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Dual Intuitionistic Logic and a Variety of Negations: The Logic of Scientific Research

Abstract. We consider a logic which is *semantically dual* (in some precise sense of the term) to intuitionistic. This logic can be labeled as "falsification logic": it embodies the Popperian methodology of scientific discovery. Whereas intuitionistic logic deals with constructive truth and non-constructive falsity, and Nelson's logic takes both truth and falsity as constructive notions, in the falsification logic truth is essentially non-constructive as opposed to falsity that is conceived constructively. We also briefly clarify the relationships of our falsification logic to some other logical systems.

Keywords: Verification, falsification, dual intuitionistic logic, kite of negations

1. Introduction

There is a (not so long) tradition in the literature of considering logics which are in one sense or another dual to intuitionistic. Thus, Goodman [10] uses algebraic methods to construct "logic of contradiction" which he also calls "anti-intuitionistic logic" that rests on Brouwerian algebra dual to Heyting algebra. He presents a sequent calculus with a "singleton on the left" restriction dually to a "singleton on the right" restriction characteristic to a Gentzen-type intuitionistic sequent calculus. A calculus with an analogues restriction has been introduced by Czermak [1]. Smirnov [26] analyses Goodman's logic as well as some of its possible modifications. In [27] Urbas provides an extended analysis of such sort of calculi and proves cut-elimination for some of them. Kamide [13] points out a correspondence between Goodman's logic and Nelson's constructive logic. Rauszer [20]-[22] investigates several extensions of intuitionistic logics by means of "dual operators" using algebraic and model-theoretic methods. The issue of "dual intuitionistic logics" has been addressed and generalized recently by Goré [11] within a natural framework of display calculus.

In the present paper we also arrive at the logic which appears to be dual to intuitionistic by starting with a philosophical motivation that relates log-

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ical systems to certain conceptions in methodology of science. We present an appropriate semantic construction that formally grasps the basic philosophical intuitions. This semantic structure is axiomatized by a suitable (first-degree) consequence system. In the last section we briefly clarify the relationships of our falsification logic to some other logical systems and show the place of dual intuitionistic negation within a variety of constructive and non-constructive negation operators.

2. A. Grzegorczyk and K. Popper: verification and falsification in scientific research

In his seminal paper of 1964 [12] Grzegorczyk proposed an interesting philosophical interpretation of intuitionistic logic. According to this interpretation

> "intuitionistic logic can be understood as the logic of scientific research (rather positivistically conceived) ... Scientific research (e.g. an experimental investigation) consists of the successive enrichment of the set of data by new established facts obtained by means of our method of inquiry. When making inquiries we question Nature and offer her a set of possible answers. Nature chooses one of them" [12, p. 596].

Such an understanding can easily be adopted to usual Kripke-style semantics for intuitionistic logic (see [14]). Consider a standard *Kripke-model* which is a triple $(S, \leq, \Vdash_T)$, where S is a set of states, $\leq$ is a reflexive and transitive relation on S, and $\Vdash_T$ – a specific *forcing relation* between elements of S and sentences of our language, satisfying the following hereditary condition (for every α, $\beta \in S$ and for every atomic sentence p):

Condition 2.1 (Heredity of constructive truth).
$\alpha \Vdash_T p$ and $\alpha \leq \beta \Rightarrow \beta \Vdash_T p$.

Relation $\Vdash_T$ stands for the intuitionistic notion of truth, i.e. expression "$\alpha \Vdash_T A$" means "state α forces (constructive) truth of proposition A", or simply "A is constructively true in the state α". We have the following standard definitions for compound formulas:

Definition 2.2 (*Intuitionistic connectives: truth*).
$\alpha \Vdash_T A \wedge B \Leftrightarrow \alpha \Vdash_T A$ and $\alpha \Vdash_T B$;
$\alpha \Vdash_T A \vee B \Leftrightarrow \alpha \Vdash_T A$ or $\alpha \Vdash_T B$;
$\alpha \Vdash_T {\sim}A \Leftrightarrow \forall \beta \geq \alpha (\beta \nVdash_T A)$;
$\alpha \Vdash_T A \supset B \Leftrightarrow \forall \beta \geq \alpha (\beta \Vdash_T A \Rightarrow \beta \Vdash_T B)$.

A simple proof by induction expands the hereditary condition to any formula of our language.

Taking into account the "positivistic methodological attitude" by Grzegorczyk [12, p. 598] one can informally treat S as a set of "experimental data"[1] or a set of "physical facts" (cf. [9, p. 220]). Then "$\alpha \Vdash_T A$" (where $\alpha \in S$) would mean "the collection of facts α *confirms* sentence A", or "experimental data α *verify* sentence A". As usual, relation $\leq$ stands for a possible time-relation between collections of experimental data, i.e. between stages of scientific research. In accordance with such an understanding the notion of truth employed in intuitionistic logic is often explicated as *verification* (or *verifiability*) (see, e.g. [19]). Condition 2.1 reflects the constructive character of this notion (constructive truth must be preserved forward).

Grzegorczyk considered his interpretation "philosophically plausible". But is a verificationistic interpretation of intuitionistic logic as a logic of experimental science really plausible? Such an interpretation is apparently at odds with a traditional understanding of intuitionistic logic (in the "orthodox intuitionism" by Brouwer and Heyting) as "the logic of mathematics". It seems also not to fit well the treatment of intuitionistic truth as constructive provability (known also as Brouwer-Heyting-Kolmogorov interpretation, see [28, p. 10]). Condition 2.1 means that constructive knowledge is developing *cumulatively*: a sentence once true, remains true and can never be false. That is, knowledge (true information) is always growing. This view perfectly suits the conception of mathematical knowledge, which deals with a certain kind of abstract objects and treats "verifications" as *mathematical proofs*.

However in natural science, e.g. in physics, chemistry etc. the situation seems to be quite different. Physical reality (the "real world") is subject to constant change and new experimental data often may lead us to modify our beliefs obtained on the base of previously conducted experiments. New physical facts not always just confirm earlier observations; facts that verify this or that sentence today, tomorrow may well become out-of-date or even cease. Generally speaking, for empirical science verification as constructive provability seems to be too strong notion and it is philosophically doubtful whether intuitionistic truth can represent "positivistic" conception of verification. That is, intuitionistic logic with constructive truth and hereditary condition can hardly play the role of a logic of scientific (experimental) research.

[1] Grzegorczyk uses here symbol J, calling it "the information set, i.e. the set of all possible experimental data" [12, p. 596].

We recall here the criticism on positivistic methodology by Karl Popper [17], [18], who questioned verificationism as the basis of empirical science. According to Popper the belief that we can start our scientific research with pure observations alone "is absurd" (see [18, p. 46]). The Popperian methodology of critical rationalism emphasizes the *priority of falsification (or refutation) over verification* in a scientific inquiry: "observations and experiments function in science as *tests* of our conjectures or hypotheses, i.e. as attempted refutations" [18, p. 53]. In a course of research we do not just "collect" experimental data, but rather make various conjectures (as merely temporary acceptable) and try to falsify them. If we succeed, we exclude the conjecture from the set of acceptable sentences (mark it as unacceptable). If we cannot refute our conjecture, we accept it and preserve it in our theory until refutation is found. Thus, a real examination of a conjecture consists not in its verification but in its falsification. The set of acceptable sentences (conjectures) is shrinking, whereas the set of refuted sentences is growing.

3. Truth and falsity in constructive logic

For Popper the notion of falsity (falsification, refutation) is more important in scientific research than the notion of truth (verification, proof).[2] This view is scarcely reflected in intuitionistic semantics. On the contrary, in intuitionistic logic the notion of falsity has a "subordinate" status, i.e. intuitionistic logic essentially rests upon a certain disparity between verification and falsification in favor of verification. Unlike intuitionistic truth, intuitionistic falsity is a non-constructive notion representing simply a non-truth of a sentence (where "truth" is of course the intuitionistic truth).[3] One can make this explicit by defining a new forcing relation ($\Vdash_f$) as follows:

DEFINITION 3.1 (*Non-constructive falsity*).
$\alpha \Vdash_f A \Leftrightarrow (\alpha \nVdash_T A)$.

Expression "$\alpha \Vdash_f A$" means "α forces (intuitionistic) falsity of A", or simply "A is intuitionistically false in the state α". Informal meaning of this expression is then: "A is *not proved* at the state α". Another possible reading of this expression is "A is *rejectable* at the state α". That is, experimental data α allow us to reject A so far, although it is still possible that A can

[2] As Wansing put it: "according to [Popper's philosophy of science] *falsification* is even the more important epistemological principle as compared to *verification*" [28, p. 14].

[3] Kripke involves this understanding of intuitionistic falsity by defining his intuitionistic models in [14, p. 94].

be proved later. Obviously $\Vdash_f$ does not satisfy the hereditary condition. Instead one can easily prove the following lemma expressing the fact that intuitionistic falsity is preserved in a "backward" direction:

LEMMA 3.2 (Backward heredity of nonconstructive falsity).
$\beta \Vdash_f p$ *and* $\alpha \leq \beta \Rightarrow \alpha \Vdash_f p$.

If a sentence is not proved now, then it has been never proved before (which does not exclude a possibility of proving it sometime in the future).

By definitions 2.2 and 3.1 we easily get the falsity conditions for intuitionistic connectives:

DEFINITION 3.3 (*Intuitionistic connectives: falsity*).
$\alpha \Vdash_f A \wedge B \Leftrightarrow \alpha \Vdash_f A$ or $\alpha \Vdash_f B$;
$\alpha \Vdash_f A \vee B \Leftrightarrow \alpha \Vdash_f A$ and $\alpha \Vdash_f B$;
$\alpha \Vdash_f {\sim}A \Leftrightarrow \exists \beta \geq \alpha (\beta \nVdash_f A)$;
$\alpha \Vdash_f A \supset B \Leftrightarrow \exists \beta \geq \alpha (\beta \Vdash_T A$ and $\beta \Vdash_f B)$.

Clearly, intuitionistic falsity ($\Vdash_f$) is not a quite good candidate for semantic representation of falsification. To really falsify a sentence we have to *refute* it, but refutation means of course something more than a simple absence of a proof.[4]

There is the logic, where the notion of falsity is used in a strong (constructive) sense – the logic of "constructible falsity" introduced by Nelson ([16], cf. also [30]). A *Nelson-model* is a quadruple $(S, \leq, \Vdash_T, \Vdash_F)$, where S, $\leq$, and $\Vdash_T$ are the same as in Kripke-models. But in a Nelson-model one introduces (side by side with $\Vdash_T$) another primitive forcing relation $\Vdash_F$ which stands for constructive falsity. Being a constructive notion $\Vdash_F$ should be persistent forwards (into the future), and the corresponding hereditary condition is:

CONDITION 3.4 (Heredity of constructive falsity).
$\alpha \Vdash_F p$ and $\alpha \leq \beta \Rightarrow \beta \Vdash_F p$.

Moreover, the following condition must hold for $\Vdash_T$ and $\Vdash_F$, forbidding so-called "over-determined valuations" (although the "under-determined valuations" are allowed):

CONDITION 3.5 (Consistency).
$\forall \alpha \in S \, \forall p \, (\alpha \nVdash_T p$ or $\alpha \nVdash_F p)$.

[4]Cf. the following remark by Wansing: "A propositional variable p is verified at $a \in I$, i.e. $a \in v(p)$, iff there is enough information at a to prove p. Thus, $a \notin v_0(p)$ does not mean that p is falsified at a, it merely says p is not verified at a" [28, p. 5].

For compound formulas we have the following definitions:

DEFINITION 3.6 (*Nelson's connectives*).

$\alpha \Vdash_T A \wedge B \Leftrightarrow \alpha \Vdash_T A$ and $\alpha \Vdash_T B$;
$\alpha \Vdash_F A \wedge B \Leftrightarrow \alpha \Vdash_F A$ or $\alpha \Vdash_F B$;

$\alpha \Vdash_T A \vee B \Leftrightarrow \alpha \Vdash_T A$ or $\alpha \Vdash_T B$;
$\alpha \Vdash_F A \vee B \Leftrightarrow \alpha \Vdash_F A$ and $\alpha \Vdash_F B$;

$\alpha \Vdash_T {\sim}A \Leftrightarrow \alpha \Vdash_F A$;
$\alpha \Vdash_F {\sim}A \Leftrightarrow \alpha \Vdash_T A$;

$\alpha \Vdash_T A \supset B \Leftrightarrow \forall\beta \geq \alpha\, (\beta \Vdash_F A$ or $\beta \Vdash_T B)$;
$\alpha \Vdash_F A \supset B \Leftrightarrow \alpha \Vdash_T A$ and $\alpha \Vdash_F B$.

Nelson's constructive falsity reflects Popperian idea of falsification very well. "$\alpha \Vdash_F A$" – "α forces (constructive) falsity of A" – represents a presence of a disproof (or refutation) for A rather than a simple absence of its proof. Informally this means that A is (constructively) *refuted* at the state α, or "experimental data α *refute* sentence A".

However, Nelson's logic is not "purely" falsificationistic, as it still possesses a strong – constructive – notion of truth which is entirely equal ("symmetric") to the notion of falsity.[5] But the Popperian methodology "is based upon an *asymmetry* between verifiability and falsifiability" [17, p. 41] (in favor of the latter). Thus, Nelson's logic as such is not suitable to serve as the "logic of scientific research" conceived by Popper.

4. Falsification logic: a semantic approach

Let us take the above mentioned asymmetry seriously and proceed further along the Popperian idea that falsification has priority over verification in a scientific research. To incorporate this idea we have to *exclude* constructive truth from our semantic models and consider the notion of constructive falsity (as refutation) as *the only* starting point of a *falsification logic* (**FL**)[6].

Thus, a *falsificationistic model* is a triple $(S, \leq, \Vdash_F)$, where S, $\leq$, and $\Vdash_F$ are as above (subject to the suitable conditions). That is, a falsificationistic model is just a dual Kripke-model for intuitionistic logic. S can again be interpreted as a set of experimental data or physical facts that we have at our disposal in a certain moment. In the course of scientific research we

[5] One can find more detailed considerations on the truth values in various constructive logics (as well as on the notion of non-constructive truth introduced below) in [23].

[6] We will freely use the expressions "falsification logic" and "falsificationistic logic" interchangeably.

are making various conjectures or hypotheses and try to falsify them by the way of comparison with the present experimental data or facts. A falsified conjecture, e.g. a conjecture that does not correspond ("contradicts") to facts is considered refuted and has to be rejected for ever (condition 3.4). The set of refuted statements can only be accumulated in the course of time, i.e. it is constantly growing.[7]

Within a falsificationistic model a counterpart notion of non-constructive truth can be defined as follows:

DEFINITION 4.1 (*Non-constructive truth*).
$\alpha \Vdash_t A \Leftrightarrow \alpha \nVdash_F A.$

Again, it is not difficult to show that $\Vdash_t$, being not always persistent into the future, satisfies backward heredity:

LEMMA 4.2 (Backward heredity of non-constructive truth).
$\beta \Vdash_t p$ *and* $\alpha \leq \beta \Rightarrow \alpha \Vdash_t p.$

Expression "$\alpha \Vdash_t A$" means informally "A is not refuted (falsified) at the state α", or "experimental data α do not refute sentence A". Taking into account that expression "$\alpha \Vdash_F A$", in its turn, may be interpreted as "A is unacceptable within data α", we might consider $\Vdash_t$ in a positive mode saying that a sentence can be accepted on a basis of the present experimental data (physical facts). Expression "$\alpha \Vdash_t A$" means then "data α *allow us to accept* sentence A", or "A is *acceptable* in α".

It seems that not only constructive falsity ($\Vdash_F$) appears a natural semantic counterpart for falsification, but also the non-constructive truth ($\Vdash_t$) expresses the idea of empirical verification much better than $\Vdash_T$ does. Indeed, within empirical science the requirement for constructive provability of all the statements is much too strong. In some minimal sense a hypothesis (conjecture) can be considered verified if our attempts to refute it do not succeed, i.e. if we could not find any empirical (factual) counterexample for this hypothesis. In this case we have sufficient reasons for *accepting* it, at least until convincing evidence *against* the hypothesis are found.

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[7]It is interesting to compare our falsificationistic models with a claim by Goodman that his anti-intuitionistic logic has a semantics "based on the proper Kripke structures. ... In the intuitionistic case, the Kripke structures have the property that any formula, once true remains true. The anti-intuitionistic Kripke structures, on the other hand, are Popperian. That is, they have the property that any formula, once false, remains false. Instead of proving new theorems as we go along, we refuting new conjectures" [10, p. 124]. However, except this short remark Goodman gives no details about his anti-intuitionistic semantics, and as Goré points out, "annoyingly fails to give the crucial clause for satisfiability for his 'pseudo-difference' connective"[11, p. 252].

By defining truth and falsity conditions for compound formulas we have to secure the corresponding hereditary conditions to any sentence of our language. Conjunction and disjunction can be defined as usual. As to the negation operator that can be introduced within a "pure" falsificationistic model defined above, we observe that **FL** cannot just maintain Nelson-negation, as we do not have at our disposal the notion of constructive truth any more. A "pure falsificationistic" negation, being defined (explicitly or implicitly) through a non-constructive truth, is not as "straight" as Nelson-negation. For example, we cannot directly say (on the level of a semantic definition) that not-A is refuted iff A is proved - in a genuine falsificationistic (Popperian) logic we do not have proofs, but only conjectures (hypotheses).

This situation is perfectly dual to the one we have in intuitionistic logic. Intuitionistic negation cannot be defined semantically by a direct clause: "not-A is proved iff A is refuted", as the notion of refutation is simply absent in a standard intuitionistic semantics. It is instructive to employ this duality between **FL** and intuitionistic logic by defining the (non-constructive) truth and (constructive) falsity conditions for our falsificationistic negation.[8] For the sake of "visuality" we give here parallel clauses for both $\Vdash_t$ and $\Vdash_F$:

Definition 4.3 (*Falsificationistic connectives*).

$\alpha \Vdash_t A \wedge B \Leftrightarrow \alpha \Vdash_t A$ and $\alpha \Vdash_t B$;
$\alpha \Vdash_F A \wedge B \Leftrightarrow \alpha \Vdash_F A$ or $\alpha \Vdash_F B$;

$\alpha \Vdash_t A \vee B \Leftrightarrow \alpha \Vdash_t A$ or $\alpha \Vdash_t B$;
$\alpha \Vdash_F A \vee B \Leftrightarrow \alpha \Vdash_F A$ and $\alpha \Vdash_F B$;

$\alpha \Vdash_t \sim A \Leftrightarrow \exists \beta \geq \alpha \, (\beta \Vdash_F A)$;
$\alpha \Vdash_F \sim A \Leftrightarrow \forall \beta \geq \alpha \, (\beta \Vdash_t A)$.[9]

The last two clauses characterize the acceptability (and non-acceptability) conditions for falsificationist negation which is semantically dual to intuitionistic. "$\sim A$" is to be understood as "A is unacceptable". Thus we can accept non-acceptability of A iff we can show that A can be refuted sometimes in the future. And vice versa – unacceptability of A is refuted as soon

[8] Although we continue to use in definition 4.3 the same symbol for negation ($\sim$), it is obviously *neither* intuitionistic negation (definitions 2.2 and 3.3) *nor* Nelson-negation (definition 3.6). This should not cause any confusion as in what follows, it will always be clear from a context, which negation we deal with in each case.

[9] Note that definition 4.3 says nothing about implication. It is not by chance. As we will see, it is impossible to define in **FL** any implicative connective such that both *modus ponens* and heredity (backward for $\Vdash_F$ and forward for $\Vdash_t$) hold for it (cf. theorem 4.7 below).

we succeed to demonstrate that from now on A will always be acceptable.[10]

Falsificationistic validity (*f-validity*, for short) of a sentence can be naturally defined in terms of acceptability: a sentence A is *f-valid* iff it is always acceptable, i.e. for every model $(S, \leq, \Vdash_F)$[11] and for every $\alpha \in S$ we have $\alpha \Vdash_t A$. Dually let us call a sentence *falsificationally refuted* (*f-refuted*) iff it is refuted (i.e. false) in any state from any falsificationistic model.

Now we consider in more detail some characteristic features of **FL**. As Goodman [10, p. 122] and Urbas [27, p. 440] pointed out the dual-intuitionistic logic should validate exactly the set of theorems of classical logic. This holds also for our falsificationistic models.

Theorem 4.4 (Consistency and completeness).
The set of f-valid formulas is exactly the set of theorems of classical logic.

Proof. In what follows $A \supset B$ is an abbreviation for $\sim A \vee B$. Consistency can be easily established by direct examination of the fact that all the axioms of some standard propositional calculus of classical logic are f-valid and *modus ponens* preserves f-validity.

As to completeness, it can be proved by constructing a suitable canonical model. As usual a *theory* is a set of sentences x closed under provable implication (if $A \in x$ and $\vdash A \supset B$, then $B \in x$) and conjunction (if $A \in x$ and $B \in x$, then $A \wedge B \in x$). Note that the set of theorems of classical logic is a theory. A theory x is *prime* iff the following holds: if $A \vee B \in x$, then $A \in x$ or $B \in x$. A theory is *trivial* iff it contains all the sentences of our language.

By Lindenbaum lemma (cf. the proof of lemma 5.2 below) for any sentence A and for every theory x such that $A \notin x$, there is a prime theory x' such that $A \notin x'$.

Let the canonical model be a triple $(T^c, \leq^c, \Vdash_t^c)$, where T^c is the set of all non-trivial prime theories, $x \leq^c y$ is defined as $y \subseteq x$, and $\Vdash_t^c$ (canonical

[10] Our definition of falsificationist negation differs in some important respect from the definition for "Brouwerian negation" proposed by Rauszer [22, p. 36]. Rauszer conceived her negation (as well as the connective of "Brouwerian implication" or "pseudo-difference") on a basis of an extension of intuitionistic logic and she extends correspondingly the usual Kripke-models. That is, definitions in [22] employ standard intuitionistic truth values and the truth of negated sentences is preserved forward as by usual intuitionistic connectives. Therefore Rauszer reverses the accessibility relation in definition for Brouwerian negation.

[11] Clearly, one can equivalently define a falsificationistic model with $\Vdash_t$ as primitive, i.e. as a triple $(S, \leq, \Vdash_t)$. In this case one postulate for $\Vdash_t$ the hereditary condition (lemma 4.2). $\Vdash_F$ can be then defined as $\nVdash_t$, and condition 3.4 becomes provable. In what follows we will have in mind just this formulation of falsificationistic models.

valuation) is defined as follows: $x \Vdash_t^c p \Leftrightarrow p \in x$.[12] It is not difficult to show by induction that the canonical valuation is extended to any sentence of our language. We consider here only the case with sentences of the form $\sim A$.

Let $\sim A \in x$. Then $A \notin x$ (otherwise x would be trivial). By reflexivity of $\subseteq$ we have: $\exists y\,(y \subseteq x$ and $A \notin y)$. By definition of $\leq^c$ and inductive hypothesis: $\exists y\,(x \leq^c y$ and $y \nVdash_t^c A)$. Hence, $x \Vdash_t^c \sim A$.

Let $\sim A \notin x$. Then for any $y \subseteq x$, $\sim A \notin y$. As $\sim A \vee A$ is a theorem of classical logic, we get $\sim A \vee A \in y$. Hence (because y is prime), $\sim A \in y$ or $A \in y$. Thus, $A \in y$, i.e. $\forall y\,(y \subseteq x \Rightarrow A \in y)$. By definition of $\leq^c$ and inductive hypothesis we get $\forall y\,(x \leq^c y \Rightarrow A \in y)$ and so $x \Vdash_F^c \sim A$.

Thus, $(T^c, \leq^c, \Vdash_t^c)$ is a falsificationistic model indeed.

Consider now an arbitrary sentence A such that $A \notin x^L$ (where x^L is the set of theorems of classical logic). As was noted above x^L is a theory. By Lindenbaum lemma x^L can be extended to a prime theory x' such that $A \notin x'$. x' is non-trivial, hence $x' \in T^c$. By definition of canonical valuation we have $x' \nVdash_t^c A$, that is A is not f-valid. ■

Corollary 4.5 (Dual Glivenko).
A sentence A is f-valid iff it is classically valid.

Although falsificationistic logic and classical logic are coincident in the sets of their valid formulas, they nevertheless differ in the sets of refuted propositions. Namely, not every classical contradiction appears f-refuted. For example, it is easy to check that there may well be S and $\alpha \in S$ such that $\alpha \Vdash_t A \wedge \sim A$. (At the same time formula $\sim (A \wedge \sim A)$ remains f-valid!) This allows one to consider **FL** a kind of *paraconsistent* logic (cf. [26], [27, p. 441]).

However the most important aspect in which **FL** differs from classical logic lays in consequence relation. Let us define this relation for falsificationistic models in a standard way:

Definition 4.6 (*Falsificationistic consequence*).
$A \models B \Leftrightarrow \forall S \forall \alpha \in S\,(\alpha \Vdash_t A \Rightarrow \alpha \Vdash_t B)$.

If we have $A \models B$, then we say that $A \models B$ is a valid falsificationistic consequence (f-consequence). Now it is not difficult to see that, e.g. the following statements are *not* valid: $A \models \sim\sim A$; $A \wedge (\sim A \vee B) \models B$; $A \wedge \sim A \models B$; $\sim A \wedge \sim B \models \sim (A \vee B)$. An interesting effect of this is that one cannot define within **FL** anything like "falsificationist implication" which

[12] Obviously $\Vdash_F^c$ is defined then as follows: $x \Vdash_F^c p \Leftrightarrow p \notin x$.

would satisfy both deduction theorem and backward heredity. This fact is established in the following theorem:

THEOREM 4.7 (Indefinability of implication).
*It is impossible to define in **FL** any connective* $*$ *such that* $\models A * B \Leftrightarrow A \models B$.

PROOF. We modify as appropriate an algebraic proof from [10] and a syntactic proof from [27]. Assume one could define within **FL** such a connective $*$. Then it would be definable in classical logic as well. As $A \models A$ is a valid f-consequence, $A * A$ would be f-valid and hence classically valid. But then $A * {\sim}{\sim}A$ would also be classically valid and so f-valid. As a result $A \models {\sim}{\sim}A$ would appear valid f-consequence what is impossible. ∎

Some authors (see, e.g. [10], [20]-[22]) formulate dual (or anti-) intuitionistic logic with so-called "anti-implication" or "pseudo-difference" as a primitive connective (let us mark it with $\div$) which is dual to intuitionistic implication (Wolter [29] calls it *coimplication*). Negation can be defined then via anti-implication and constant $\top$[13]: ${\sim}A \Leftrightarrow \top \div A$. Goodman [10, p. 121] proposes to read $\div$ as "but not", i.e. $A \div B$ would mean then "A but not B". Urbas [27, p. 451] remarks that such an interpretation is not quite satisfactory as it suggests an equivalence with $A \wedge {\sim}B$ which generally does not hold. Instead, Urbas suggests the $A \div B$ be interpreted as "A excludes B".

In the present paper we explicitly content our consideration with a "first-degree level", thus concentrating on a set of binary consequence statements of the form (A, B) where both A and B may contain only $\vee, \wedge$ and $\sim$. That is, neither A nor B include any implications or co-implications. As we believe, it allows us to focus our attention on various kinds of constructive negations and their duals. We leave the issue of extending **FL** to richer languages (possibly containing implication-like connectives) for a future work.

5. Falsification logic: an axiomatization

It is quite remarkable that **FL** and classical logic — being identical in the set of their valid propositions — have different consequence relations. It can be viewed as another argument in favor of understanding logic not as a set of theorems, but a set of correct consequences. Thus, we construct falsification logic as a (first degree) consequence system $(L, \vdash)$, where L is the set of sentences of the language of **FL** (with $\wedge$, $\vee$ and $\sim$), and $\vdash$ (consequence) is a binary relation over L satisfying the following postulates and rules:

[13] Satisfying the condition $A \models \top$, where A is any sentence of our language.

$a1.\ A \vdash A$
$a2.\ A \wedge B \vdash A$
$a3.\ A \wedge B \vdash B$
$a4.\ A \vdash A \vee B$
$a5.\ B \vdash A \vee B$
$a6.\ A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$
$a7.\ {\sim}{\sim}A \vdash A$
$a8.\ B \vdash A \vee {\sim}A$

$r1.\ A \vdash B,\ B \vdash C / A \vdash C$
$r2.\ A \vdash B,\ A \vdash C / A \vdash B \wedge C$
$r3.\ A \vdash C,\ B \vdash C / A \vee B \vdash C$
$r4.\ A \vdash B / {\sim}B \vdash {\sim}A$
$r5.\ A \vdash B \vee C / {\sim}C \vdash {\sim}A \vee B.$[14]

Note that if we replace $a7$, $a8$ and $r5$ by their duals:

(1) $A \vdash {\sim}{\sim}A$
(2) $A \wedge {\sim}A \vdash B$

and

(3) $A \wedge B \vdash C / A \wedge {\sim}C \vdash {\sim}B$

respectively, we get an (implication-free) intuitionistic consequence system.

For the sake of illustration we present here the proof in **FL** of the consequence $B \vdash {\sim}(A \wedge {\sim}A)$:

1. $A \wedge {\sim}A \vdash A$ $(a2)$
2. $A \wedge {\sim}A \vdash {\sim}A$ $(a3)$
3. ${\sim}A \vdash {\sim}(A \wedge {\sim}A)$ $(1;\ r4)$
4. ${\sim}{\sim}A \vdash {\sim}(A \wedge {\sim}A)$ $(2;\ r4)$
5. ${\sim}A \vee {\sim}{\sim}A \vdash {\sim}(A \wedge {\sim}A)$ $(3,\ 4;\ r3)$
6. $B \vdash {\sim}A \vee {\sim}{\sim}A$ $(a8)$
7. $B \vdash {\sim}(A \wedge {\sim}A)$ $(5,\ 6;\ r1)$

It remains to prove the adequacy of the consequence system **FL** to proposed semantics.

THEOREM 5.1 (Consistency).
If $A \vdash B$, then $A \models B$.

[14] The rule $r5$ was absent in an earlier version of this paper (see [25]), although it is needed for completeness. I am grateful to Chunlai Zhou and Michael Dunn for pointing this out to me.

PROOF. We leave to an "interested reader" a direct check that all the schemes $a1$-$a8$ represent valid consequences, and that the rules $r1$-$r5$ preserve such validity. ■

Completeness can be proved by a canonical model construction. We define a *theory* as a set of sentences closed under $\vdash$ (i.e. for every theory x, if $A \in x$ and $A \vdash B$, then $B \in x$) and $\wedge$ (if $A \in x$ and $B \in x$, then $A \wedge B \in x$). A prime and trivial theory are defined as in proof of theorem 4.4.

LEMMA 5.2 (Lindenbaum).
For any A and B, if $A \nvdash B$, then there exists a prime theory x such that $A \in x$ and $B \notin x$.

PROOF. The proof is quite standard. We reproduce here a variant of an analogues proof from [6, p. 13] adopting it to **FL**. Define $x_0 = \{C : A \vdash C\}$. It is not difficult to see that x_0 is a theory, $A \in x_0$ and $B \notin x_0$. We enumerate all the sentences of our language: $A_1, A_2, A_3, ...$ and then construct a sequence of theories starting from x_0 and defining $x_{n+1} = x_n + A_{n+1}$ if $B \notin x_n + A_{n+1}$, otherwise $x_{n+1} = x_n$ ($y + C$ is the smallest theory we get by closing $y \cup \{C\}$ by $\vdash$ and $\wedge$). We can now define a theory x we are seeking for as the union of all the x_n belonging to the sequence of theories so constructed. It is easy to see that x is a theory. Moreover, it is maximal theory such that $B \notin x$. One can show that it is also prime. Namely, assume $D \vee E \in x$ and yet $D \notin x$ and $E \notin x$. Consider theories $x + D$ and $x + E$. Since both these theories are proper extensions of x we have $B \in x + D$ and $B \in x + E$. Hence, there exist $C_1, ..., C_i \in x$ such that $C_1 \wedge ... \wedge C_i \wedge D \vdash B$ and $C_1 \wedge ... \wedge C_i \wedge E \vdash B$. Applying $r3$ and $a6$ we get $C_1 \wedge ... \wedge C_i \wedge (D \vee E) \vdash B$. But this would mean that $B \in x$. A contradiction. ■

Lemma 5.2 states in effect that any theory of a certain kind can be extended to a *prime* theory of the same kind. We formulate now a lemma which is in a sense dual, saying that any prime theory of a certain kind can be in a certain way *contracted* while preserving primeness.

LEMMA 5.3 (Contraction).
Let x be a non-trivial prime theory such that $A \in x$ and $\sim A \in x$. Then there exists a prime theory y such that $y \subseteq x$ and $A \notin y$.

PROOF. We make use of "Dual-to-Lindenbaum" lemma by Łukowski (see [15, pp. 66-67]). First, notice that $\sim A \nvdash A$. Otherwise it would be not difficult to show (using $a1$, $a7$ and $a8$) that $\sim A \vdash B$ for any B which would contradict the assumption of non-triviality of x. Note also that $a1 - a8$, $r1 -$

$r5$ determine a Tarski-style operation of deductive closure C on L. Following Łukowski, we can uniquely define a counterpart elimination operation E (see [15, p. 60]) that allows us to eliminate sentences from a given theory. Define an E-theory as a set of sentences closed under E. The notion of a relatively minimal E-theory see in [15, p. 66]. Consider a theory $y_0 = \{B \in x : B \nvdash A\}$. Clearly, $A \notin y_0$ and $\sim A \in y_0$. Moreover, by Łukowski's lemma [15, pp. 66-67] there exists an E-theory y minimal relative to $\sim A$ such that $y \subseteq y_0$. Since y is a relatively minimal E-theory, it is also a maximal theory. Taking into account that any maximal theory is prime, we get our lemma.[15] ■

Let the canonical model $(T, \leq, \Vdash_t)$ and canonical valuation be defined as in the proof of theorem 4.4 *mutatis mutandis*. Then we have

LEMMA 5.4 (Canonical model).
$(T^c, \leq^c, \Vdash^c_t)$ *is a falsificationistic model.*

PROOF. It is easy to see that $\leq^c$ is a partial order and that backward heredity holds for $\Vdash^c_t$. We have to show that definition of $\Vdash^c_t$ holds for any sentence of our language. This can be done by induction of formula construction. The cases with $\vee$ and $\wedge$ are straightforward. For negation consider any sentence $\sim A$ and let the definition of canonical valuation holds for A (inductive hypothesis).

Let $\sim A \in x$. As to A, either $A \notin x$, or $A \in x$. In the first case, by reflexivity of $\subseteq$, $\exists y\,(y \subseteq x \text{ and } A \notin y)$. In the second case the same holds by lemma 5.3. By definition of $\leq^c$ and inductive hypothesis we get $\exists y\,(x \leq^c y \text{ and } y \nVdash^c_t A)$. Hence, $x \Vdash^c_t \sim A$.

Let $\sim A \notin x$. Then for any $y \subseteq x$ we have $\sim A \notin y$. Note that $\sim A \vee A$ belongs to any theory, i.e. $\sim A \vee A \in y$, and by the primeness of y, $\sim A \in y$ or $A \in y$. Hence $A \in y$. That is, $\forall y\,(y \subseteq x \Rightarrow A \in y)$. By definition of $\leq^c$ and inductive hypothesis we get $\forall y\,(x \leq^c y \Rightarrow A \in y)$ and so $x \Vdash^c_F \sim A$. ■

THEOREM 5.5 (Completeness).
If $A \models B$, *then* $A \vdash B$.

PROOF. We argue by contraposition. Suppose $A \nvdash B$. Then by lemma 5.2 there exists a prime theory x such that $A \in x$ and $B \notin x$. x is non-trivial, hence $x \in T^c$. By definition of canonical valuation $x \Vdash^c_t A$ and $x \nVdash^c_t B$, i.e. $A \nvDash B$. ■

[15] Contraction lemma is dual to Lindenbaum only to a certain respect, or better to say in a certain sense of "duality". Like Lindenbaum it deals with prime theories, but looking *into* them rather than *over* them. It may be therefore more appropriate to label it as "reverse Lindenbaum lemma". As opposed to this, Łukowski's lemma is dual Lindenbaum in a deep sense of the term.

6. The united kite of negations. Minimal falsificationistic negation

Dunn in [4] and [5] considers several kinds of negations, starting from what he calls "subminimal negation" and up to the classical one. Namely, let a (positive) consequence system $\mathfrak{S}$ be determined by the axioms and rules $a1 - a6$, $r1 - r3$ from the previous section, and let us consider the following extra schemata for negation:

Contraposition: $A \vdash B / {\sim}B \vdash {\sim}A$;
Constructive contraposition: $A \vdash {\sim}B / B \vdash {\sim}A$;
Classical contraposition: ${\sim}A \vdash B / {\sim}B \vdash A$;
Conjunctive contraposition: $A \wedge B \vdash C / A \wedge {\sim}C \vdash {\sim}B$;
Disjunctive contraposition: $A \vdash B \vee C / {\sim}C \vdash {\sim}A \vee B$;
Constructive double negation: $A \vdash {\sim}{\sim}A$;
Classical double negation: ${\sim}{\sim}A \vdash A$;
Absurdity: $A \wedge {\sim}A \vdash B$;
Negated absurdity: $A \wedge {\sim}A \vdash {\sim}B$;
Triviality: $B \vdash A \vee {\sim}A$;
Negated triviality: ${\sim}B \vdash A \vee \sim A$.

Then we have:

Subminimal negation (**Sub**) = $\mathfrak{S}$ + *contraposition.*

As Dunn points out, subminimal negation has been introduced by Hazen [7], who in turn follows a suggestion of Humberstone. Note, that Dunn's considerations do not generally presuppose the presence of conjunction and disjunction. Dunn [4, pp. 5, 20] remarks that if we have conjunction and disjunction it can be useful to require for subminimal negation to satisfy an additional axiom ${\sim}A \wedge {\sim}B \vdash \sim (A \vee B)$ and he calls the operator so defined *preminimal negation.* It is just the negation of Došen's system **N** from [2] and [3].

Quasi-minimal negation (**Qua**) = **Sub** + *constructive double negation.* Incidentally, ${\sim}A \wedge {\sim}B \vdash \sim (A \vee B)$ becomes then derivable (together with two others intuitionistically acceptable De Morgan laws). Another way to get **Qua** is to directly extend $\mathfrak{S}$ by the rule of *constructive contraposition.*

It is interesting to observe that what we call here "quasi-minimal"[16] negation is called just "minimal" in [4] and [5]. However, the latter label is usually reserved for the negation operator of famous *minimal logic* of

[16] The label first suggested by J.M. Dunn (personal communication).

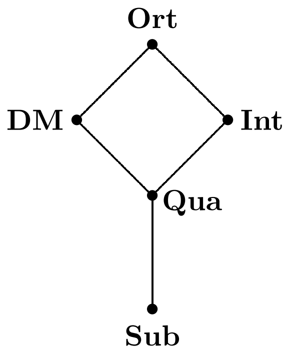

Figure 1. Dunn's Kite of Negations

Johansson [8], a characteristic feature of which demands that although a contradiction does not imply any sentence it does imply the negation of any sentence. Quasi-minimal negation appears not to conform to this demand.

Intuitionistic negation (**Int**) = **Qua** + *absurdity.* Note, that this formulation is from [4], where *absurdity* is taken in the form: $A \vdash B, A \vdash {\sim}B / A \vdash C$, as Dunn does not necessarily has conjunction in his logics. In the full language we have to postulate for intuitionistic negation to satisfy additionally *conjunctive contraposition*[17] also known under the name *antilogism.*

De Morgan negation (**DM**) = **Qua** + *classical double negation.* De Morgan negation can alternatively be defined by requiring that it satisfies both *constructive contraposition* and its dual (*classical contraposition*).

Ortho negation (**Ort**) = **Qua** + *classical double negation* + *absurdity.* Ortho negation is just *classical* (Boolean) negation, if the whole framework of $\mathfrak{S}$ is present. But as Dunn [5, p. 26] remarks, it can also be more generally studied in the (non-distributive) framework of quantum logic.

Dunn considers also a possibility of having in one system a pair of different negation operators connected by a so-called *Galois property.* He calls such a pair *Galois connected negations* or *split negation*, placing it between subminimal and quasi-minimal negations. We can ignore this aspect here as we will always presuppose a context with one negation operator within one logical system.

The relationships among negations introduced above can be summarized in a form of a diagram (resembling a child's kite, whence its name) as shown in Figure 1.[18] Moreover, the above observation concerning quasi-minimal

[17] I am grateful to Dunn and Zhou for turning my attention to this fact.

[18] We omit here preminimal negation mentioned above which should be placed between

negation suggests a natural extension of this construction by minimal negation proper (*à la* Johansson):

Minimal negation (**Min**) = **Qua** + *conjunctive contraposition*. That is, **Min** is just **Int** without unrestricted *absurdity*.

In **Min** *negated absurdity* can easily be derived by a one-step application of *conjunctive contraposition* to $a2$. If our language does not include conjunction and disjunction, *negated absurdity* has to be postulated instead of *conjunctive contraposition*, e.g. in the form $A \vdash B$, $A \vdash {\sim}B / A \vdash {\sim}C$. Then we can get intuitionistic negation directly from minimal negation by replacing *negated absurdity* by the unrestricted one.

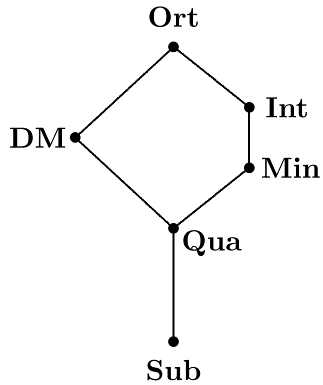

Figure 2. Extended Kite of Negations

Min is naturally settled between quasi-minimal and intuitionistic negations thus producing (a bit lopsided) "extended kite of negations" (Figure 2). It is interesting to observe that some of the negations in Figure 2 allow dual counterparts. Namely, we can define:

Dual quasi-minimal negation (**D-Qua**) = **Sub** + *classical double negation*. **D-Qua** is a departure from subminimal negation in a different direction as **Qua**. It can also be obtained by extending system $\mathfrak{S}$ simply with *classical contraposition*.

Analogously as with quasi-minimal negation, we face at this point an interesting alternative: either to turn to De Morgan negation by adding *constructive double negation* to **D-Qua**, or to continue our way towards dual minimal negation and, further, dual intuitionistic negation:

Sub and **Qua**.

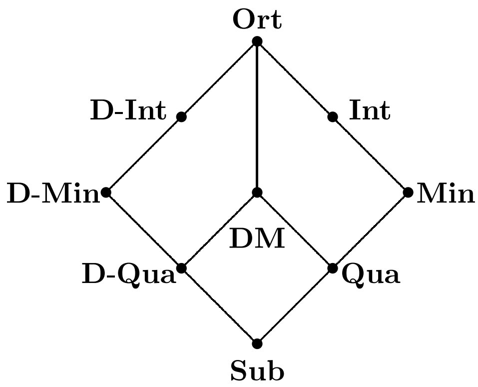

Figure 3. United Kite of Negations

Dual minimal negation (**D-Min**) = **D-Qua** + *disjunctive contraposition.* (*Negated triviality* is derivable in **D-Min**).

Dual intuitionistic negation (**D-Int**) = **D-Min** + *triviality.* In this way we just regain our falsification logic **FL**.

Then once again we end up with ortho negation by extending dual intuitionistic (falsificationistic) negation with *constructive double negation.*

It is easy to see that replacing of **Qua**, **Min** and **Int** in Figure 2 by **D-Qua**, **D-Min** and **D-Int** respectively gives us the "dual kite of negations". Both kites can be united within a joint diagram (Figure 3) which reflects the relationships between all the negations introduced above.

Note, that as dual intuitionistic negation appears to be virtually coincident with falsificationistic negation, **D-Min** can be regarded as *minimal falsificationistic negation.*[19] The model for **D-Int** (i.e. a falsificationistic model) immediately suggests an analogous construction for minimal falsification logic that can be obtained by an appropriate "dualizing" of a semantics for minimal logic (see, e.g. [24]).

That is, a *minimal falsificationistic model* is a quadruple $(S, N, \leq, \Vdash_F)$, where S, $\leq$, and $\Vdash_F$ are as in falsificationistic model above, and $N \subseteq S$. We take again definition 4.1 for $\Vdash_t$ and observe that lemma 4.2 holds. The crucial clauses for **D-Min** are then as follows:

Definition 6.1 (*Minimal falsificationistic negation*).
$\alpha \Vdash_t {\sim}A \Leftrightarrow \exists\beta \geq \alpha\, (\beta \in N \text{ and } \beta \Vdash_F A)\,;$

[19]In the presence of anti-implication this would imply definition ${\sim}A \Leftrightarrow t \div A$, with no further conditions on t (which would mean that t could be just some arbitrary chosen propositional variable).

$\alpha \Vdash_F \sim A \Leftrightarrow \forall \beta \geq \alpha\, (\beta \in N \Rightarrow \beta \Vdash_t A)$.

A direct check shows that although unrestricted *triviality* is not valid under this definition, *negated triviality* is.

Informally N can be interpreted as a set of *normal states*, e.g. a set of experimental data which are consistent and can serve as a reliable source for scientific research. Then definition 6.1 says that $\sim A$ is acceptable in α iff A is refuted in some *normal* state β accessible from α. And vice versa, $\sim A$ is refuted in α iff we can show that A will be acceptable in any future state provided this state is normal.

Summing up, United Kite of Negations presents a general structure which allows us to locate our falsification logic among a broad variety of logical systems. We leave for a future work developing a uniform semantic construction for the whole United Kite.

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References

[1] Czermak, J., 'A remark on Gentzen's calculus of sequents', *Notre Dame Journal of Formal Logic* 18:471–474, 1977.

[2] Došen, K., 'Negation as a modal operator', *Reports on Mathematical Logic* 20:15–27, 1986.

[3] Došen, K., 'Negation in the light of modal logic', in D. M. Gabbay and H. Wansing, (eds.), *What is Negation?*, Applied Logic Series, 13, pp. 77–86, Kluwer Academic Publishers, Dordrecht, 1999.

[4] Dunn, J. M., 'Generalized ortho negation', in H. Wansing, (ed.), *Negation. A Notion in Focus*, pp. 3–26, De Gruyter, Berlin, 1995.

[5] Dunn, J. M., 'A Comparative study of various model-theoretic treatments of negation: a history of formal negation', in D. M. Gabbay and H. Wansing, (eds.), *What is Negation?*, Applied Logic Series, 13, pp. 23–51, Kluwer Academic Publishers, Dordrecht, 1999.

[6] Dunn, J. M., 'Partiality and its dual', *Studia Logica* 65:5–40, 2000.

[7] Hazen, A., *Subminimal Negation.* Unpublished Manuscript, 1992.

[8] Johansson, I., 'Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus', *Compositio Mathematika* 4:119–136, 1937.

[9] Fitting, M., *Proof Methods for Modal and Intuitionistic Logics.* (Synthese Library; v. 169) D. Reidel Publishing Company, Dordrecht, Boston, Lancaster, 1983.

[10] Goodman, N., 'The logic of contradiction', *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* 27:119–126, 1981.

[11] Goré, R., 'Dual intuitionistic logic revisited', in R. Dyckhoff, (ed.), *Proceedings Tableaux 2000* (Lecture Notes in Artificial Intelligence 1847), pp. 252–267, Springer Verlag, Berlin, 2000.

[12] Grzegorczyk, A., 'A philosophically plausible formal interpretation of intuitionistic Logic', *Indagationes Matimaticae* 26:596–601, 1964.

[13] Kamide, N., 'A note on dual-intuitionistic logic', *Mathematical Logic Quarterly*, 49:519–524, 2003.

[14] Kripke, S., 'Semantical analysis of intuitionistic logic I', in J.N. Crossley and M.A.E. Dumment, (eds.), *Formal Systems and Recursive Functions*, pp. 92–130, North-Holland Publishing Company, Amsterdam, 1965.

[15] Łukowski, P., 'A deductive-reductive form of logic: general theory and intuitionistic case', *Logic and Logical Philosophy* 10:59–78, 2002.

[16] Nelson, D., 'Constructible falsity', *Journal of Symbolic Logic* 14:16–26, 1949.

[17] Popper, K., *The Logic of Scientific Discovery*, Routledge, London and New York, 1992.

[18] Popper, K., *Conjectures and Refutations. The Growth of Scientific Knowledge*, Routlege, London, 1989.

[19] Rabinowicz, W., 'Intuitionistic truth', *Journal of Philosophical Logic* 14:191–228, 1985.

[20] Rauszer, C., 'Semi-Boolean algebras and their applications to intuitionistic logic with dual operations', *Fundamenta Mathematicae* LXXXIII:219–249, 1974.

[21] Rauszer, C., 'A formalization of the propositional calculus of H-B-logic', *Studia Logica* XXXIII:23–34, 1974.

[22] Rauszer, C., 'An algebraic and Kripke-style approach to a certain extension of intuitionistic logic', *Dissertationes Mathematicae*, CLXVII, PWN, Warszawa, 1980.

[23] Shramko, Y., J. M. Dunn, and T. Takenaka, 'The trilaticce of constructive truth values', *Journal of Logic and Computation* 11:761–788, 2001.

[24] Shramko, Y., 'Semantics for constructive negations', in H. Wansing (ed.) *Essays on Non-Classical Logic*, pp. 187–217, World Scientific, New Jersey – London – Singapore – Hong Kong, 2001.

[25] Shramko, Y., 'The logic of scientific research' (In Russian), in *Logic and V.E.K.: Festschrift dedicated to 90th birthday of Prof. E.K. Voishvillo*, pp. 237–249, Sovremennye tetradi, Moscow, 2003. (The On-line version see: Logical Studies, Online Journal, No. 10, 2003, http://www.logic.ru/Russian/LogStud/10/No10-20.html, ISBN 5-85593-140-4).

[26] Smirnov, V. A., 'On a system of paraconsistent logic' (in Russian), in V. A. Smirnov and A.S. Karpenko, (eds.), *Many-valued, Relevant ans Paraconsistent Logics*, pp. 129–133, Moscow, 1984, (Reprinted in V.A. Smirnov, *A Theory of Logical Inference*, pp. 293–298, ROSSPEN, Moscow, 1999).

[27] Urbas, I., 'Dual-intuitionistic logic', *Notre Dame Journal of Formal Logic* 37:440–451, 1996.

[28] Wansing, H., *The Logic of Information Structures*, Springer-Verlag, Berlin Heidelberg, 1993.

[29] Wolter, F., 'On logics with coimplication', *Journal of Philosophical Logic* 27:353–387, 1998.

[30] Zaslavskij, I. D., *Symmetrical Constructive Logic* (in Russian), Erevan, 1978.

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