Sergei P. Odintsov*

The Class of Extensions of Nelson's Paraconsistent Logic

Abstract. The article is devoted to the systematic study of the lattice $\mathcal{E}\mathbf{N4}^{\perp}$ consisting of logics extending $\mathbf{N4}^{\perp}$. The logic $\mathbf{N4}^{\perp}$ is obtained from paraconsistent Nelson logic $\mathbf{N4}$ by adding the new constant $\perp$ and axioms $\perp \to p$, $p \to \sim\perp$. We study interrelations between $\mathcal{E}\mathbf{N4}^{\perp}$ and the lattice of superintuitionistic logics. Distinguish in $\mathcal{E}\mathbf{N4}^{\perp}$ basic subclasses of explosive logics, normal logics, logics of general form and study how they are relate.

Keywords: paraconsistent logic, strong negation, **N4**-lattice, lattice of logics.

1. Introduction

This article continues the study of extensions of Nelson's paraconsistent logic started in [10, 11] and in what follows we assume the acquaintance of [11]. In [10], we have introduced different kinds of semantics for Nelson's logic **N4**, in particular, we have defined the variety $\mathcal{V}_{\mathbf{N4}}$ of **N4**-lattices such that the lattice $\mathcal{E}\mathbf{N4}$ of **N4**-extensions is dually isomorphic to the lattice $Sub(\mathcal{V}_{\mathbf{N4}})$ of subvarieties of the variety $\mathcal{V}_{\mathbf{N4}}$. We have also proved that **N4**-lattices can be represented as twist-structures over implicative lattices. The article [11] contains the origins of algebraic theory of **N4**-lattices. In this way, the articles [10] and [11] provide the necessary semantical basis for the study of $\mathcal{E}\mathbf{N4}$. But we will not do it and slightly change the object of investigation. The logic **N3**, the explosive extension of **N4**, is usually considered in the language with two negations, strong $\sim$ and intuitionistic $\neg$, despite the fact that $\neg$ can be defined through $\sim$ as follows: $\neg\varphi \leftrightarrow \varphi \to \sim\varphi$. The last equivalence explains also the name "strong negation". If we pass from **N3** to **N4**, the above relation will not define neither intuitionistic nor minimal negation. Due to this reason the logic **N4** is usually considered in the language with the strong negation $\sim$ only. However, as we will see, the presence of intuitionistic negation is desirable in this setting and the semantics in terms

* *The author acknowledges support by the Alexander von Humboldt-Stiftung and by Counsil for Grants under RF President, project NSh – 2112.2003.1.*

Special Issue Negation in Constructive Logic
Edited by **Heinrich Wansing, Sergei Odintsov, and Yaroslav Shramko**

of twist-structures allows to obtain a philosophically plausible combination of paraconsistent strong negation and explosive intuitionistic negation. The interpretation of Nelson's logic as a logic of information structures has a long tradition, see, e.g., [20, 8]. The presentation of **N4**-lattices in terms of twist-structures over implicative lattices allows to consider the logic **N4** as an information superstructure over positive logic and arbitrary **N4**-extension as an information superstructure over some extension of positive logic. Recall that implicative lattices provide semantics for positive logic and its extensions. Thus, the underlying theory is positive. It is natural to consider an information logic over logic considered in the full language containing negation. It is also natural to assume that this logic describing one or another object domain is explosive and only the information superstructure is paraconsistent. According to these considerations the replacement of implicative lattices by Heyting algebras in the definition of twist-structures will results in the logic which can be considered as an information superstructure over intuitionistic logic. This resulting logic is denoted $\mathbf{N4}^{\perp}$ and the lattice of its extensions $\mathcal{E}\mathbf{N4}^{\perp}$ will be the main object of investigations in this article.

2. Preliminaries

We will consider logics in the propositional languages $\mathcal{L} := \{\vee, \wedge, \rightarrow, \sim\}$ with the symbol $\sim$ for strong negation and $\mathcal{L}^{\perp} := \mathcal{L} \cup \{\perp\}$ with additional symbol for the constant of absurdity. The connectives of equivalence $\leftrightarrow$ and of strong equivalence $\Leftrightarrow$ are defined as follows: $\varphi \leftrightarrow \psi \rightleftharpoons (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ and $\varphi \Leftrightarrow \psi \rightleftharpoons (\varphi \leftrightarrow \psi) \wedge (\sim \psi \leftrightarrow \sim \varphi)$. By a *logic* we mean a set of formulas closed under the rules of substitution and *modus ponens*. By For ($For^{\perp}$) we denote the trivial logic, i.e., the set of all formulas of the language $\mathcal{L}$ ($\mathcal{L}^{\perp}$). For a logic L and a set of formulas X, denote $L + X$ the least extension of L containing X. By $+$ we denote also the join operation in the lattice of logics. Logics will be defined usually via Hilbert-style deductive systems with the only rules of substitution and *modus ponens*. In this way, to define a logic it is enough to give its axioms. The paraconsistent Nelson's logic **N4** is a logic in the language $\mathcal{L}$ characterized by the axioms of positive logic and the following axioms for strong negation:

A1. $\sim\sim p \leftrightarrow p$

A2. $\sim(p \vee q) \leftrightarrow (\sim p \ \wedge \sim q)$

A3. $\sim(p \wedge q) \leftrightarrow (\sim p \ \vee \sim q)$

A4. $\sim(p \rightarrow q) \leftrightarrow (p \ \wedge \sim q)$

To obtain the explosive Nelson's logic **N3** one should add to the list of **N4**-axioms the axiom

$$\text{A5. } \sim p \to (p \to q)$$

The logic $\mathbf{N4}^{\perp}$ is a logic in the language $\mathcal{L}^{\perp}$ determined by the axioms of **N4** and the two additional axioms for the constant $\perp$:

$$\text{A6. } \perp \to p \text{ and A7. } p \to \sim \perp.$$

Due to axiom A6 the intuitionistic negation can be defined in $\mathbf{N4}^{\perp}$ as $\neg\varphi := \varphi \to \perp$. If we put $\perp := \sim(p_0 \to p_0)$, we can prove

$$\mathbf{N3} \vdash \perp \to p,\ p \to \sim \perp.$$

In particular, the intuitionistic negation is definable in **N3**. This is the reason, why we do not distinguish the logics **N3** and $\mathbf{N3}^{\perp}$.

We say that a formula φ in the language $\mathcal{L}$ or $\mathcal{L}^{\perp}$ is in *negation normal form* (*nnf*) if it contains $\sim$ only before atomic formulas. The following translation $\overline{(\cdot)}$ sends every formula φ to a formula in negation normal form, where $p \in Prop$ and $\diamond \in \{\vee, \wedge, \to\}$:

$$\begin{array}{rclrcl}
\overline{p} & = & p & \overline{\sim p} & = & \sim p \\
\overline{\sim\sim\varphi} & = & \overline{\varphi} & \overline{\varphi \diamond \psi} & = & \overline{\varphi} \diamond \overline{\psi} \\
\overline{\sim(\varphi \vee \psi)} & = & \overline{\sim\varphi} \wedge \overline{\sim\psi} & \overline{\sim(\varphi \wedge \psi)} & = & \overline{\sim\varphi} \vee \overline{\sim\psi} \\
\overline{\sim(\varphi \to \psi)} & = & \overline{\varphi} \wedge \overline{\sim\psi} & & &
\end{array}$$

PROPOSITION 2.1. *For any* $\varphi \in For(For^{\perp})$, $\mathbf{N4}(\mathbf{N4}^{\perp}) \vdash \varphi \leftrightarrow \overline{\varphi}$.

The *proof* easily follows from strong negation axioms A1–A4.

An important peculiarity of Nelson's systems **N3**, **N4** and $\mathbf{N4}^{\perp}$ is that the provable equivalence is not a congruence relation. However, the presence of axioms of positive logic means that the provable equivalence is a congruence relation with respect to positive connectives. More exactly, for any formulae φ_0, φ_1 and positive formula $\psi(p)$ with a propositional parameter, the provability of $\varphi_0 \leftrightarrow \varphi_1$ in **N3** (or **N4**, $\mathbf{N4}^{\perp}$) implies the provability of $\psi(\varphi_0) \leftrightarrow \psi(\varphi_1)$ in that logic. The next proposition easily follows from the existence of *nnf* for formulas in Nelson's logics and shows that these logics are closed under a weak form of replacement rule.

PROPOSITION 2.2. *The logics* **N4** *and* $\mathbf{N4}^{\perp}$ *are closed under the weak replacement rule*

$$\frac{\varphi_0 \leftrightarrow \varphi_1 \quad \sim\varphi_0 \leftrightarrow \sim\varphi_1}{\psi(\varphi_0) \leftrightarrow \psi(\varphi_1)}.$$

It follows immediately from this proposition that the provable strong equivalence $\Leftrightarrow$ will be a congruence relation in **N4** and $\mathbf{N4}^{\perp}$ as well as in **N3**.

As concerns the semantical notation, a *matrix* is, as usual, a pair $\mathcal{M} = \langle \mathcal{A}, D^{\mathcal{A}} \rangle$, where $\mathcal{A}$ is an algebra and $D^{\mathcal{A}} \subseteq A$ the set of distinguished elements. In case when $D^{\mathcal{A}} = \{1\}$ is one-element, we write $\langle \mathcal{A}, 1 \rangle$ instead of $\langle \mathcal{A}, \{1\} \rangle$ and identify, in fact, a matrix with an algebra in a language with additional constant 1. A *valuation* in an algebra is defined in a standard way. A formula φ is said to *be true* on a matrix $\mathcal{M} = \langle \mathcal{A}, D^{\mathcal{A}} \rangle$, $\mathcal{M} \models \varphi$, if for any $\mathcal{A}$-valuation, $v(\varphi) \in D^{\mathcal{A}}$. An identity $\varphi = \psi$ is *true* on an algebra $\mathcal{A}$, $\mathcal{A} \models \varphi = \psi$, if $v(\varphi) = v(\psi)$ for any $\mathcal{A}$-valuation. The set $Th(\mathcal{M}) \rightleftharpoons \{\varphi : \mathcal{M} \models \varphi\}$ is a *theory* of matrix $\mathcal{M}$ and the set $Eq(\mathcal{A}) \rightleftharpoons \{\varphi = \psi : \mathcal{A} \models \varphi = \psi\}$ is an *equational theory* of an algebra $\mathcal{A}$. For a class $\mathcal{K}$ of matrices (algebras), we define $Th(\mathcal{K}) \rightleftharpoons \bigcap\{Th(\mathcal{M}) : \mathcal{M} \in \mathcal{K}\}$ $(Eq(\mathcal{K}) \rightleftharpoons \bigcap\{Eq(\mathcal{A}) : \mathcal{A} \in \mathcal{K}\})$.

For a class of algebras $\mathcal{K}$, we denote $\mathbf{I}\mathcal{K}$ the class of all algebras isomorphic to elements of $\mathcal{K}$. For a Heyting algebra $\mathcal{A}$, we denote by $F_d(\mathcal{A})$ its filter of dense elements and by $\mathcal{F}(\mathcal{A})$ the lattice of filters on $\mathcal{A}$. If $X \subseteq |\mathcal{A}|$, denote $\langle X \rangle$ the filter generated by X.

Definition 2.1. Let $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, 1 \rangle$ $(= \langle A, \vee, \wedge, \rightarrow, 0, 1 \rangle)$ be an implicative lattice (Heyting algebra with the least element 0).

1. A *full twist-structure* over $\mathcal{A}$ is an algebra

$$\mathcal{A}^{\bowtie} = \langle A \times A, \vee, \wedge, \rightarrow, \sim \rangle (\mathcal{A}^{\bowtie} = \langle A \times A, \vee, \wedge, \rightarrow, \sim, \perp, 1 \rangle)$$

   with twist-operations defined for $(a, b), (c, d) \in A \times A$ as follows:

$$(a, b) \vee (c, d) := (a \vee c, b \wedge d), \ (a, b) \wedge (c, d) := (a \wedge c, b \vee d)$$
$$(a, b) \rightarrow (c, d) := (a \rightarrow c, a \wedge d), \ \sim(a, b) := (b, a),$$
$$(\perp := (0, 1), 1 := (1, 0)).$$

2. A *twist-structure* over $\mathcal{A}$ is an arbitrary subalgebra $\mathcal{B}$ of the full twist-structure $\mathcal{A}^{\bowtie}$ such that $\pi_1(\mathcal{B}) = \mathcal{A}$ (in which case also $\pi_2(\mathcal{B}) = \mathcal{A}$), where π_i, $i = 1, 2$, is a projection of a direct product onto the ith coordinate.

3. The class of all twist-structures over $\mathcal{A}$ is denoted $S^{\bowtie}(\mathcal{A})$.

A valuation into a twist-structure $\mathcal{B}$ is defined in a usual way as a homomorphism of the algebra of formulae into $\mathcal{B}$. The relation $\mathcal{B} \models_{\bowtie} \varphi$, where φ is a formula of a respective language, means that $\pi_1 v(\varphi) = 1$ for any

$\mathcal{B}$-valuation v. For a formula $\varphi \in For(For^{\perp})$, the relation $\models_{\bowtie} \varphi$ ($\models^{\perp}_{\bowtie} \varphi$) means that $\mathcal{B} \models_{\bowtie} \varphi$ for any twist-structure $\mathcal{B}$ over implicative lattice (Heyting algebra).

For the logic **N4**, the completeness theorem in terms of twist-structures was proved in [10] in an indirect way. First, we have proved the completeness of **N4** wrt Fidel-structures [6], and then established a one-to-one correspondence between Fidel-structures and twist-structures. Below we give a short direct proof of this statement for both **N4** and $\mathbf{N4}^{\perp}$.

THEOREM 2.2. *Let* $\varphi \in For(For^{\perp})$. *Then*

$$\mathbf{N4} \vdash \varphi \;\Leftrightarrow\; \models_{\bowtie} \varphi \quad (\mathbf{N4}^{\perp} \vdash \varphi \;\Leftrightarrow\; \models^{\perp}_{\bowtie} \varphi).$$

PROOF. The correctness in both cases can be verified directly. The completeness we prove only for $\mathbf{N4}^{\perp}$.

Let $|\varphi| := \{\psi | \ \mathbf{N4}^{\perp} \vdash \varphi \leftrightarrow \psi\}$ and $L_{\mathbf{N4}^{\perp}} := \{|\varphi| \mid \varphi \in For^{\perp}\}$. Consider the structure

$$\mathcal{L}_{\mathbf{N4}^{\perp}} := \langle L_{\mathbf{N4}^{\perp}}, \vee, \wedge, \to, 0, 1\rangle,$$

where $|\varphi\tau\psi| := |\varphi|\tau|\psi|$ for $\tau \in \{\vee, \wedge, \to\}$, $0 := |\bot|$, and $1 := |\sim\bot|$. Due to the axioms of positive logic and axioms A6 and A7 it is a Heyting algebra. Note that $|\varphi| = 1$ iff $\mathbf{N4}^{\perp} \vdash \varphi$. Further, consider the full twist-structure $(\mathcal{L}_{\mathbf{N4}^{\perp}})^{\bowtie}$ and its subset

$$A := \{(|\varphi|, |\sim\psi|) \mid \psi \in |\varphi|\}.$$

It follows immediately from the axioms of strong negation that this set is closed under twist-operations. The twist-structure over $\mathcal{L}_{\mathbf{N4}^{\perp}}$ with the universe A we denote $\mathcal{L}^{\bowtie}_{\mathbf{N4}^{\perp}}$.

Consider an $\mathcal{L}^{\bowtie}_{\mathbf{N4}^{\perp}}$-valuation v such that $v(p) = (|p|, |\sim p|)$. It follows by induction on the structure of formulas that $\pi_1 v(\varphi) = |\varphi|$ and $\pi_2 v(\varphi) = |\sim\varphi|$ for any formula φ. In this way, the valuation v refutes all formulas non-provable in $\mathbf{N4}^{\perp}$. ∎

Remark. It was noted that the provable strong equivalence $\Leftrightarrow$ has congruence properties, therefore, on the set $L^{*}_{\mathbf{N4}^{\perp}} := \{|\varphi|^{*} \ |\varphi \in For^{\perp}\}$, where $|\varphi|^{*} := \{\psi | \ \mathbf{N4}^{\perp} \vdash \varphi \Leftrightarrow \psi\}$, we can define the structure of Lindenbaum algebra in the usual way. Denote this algebra $\mathcal{L}^{*}_{\mathbf{N4}^{\perp}}$. Notice that the mapping $|\varphi|^{*} \mapsto (|\varphi|, |\sim\varphi|)$ is an isomorphism of $\mathcal{L}^{*}_{\mathbf{N4}^{\perp}}$ and $\mathcal{L}^{\bowtie}_{\mathbf{N4}^{\perp}}$.

3. $\mathbf{N4}^{\perp}$-lattices

In this section we adopt to the logic $\mathbf{N4}^{\perp}$ the lattice-theoretic semantics developed for **N4** in [10, 11].

Definition 3.1. An algebra $\mathcal{A} = \langle A, \vee, \wedge, \to, \sim, \perp, 1\rangle$ is an $\mathbf{N4}^{\perp}$-*lattice* if the following hold.

1. The reduct $\langle A, \vee, \wedge, \sim, \perp, 1\rangle$ is a bounded De Morgan algebra, i.e., $\langle A, \vee, \wedge, \perp, 1\rangle$ is a bounded distributive lattice and the following identities hold: $\sim(p \vee q) = \ \sim p \ \wedge \sim q$ and $\sim\sim p = p$.
2. The relation $\preceq$, where $a \preceq b$ denotes $(a \to b) \to (a \to b) = a \to b$, is a preordering on $\mathcal{A}$.
3. The relation $\approx$, where $a \approx b$ if and only if $a \preceq b$ and $b \preceq a$, is a congruence relation with respect to $\vee, \wedge, \to$ and the quotient-algebra $\mathcal{A}_{\bowtie} := \langle A, \vee, \wedge, \to, \perp, 1\rangle / \approx$ is a Heyting algebra.
4. For any $a, b \in A$, $\sim(a \to b) \approx a \ \wedge \sim b$.
5. For any $a, b \in A$, $a \leq b$ if and only if $a \preceq b$ and $\sim b \preceq \sim a$, where $\leq$ is a lattice ordering on $\mathcal{A}$.

The only difference of $\mathbf{N4}^{\perp}$-lattices from $\mathbf{N4}$-lattices defined in [10] is that $\mathbf{N4}^{\perp}$-lattices are bounded and $\mathcal{A}_{\bowtie}$ is a Heyting algebra, not an implicative lattice as in the case of $\mathbf{N4}$-lattices. A bounded implicative lattice can be turned into a Heyting algebra, therefore, the following statement holds.

Proposition 3.1. *An algebra* $\mathcal{A} = \langle A, \vee, \wedge, \to, \sim, \perp, 1\rangle$ *is an* $\mathbf{N4}^{\perp}$*-lattice iff* $\langle A, \vee, \wedge, \to, \sim\rangle$ *is a bounded* $\mathbf{N4}$*-lattice and* $\perp$ *and* 1 *are the least and the greatest elements.*

The next two statements establish the equivalence of the notions of $\mathbf{N4}^{\perp}$-lattices and of twist-structures over Heyting algebras. They can be proved exactly so as in the case of $\mathbf{N4}$-lattices and twist-structures over implicative lattices [10].

Proposition 3.2. *Let* $\mathcal{A}$ *be a Heyting algebra. If* $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$*, then* $\mathcal{B}$ *is an* $\mathbf{N4}^{\perp}$*-lattice, moreover, the following facts are true. Let* $(a, b), (c, d) \in |\mathcal{B}|$.

a) $(a, b) \preceq (c, d)$ *if and only if* $a \leq c$.

b) $(a, b) \approx (c, d)$ *if and only if* $a = c$.

c) $(a, b) \leq (c, d)$ *if and only if* $a \leq c$ *and* $d \leq b$.

d) The mapping $[(a, b)]_{\approx} \mapsto a$ *determines an isomorphism of implicative lattices* $\mathcal{B}_{\bowtie}$ *and* $\mathcal{A}$.

Proposition 3.3. *Every* $\mathbf{N4}^{\perp}$*-lattice* $\mathcal{A}$ *is isomorphic to a twist-structure over* $\mathcal{A}_{\bowtie}$ *and the isomorphism is given by the rule*

$$a \mapsto ([a]_{\approx}, [\sim a]_{\approx}).$$

For an $\mathbf{N4}^{\perp}$-lattice $\mathcal{A}$, we put $D^{\mathcal{A}} := \{a \in A |\ a \to a = a\}$ and define a matrix $M(\mathcal{A}) := \langle \mathcal{A}, D^{\mathcal{A}} \rangle$. For $\varphi \in For^{\perp}$, we define

$$\mathcal{A} \models \varphi \ \Leftrightarrow\ M(\mathcal{A}) \models \varphi.$$

Let $\mathcal{A}$ be a Heyting algebra and $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$. It can be easily checked using the definition of twist-operations that $D^{\mathcal{B}} = \{(1, a)|\ (1, a) \in |\mathcal{B}|\}$. Thus, the validity of formulas on a twist-structure $\mathcal{B}$ coincides with the validity of formulas on $\mathcal{B}$ considered as an $\mathbf{N4}^{\perp}$-lattice:

$$\mathcal{B} \models \varphi \ \Leftrightarrow\ \mathcal{B} \models^{\perp}_{\bowtie} \varphi.$$

In this way, we infer from Theorem 2.2 the completeness of $\mathbf{N4}^{\perp}$ wrt $\mathbf{N4}^{\perp}$-lattices.

Theorem 3.2. *For any* $\varphi \in For^{\perp}$*,* $\mathbf{N4}^{\perp} \vdash \varphi$ *if and only if* $\mathcal{A} \models \varphi$ *for any* $\mathbf{N4}^{\perp}$*-lattice* $\mathcal{A}$*.*

In [10], it was proved that **N4**-lattices form a variety determined by the identities of de Morgan algebras and the set of identities 1^N–11^N. Since $\mathbf{N4}^{\perp}$-lattices are exactly bounded **N4**-lattices, they also form a variety $\mathcal{V}_{\mathbf{N4}^{\perp}}$ determined by the identities of de Morgan algebras, the set of identities 1^N–11^N and the new identity:

12^N. $\perp \leq p$

The mappings $Var : \mathcal{E}\mathbf{N4}^{\perp} \to Sub(\mathcal{V}_{\mathbf{N4}^{\perp}})$ and $L : Sub(\mathcal{V}_{\mathbf{N4}^{\perp}}) \to \mathcal{E}\mathbf{N4}^{\perp}$ are defined as follows. For any $L \in \mathcal{E}\mathbf{N4}^{\perp}$, define a variety

$$Var(L) := \{\mathcal{A} \mid \varphi \to \varphi = \varphi \in Eq(\mathcal{A}) \text{ for all } \varphi \in L\}.$$

Clearly, $Var(L) \in Sub(\mathcal{V}_{\mathbf{N4}^{\perp}})$. For any $V \in Sub(\mathcal{V}_{\mathbf{N4}^{\perp}})$, define a set of formulae

$$L(V) := \{\varphi \mid \varphi \to \varphi = \varphi \in Eq(V)\}.$$

Then $L \in \mathcal{E}\mathbf{N4}^{\perp}$. Repeating the reasoning from [10] we prove

Theorem 3.3. *The mappings* Var *and* L *are mutually inverse dual lattice isomorphisms between* $\mathcal{E}\mathbf{N4}^{\perp}$ *and* $Sub(\mathcal{V}_{\mathbf{N4}^{\perp}})$*.*

Notice that $\mathbf{N4}^{\perp}$-lattices have two lattice operations, therefore (see, e.g. [5]), the variety $\mathcal{V}_{\mathbf{N4}^{\perp}}$ is congruence distributive. From this fact we infer

PROPOSITION 3.4. *The lattice $\mathcal{E}\mathbf{N4}^{\perp}$ is distributive.*

The following representation of twist-structures will be useful.

For a Heyting algebra $\mathcal{A}$, let ∇ be a filter on $\mathcal{A}$ such that $F_d(\mathcal{A}) \subseteq \nabla$, and let Δ be an ideal on $\mathcal{A}$. Then it can be easily checked that the set

$$Tw(\mathcal{A}, \nabla, \Delta) = \{(a, b)|\ a, b \in A, a \vee b \in \nabla, a \wedge b \in \Delta\}$$

is closed under the twist-operations. The twist-structure from $S^{\bowtie}(\mathcal{A})$ with such unverse we denote also $Tw(\mathcal{A}, \nabla, \Delta)$. It turns out that structures of the form $Tw(\mathcal{A}, \nabla, \Delta)$ exhaust the set $S^{\bowtie}(\mathcal{A})$.

PROPOSITION 3.5. *Let $\mathcal{A}$ be a Heyting algebra and $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$. We define*

$$I(\mathcal{B}) := \{a \vee \sim a|\ a \in B\},\ \nabla(\mathcal{B}) := \pi_1(I(\mathcal{B})),\ \Delta(\mathcal{B}) := \pi_2(I(\mathcal{B})).$$

Then $F_d(\mathcal{A}) \subseteq \nabla(\mathcal{B})$ is a filter on $\mathcal{A}$ and $\Delta(\mathcal{B})$ is an ideal on $\mathcal{A}$. Moreover,

$$\mathcal{B} = Tw(\mathcal{A}, \nabla(\mathcal{B}), \Delta(\mathcal{B})).$$

This is an immediate consequence of Theorem 3.1 in [11].

For a twist-structure $\mathcal{B}$ and $(a, b) \in |\mathcal{B}|$, we have $(a, b) \vee \sim(a, b) = (a \vee b, a \wedge b)$. Therefore, it follows from Proposition 3.5 that

$$\nabla(\mathcal{B}) = \{a \vee b|\ (a, b) \in \mathcal{B}\} \text{ and } \Delta(\mathcal{B}) = \{a \wedge b|\ (a, b) \in \mathcal{B}\}.$$

Taking into account these representations and the intuition that for $(a, b) \in |\mathcal{B}|$, the element b is one of possible negations (counterexamples) of a we call $\nabla(\mathcal{B})$ a *filter of completions* of $\mathcal{B}$ and $\Delta(\mathcal{B})$ an *ideal of contradictions* of $\mathcal{B}$. For arbitrary $\mathbf{N4}^{\perp}$-lattices these notions can be defined as follows. Let $\mathcal{A}$ be an $\mathbf{N4}^{\perp}$-lattice. We define

$$\nabla(\mathcal{A}) := \{[a \vee \sim a]_{\approx}|\ a \in A\} \text{ and } \Delta(\mathcal{A}) := \{[a \wedge \sim a]_{\approx}|\ a \in A\}.$$

Then

$$\mathcal{A} \cong Tw(\mathcal{A}_{\bowtie}, \nabla(\mathcal{A}), \Delta(\mathcal{A})).$$

As well as in the case of $\mathbf{N4}$-lattices [11] we define a *special filter* on an $\mathbf{N4}^{\perp}$-lattice $\mathcal{A}$ as a nonempty subset $\nabla \subseteq A$ such that: 1. $a \in \nabla$ and $b \in \nabla$ imply $a \wedge b \in \nabla$; 2. $a \in \nabla$ and $a \preceq b$ imply $b \in \nabla$. This notion corresponds to special filters of the first kind on N-lattices introduced originally by H. Rasiowa [13]. It is obvious that the set of all special filters on $\mathcal{A}$ forms a lattice, which we denote $\mathcal{F}^s(\mathcal{A})$. For a homomorphism $h : \mathcal{A} \to \mathcal{B}$ of $\mathbf{N4}^\perp$-lattices we define a *kernel* $Ker(h) := h^{-1}(D^{\mathcal{B}})$. As we can see from the following proposition special filters are in a one-to-one correspondence with congruences on an $\mathbf{N4}^\perp$-lattice and the least special filter on $\mathcal{A}$ coincides with $D^{\mathcal{A}}$.

PROPOSITION 3.6. *1. For any $\mathbf{N4}^\perp$-lattice $\mathcal{A}$, $D^{\mathcal{A}}$ is a special filter.*

2. *Let $h : \mathcal{A} \to \mathcal{B}$ be an epimorphism of $\mathbf{N4}^\perp$-lattices. Then $Ker(h)$ is a special filter. For any $a, b \in |\mathcal{A}|$, $h(a) = h(b)$ if and only if $a \Leftrightarrow b \in Ker(h)$.*
3. *Let $\mathcal{A}$ be an $\mathbf{N4}^\perp$-lattice and ∇ a special filter on $\mathcal{A}$. Then $D^{\mathcal{A}} \subseteq \nabla$. The relation $\approx_\nabla$, $a \approx_\nabla b$ is equivalent to $a \Leftrightarrow b \in \nabla$, is a congruence relation on $\mathcal{A}$ and $\nabla = Ker(h)$, where $h : \mathcal{A} \to \mathcal{A}/\approx_\nabla$ is a canonical epimorphism.*

This follows from Proposition 4.2 in [11]. Let $\mathcal{A}$ be a Heyting algebra and $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$. For $\nabla \in \mathcal{F}(\mathcal{A})$, put $\nabla^{\bowtie} := \pi_1^{-1}(\nabla) = \{(a, b) \in A^2 | \ a \in \nabla\}$. For any $\nabla \in \mathcal{F}^s(\mathcal{B})$, put $\nabla_{\bowtie} := \pi_1(\nabla)$.

PROPOSITION 3.7. *Let $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$. The lattices $\mathcal{F}(\mathcal{A})$ and $\mathcal{F}^s(\mathcal{B})$ are isomorphic and the mappings $\nabla \mapsto \nabla^{\bowtie}$, $\nabla \in \mathcal{F}(\mathcal{A})$, and $\nabla \mapsto \nabla_{\bowtie}$, $\nabla \in \mathcal{F}^s(\mathcal{B})$, determine mutually inverse isomorphisms.*

This is a variant of Proposition 4.3 in [11].

COROLLARY 3.1. *Let $\mathcal{A}$ be an $\mathbf{N4}^\perp$-lattice.*

1. *$Con(\mathcal{A}) \cong Con(\mathcal{A}_{\bowtie})$.*
2. *$\mathcal{A}$ is subdirectly irreducible if and only if the Heyting algebra $\mathcal{A}_{\bowtie}$ is subdirectly irreducible.*

PROPOSITION 3.8. *Let $\mathcal{B}$ be a Heyting algebra, F a filter on $\mathcal{B}$. Let $\mathcal{A} \in S^{\bowtie}(\mathcal{B})$ and $\mathcal{A} = Tw(\mathcal{B}, \nabla, \Delta)$. Then*

$$\mathcal{A}/F^{\bowtie} \cong Tw(\mathcal{B}/F, \nabla/F, \Delta/F).$$

PROOF. Define a mapping $h : \mathcal{A}/F^{\bowtie} \to (\mathcal{B}/F)^{\bowtie}$ as follows. For any $(a, b) \in |\mathcal{A}|$, $h((a, b)/F^{\bowtie}) := (a/F, b/F)$. Clearly, h is a homomorphism. Check that this is a monomorphism. The equality $(a/F, b/F) = (c/F, d/F)$ is equivalent to $(a \leftrightarrow c) \wedge (b \leftrightarrow d) \in F$. At the same time, $(a, b)/F^{\bowtie} = (c, d)/F^{\bowtie}$ is equivalent by Proposition 3.6 to $(a, b) \Leftrightarrow (c, d) \in F^{\bowtie}$, which is equivalent in turn to $(a \leftrightarrow c) \wedge (b \leftrightarrow d) \in F$. Thus, h is a monomoprhism and it remains to check that

$$h(\mathcal{A}/F^{\bowtie}) = Tw(\mathcal{B}/F, \nabla/F, \Delta/F),$$

i.e., $\nabla(h(\mathcal{A}/F^{\bowtie})) = \nabla/F$ and $\Delta(h(\mathcal{A}/F^{\bowtie})) = \Delta/F$. The property "to be presented as $a \vee \sim a$" is preserved and reflected by any homomorphism. Therefore, $(a/F, b/F) \in I(h(\mathcal{A}/F^{\bowtie}))$ is equivalent to $(a, b) \in I(\mathcal{A})$, which immediately implies the desired equalities. ■

4. $\mathcal{E}$N4$^{\perp}$ and Int

We start the investigation of the class $\mathcal{E}\mathbf{N4}^{\perp}$ with the question on the interrelation between a logic in $\mathcal{E}\mathbf{N4}^{\perp}$ and its intuitionistic fragment. We define a mapping σ from $\mathcal{E}\mathbf{N4}^{\perp}$ into the class Int of extensions of intuitionistic logic **Li** so that $\sigma(L)$ is simply a $\langle\vee, \wedge, \rightarrow, \perp\rangle$-fragment of L. The restriction of σ to the class $\mathcal{E}\mathbf{N3}$ was investigated by Kracht and Sendlewski [9, 17].

First, we point out that the logic $\sigma(L)$ is determined by the underlying Heyting algebras of L-models. For a class $\mathcal{K}$ of $\mathbf{N4}^{\perp}$-lattices, we put

$$\mathcal{K}_{\bowtie} := \{\mathcal{A}_{\bowtie} | \ \mathcal{A} \in \mathcal{K}\}.$$

PROPOSITION 4.1. *For any $L \in \mathcal{E}\mathbf{N4}^{\perp}$ and class $\mathcal{K}$ of $\mathbf{N4}^{\perp}$-lattices, if $L = L\mathcal{K}$, then $\sigma(L) = L\mathcal{K}_{\bowtie}$.*

PROOF. We may assume that $\mathcal{K} = \mathbf{I}\mathcal{K}$. Due to Proposition 3.3 it is enough to consider twist-structures in $\mathcal{K}$. Let $\mathcal{B} \in \mathcal{K}$ and $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$. Assume that $\mathcal{A}(\cong \mathcal{B}_{\bowtie})$ is not a model of $\sigma(L)$. Let $\varphi \in \sigma(L)$ and an $\mathcal{A}$-valuation v are such that $v(\varphi) \neq 1$. For any propositional variable p, there is an element $b_p \in A$ with $(v(p), b_p) \in |\mathcal{B}|$. Let v' be a $\mathcal{B}$-valuation such that $v'(p) = (v(p), b_p)$ for any p. It follows from the definition of twist-operations that $\pi_1 v'(\psi) = v(\psi)$ for any ψ. Thus, $\pi_1 v'(\varphi) \neq 1$ and $\mathcal{B} \not\models \varphi$, which conflicts with the assumption $\mathcal{B} \models \sigma(L) \subseteq L$.

We have thus proved the inclusion $\sigma(L) \subseteq L\mathcal{K}_{\bowtie}$. To check the inverse inclusion take an intuitionistic formula $\varphi \notin \sigma(L)$. Let $\mathcal{B} \in \mathcal{K}$, $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$ and $\mathcal{B}$-valuation v be such that $\pi_1 v(\varphi) \neq 1$. Then $\pi_1 v$ is an $\mathcal{A}$-valuation proving that $\mathcal{A} \not\models \varphi$. ■

The following fact stated in the last proof we distinguish as a separate lemma.

LEMMA 4.1. *Let $\mathcal{A}$ be an $\mathbf{N4}^{\perp}$-lattice and φ an intuitionistic formula. Then*

$$\mathcal{A} \models \varphi \text{ if and only if } \mathcal{A}_{\bowtie} \models \varphi.$$

Now we study the inverse image $\sigma^{-1}(L)$ for any $L \in \mathsf{Int}$, i.e., the class of all conservative extensions of L in $\mathcal{E}\mathbf{N4}^{\perp}$. We show that $\sigma^{-1}(L)$ forms an interval in the lattice $\mathcal{E}\mathbf{N4}^{\perp}$ and consider the mappings sending L to the end-points of the interval $\sigma^{-1}(L)$. Let for any $L \in \mathsf{Int}$,

$$\eta(L) := \mathbf{N4}^{\perp} + L,$$

i.e., $\eta(L)$ is obtained by extending the language and adding the strong negation axioms to L, and

$$\eta^{\circ}(L) = \eta(L) + \{\sim p \to (p \to q),\ \neg\neg(p \vee \sim p)\}.$$

Logics in $\mathcal{E}\mathbf{N4}^{\perp}$ having the form $\eta(L)$ we call *special* or *s*-logics, and logics of the form $\eta^{\circ}(L)$ are called *normal explosive* or *ne*-logics. Note that *ne*-logics extend **N3** and they were originally introduced by V. Goranko [7].

Prior to prove that $\eta(L)$ and $\eta^{\circ}(L)$ are the end-points of $\sigma^{-1}(L)$ we describe models of $\eta(L)$ and $\eta^{\circ}(L)$ and translations of these logics into L.

PROPOSITION 4.2. *Let $L \in \mathsf{Int}$ and $\mathcal{A}$ be an $\mathbf{N4}^{\perp}$-lattice.*

1. $\mathcal{A} \models \eta(L)$ *iff* $\mathcal{A}_{\bowtie} \models L$.
2. $\mathcal{A} \models \eta^{\circ}(L)$ *iff* $\mathcal{A}_{\bowtie} \models L$, $\nabla(\mathcal{A}) = F_d(\mathcal{A}_{\bowtie})$ *and* $\Delta(\mathcal{A}) = \{0\}$.

PROOF. 1. The direct implication follows by Lemma 4.1. The inverse implication follows from the same lemma and the fact that the axioms of $\mathbf{N4}^{\perp}$ hold in any $\mathbf{N4}^{\perp}$-lattice.

2. If $\eta^{\circ}(L) \models \mathcal{A}$, then $\mathcal{A} \models \sim p \to (p \to q)$. By Proposition 3.3 we may assume that $\mathcal{A}$ is a twist-structure. For $(a, b), (c, d) \in |\mathcal{A}|$,

$$\sim(a, b) \to ((a, b) \to (c, d)) = ((a \wedge b) \to c, a \wedge b \wedge d) = (1, a \wedge b \wedge d),$$

i.e., $a \wedge b = 0$ for any $(a, b) \in |\mathcal{A}|$. This means exactly that $\Delta(\mathcal{A}) = \{0\}$. Similarly, a direct computation shows that for a twist-structure $\mathcal{A} \in S^{\bowtie}(\mathcal{B})$, the condition $\mathcal{A} \models \neg\neg(p \vee \sim p)$ is equivalent to $\neg\neg(a \vee b) = 1$ for any $(a, b) \in \mathcal{B}$, i.e., $\nabla(\mathcal{A}) \subseteq F_d(\mathcal{B})$. The inclusion $F_d(\mathcal{B}) \subseteq \nabla(\mathcal{A})$ holds for any twist-structure, which proves the direct implication.

Now we assume that the right hand conditions of the equivalence are satisfied. Due to Lemma 4.1 the first condition implies $\mathcal{A} \models L$. We have $\mathcal{A} \models \sim p \to (p \to q)$ by $\Delta(\mathcal{A}) = \{0\}$. Finally, in view of the above considerations, $\nabla(\mathcal{A}) = F_d(\mathcal{A}_{\bowtie})$ implies $\mathcal{A} \models \neg\neg(p \vee \sim p)$. ■

For a Heyting algebra $\mathcal{A}$, we define $\mathcal{A}^{\bowtie}_{\circ} := Tw(\mathcal{A}, F_d(\mathcal{A}), \{0\})$. This is the least twist-structure over $\mathcal{A}$. For any $\mathcal{B} \in S^{\bowtie}$, $\mathcal{A}^{\bowtie}_{\circ} \leq \mathcal{B}$. Namely such lattices are up to isomorphism models of *ne*-logics. If an $\mathbf{N4}^{\perp}$-lattice $\mathcal{A}$ is isomorphic to a lattice of the form $\mathcal{B}^{\bowtie}_{\circ}$, we call it a *normal* **N3**-lattice. In what follows, $\mathbf{N4}^{\perp}$-lattices isomorphic to full twist-structures are called *special* $\mathbf{N4}^{\perp}$-lattices.

Further, we define translations of $\eta^{\circ}(L)$ and $\eta(L)$ into L. To any formula φ we assign intuitionistic formulas $\varphi_{\bowtie}$ and $\varphi^{\circ}_{\bowtie}$ defined as follows. Let $\varphi = \varphi(p_0, \ldots, p_n)$. For any $k \leq n$, we put $\overline{p_k} := p_{n+k+1}$. If φ is in *nnf*, then $\varphi_{\bowtie}$ is the result of replacement of every occurrence of $\sim p$ in φ by $\overline{p}$. If φ is not in *nnf*, then $\varphi_{\bowtie} := (\overline{\varphi})_{\bowtie}$. We define formulas $\widetilde{\varphi}$ and $\varphi^{\circ}_{\bowtie}$ as follows:

$$\widetilde{\varphi} := \bigwedge_{p \in var(\varphi)} \neg(p \wedge \overline{p}) \wedge \bigwedge_{p \in var(\varphi)} \neg\neg(p \vee \overline{p}), \quad \varphi^{\circ}_{\bowtie} := \widetilde{\varphi} \to \varphi_{\bowtie}.$$

Proposition 4.3. *Let $\mathcal{A}$ be a Heyting algebra and $\varphi \in For^{\perp}$. The following equivalences hold:*

$$\mathcal{A}^{\bowtie} \models \varphi \;\Leftrightarrow\; \mathcal{A} \models \varphi_{\bowtie} \;\; \textit{and} \;\; \mathcal{A}^{\bowtie}_{\circ} \models \varphi \;\Leftrightarrow\; \mathcal{A} \models \varphi^{\circ}_{\bowtie}.$$

Proof. Below we assume that φ is in *nnf*.

To prove the first equivalence for any $\mathcal{A}^{\bowtie}$-valuation v, we take an $\mathcal{A}$-valuation $v_{\bowtie}$ such that $v_{\bowtie}(p) := \pi_1 v(p)$ and $v_{\bowtie}(\overline{p}) := \pi_2 v(p)$ for $p \in var(\varphi)$. For any $\mathcal{A}$-valuation v, we define $\mathcal{A}^{\bowtie}$-valuation $v^{\bowtie}$ such that $v^{\bowtie}(p) := (v(p), v(\overline{p}))$ for $p \in var(\varphi)$. Taking into account the fact that the action of intuitionistic operations on a twist-structure is agreed with their action on the first component and that $\pi_1 v(\sim\varphi) = \pi_2 v(\varphi)$ we obtain

$\pi_1 v(\varphi) = v_{\bowtie}(\varphi_{\bowtie})$ for any $\mathcal{A}^{\bowtie}$-valuation v and

$\pi_1 (v')^{\bowtie}(\varphi) = v'(\varphi)$ for any $\mathcal{A}$-valuation v'.

The desired equivalence follows from these relations.

For the proof of the second equivalence let $\mathcal{A} \models \varphi^{\circ}_{\bowtie}$ and v be an $\mathcal{A}^{\bowtie}_{\circ}$-valuation. By definition of $\mathcal{A}^{\bowtie}_{\circ}$, $\pi_1 v(p) \wedge \pi_2 v(p) = 0$ and $\pi_1 v(p) \vee \pi_2 v(p) \in F_d(\mathcal{A})$ for any $p \in var(\varphi)$, which implies $v_{\bowtie}(\widetilde{\varphi}) = 1$, whence, $v_{\bowtie}(\varphi_{\bowtie}) = 1$. As was stated in the previous item $\pi_1 v(\varphi) = v_{\bowtie}(\varphi_{\bowtie}) = 1$.

Assume now $\mathcal{A}^{\bowtie}_{\circ} \models \varphi$ and $v(\varphi^{\circ}_{\bowtie}) \neq 1$ for some $\mathcal{A}$-valuation v. Then $v(\widetilde{\varphi}) \not\leq v(\varphi_{\bowtie})$. Put $F = \langle v(\widetilde{\varphi}) \rangle$ and consider the quotient algebra $\mathcal{A}/F$ and the quotient valuation $v' := v/F$. We have $v'(\widetilde{\varphi}) = 1$ and $v'(\varphi_{\bowtie}) \neq 1$. The equality $v'(\widetilde{\varphi}) = 1$ implies that $\pi_1 (v')^{\bowtie}(p) \wedge \pi_2 (v')^{\bowtie}(p) = 0$ and $\pi_1 (v')^{\bowtie}(p) \vee \pi_2 (v')^{\bowtie}(p) \in F_d(\mathcal{A})$ for any $p \in var(\varphi)$, i.e., we may assume that $(v')^{\bowtie}$ is an

$(\mathcal{A}/F)^{\bowtie}_{\circ}$-valuation. Arguing as in the previous item from $v'(\varphi_{\bowtie}) \neq 1$ we infer $(\mathcal{A}/F)^{\bowtie}_{\circ} \not\models \varphi$. It remains to note that $(\mathcal{A}/F)^{\bowtie}_{\circ} \cong (\mathcal{A})^{\bowtie}_{\circ}/F^{\bowtie}$ by Proposition 3.8. ■

COROLLARY 4.1. *Let* $L \in \mathsf{Int}$ *and* $\varphi \in For^{\perp}$. *The following equivalences hold:*

$$\varphi \in \eta(L) \;\Leftrightarrow\; \varphi_{\bowtie} \in L \;\; \text{and} \;\; \varphi \in \eta^{\circ}(L) \;\Leftrightarrow\; \varphi^{\circ}_{\bowtie} \in L.$$

PROOF. Let $\varphi \in \eta(L)$ and $\mathcal{A} \models L$. Then $\mathcal{A}^{\bowtie} \models \eta(L)$ by Proposition 4.2, and $\mathcal{A} \models \varphi_{\bowtie}$ by the last proposition. Conversely, let $\varphi_{\bowtie} \in L$ and $\mathcal{A} \models \eta(L)$. Then $\mathcal{A}_{\bowtie} \models L$ by Proposition 4.2, and $(\mathcal{A}_{\bowtie})^{\bowtie} \models \eta(L)$. Since $\mathcal{A} \hookrightarrow (\mathcal{A}_{\bowtie})^{\bowtie}$, we have $\mathcal{A} \models \varphi$.

The second equivalence follows similarly from the second equivalence of Proposition 4.2. ■

COROLLARY 4.2. *For any* $L \in \mathsf{Int}$, *the logics* $\eta(L)$ *and* $\eta^{\circ}(L)$ *are conservative extensions of* L.

PROOF. The intuitionistic formulas are $\sim$-free, therefore, $\varphi_{\bowtie} = \varphi$ for any such formula. In this way, the conservativeness of $\eta(L)$ follows from Corollary 4.1.

Let φ be an intuitionistic formula and $\varphi \in \eta^{\circ}(L)$. Then $\varphi^{\circ}_{\bowtie} \in L$. Take a Heyting algebra $\mathcal{A} \models L$ and $\mathcal{A}$-valuation v. Let $\mathcal{A}$-valuation v' be such that $v'(p) = v(p)$ and $v'(\overline{p}) = \neg v(p)$ for any $p \in var(p)$. Then $v'(\widetilde{\varphi}) = 1$ and $v'(\varphi_{\bowtie}) = v(\varphi)$. Taking into account $\mathcal{A} \models \varphi^{\circ}_{\bowtie}$ we obtain $v(\varphi) = 1$. ■

For $L \in \mathcal{E}\mathbf{N4}^{\perp}$, denote $Mod(L) := \{\mathcal{A} |\ \mathcal{A} \models L\}$.

Let $\mathcal{K}$ be a class of Heyting algebras. Put

$$\mathcal{K}^{\bowtie} := \{\mathcal{A}^{\bowtie} |\ \mathcal{A} \in \mathcal{K}\} \text{ and } \mathcal{K}^{\bowtie}_{\circ} := \{\mathcal{A}^{\bowtie}_{\circ} |\ \mathcal{A} \in \mathcal{K}\}.$$

Proposition 4.2 states, in fact, that

$$\eta(L) = L(Mod(L))^{\bowtie} \text{ and } \eta^{\circ}(L) = L(Mod(L))^{\bowtie}_{\circ}.$$

The proposition below generalizes this result.

PROPOSITION 4.4. *Let* $L \in \mathsf{Int}$ *and* $L = L\mathcal{K}$. *Then*

$$\eta(L) = L\mathcal{K}^{\bowtie} \;\text{and}\; \eta^{\circ}(L) = L\mathcal{K}^{\bowtie}_{\circ}.$$

Proof. The inclusions $\eta(L) \subseteq L\mathcal{K}^{\bowtie}$ and $\eta^\circ(L) \subseteq L\mathcal{K}_\circ^{\bowtie}$ follow from Proposition 4.2.

Let $\varphi \notin \eta(L)$. Then $\varphi_{\bowtie} \notin L$ and there is $\mathcal{A} \in \mathcal{K}$ such that $\mathcal{A} \not\models \varphi_{\bowtie}$. It follows by Proposition 4.3 that $\mathcal{A}^{\bowtie} \not\models \varphi$, i.e., $\varphi \notin L\mathcal{K}^{\bowtie}$.

Similarly, we use Proposition 4.3 to prove the second equality. ■

Taking into account Proposition 4.1 the last proposition can be reworded as follows.

Corollary 4.3. *Let $L \in \mathcal{E}\mathbf{N4}^{\perp}$.*

1. *L is a special logic if and only if L is determined by some family of special $\mathbf{N4}^{\perp}$-lattices.*
2. *L is an* ne-*logic if and only if L is determined by some family of normal $\mathbf{N3}$-lattices.*

Note that Item 2 of this Corollary was stated in [7].

Proposition 4.5. *For any $L \in \mathsf{Int}$, $\sigma^{-1}(L) = [\eta(L), \eta^\circ(L)]$.*

Proof. If $L_1 \in \sigma^{-1}(L)$ and an $\mathbf{N4}^{\perp}$-lattice $\mathcal{A}$ is such that $\mathcal{A} \models L_1$, then $\mathcal{A}_{\bowtie} \models L$ by Lemma 4.2, and $\mathcal{A} \models \eta(L)$ by Item 1 of Proposition 4.3. Thus, $\eta(L) \subseteq L_1$.

For any $\mathcal{A} \in Mod(L_1)$, $(\mathcal{A}_{\bowtie})_\circ^{\bowtie}$ embeds into $\mathcal{A}$ and belongs to $Mod(L_1)$. Let $\mathcal{K} = \{\mathcal{A}_{\bowtie} | \ \mathcal{A} \in Mod(L_1)\}$. Then $\mathcal{K}_\circ^{\bowtie} \subseteq Mod(L_1)$. By Proposition 4.1, $L = L\mathcal{K}$, and $\eta^\circ(L) = L\mathcal{K}_\circ^{\bowtie}$ by Proposition 4.4. We have thus proved $L_1 \subseteq \eta^\circ(L)$.

That $\eta(L)$ and $\eta^\circ(L)$ belong to $\sigma^{-1}(L)$ was stated in Corollary 4.2. ■

Remark. An analog of this statement does not hold for the lattice of **N4**-extensions. We can define $\sigma^p(L)$ as a positive fragment of $L \in \mathcal{E}\mathbf{N4}$. It is true that $\mathbf{N4} + L$ is the least element of $(\sigma^p)^{-1}(L)$, where L is some extension of positive logic. In general case $(\sigma^p)^{-1}(L)$ has not the greatest element. This is due to the fact that the family of intermediate logics with the same positive fragment not necessarily has the greatest element.

Following M. Kracht [9] we determine another useful characterization of special logics.

Proposition 4.6. *Let $L \in \mathcal{E}\mathbf{N4}^{\perp}$.*

1. *L is a special logic if and only if all rules of the form $\psi/\psi_{\bowtie}$ are admissible in L.*
2. *L is an* ne-*logic if and only if $L \in \mathcal{E}\mathbf{N3}$ and $\neg\neg(p \vee {\sim} p) \in L$.*

PROOF. 1. Let L be a special logic. According to Corollary 4.3 L is determined by some family of special $\mathbf{N4}^{\perp}$-lattices. Now, admissibility of rules $\psi/\psi_{\bowtie}$ follows from Proposition 4.3.

Conversely, assume that all rules of the form $\psi/\psi_{\bowtie}$ are admissible in L.

LEMMA 4.2. *Rules of the form $\psi_{\bowtie}/\psi$ are admissible in any $L \in \mathcal{E}\mathbf{N4}^{\perp}$.*

PROOF. Let $\mathcal{A}$ be an $\mathbf{N4}^{\perp}$-lattice. If $\mathcal{A} \models \psi_{\bowtie}$, then $\mathcal{A}_{\bowtie} \models \psi_{\bowtie}$ by Lemma 4.1, and $(\mathcal{A}_{\bowtie})^{\bowtie} \models \psi$ by Proposition 4.3. Since $\mathcal{A}$ embeds into $(\mathcal{A}_{\bowtie})^{\bowtie}$, we conclude $\mathcal{A} \models \psi$. ∎

Taking into account this lemma and Corollary 4.1 we obtain for any φ, $\varphi \in L$ if and only if $\varphi_{\bowtie} \in L$ if and only if $\varphi \in \eta(\sigma(L))$. Thus, $L = \eta(\sigma(L))$ and L is a special logic.

2. If L is an *ne*-logic then, $\sim p \to (p \to q)$ and $\neg\neg(p \vee \sim p)$ belong to L by definition.

Let $L \in \mathcal{E}\mathbf{N3}$, $\neg\neg(p \vee \sim p) \in L$, and $\mathcal{A} \models L$. In this case, $\Delta(\mathcal{A}) = \{0\}$. The validity of $\neg\neg(p \vee \sim p)$ is equivalent to $\nabla(\mathcal{A}) = F_d(\mathcal{A})$. We have thus proved that any model of L is a normal $\mathbf{N3}$-lattice. By Corollary 4.3 L is an *ne*-logic. ∎

Denote $\mathbf{N3}^{\circ} := \mathbf{N3} + \{\neg\neg(p \vee \sim p)\}$.

PROPOSITION 4.7. *1. $\sigma : \mathcal{E}\mathbf{N4}^{\perp} \to \mathsf{Int}$ is a lattice epimorphism commuting with infinite meets and joins.*

2. $\eta : \mathsf{Int} \to \mathcal{E}\mathbf{N4}^{\perp}$ is a lattice monomorphism commuting with infinite meets and joins.

3. η° is a lattice isomorphism of Int and $\mathcal{E}\mathbf{N3}^{\circ}$.

PROOF. 1. It follows immediately from the definition.

2. That η is one-to-one follows from Corollary 4.2. It can be easily checked that η commutes with infinite joins. Let $L^* := \bigcap_{i\in I} L_i$. Since $\sigma(\bigcap_{i\in I} \eta(L_i))$ is obviously equal to L^*, the equality $\eta(L^*) = \bigcap_{i\in I} \eta(L_i)$ will follow from the fact that $\bigcap_{i\in I} \eta(L_i)$ is a special logic. Each of $\eta(L_i)$ is closed under all rules of the form $\psi/\psi_{\bowtie}$ by Proposition 4.6. Obviously, the intersection $\bigcap_{i\in I} \eta(L_i)$ is also closed under such rules, and it is also a special logic by the same proposition.

3. η° embeds Int into $\mathcal{E}\mathbf{N3}^{\circ}$ by Corollary 4.2. That η° is onto follows from Item 2 of Proposition 4.6. ∎

We have thus presented the class $\mathcal{E}\mathbf{N4}^{\perp}$ as a union of disjoint intervals of the form $\sigma^{-1}(L)$:

$$\mathcal{E}\mathbf{N4}^{\perp} = \bigcup_{L \in \mathsf{Int}} [\eta(L), \eta^{\circ}(L)].$$

Now, we establish interrelations between these intervals. It turns out that if $L_1 \subseteq L_2$, then $\sigma^{-1}(L_2)$ is embedded into $\sigma^{-1}(L_1)$ as upper subinterval, at the same time, $\sigma^{-1}(L_2)$ is a homomorphic image of $\sigma^{-1}(L_1)$.

Let $L_1, L_2 \in \mathsf{Int}$ and $L_1 \subseteq L_2$. We define two mappings $r_{L_2,L_1}\colon \sigma^{-1}(L_2) \to \mathcal{E}\mathbf{N4}^{\perp}$ and $e_{L_1,L_2} : \sigma^{-1}(L_1) \to \mathcal{E}\mathbf{N4}^{\perp}$ as follows:

$$r_{L_2,L_1}(L) = L \cap \eta^{\circ}(L_1) \text{ and } e_{L_1,L_2}(L) = L + \eta(L_2).$$

Proposition 4.8. *Let $L_1, L_2 \in \mathsf{Int}$ and $L_1 \subseteq L_2$. The following facts hold.*

1. *For any $L \in \sigma^{-1}(L_2)$, we have $e_{L_1,L_2} r_{L_2,L_1}(L) = L$.*
2. *For any $L \in \sigma^{-1}(L_2)$, we have*

$$r_{L_2,L_1} e_{L_1,L_2}(L) = L + r_{L_2,L_1}(\eta(L_2)).$$

3. *e_{L_1,L_2} is a lattice epimorphism from $\sigma^{-1}(L_1)$ onto $\sigma^{-1}(L_2)$.*
4. *r_{L_2,L_1} is a lattice monomorphism from $\sigma^{-1}(L_2)$ into $\sigma^{-1}(L_1)$ and*

$$r_{L_2,L_1}(\sigma^{-1}(L_2)) = [r_{L_2,L_1}(\eta(L_2)), \eta^{\circ}(L_1)].$$

5. *For any $L_3 \in \mathcal{E}\mathbf{N4}^{\perp}$ such that $L_2 \subseteq L_3$, we have*

$$e_{L_2,L_3} e_{L_1,L_2} = e_{L_1,L_3} \text{ and } r_{L_2,L_1} r_{L_3,L_2} = r_{L_3,L_1}.$$

Proof. 1. Let $L \in \sigma^{-1}(L_2)$. We calculate

$$e_{L_1,L_2} r_{L_2,L_1}(L) = (L + \eta(L_2)) \cap (\eta^{\circ}(L_1) + \eta(L_2)).$$

We have $L + \eta(L_2) = L$ since $\eta(L_2)$ is the least element of $\sigma^{-1}(L_2)$. Consider $L' := \eta^{\circ}(L_1) + \eta(L_2)$. Due to homomorphism properties of σ we have $\sigma(L') = L_2$. Moreover, $L' \in \mathcal{E}\mathbf{N3}^{\circ}$ since $\eta^{\circ}(L_1) \subseteq L'$. By Item 3 of Proposition 4.7 we conclude $L' = \eta^{\circ}(L_2)$. Finally, $L + \eta^{\circ}(L_2) = L$ since $\eta^{\circ}(L_2)$ is the greatest element of $\sigma^{-1}(L_2)$.

2. Again, for $L \in \sigma^{-1}(L_1)$, we have

$$r_{L_2,L_1} e_{L_1,L_2}(L) = (L \cap \eta^{\circ}(L_1)) + (\eta(L_2) \cap \eta^{\circ}(L_1)),$$

where the first conjunction term is equal to L since $\eta^{\circ}(L_1)$ is the greatest element of $\sigma^{-1}(L_1)$, and the second disjunction term is exactly $r_{L_2,L_1}(\eta(L_2))$.

3. That e_{L_1,L_2} is a lattice homomorphism follows from the distributivity of $\mathcal{E}\mathbf{N4}^{\perp}$. Let $L \in \sigma^{-1}(L_1)$. Since σ is a homomorphism, we have

$$\sigma e_{L_1,L_2}(L) = \sigma(L) + \sigma\eta(L_2) = L_1 + L_2 = L_2.$$

Thus, the range of e_{L_1,L_2} is contained in $\sigma^{-1}(L_2)$. Item 1 implies that e_{L_1,L_2} is onto.

4. As above, we use distributivity of $\mathcal{E}\mathbf{N4}^{\perp}$ and homomorphism property of σ to prove that r_{L_2,L_1} is a lattice homomorphism from $\sigma^{-1}(L_2)$ into $\sigma^{-1}(L_1)$. If $r_{L_2,L_1}(L) = r_{L_2,L_1}(L')$, then applying the formula of Item 1 we obtain $L = L'$. In this way, r_{L_2,L_1} is a monomorphism. The equality $r_{L_2,L_1}(\sigma^{-1}(L_2)) = [r_{L_2,L_1}(\eta(L_2)), \eta^{\circ}(L_1)]$ follows from Item 2.

5. This follows immediately from definitions. ■

5. The interval $\sigma^{-1}(\mathbf{Lk})$

It was proved in the previous section that the lattice $\mathcal{E}\mathbf{N4}^{\perp}$ decomposes into a union of disjoint intervals $\sigma^{-1}(L)$, where $L \in \mathsf{Int}$ and that $\sigma^{-1}(L_1)$ is isomorphic to an upper subinterval of $\sigma^{-1}(L_2)$, whenever $L_2 \subseteq L_1$. Since the classical logic $\mathbf{Lk}$ is the greatest non-trivial point of Int, $\sigma^{-1}(\mathbf{Lk})$ is an upper part of any interval of the form $\sigma^{-1}(L)$ for any intermediate L. Due to this reason we start the study of the structure of $\mathcal{E}\mathbf{N4}^{\perp}$ with the description of the interval $\sigma^{-1}(\mathbf{Lk})$.

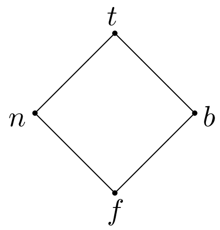
First, we consider subdirectly irreducible models of logics in $\sigma^{-1}(\mathbf{Lk})$. According to Corollary 3.1 any such model is isomorphic to an element of $S^{\bowtie}(\mathcal{A})$, where $\mathcal{A}$ is a subdirectly irreducible model of $\mathbf{Lk}$. Any subdirectly irreducible model of $\mathbf{Lk}$ is isomorphic to the two-element Boolean algebra $\mathbf{2}$. Therefore, we have exactly four subdirectly irreducible models of logics in $\sigma^{-1}(\mathbf{Lk})$:

$$\mathbf{2}^{\bowtie} = Tw(\mathbf{2}, \{0,1\}, \{0,1\}),\ \ \mathbf{2}^{\bowtie}_{\circ} = Tw(\mathbf{2}, \{1\}, \{0\}),$$

$$\mathbf{2}^{\bowtie}_{R} = Tw(\mathbf{2}, \{1\}, \{0,1\}),\ \ \mathbf{2}^{\bowtie}_{L} = Tw(\mathbf{2}, \{0,1\}, \{0\}).$$

These lattices define logics closely related with the well known finite-valued logics. Consider matrices of the form $M(\mathcal{A}) = \langle \mathcal{A}, D^{\mathcal{A}} \rangle$ corresponding to the above defined $\mathbf{N4}^{\perp}$-lattices. The lattice $\mathbf{2}^{\bowtie}_{\circ}$ is two-element. Its elements $(0,1)$ and $(1,0)$ can be identified with the classical truth-values f and t, respectively. The strong negation $\sim$ coincides with the classical one and $D^{\mathcal{A}} = \{t\}$ in this case. Thus, the matrix $M(\mathbf{2}^{\bowtie}_{\circ})$ defines, in fact, the classical logic with the additional constant $\perp$ such that $\sim p \leftrightarrow p \to \perp$. We denote $\widetilde{\mathbf{Lk}} := L\mathbf{2}^{\bowtie}_{\circ}$.

The matrix $M(\mathbf{2}^{\bowtie})$ has additional truth values $b := (1,1)$ and $n := (0,0)$. It has two distinguished values, $D^{\mathbf{2}^{\bowtie}} = \{t, b\}$, and the following lattice structure.



In fact, $M(2^{\bowtie})$ can be considered as a four-valued Belnap's matrix ([3, 4]) enriched with the weak implication $\to$ and the constant $\perp$ interpreted as f. We denote $\mathbf{B}_4^{\to} := L\mathbf{2}^{\bowtie}$.

The logic of $\mathbf{2}_R^{\bowtie}$ differs from the logic RM_3 [1, 2], the greatest non-classical extension of relevance logic RM, only by the additional constant $\perp$. Indeed, the logic RM_3 can be defined via the matrix $M = \langle \{f, t, b\}, \vee, \wedge, \supset, \sim, \{t, b\} \rangle$, where $f \leq b \leq t$, $\vee$ and $\wedge$ are the usual lattice join and meet, $\sim$ is an order reversing involution, and the implication is defined by the rule:

$$a \supset b := \begin{cases} b, & \text{if } a \in D^M \\ t, & \text{if } a \notin D^M. \end{cases} \tag{i}$$

We have $|\mathbf{2}_R^{\bowtie}| = \{f, t, b\}$ and $D^{\mathbf{2}_R^{\bowtie}} = \{t, b\}$. The lattice structures of M and $\mathbf{2}_R^{\bowtie}$ are identical, the strong negation is also an order reversing involution, finally, a direct computation shows that $\supset$ coincides with the weak implication of $\mathbf{2}_R^{\bowtie}$. Thus, $L\mathbf{2}_R^{\bowtie}$ is RM_3 enriched with $\perp$. We denote $RM_3^{\perp} := L\mathbf{2}_R^{\bowtie}$. Note that this logic under the notation Ł3w (weak Łukasiewicz logic) is also studied and axiomatized on a different logical base by D. Vakarelov [19].

It was pointed out in [18] that the logic of $\mathbf{2}_L^{\bowtie}$ is definitionally equivalent to the three valued logic Ł$_3$ of Łukasiewicz. We have $|\mathbf{2}_L^{\bowtie}| = \{f, n, t\}$. Let us set

$$a \supset b := (a \to b) \wedge ({\sim} b \to {\sim} a)$$

and calculate the truth-tables for $\supset$ and $\sim$ on $\{f, n, t\}$:

$\supset$	f	n	t
f	t	t	t
n	n	t	t
t	f	n	t

$\sim$	
f	t
n	n
t	f

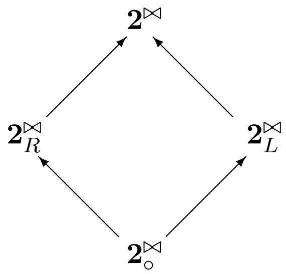
Thus, $\langle \{f, n, t\}, \supset, \sim, \{t\} \rangle$ is the well known matrix for Ł$_3$. All operations of $\mathbf{2}_L^{\bowtie}$ can be defined through $\supset$ and $\sim$ as follows:

$$\neg a := a \supset \sim a,\ \ a \vee b := (a \supset b) \supset b,$$

$$a \wedge b := \sim(\sim a \vee \sim b),\ \ a \to b := a \supset (a \supset b).$$

Due to this reason we denote $L\mathbf{2}_L^{\bowtie}$ as Ł$_3$.

Note that the $\mathbf{N4}^{\perp}$-lattices $\mathbf{2}^{\bowtie}$, $\mathbf{2}_{\circ}^{\bowtie}$, $\mathbf{2}_R^{\bowtie}$, and $\mathbf{2}_L^{\bowtie}$ have no non-trivial homomorphic images and are embedded one into another as follows

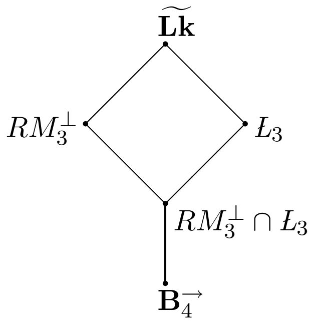


Thus, the interval $\sigma^{-1}(\mathbf{Lk})$ contains exactly five logics determined by the following sets of subdirectly irreducible $\mathbf{N4}^{\perp}$-lattices:

$$\{\mathbf{2}^{\bowtie}\},\ \{\mathbf{2}_R^{\bowtie}, \mathbf{2}_L^{\bowtie}\},\ \{\mathbf{2}_R^{\bowtie}\},\ \{\mathbf{2}_L^{\bowtie}\},\ \{\mathbf{2}_{\circ}^{\bowtie}\}.$$

We have thus proved the following

Proposition 5.1. *The interval $\sigma^{-1}(\mathbf{Lk})$ has the following structure:*



In particular, $\eta(\mathbf{Lk}) = \mathbf{B}_4^{\to}$ and $\eta^{\circ}(\mathbf{Lk}) = \widetilde{\mathbf{Lk}}$.

Taking into account that $L \subseteq \eta^{\circ}\sigma(L)$ for any $L \in \mathcal{E}\mathbf{N4}^{\perp}$ and that $\eta^{\circ}(L_1) \subseteq \eta^{\circ}(L_2)$ whenever $L_1 \subseteq L_2$ we conclude

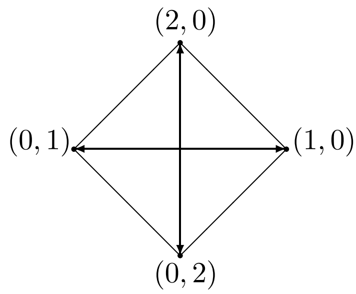
Corollary 5.1. *1. Any non-trivial extension of $\mathbf{N4}^{\perp}$ is contained in the logic $\widetilde{\mathbf{Lk}}$.*

2. The logic $\mathbf{N4}^{\perp}$ has no contradictory non-trivial extension.

We have thus pointed out the first essential difference of the structure of $\mathcal{E}\mathbf{N4}^{\perp}$ from the structure of Jhn, the class of non-trivial extensions of the minimal logic [12]. The minimal logic has the whole class of contradictory extensions isomorphic to the class of extensions of positive logic, whereas in the case of $\mathbf{N4}^{\perp}$, adding to $\mathbf{N4}^{\perp}$ a contradictory scheme leads to a trivialization. Further, unlike Jhn the class $\mathcal{E}\mathbf{N4}^{\perp} \setminus \{For^{\perp}\}$ of non-trivial $\mathbf{N4}^{\perp}$-extensions forms a lattice with the unit element $\widetilde{\mathbf{Lk}}$.

6. The lattice structure of $\mathcal{E}$N4$^{\perp}$

Our next step is to describe the predecessors of $\widetilde{\mathbf{Lk}}$ in $\mathcal{E}\mathbf{N4}^{\perp}$. We know of two predecessors. It follows from Proposition 5.1 that $RM_3^{\perp}$ and Ł$_3$ are the predecessors of $\widetilde{\mathbf{Lk}}$ in $\mathcal{E}\mathbf{N4}$. A further example provides the twist structure $\mathbf{3}_{\circ}^{\bowtie}$, where $\mathbf{3}$ is a three-element linearly ordered Heyting algebra, $|\mathbf{3}| = \{0, 1, 2\}$, $0 \leq 1 \leq 2$. Since $F_d(\mathbf{3}) = \{1, 2\}$, the lattice $\mathbf{3}_{\circ}^{\bowtie}$ has four elements $(0, 1)$, $(1, 0)$, $(0, 2)$, and $(2, 0)$. The lattice structure and the action of strong negation on $\mathbf{3}_{\circ}^{\bowtie}$ are presented on the diagram below.



We can see that $\mathbf{3}_{\circ}^{\bowtie}$ and $\mathbf{2}^{\bowtie}$ are isomorphic as lattices, but the negation acts on $\mathbf{3}_{\circ}^{\bowtie}$ in another way.

Proposition 6.1. *In the lattice $\mathcal{E}\mathbf{N4}^{\perp}$, $\widetilde{\mathbf{Lk}}$ has exactly three predecessors: $RM_3^{\perp}$, Ł$_3$, and $L\mathbf{3}_{\circ}^{\bowtie}$. Each non-trivial element of $\mathcal{E}\mathbf{N4}^{\perp}$ different from $\widetilde{\mathbf{Lk}}$ is contained in one of them.*

Proof. Let $L \in \mathcal{E}\mathbf{N4}^{\perp}$ and $L \neq \widetilde{\mathbf{Lk}}$. Assume L is not contained in $RM_3^{\perp}$ or in Ł$_3$. By Proposition 5.1 $\sigma(L) \neq \mathbf{Lk}$ in this case. It follows that L has a model $\mathcal{A} = Tw(\mathcal{B}, \nabla, \Delta)$, where $\mathcal{B}$ is not a Boolean algebra, in which case $\nabla \neq \{1\}$. Take an element a in $\mathcal{B}$ such that $a \neq 0, 1$ and $a \in \nabla$, and consider the twist-structure $\mathcal{A}_0 := Tw(\{0, a, 1\}, \{a, 1\}, \{0\})$. Clearly, $\mathcal{A}_0 \leq \mathcal{A}$ and $\mathcal{A}_0 \cong \mathbf{3}_{\circ}^{\bowtie}$. Thus, $L \models \mathbf{3}_{\circ}^{\bowtie}$ and $L\mathbf{3}_{\circ}^{\bowtie} \subseteq L$.

The logic $L\mathbf{3}^{\bowtie}_{\circ}$ is incomparable with $RM_3^{\perp}$ and Ł$_3$ because on one hand the intuitionistic fragment of $L\mathbf{3}^{\bowtie}_{\circ}$ is not classical and on the other hand $\sim p \to (p \to q) \in L\mathbf{3}^{\bowtie}_{\circ} \setminus RM_3^{\perp}$ and $\neg\neg(p \vee \sim p) \in L\mathbf{3}^{\bowtie}_{\circ} \setminus$ Ł$_3$. ■

Recall that an element x of a lattice $\mathcal{A}$ is called a *splitting element* if there exists a y such that for every element $z \in A$, either $z \leq x$ or $y \leq z$. If x is a splitting element, the corresponding y is called the *splitting* of $\mathcal{A}$ by x and is denoted by $\mathcal{A}/x$. We write $\mathcal{A}/\{x, y\}$ for $\mathcal{A}/x \vee \mathcal{A}/y$.

Proposition 6.2. *The logics $RM_3^{\perp}$ and Ł$_3$ are splitting elements in the lattice $\mathcal{E}\mathbf{N4}^{\perp}$ and the following holds:*

$$\mathcal{E}\mathbf{N4}^{\perp}/RM_3^{\perp} = \mathbf{N3},\ \mathcal{E}\mathbf{N4}^{\perp}/Ł_3 = \mathbf{N4}^{\perp} + \neg\neg(p \vee \sim p),\ \mathcal{E}\mathbf{N4}^{\perp}/L\mathbf{3}^{\bowtie}_{\circ} = \mathbf{B}_4^{\rightarrow}$$

and

$$\mathcal{E}\mathbf{N4}^{\perp}/\{RM_3^{\perp}, Ł_3\} = \mathbf{N3}^{\circ}.$$

Proof. If $L \in \mathcal{E}\mathbf{N3}$, then $\Delta(\mathcal{A}) = \{0\}$ for any $\mathcal{A} \models L$. Therefore, $\mathbf{2}^{\bowtie}_{R}$ is not a model of L and $L \not\subseteq RM_3^{\perp}$. If $L \notin \mathcal{E}\mathbf{N3}$, there is $\mathcal{A} = Tw(\mathcal{B}, \nabla, \Delta)$ such that $\mathcal{A} \models L$ and $\Delta \neq \{0\}$. Choose an $a \in \Delta$ such that $a \neq 0$ and consider the quotient $\mathcal{A}_0 := \mathcal{A}/\langle a \rangle^{\bowtie}$. By Proposition 3.8

$$\mathcal{A}_0 \cong \mathcal{A}_1 := Tw(\mathcal{B}/\langle a \rangle, \nabla/\langle a \rangle, \Delta/\langle a \rangle).$$

Since $a \in \Delta$, $\Delta/\langle a \rangle = \mathcal{B}/\langle a \rangle$. Consequently, $\mathcal{A}_1$ contains a subalgebra $Tw(\{0/\langle a \rangle, 1/\langle a \rangle\}, \{1/\langle a \rangle\}, \{0/\langle a \rangle, 1/\langle a \rangle\})$, which is isomorphic to $\mathbf{2}^{\bowtie}_{R}$. Thus, $L \subseteq RM_3^{\perp}$ and we have proved the equality $\mathcal{E}\mathbf{N4}^{\perp}/RM_3^{\perp} = \mathbf{N3}$.

If $\neg\neg(p \vee \sim p) \in L$, then every model of L is up to isomorphism of the form $Tw(\mathcal{B}, F_d(\mathcal{B}), \Delta)$. Obviously, $\mathbf{2}^{\bowtie}_{L}$ does not satisfy this condition since $\nabla(\mathbf{2}^{\bowtie}_{L}) = \{0, 1\}$. Therefore, $L \not\models \mathbf{2}^{\bowtie}_{L}$, and L is not contained in Ł$_3$.

If $\neg\neg(p \vee \sim p) \notin L$, then L has a model $\mathcal{A} = Tw(\mathcal{B}, \nabla, \Delta)$ such that $\nabla \neq F_d(\mathcal{B})$. Passing to the quotient $\mathcal{A}_0 := \mathcal{A}/(F_d(\mathcal{B}))^{\bowtie}$ we obtain a model of L such that $\mathcal{A}_0 \cong Tw(\mathcal{B}_0, \nabla_0, \Delta_0)$, $\mathcal{B}_0$ is a Boolean algebra and $\nabla_0 \neq \{1\}$. Let $a \in \nabla_0$ and $a \neq 1$. Let $\overline{a}$ be a Boolean complement of a in $\mathcal{B}_0$. By Proposition 3.8 we have

$$\mathcal{A}_0/\langle \overline{a} \rangle^{\bowtie} \cong \mathcal{A}_2 := Tw(\langle a \rangle, \langle a \rangle, \Delta_1).$$

Since $a \neq 1$, the subalgebra $Tw(\{1, a\}, \{1, a\}, \{a\})$ of $\mathcal{A}_2$ is isomorphic to $\mathbf{2}^{\bowtie}_{L}$. Therefore, $L \models \mathbf{2}^{\bowtie}_{L}$. We have thus proved $\mathcal{E}\mathbf{N4}^{\perp}/$Ł$_3 = \mathbf{N4}^{\perp} + \neg\neg(p \vee \sim p)$.

To prove $\mathcal{E}\mathbf{N4}^{\perp}/L\mathbf{3}^{\bowtie}_{\circ} = \mathbf{B}_4^{\rightarrow}$ we notice that $L\mathbf{3}^{\bowtie}_{\circ} = \eta^{\circ}(L\mathbf{3})$ according to Proposition 4.4. Thus, if $\sigma(L) \subseteq L\mathbf{3}$, then $L \subseteq \eta^{\circ}(L) \subseteq \eta^{\circ}(L\mathbf{3})$. It is well known that $L\mathbf{3}$ is the greatest intermediate logic different from $\mathbf{Lk}$. Consequently, if $\sigma(L) \not\subseteq L\mathbf{3}$, then $\sigma(L) = \mathbf{Lk}$ and L extends $\eta(\mathbf{Lk})$. It remains to notice that $\eta(\mathbf{Lk}) = \mathbf{B}_4^{\rightarrow}$ by Proposition 5.1.

The last equality immediately follows from the first and the second. ■

We denote $\mathbf{N4}^{\mathsf{N}} := \mathbf{N4}^{\perp} + \{\neg\neg(p \vee \sim p)\}$ and distinguish in $\mathcal{E}\mathbf{N4}^{\perp}$ the following subclasses:

$$\mathsf{Exp} := \{L \in \mathcal{E}\mathbf{N4}^{\perp} \mid \sim p \to (p \to q) \in L\},$$
$$\mathsf{Nor} := \{L \in \mathcal{E}\mathbf{N4}^{\perp} \mid \neg\neg(p \vee \sim p) \in L\},$$
$$\mathsf{Gen} := \mathcal{E}\mathbf{N4}^{\perp} \setminus (\mathsf{Exp} \cup \mathsf{Nor}).$$

Let $L \in \mathcal{E}\mathbf{N4}^{\perp}$. We say that L is *explosive* if $L \in \mathsf{Exp}$, we call L *normal* if $L \in \mathsf{Nor}$. Finally, if $L \in \mathsf{Gen}$, we say that L is a logic of *general form.* In the proposition below, we collect a series of simple facts on the introduced classes.

PROPOSITION 6.3. *1.* $\mathsf{Exp} \cap \mathsf{Nor} = \mathcal{E}\mathbf{N3}^{\circ}$ *and* $L\mathbf{3}_{\circ}^{\bowtie}$ *is the greatest element in* $\mathsf{Exp} \cap \mathsf{Nor}$ *different from* $\widetilde{\mathbf{Lk}}$.

2. $\mathsf{Exp} = \mathcal{E}\mathbf{N3}$.

3. $\mathsf{Exp} \setminus \mathsf{Nor} = [\mathbf{N3}, Ł_3]$

4. $L \in \mathsf{Exp}$ *iff* $L \not\subseteq RM_3^{\perp}$ *iff for any* $\mathcal{A} \in Mod(L)$, $\Delta(\mathcal{A}) = \{0\}$.

5. $\mathsf{Nor} = \mathcal{E}\mathbf{N4}^{\mathsf{N}}$.

6. $\mathsf{Nor} \setminus \mathsf{Exp} = [\mathbf{N4}^{\mathsf{N}}, RM_3^{\perp}]$

7. $L \in \mathsf{Nor}$ *iff* $L \not\subseteq Ł_3$ *iff for any* $\mathcal{A} \in Mod(L)$, $\nabla(\mathcal{A}) = F_d(\mathcal{A}_{\bowtie})$.

8. $\mathsf{Gen} = [\mathbf{N4}^{\perp}, RM_3^{\perp} \cap Ł_3]$.

All statements of this proposition easily follow from Propositions 6.1 and 6.2. Figure 1 below presents the structure of $\mathcal{E}\mathbf{N4}^{\perp} \setminus \{For^{\perp}\}$ based on the information given in the last proposition.

Our plans for the rest of the article are as follows. First, we consider the restrictions of the operator σ to the classes Exp and Nor and point out the full analogy with the situation described above. For any $L \in \mathsf{Int}$, the inverse image of L with respect to the corresponding restriction of σ forms an interval in the class Exp (Nor), and the end-points of such an interval can be translated into L. Further, we study interrelations of the class Gen and classes Exp and Nor.

Denote

$$\sigma^3 := \sigma\restriction_{\mathsf{Exp}} \quad \text{and} \quad \sigma^n := \sigma\restriction_{\mathsf{Nor}}.$$

The mappings $\eta^3 : \mathsf{Int} \to \mathsf{Exp}$ and $\eta^n : \mathsf{Int} \to \mathsf{Nor}$ are defined as follows. For every $L \in \mathsf{Int}$,

$$\eta^3(L) := \eta(L) + \{\sim p \to (p \to q)\} \text{ and } \eta^n(L) := \eta(L) + \{\neg\neg(p \vee \sim p)\}.$$

Clearly, $\eta^\circ(L) = \eta^3(L) + \eta^n(L)$.

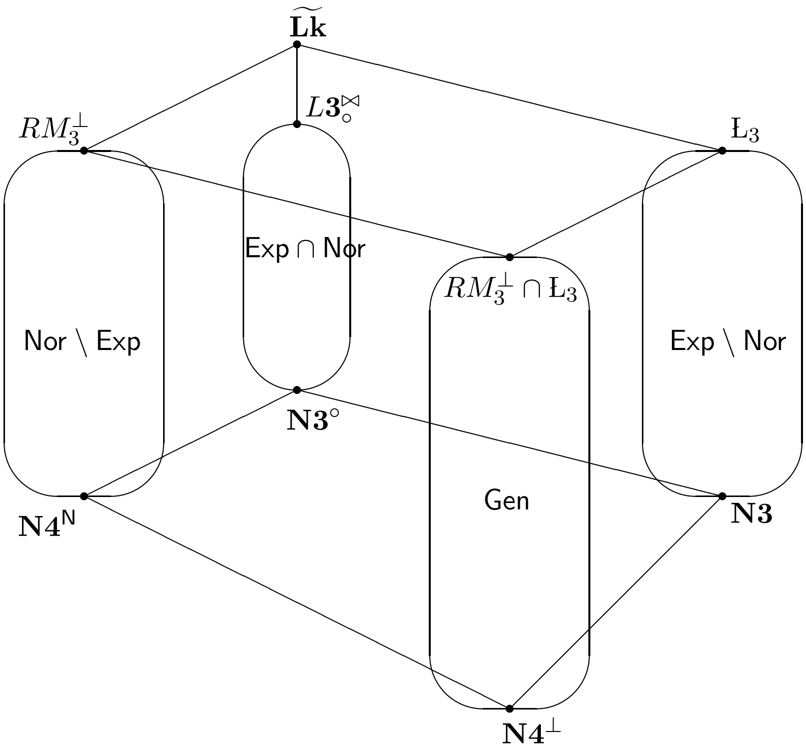


Figure 1.

Logics in Exp having the form $\eta^3(L)$ we call *special explosive* or *se*-logics, and logics in Nor of the form $\eta^n(L)$ are called *special normal* or *sn*-logics. Models of *se*- and *sn*-logics are described in the following

PROPOSITION 6.4. *Let* $L \in \mathsf{Int}$ *and* $\mathcal{A}$ *be an* $\mathbf{N4}^\perp$*-lattice.*

1. $\mathcal{A} \models \eta^3(L)$ *iff* $\mathcal{A}_{\bowtie} \models L$ *and* $\Delta(\mathcal{A}) = \{0\}$.
2. $\mathcal{A} \models \eta^n(L)$ *iff* $\mathcal{A}_{\bowtie} \models L$ *and* $\nabla(\mathcal{A}) = F_d(\mathcal{A}_{\bowtie})$.

The *proof* is analogous to Proposition 4.2.

In what follows we omit the proofs if they can be obtained similarly to the previous results.

For a Heyting algebra $\mathcal{A}$, we define

$$\mathcal{A}_3^{\bowtie} := Tw(\mathcal{A}, \mathcal{A}, \{0\}) \text{ and } \mathcal{A}_n^{\bowtie} := Tw(\mathcal{A}, F_d(\mathcal{A}), \mathcal{A}).$$

If an $\mathbf{N4}^{\perp}$-lattice $\mathcal{A}$ is isomorphic to a lattice of the form $\mathcal{B}_3^{\bowtie}$, we call it a *special* $\mathbf{N3}$-lattice. If an $\mathbf{N4}^{\perp}$-lattice $\mathcal{A}$ is isomorphic to a lattice of the form $\mathcal{B}_n^{\bowtie}$, we call it a *special normal* $\mathbf{N4}^{\perp}$-lattice.

We define formulas $\varphi_{\bowtie}^3$ and $\varphi_{\bowtie}^n$ as follows:

$$\varphi_{\bowtie}^3 := \bigwedge_{p \in var(\varphi)} \neg(p \wedge \overline{p}) \to \varphi_{\bowtie}, \quad \varphi_{\bowtie}^n := \bigwedge_{p \in var(\varphi)} \neg\neg(p \vee \overline{p}) \to \varphi_{\bowtie}.$$

PROPOSITION 6.5. *Let $L \in \mathsf{Int}$ and $\varphi \in For^{\perp}$. The following equivalences hold:*

$$\varphi \in \eta^3(L) \;\Leftrightarrow\; \varphi_{\bowtie}^3 \in L \;\text{ and }\; \varphi \in \eta^n(L) \;\Leftrightarrow\; \varphi_{\bowtie}^n \in L.$$

PROPOSITION 6.6. *Let $L \in \mathcal{E}\mathbf{N4}^{\perp}$.*

1. *L is a special explosive logic if and only if L is determined by some family of special $\mathbf{N3}$-lattices.*
2. *L is a special normal logic if and only if L is determined by some family of special normal $\mathbf{N4}^{\perp}$-lattices.*

PROPOSITION 6.7. *For any $L \in \mathsf{Int}$,*

$$(\sigma^3)^{-1}(L) = [\eta^3(L), \eta^{\circ}(L)] \text{ and } (\sigma^n)^{-1}(L) = [\eta^n(L), \eta^{\circ}(L)].$$

PROPOSITION 6.8.

1. *$\sigma^3 : \mathsf{Exp} \to \mathsf{Int}$ and $\sigma^n : \mathsf{Nor} \to \mathsf{Int}$ are lattice epimorphisms commuting with infinite meets and joins.*
2. *$\eta^3 : \mathsf{Int} \to \mathsf{Exp}$ and $\eta^n : \mathsf{Int} \to \mathsf{Nor}$ are lattice monomorphisms commuting with infinite meets and joins.*

Note that the results on σ^3 and η^3 are known from [7, 9].

7. Explosive and normal counterparts

The decomposition of $\mathcal{E}\mathbf{N4}^{\perp}$ into the classes Exp, Nor, and Gen is very similar to the decomposition of the class Jhn of non-trivial extensions of minimal logic into subclasses of intermediate, negative, and of proper paraconsistent logics [12]. Our next step is to define explosive and normal counterparts for logics in $\mathcal{E}\mathbf{N4}^{\perp}$ in exactly the same way as we have defined intuitionistic and negative counterparts for extensions of the minimal logic.

The mappings $(-)_{exp} : \mathcal{E}\mathbf{N4}^{\perp} \to \mathsf{Exp}$, $(-)_{nor} : \mathcal{E}\mathbf{N4}^{\perp} \to \mathsf{Nor}$, and $(-)_{ne} : \mathcal{E}\mathbf{N4}^{\perp} \to \mathsf{Exp} \cap \mathsf{Nor}$ are defined by the rules:

$$L_{exp} := L + \mathbf{N3}, \quad L_{nor} := L + \mathbf{N4}^{\mathsf{N}}, \quad L_{ne} := L_{exp} + L_{nor},$$

where $L \in \mathcal{E}\mathbf{N4}^{\perp}$. Call the logic L_{exp} an *explosive counterpart* of L, L_{nor} a *normal counterpart* of L, and L_{ne} a *normal explosive counterpart* of L. Thus, by definition the explosive (normal) counterpart of a logic $L \in \mathcal{E}\mathbf{N4}^{\perp}$ is the least explosive (normal) logic containing L.

Notice that the logic $\mathbf{N4}^{\perp}$ has the following counterparts:

$$\mathbf{N4}^{\perp}_{exp} = \mathbf{N3}, \quad \mathbf{N4}^{\perp}_{nor} = \mathbf{N4}^{\mathsf{N}}, \quad \mathbf{N4}^{\perp}_{ne} = \mathbf{N3}^{\circ}.$$

Further simple properties of counterparts are collected in the proposition below.

PROPOSITION 7.1. *1. $(-)_{exp}$, $(-)_{nor}$, and $(-)_{ne}$ are lattice epimorphisms.*

2. $L \in \mathsf{Exp}$ iff $L = L_{exp}$ iff $L_{nor} = L_{ne}$.

3. $L \in \mathsf{Nor}$ iff $L = L_{nor}$ iff $L_{exp} = L_{ne}$.

4. $L \in \mathcal{E}\mathbf{N3}^{\circ}$ iff $L_{exp} = L_{nor}$

5. $\sigma(L) = \sigma(L_{exp}) = \sigma(L_{nor}) = \sigma(L_{ne})$ for every L.

6. $L_{ne} = \eta^{\circ}\sigma(L)$ for every L.

PROOF. Item 1 follows from the distributivity of the lattice $\mathcal{E}\mathbf{N4}^{\perp}$. Items 2 and 3 hold by the definition of counterparts. Item 4 follows from the relation $\mathbf{N3}^{\circ} = \mathbf{N3}+\mathbf{N4}^{\mathsf{N}}$. Homomorphism properties of σ and the equalities $\sigma(\mathbf{N3}) = \sigma(\mathbf{N4}^{\mathsf{N}}) = \sigma(\mathbf{N3}^{\circ})$ imply Item 5.

We prove the last item. By Item 5 $\sigma(L_{ne}) = \sigma(L)$ and $L_{ne} \in \mathcal{E}\mathbf{N3}^{\circ}$ by definition. Thus the equality $L_{ne} = \eta^{\circ}\sigma(L)$ follows from the fact that η° is a lattice isomorphism of Int and $\mathcal{E}\mathbf{N3}^{\circ}$ stated in Proposition 4.7. ■

Semantical characterization of logics $\mathbf{N3}$, $\mathbf{N4}^{\mathsf{N}}$, and $\mathbf{N3}^{\circ}$ and the definition of counterparts allow to characterize models of counterparts as follows.

PROPOSITION 7.2. *Let $L \in \mathcal{E}\mathbf{N4}$ and $\mathcal{A}$ be an $\mathbf{N4}^{\perp}$-lattice.*

1. $\mathcal{A} \models L_{exp}$ if and only if $\mathcal{A} \models L$ and $\Delta(\mathcal{A}) = \{0\}$.

2. $\mathcal{A} \models L_{nor}$ if and only if $\mathcal{A} \models L$ and $\nabla(\mathcal{A}) = \mathcal{F}_d(\mathcal{A})$.

3. $\mathcal{A} \models L_{ne}$ if and only if $\mathcal{A} \models L$, $\Delta(\mathcal{A}) = \{0\}$, and $\nabla(\mathcal{A}) = \mathcal{F}_d(\mathcal{A})$.

We study now how counterparts can be defined in an original logic L. For any formula φ, we put

$$\varphi^e := \bigwedge_{p\in var(\varphi)} \neg(p \wedge \sim p), \quad \varphi^n := \bigwedge_{p\in var(\varphi)} \neg\neg(p \vee \sim p),$$

and

$$\varphi^{exp} := \varphi^e \to \varphi, \quad \varphi^{nor} := \varphi^n \to \varphi, \quad \varphi^{ne} := (\varphi^e \wedge \varphi^n) \to \varphi.$$

Let $\mathcal{A}$ be a twist-structure, $\mathcal{A} = Tw(\mathcal{B}, \nabla, \Delta)$. We associate with $\mathcal{A}$ the following substructures:

$$\mathcal{A}_{exp} = Tw(\mathcal{B}, \nabla, \{0\}), \quad \mathcal{A}_{nor} = Tw(\mathcal{B}, F_d(\mathcal{B}), \Delta),$$

$$\mathcal{A}_{ne} = Tw(\mathcal{B}, F_d(\mathcal{B}), \{0\}).$$

It can be easily seen that if $\mathcal{A}$ is a model of L, then $\mathcal{A}_{exp}$, $\mathcal{A}_{nor}$, and $\mathcal{A}_{ne}$ are models of explosive, normal, and normal explosive counterpart of L, respectively. The validity of formulas on $\mathcal{A}_{exp}$, $\mathcal{A}_{nor}$, and $\mathcal{A}_{ne}$ can be simulated in $\mathcal{A}$ as follows.

PROPOSITION 7.3. *Let $\mathcal{A} \in S^{\bowtie}(\mathcal{B})$ and $\varphi \in For^{\perp}$. The following equivalences hold:*

$$\mathcal{A}_{exp} \models \varphi \;\Leftrightarrow\; \mathcal{A} \models \varphi_{exp}, \; \mathcal{A}_{nor} \models \varphi \;\Leftrightarrow\; \mathcal{A} \models \varphi_{nor}, \; \mathcal{A}_{ne} \models \varphi \;\Leftrightarrow\; \mathcal{A} \models \varphi_{ne}.$$

PROPOSITION 7.4. *Let $L \in \mathcal{E}\mathbf{N4}^{\perp}$ and $\varphi \in For^{\perp}$. The following equivalences hold:*

$$\varphi \in L_{exp} \;\Leftrightarrow\; \varphi_{exp} \in L, \; \varphi \in L_{nor} \;\Leftrightarrow\; \varphi_{nor} \in L, \; \varphi \in L_{ne} \;\Leftrightarrow\; \varphi_{ne} \in L$$

These two propositions can be proved similarly to Proposition 4.3 and Corollary 4.1.

Let $\mathcal{K}$ be a class of twist-structures and $\tau \in \{exp, nor, ne\}$. We put

$$\mathcal{K}_\tau := \{\mathcal{A}_\tau |\; \mathcal{A} \in \mathcal{K}\}.$$

PROPOSITION 7.5. *Let $\mathcal{K}$ be a class of twist-structures, $L = L\mathcal{K}$, and $\tau \in \{exp, nor, ne\}$. Then $L_\tau = L\mathcal{K}_\tau$.*

PROOF. Consider the case of explosive counterpart. Obviously, $L_{exp} \subseteq L\mathcal{K}_{exp}$. If $\varphi \notin L_{exp}$, then $\varphi_{exp} \notin L$ by the previous proposition, and there is $\mathcal{A} \in \mathcal{K}$ such that $\mathcal{A} \not\models \varphi_{exp}$. By Proposition 7.3 $\mathcal{A}_{exp} \not\models \varphi$. ∎

In the end of this section, we consider classes of logics having given explosive and normal logics as explosive and normal counterparts. According to Proposition 7.1 explosive and normal counterparts of a logic have the same intuitionistic fragment. Therefore, for given $L_1 \in \mathsf{Exp}$ and $L_2 \in \mathsf{Nor}$, there is a logic L with $L_{exp} = L_1$ and $L_{nor} = L_2$ only if $\sigma(L_1) = \sigma(L_2)$.

For $L_1 \in \mathsf{Exp}$ and $L_2 \in \mathsf{Nor}$ with $\sigma(L_1) = \sigma(L_2)$, we define the family of logics

$$Spec(L_1, L_2) := \{L \in \mathcal{E}\mathbf{N4}^{\perp} |\ L_{exp} = L_1 \text{ and } L_{nor} = L_2\}$$

and the logic

$$L_1 * L_2 := \mathbf{N4}^{\perp} + \{\varphi_{exp} |\ \varphi \in L_1\} \cup \{\varphi_{nor} |\ \varphi \in L_2\}.$$

First of all we note that all logics of $Spec(L_1, L_2)$ have the same intuitionistic fragment and that if one of logics L_1 or L_2 belongs to the intersection $\mathsf{Exp} \cap \mathsf{Nor}$, the class is one-element.

PROPOSITION 7.6. *Let $L_1 \in \mathsf{Exp}$ and $L_2 \in \mathsf{Nor}$ be such that $\sigma(L_1) = \sigma(L_2) = L$.*

1. *$Spec(L_1, L_2) \subseteq \sigma^{-1}(L)$.*
2. *If $L_2 \in \mathsf{Exp}$, then $Spec(L_1, L_2) = \{L_1\}$.*
3. *If $L_1 \in \mathsf{Nor}$, then $Spec(L_1, L_2) = \{L_2\}$.*

PROOF. 1. If $L' \in Spec(L_1, L_2)$, then $L'_{exp} = L_1$ and

$$L = \sigma(L'_{exp}) = \sigma(L') + \sigma(\mathbf{N3}) = \sigma(L') + \mathbf{Li} = \sigma(L').$$

2. If $L_2 \in \mathsf{Exp} \cap \mathsf{Nor} = \mathcal{E}\mathbf{N3}^{\circ}$, then $L_2 = \eta^{\circ}(L)$. Since $\eta^{\circ}(L)$ is the greatest element of $\sigma^{-1}(L)$, we obtain $L_1 \subseteq L_2$.

Let $L' \in Spec(L_1, L_2)$. Then we have $L' + \mathbf{N4}^{\mathsf{N}} = L_2 \supseteq L_1 \supseteq \mathbf{N3}$. We claim that $\mathbf{N3} \subseteq L'$ in this case. Assume that L' is not explosive. Then there is a twist-structure $\mathcal{A} \models L'$ with $\Delta(\mathcal{A}) \neq \{0\}$. By Proposition 7.2, $\mathcal{A}_{nor} \models L_2 = L'_{nor}$ and $\Delta(\mathcal{A}_{nor}) = \Delta(\mathcal{A}) \neq \{0\}$. The latter is impossible, since L_2 is explosive.

3. From $L_1 \in \mathsf{Nor}$ we conclude $L_2 \subseteq L_1$ and finish the proof similarly to the previous item. ■

In case when neither of L_1 or L_2 is normal explosive, the class $Spec(L_1, L_2)$ forms an interval in the lattice $\mathcal{E}\mathbf{N4}^{\perp}$ containing at least two points.

PROPOSITION 7.7. *Let $L_1 \in \mathsf{Exp}$ and $L_2 \in \mathsf{Nor}$ be such that $\sigma(L_1) = \sigma(L_2)$. Then $Spec(L_1, L_2) = [L_1 * L_2, L_1 \cap L_2]$. If $L_1 \notin \mathsf{Nor}$ and $L_2 \notin \mathsf{Exp}$, then $L_1 * L_2 \neq L_1 \cap L_2$.*

PROOF. It follows by definition that $L \subseteq L_{exp} \cap L_{nor}$ for all $L \in \mathcal{E}\mathbf{N4}^{\perp}$, i.e., $L \subseteq L_1 \cap L_2$ for all $L \in Spec(L_1, L_2)$. Let us check that $L_1 \cap L_2 \in Spec(L_1, L_2)$. We calculate

$$(L_1 \cap L_2)_{exp} = (L_1 + \mathbf{N3}) \cap (L_2 + \mathbf{N3}) = L_1 \cap \eta^{\circ}\sigma(L_1) = L_1.$$

That $(L_1 \cap L_2)_{nor} = L_2$ can be checked similarly.

We have thus proved that $L_1 \cap L_2$ is the greatest point of $Spec(L_1, L_2)$. Let us consider the logic $L_1 * L_2$. If $L \in Spec(L_1, L_2)$, then

$$\{\varphi_{exp} |\ \varphi \in L_1\} \cup \{\varphi_{nor} |\ \varphi \in L_2\} \subseteq L$$

by Proposition 7.4, i.e., $L_1 * L_2 \subseteq L$. It remains to verify that $L_1 * L_2 \in Spec(L_1, L_2)$.

Let $L^* := L_1 * L_2$. The inclusions $L_1 \subseteq L^*_{exp}$ and $L_2 \subseteq L^*_{nor}$ follow from the definition of $L_1 * L_2$ and Proposition 7.4. Let us prove the inverse inclusions.

By definition

$$L^*_{exp} = \mathbf{N3} + \{\varphi_{exp} |\ \varphi \in L_1\} \cup \{\varphi_{nor} |\ \varphi \in L_2\} = L_1 + \{\varphi_{ne} |\ \varphi \in L_2\}.$$

The last equality is due to the fact that for any φ,

$$\mathbf{N3} \vdash \varphi \leftrightarrow \varphi_{exp} \text{ and } \mathbf{N3} \vdash \varphi_{nor} \leftrightarrow \varphi_{ne},$$

which follows from $\neg(p \wedge {\sim} p) \in \mathbf{N3}$. By Proposition 7.4 formulas of the form φ_{ne} belong to L^*_{exp} if and only if $\varphi \in (L^*_{exp})_{ne}$. The equality $\sigma(L_1) = \sigma(L_2)$ implies $L_2 \subseteq (L_2)_{ne} = (L_1)_{ne}$, whence, $\varphi \in L_2$ implies $\varphi_{ne} \in L_1$. Thus, $L^*_{exp} = L_1$. That $L^*_{nor} = L_2$ can be proved similarly.

Assume $L_1 \notin \mathsf{Nor}$ and $L_2 \notin \mathsf{Exp}$ and prove that in this case

$$({\sim} p \to (p \to q)) \vee \neg\neg(r \vee {\sim} r) \in (L_1 \cap L_2) \setminus (L_1 * L_2).$$

Since ${\sim} p \to (p \to q) \in L_1$ and $\neg\neg(r \vee {\sim} r) \in L_2$, the disjunction of these formulas belongs to the intersection $L_1 \cap L_2$. To prove that $({\sim} p \to (p \to q)) \vee \neg\neg(q \vee {\sim} q) \notin L_1 * L_2$ we notice that on one hand $L_1 * L_2 \subseteq \mathbf{B}_4^{\to} = \text{Ł}_3 * RM_3^{\perp}$ and on the other hand $({\sim} p \to (p \to q)) \vee \neg\neg(q \vee {\sim} q) \notin \mathbf{B}_4^{\to}$. ■

As a consequence we obtain a semantical characterization of logics of the form $L_1 * L_2$.

COROLLARY 7.1. *Let $L_1 \in \mathsf{Exp}$ and $L_2 \in \mathsf{Nor}$ be such that $\sigma(L_1) = \sigma(L_2)$. For every $\mathbf{N4}^{\perp}$-lattice $\mathcal{A}$ holds the equivalence*

$$\mathcal{A} \models L_1 * L_2 \ \Leftrightarrow\ \mathcal{A}_{exp} \models L_1 \text{ and } \mathcal{A}_{nor} \models L_2.$$

PROOF. The direct implication follows from the fact that $L_1 * L_2 \in Spec(L_1, L_2)$ stated above. The inverse implication follows from the definition of $L_1 * L_2$. ∎

8. Conclusion

The goal of this paper was to show that if we pass from the explosive logic **N3** to its paraconsistent analogue $\mathbf{N4}^{\perp}$, then the class of extensions extends in a regular way. The above results demonstrate that the lattice $\mathcal{E}\mathbf{N4}^{\perp}$ has natural and non-trivial structure modulo the lattice of **N3**-extensions. Despite essential differences this picture is similar to the structure of the lattice of extensions of Johansson logic over the lattice of superintuitionistic logics [12]. Moreover, we have studied in details the interrelation between the class of $\mathbf{N4}^{\perp}$-extensions and the well studied class of intermediate logics, which provides basis for the investigation of transfer problems for the class of $\mathbf{N4}^{\perp}$-extensions. Transfer problems for **N3**-extensions were studied in [7, 9, 16, 17].

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Sergei P. Odintsov
Sobolev Institute of Mathematics
Koptyug prosp. 4
630090 Novosibirsk, Russia
odintsov@math.nsc.ru