A. A., 'Constructive Logic', (in Russian), *Uspekhi Matematicheskih Nauk* 5:187–188, 1950.

[13] Nelson, D., 'Constructible falsity', *Journal of Symbolic Logic*, 14:16-26, 1949.

[14] Odintsov, S. P., 'Algebraic semantics for paraconsistent Nelson's Logic', *Journal of Logic and Computation*, 13(4):453–468, 2003.

[15] Odintsov, S. P., 'On the Representation of N4-Lattices', *Studia Logica*, 76(3):385-405, 2004.

[16] Pearce, D. and G. Wagner, 'Reasoning with negative information, I: Strong negation in logic programs', *Language, Knowledge and Intensionality (Acta Filosophica Fenica)*, Helsinki, 49:405–439, 1990.

[17] Pearce, D., and G. Wagner, 'Logic programming with strong negation', in P. Schroeder-Heister, (ed.), *Extensions of Logic Programming*, Lecture Notes in Artificial Intelligence, No. 475, pp. 311–326, Springer-Verlag, Berlin, 1991.

[18] Pynko, A. P., 'Functional completeness and axiomatizability within Belnap's four-valued logic and its expansions', *Journal of Applied Non-Classical Logics*, 9:61–105, 1999.

[19] Rasiowa, H., 'N-lattices and constructive logic with strong negation', *Fundamenta Mathematicae*, 46:61–80, 1958.

[20] Rasiowa, H., *An algebraic approach to non-classical logic*, North-Holland Publishing Company, Amsterdam, London, 1974.

[21] Sendlewski, A., 'Some investigations of varieties of N-lattices', *Studia Logica*, 43:257–280, 1984.

[22] Sendlewski, A., 'Nelson algebras through Heyting ones', *Studia Logica*, 49:106–126, 1990.

[23] Vakarelov, D., 'Ekstensionalnye Logiki', (in Russian), *Doklady BAN* , 25:1609-1612, 1972.

[24] Vakarelov, D., 'Models for constructive logic with strong negation', *V Balkan Mathematical Congress, Abstracts*, Beograd, 1974, 298.

[25] Vakarelov, D., 'Obobschennye reshetki Nelsona', , *Chetvertaya Vsesoyuznaya Conferenciya po Matematicheskoy Logike, tezisy dokladov i soobschtenii*, Kishinev, 1976.

[26] Vakarelov, D., *Theory of Negation in Certain Logical Systems. Algebraic and Semantical Approach*, Ph.D. dissertation, University of Warsaw, 1976.

[27] Vakarelov, D., 'Notes on N-lattices and constructive logic with strong negation', *Studia Logica* 36:109–125, 1977.

[28] Vakarelov, D., 'Intuitive Semantics for Some Three-valued Logics Connected with Information, Contrariety and subcontrariety', *Studia Logica*, 48(4):565–575, 1989.

[29] Vakarelov, D., 'Consistency, Completeness and Negation', in Gr. Priest, R. Routley and J. Norman, (eds.), *Paraconsistent Logic. Essays on the Inconsistent*, pp. 328–363, Analiytica, Philosophia Verlag, Munhen, 1989.

[30] Vorob'ev, N. N., 'Constructive propositional calculus with strong negation', (in Russian), *Doklady Academii Nauk SSSR*, 85:456–468, 1952.

[31] Vorob'ev, N. N., 'The problem of provability in constructive propositional calculus with strong negation', (in Russian), *Doklady Academii Nauk SSSR*, 85:689–692, 1952.

[32] Vorob'ev, N. N., 'Constructive propositional calculus with strong negation', (in Russian), *Transactions of Steklov's institute*, 72:195–227, 1964.

Dimiter Vakarelov
Department of Mathematical Logic with Laboratory for Applied Logic
Faculty of Mathematics and Computer Science
Sofia University
blvd James Bouchier 5
1126 Sofia
Bulgaria
dvak@fmi.uni-sofia.bg