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# First Order Theories for Partial Models

**Abstract.** We investigate first order sentences valid in completions of a given partial algebraic structure - a partial model. We give semantic and syntactic description of the set of all sentences valid in every completion of the given partial model - first order theory of this model.

*Keywords*: partial algebra, partial model, extension, possible sets of sentences, infallible sets of sentences, standard models, Scott's domain, default logic

## 1. Introduction and overview

We assume that there is a unique real world, which we are not able to discover completely. Only fragments of this world are cognizable. By investigating the fragments we try to describe the real world.

We decided to represent the cognizable part of the world as a partial first order model, which is just a partial algebra enriched in predicates.The properties of partial algebras can be easily imported to partial models. We investigate extensions and completions of the given partial model or, more generally, a family of partial models and describe a first order theory for partial models as a set of infallible sentences i.e. the set of all first order sentences valid in every completion of the given partial model. On the other hand we find for the given consistent theory its partial part in a purely syntactic way and construct a unique standard family of partial models satisfying this partial theory.

Second section includes some facts concerning partial models; we also investigate a family of partial models. Definitions and properties of possible and infallible sets of sentences are presented in Section 4, where also some algebraic properties of the class of completions are described. The three next sections contain the main results of this paper. First we describe syntactically the sets of first order sentences of the given language which generates the first order theory (the infallible set of sentences) for the given partial model. Next we construct a standard model for the given one as a direct limit of weak strictly finite submodels, i.e., the least partial model having the same theory as the latter. In Section 7. we chose an infallible part of the

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given consistent first order theory and describe a construction of standard model or family of models for this part. In the last section we use our results to show a Scott's information system and later in a kind of non-monotonic logic - default logic and autoepistemic logic.

## 2. Partial models

We consider a signature $\langle F, \Pi, n\rangle$, with at most countable and pairwise disjoint sets of function and predicate symbols and with an arity function $n\colon F \cup \Pi \to \mathcal{N}$.

We use partial algebras theory throughout the paper. For more details see [2] or [1]. The basic notion of this paper, a partial model is a structure obtained from a partial algebra by enriching the latter in relations. Almost all facts concerning partial algebras are easily imported to partial models, see here our previous papers in this topic [8] [9] or [10].

DEFINITION 2.1. *A partial model* of signature $\langle F, \Pi, n\rangle$ is a structure $\mathbf{A} = \langle A, (f^{\mathbf{A}})_{f\in F}, (r^{\mathbf{A}})_{r\in\Pi}\rangle$ such that $\langle A, (f^{\mathbf{A}})_{f\in F}\rangle$ is a partial algebra of signature $\langle F, \eta\rangle$, where $\eta = n \mid F$ (the domain of the operation $f^{\mathbf{A}} \subseteq A^{n(f)} \times A$ is denoted by $domf^{\mathbf{A}}$) and $r^{\mathbf{A}} \subseteq A^{n(r)}$. We say that $\mathbf{A}$ is a model (or a total model) of signature $\langle F, \Pi, n\rangle$ if all its operations are defined everywhere (see [2]). A relation ( or operation) is discrete if it has an empty domain. A partial model $\mathbf{A}$ is *discrete* if all its operations and relations are discrete.

Every model (even total) of a given signature is a partial model of any wider signature. Then, the additional operations and relations are discrete.

*A homomorphism* of partial models $h : \mathbf{A} \longrightarrow \mathbf{B}$ is a function $h : A \longrightarrow B$ such that, for any $f \in F$, if $\underline{a} \in domf^{\mathbf{A}}$ then $h \circ \underline{a} \in domf^{\mathbf{B}}$ and then $h(f^{\mathbf{A}}(\underline{a})) = f^{\mathbf{B}}(h \circ \underline{a})$ and for any $r \in \Pi$ and $a_1, \ldots, a_{n(r)} \in A$ if $r^{\mathbf{A}}(a_1, \ldots, a_{n(r)})$ then $r^{\mathbf{B}}(h(a_1), \ldots, h(a_{n(r)}))$.

A bijective homomorphism of partial models $h : \mathbf{A} \longrightarrow \mathbf{B}$ is *an isomorphism* if the inverse function is a homomorphism. We say then that $\mathbf{A}$ and $\mathbf{B}$ are isomorphic and write $\mathbf{A} \cong \mathbf{B}$.

For any homomorphism of partial models $h : \mathbf{A} \longrightarrow \mathbf{B}$ we define a homomorphic image $h(\mathbf{A})$ as a partial model with carrier set $h(A)$ and the structure carried from $\mathbf{A}$ i.e. for any $f \in F$ and $b_1, \ldots, b_{n(f)} \in h(\mathbf{A})$ $(b_1, \ldots, b_{n(f)}) \in domf^{h(\mathbf{A})}$ iff there exists $(a_1, \ldots, a_{n(f)}) \in domf^{\mathbf{A}}$ such that $b_1 = h(a_1), \ldots, b_{n(f)} = h(a_{n(f)})$ and then
$f^{h(\mathbf{A})}(b_1, \ldots, b_{n(f)}) = h(f^{\mathbf{A}}(a_1, \ldots, a_{n(f)})$

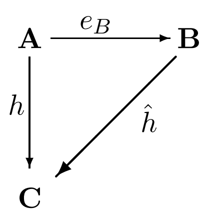
and for any $r \in \Pi$ and $b_1, \ldots, b_{n(r)} \in h(\mathbf{A})$ there exist $a_1, \ldots, a_{n(r)} \in A$ such that $b_1 = h(a_1), \ldots, b_{n(r)} = h(a_{n(r)})$ and then
$r^{h(\mathbf{A})}(b_1, \ldots, b_{n(r)})$ iff $r^{\mathbf{A}}(a_1, \ldots, a_{n(r)})$.

A partial model **B** is *an extension* of a partial model **A** iff there exists an injective homomorphism $e_B : \mathbf{A} \longrightarrow \mathbf{B}$. If **B** is total, then we say that **B** is *a completion* of **A**.

- $E(\mathbf{A})$ denotes the class of all extensions and
- $T(\mathbf{A})$ denotes the class of all completions of **A**.

We now generalize the well known notion of a free completion to partial models. For details see papers [8] or [10].

Definition 2.2. We say that a total model **B** is *a free completion* of a partial model **A** iff it is a completion of **A** generated by **A** and for every total model **C** (of the same signature) and homomorphism $h : \mathbf{A} \to \mathbf{C}$ there exists a unique homomorphism $\hat{h} : \mathbf{B} \to \mathbf{C}$ extending $h$ such that the following diagram commutes:




Fact 2.3. *For every partial model there exists a unique (up to isomorphism) free completion of this model.*

$F(\mathbf{A})$ denotes the free completion of a partial model **A**.

**A** is *a weak submodel* of **B** iff the identity embedding $id_A : A \to B$ is a homomorphism of partial models $id_A : \mathbf{A} \to \mathbf{B}$.

Hence every partial model is an extension of its weak submodel.

Fact 2.4. *For any partial model* **B** *the family of all weak submodels of* **B** *forms an algebraic lattice under the weak submodel relation (if empty submodels are allowed).*

Definition 2.5. A partial model is *finitely determined* iff it has got finitely many non-discrete operations and relations. It is *strictly finite* if it is finite and finitely determined.

The notion of direct limit will be used in the sequel. It can be found in books on partial algebra theory e.g. [2] or [1]. For partial models only the definition of homomorphism is changed.

FACT 2.6. *Every partial model* $\mathbf{B}$ *is a direct limit of the family of all its weak strictly finite submodels.*

## 3. Extensions of a family of partial models

We assume that knowledge is represented by a family of partial models. Every possible world should include (in some way) every member of the given family, as well as the entire family. Let us take the following definition:

Let $\Re = (\mathbf{A}_i)_{i\in I}$ be a family of partial models of a given fixed signature. A partial model $\mathbf{B}$ is *an extension* of $\Re$ iff there exists a family of injections $e_{iB} : \mathbf{A}_i \to \mathbf{B}$ , for every $\mathbf{A}_i \in \Re$.

$E(\Re)$, $T(\Re)$ denote the classes of extensions and completions of a family $\Re$, respectively.

FACT 3.1. $E(\Re) = \bigcap\{E(\mathbf{A}_i) : \mathbf{A}_i \in \Re\} \quad T(\Re) = \bigcap\{T(\mathbf{A}_i) : \mathbf{A}_i \in \Re\}$

*The disjoint sum* of a family $\Re = (\mathbf{A}_i)_{i\in I}$ is a partial model $\mathbf{A}$, with carrier set $\bigcup(A_i \times \{i\})$, where operations are disjoint sums of operations in the models $\mathbf{A}_i$, and analogously for relations. The disjoint sum of a family $\Re$ is denoted $\dot{\bigcup}\Re$.

THEOREM 3.2. *If there are no 0-ary operations (constants) in the signature and if* $\mathbf{B}$ *is an extension of a family* $\Re$ *then*

1. *the disjoint sum of the family* $\Re$ *is an extension of* $\Re$
2. $\mathbf{B}_\Re = \bigcup(e_{iB}(\mathbf{A}_i))_{i\in I}$ *is a weak submodel of* $\mathbf{B}$
3. *there exists an epimorphism* $h$ *from* $\dot{\bigcup}\Re$ *onto* $\mathbf{B}_\Re$ *such that for any* $i \in I \quad h\mid_{A_i}$ *is an injection into* $\mathbf{B}$.
4. *if* $\mathbf{B}$ *is a completion of* $\Re$ *then there is a total submodel of* $\mathbf{B}$ *which is a completion of* $\Re$ *and additionally, is a homomorphic image of the free completion* $F(\dot{\bigcup}\Re)$ *of the disjoint sum of* $\Re$.

DEFINITION 3.3. A partial model $\mathbf{B}$ is called *a free sum of a family* of partial models $\Re = (\mathbf{A}_i)_{i\in I}$ iff there exists a family of injections $e_{iB} : \mathbf{A}_i \to \mathbf{B}$ for every $\mathbf{A}_i \in \Re$ and if for certain partial model $\mathbf{C}$ there exists a family of injections $e_{iC} : \mathbf{A}_i \to \mathbf{C}$ then there exists a unique homomorphism $h : \mathbf{B} \to \mathbf{C}$ such that $h \circ e_{iB} = e_{iC}$.

Notice that in a signature without constants the disjoint sum is a free sum.

FACT 3.4. *For any family* $\Re$ *of partial models* $E(\Re)$ *and* $T(\Re)$ *are nonempty iff the free sum of a family* $\Re$ *exists.*

In the next section (Fact 4.3) we give a logical condition on existence of a free sum of the given family

## 4. Possible and infallible sets of sentences

We use in this section well known facts of first order logic [7]. The notion of possibility is closely related to completions of the given partial model.

Let $\mathcal{L}$ be a first order language of signature $\langle F, \Pi, n\rangle$.

The set of all sentences of the language $\mathcal{L}$ is denoted by $Sent(\mathcal{L})$.

For any $\varphi \in Sent(\mathcal{L})$

$Var(\varphi)$ denotes the set of all variables in $\varphi$

For $\Sigma \subseteq Sent(\mathcal{L})$, $Cn(\Sigma)$ denotes the closure of $\Sigma$ under classical Tarski style first order consequence. We write $\Sigma \models \phi$ iff $\phi \in Cn(\Sigma)$.

$Mod\Sigma$, where $\Sigma \subseteq Sent(\mathcal{L})$ denotes the class of all models satisfying $\Sigma$.

Assume that $\mathbf{A}$ is a given partial model in the signature of the given language $\mathcal{L}$ ( a partial model of $\mathcal{L}$, for short ).

DEFINITION 4.1. A set of sentences $\Sigma \subseteq \mathcal{L}$ is *possible* for $\mathbf{A}$ iff there is a model $\mathbf{B} \in T(\mathbf{A})$ such that $\mathbf{B} \models \Sigma$.

We now introduce the notion of an infallible set of sentences, which is in fact the theory of a partial model.

DEFINITION 4.2. The set of sentences $P_{\mathbf{A}} = \bigcap\{Th(\mathbf{B}) : \mathbf{B} \in T(\mathbf{A})\}$ is called *an infallible set of sentences* for $\mathbf{A}$. Every sentence belonging to $P_{\mathbf{A}}$ is called *an infallible sentence* for $\mathbf{A}$. We write $\mathbf{A} \models^p \varphi$ for $\varphi \in P_{\mathbf{A}}$.

The properties of possible and infallible sets of sentences are described and proved in [8], [9] or [10].

If $\Re$ is a family of partial models then we define possibility and infallibility for $\Re$ analogously.

A set of sentences $\Sigma \subseteq \mathcal{L}$ is *possible* for $\Re$ iff there is a model $\mathbf{B} \in T(\Re)$ such that $\mathbf{B} \models \Sigma$.

Similarly, *the infallible* set of sentences for $\Re$ is the set of all the sentences true in every completion of this family. $P_{\Re}$ denotes the set of sentences infallible for $\Re$ and we write $\Re \models^p \varphi$ for $\varphi \in P_{\Re}$.

FACT 4.3. *Let $\Re$ be a family of partial models then*

1. $P_{\Re} = \bigcap\{Th(\mathbf{B}) \ : \ \mathbf{B} \in T(\Re)\}$.
2. $P_{\Re} = Cn(\bigcup\{P_{\mathbf{A}_i} \ : \ \mathbf{A}_i \in \Re\})$.
3. *$P_{\Re}$ is consistent iff $T(\Re)$ is nonempty.*
4. *The free sum of $\Re$ exists iff $P_{\Re}$ is consistent.*

The next theorem will be very useful in describing the infallible set for the given partial model.

Let $\mathbf{A}$ be a partial model of a language $\mathcal{L}$ of signature $\langle F, \Pi, \eta\rangle$. We extend $\mathcal{L}$ to $\mathcal{L}_A$ by adding a set of constants $\mathcal{C} = \{c_a \colon a \in A\}$. Now, we describe the structure of $\mathbf{A}$ in $\mathcal{L}_A$. Let $\Sigma_A$ be the sum of the following sets:

$$\Sigma_1 = \{c_a \neq c_b \colon a, b \in A \ , \ a \neq b\}$$

$$\Sigma_2 = \{f(c_{a_1}, ..., c_{a_{\eta(f)}}) = c_a \ : \ f \in F \ , \\ (a_1, ..., a_{\eta(f)}) \in dom\, f^{\mathbf{A}} \ , \ f^{\mathbf{A}}(a_1, ..., a_{\eta(f)}) = a\}$$

$$\Sigma_3 = \{r(c_{a_1}, ..., c_{a_{\eta(r)}}) \colon r \in \Pi \ , \ r^{\mathbf{A}}(a_1, ..., a_{\eta(r)})\}$$

THEOREM 4.4. *Let $\mathbf{A}$ be a partial model of $\mathcal{L}$. A set $\Sigma \subseteq Sent(\mathcal{L})$ is possible for $\mathbf{A}$ iff the set $\Sigma \cup \Sigma_A$ of sentences of $\mathcal{L}_A$ is consistent.*

Let $\mathbf{A}'$ denote a partial model interpreting $\mathbf{A}$ in $\mathcal{L}_A$ i.e. $A' = A$ , $f^{\mathbf{A}} = f^{\mathbf{A}'}$ for every $f \in F$ , $r^{\mathbf{A}} = r^{\mathbf{A}'}$ for every $f \in F$ and $c_a^{\mathbf{A}'} = a$.

COROLLARY 4.5. *For any partial model $\mathbf{A}$ of $\mathcal{L}$ $P_{\mathbf{A}} = P_{\mathbf{A}'} \cap Sent(\mathcal{L})$.*

COROLLARY 4.6. $T(\mathbf{A}') = Mod\Sigma_A = ModP_{\mathbf{A}}$

**Properties of classes of completions in a language $\mathcal{L}_A$**

Some algebraic properties of $T(\mathbf{A})$ for a given partial model $\mathbf{A}$ of $\mathcal{L}$ will be discussed first. We use the notation of [3]. Thus for a class $\mathcal{K}$ of total models we use denotations:
$S(\mathcal{K})$ - the class of all models isomorphic to submodels of models in $\mathcal{K}$
$H(\mathcal{K})$ - the class of all homomorphic images of models in $\mathcal{K}$
$I(\mathcal{K})$ - the class of all isomorphic copies of models in $\mathcal{K}$
$P(\mathcal{K})$ - the class of all models isomorphic to direct products of models in $\mathcal{K}$
$P_0(\mathcal{K})$ - the class of all models isomorphic to direct products of nonempty families of models in $\mathcal{K}$
$P^r(\mathcal{K})$ - the class of all models isomorphic to reduced products of models in $\mathcal{K}$

$P_0^r(\mathcal{K})$ - the class of all models isomorphic to reduced products of nonempty families of models in $\mathcal{K}$
$P^u(\mathcal{K})$ - the class of all models isomorphic to ultraproducts of models in $\mathcal{K}$
$P_0^u(\mathcal{K})$ - the class of all models isomorphic to ultraproducts of nonempty families of models in $\mathcal{K}$

We say that $\mathcal{K}$ is *closed* under an operator $O$ iff $O(\mathcal{K}) \subseteq \mathcal{K}$.

FACT 4.7. *The class $T(\mathbf{A})$ of completions of a partial model $\mathbf{A}$ of a language $\mathcal{L}$ is closed under $I, P_0, P_0^r, P_0^u$ and, in general, is not closed under $S, H, P, P^r, P^u$.*

The non-closure under $P, P^r, P^u$ follows from the fact that the product of the empty family of models is a trivial one-element model.

The class of all completions of a partial model $\mathbf{A}$ is not closed under elementary submodels, in general. Look at the following:

EXAMPLE 4.8. Let $\mathbf{A} = \mathcal{N} \cup \{\infty\}$, that is $\mathbf{A}$ is a partial model obtained from the standard model of arithmetics $\mathcal{N}$ of signature $< +, \cdot, 0, 1 >$ by joining the additional separated element $\infty$. Such a partial model embeds into the non-standard model of arithmetics, thus the latter is a completion of $\mathbf{A}$. And $\mathcal{N}$ is an elementary submodel of the non-standard model of arithmetics, but it is not a completion of $\mathbf{A}$.

But the following holds:

THEOREM 4.9. *If $\mathbf{B} \models P_{\mathbf{A}}$ for a given partial model $\mathbf{A}$ then there exists a model which is elementarily equivalent to $\mathbf{B}$ and is a completion of $\mathbf{A}$.*

Let us consider know the class of completions of the partial model $\mathbf{A}'$ of the language $\mathcal{L}_A$ as defined after Theorem 4.4. Then

FACT 4.10. *A class $T(\mathbf{A}')$ of completions of the partial model $\mathbf{A}'$ of the language $\mathcal{L}_A$ is closed under $I, P_0, P_0^r, P_0^u, S$ and elementary submodels and, in general, is not closed under $H, P, P^r, P^u$.*

The closure under $S$ follows from the fact that all elements of $\mathbf{A}$ are constants now, so they must belong to every submodel.

For a moment let us fix our attention on languages in a signature without predicates. Then we can use the universal-algebraic terminology.

*A variety* is a class of algebras closed under $HSP$; on the other hand a variety is a class definable by a set of equations. Similarly, a quasi-variety is a class of algebras closed under $ISP^r$, and on the other hand a quasi-variety is a class definable by a set of quasi-equations. For details see e.g. [3].

Let $\mathbf{1}$ denote the (unique up to isomorphism) one-element total algebra. Then the class of completions of a partial model $\mathbf{A}'$ of a language $\mathcal{L}_A$ has the following properties:

THEOREM 4.11. *1.* $T(\mathbf{A}') \cup \{\mathbf{1}\}$ *is a quasi-variety definable by* $\Sigma_q \cup \Sigma_2$, *where* $\Sigma_q = \{c_a = c_b \Rightarrow x = y \; : \; a, b \in A, \; a \neq b\}$.

*2.* $H(T(\mathbf{A}'))$ *is a variety definable by* $\Sigma_2$.

*3.* $T(\mathbf{A}') = H(T(\mathbf{A}')) \setminus \bigcup\{Mod(c_a = c_b) \; : \; a, b \in A, \; a \neq b\}$

THEOREM 4.12. *For* $T(\mathbf{A}') \cup \{\mathbf{1}\}$ *and* $H(T(\mathbf{A}'))$ *we have that*

1. $F(\mathbf{A}')$ *is the free algebra with an empty set of generators*
2. *the free extension* $F(\mathbf{A}'\dot{\cup}X)$ *of a disjoint sum* $\mathbf{A}'\dot{\cup}X$ *is a free algebra with the set of generators* $X$

Let $\Delta$ be a set of equations in $\mathcal{L}$. Let $\mathcal{K}_\Delta$ denote the class of completions of $\mathbf{A}$ that satisfy $\Delta$, i.e., $\mathcal{K}_\Delta = T(\mathbf{A}') \cap Mod\Delta$.

THEOREM 4.13. *1.* $\mathcal{K}_\Delta \cup \mathbf{1}$ *is a quasi-variety.*

*2.* $H(\mathcal{K}_\Delta) = H(T(\mathbf{A}')) \cap Mod\Delta$ *is a variety.*

*3. the free algebras are* $F(\mathbf{A}'\dot{\cup}X)/\Delta$

*4.* $\mathcal{K}_\Delta$ *is nonempty iff* $nat_\Delta : F(\mathbf{A}') \to F(\mathbf{A}')/\Delta$ *is a homomorphism injective on* $A$.

Analogously we can define a variety of models (a logical variety) as a class of models closed under $HSP$, and logical quasi-variety as a class of models closed under $ISP^r$.

## 5. Characterizing sets of sentences

In this section we give a characterization of a given partial model by a set of existential sentences of the given language generating the theory of this model.

### Finite case

Let $\mathbf{A}$ be a strictly finite partial model of $\mathcal{L}$. We uniquely describe the structure of $\mathbf{A}$ by a certain existential sentence of $\mathcal{L}$.

Example 5.1. Let $\mathbf{A}$ be a partial model with carrier set $A = \{2, 4, 6\}$,the partial operation of addition and the relation of divisibility taken from arithmetics. Then we have the following sentence in $\mathcal{L}_{\mathcal{A}}$
$s_A^A := 2 \neq 4 \wedge 2 \neq 6 \wedge 4 \neq 6 \wedge 2 + 2 = 4 \wedge 2 + 4 = 6 \wedge 4 + 2 = 6 \wedge 2 \mid 2 \wedge 2 \mid 4 \wedge 4 \mid 4 \wedge 2 \mid 6 \wedge 6 \mid 6$, which conveys the positive information on operations and relations in $\mathbf{A}$. Notice that this sentence is a conjunction of the sentences in $\Sigma_A$.

Now, notice that the sentence
$s_A := \exists x \exists y \exists z (x \neq y \wedge x \neq z \wedge y \neq z \wedge x + x = y \wedge x + y = z \wedge y + x = z \wedge x \mid x \wedge x \mid y \wedge y \mid y \wedge x \mid z \wedge z \mid z)$ is a consequence of $s_A^A$ in $\mathcal{L}$.

Let $\mathbf{A}$ be a strictly finite ($n$-element) partial model. Then for every element of $A$ we introduce a variable symbol. Let $\Sigma_1(x_1, ..., x_n)$, $\Sigma_2(x_1, ..., x_n)$, $\Sigma_3(x_1, ..., x_n)$ be $\Sigma_1, \Sigma_2, \Sigma_3$ from Theorem 4.4 with constants replaced by the corresponding variables.

Definition 5.2. A sentence
$s_A := \exists x_1 \exists x_2 ... \exists x_n (\bigwedge \Sigma_1(x_1, ..., x_n) \wedge \bigwedge \Sigma_2(x_1, ..., x_n) \wedge \bigwedge \Sigma_3(x_1, ..., x_n))$
of the language $\mathcal{L}$ is called *the characterizing sentence* of the strictly finite partial model $\mathbf{A}$.

Definition 5.3. We say that a sentence of a given language $\mathcal{L}$ is of the $\star$-form iff it has the following form

($\star$) $s := \exists x_1 \exists x_2 ... \exists x_n (x_1 \neq x_2 \wedge x_1 \neq x_2 \wedge ... \wedge x_2 \neq x_3 \wedge ... \wedge x_{n-1} \neq x_n \wedge f_1(...) = . \wedge f_2(...) = . \wedge ... \wedge f_k(...) = . \wedge r_1(...) \wedge r_2(...) \wedge ... \wedge r_n(...))$,

where $n \in \mathcal{N}$ and $f_i(...) = .$ and $r_j(...)$ denote the conjunctions of all formulas of the form $f_i(x_{1_i}, ..., x_{n(f_i)}) = x_{n(f_i)+1}$ and $r_j(x_{1_j}, ..., x_{n(r_j)})$, respectively

Fact 5.4. *Every sentence $s$ of $\mathcal{L}$ of the $\star$-form is a characterizing sentence of some strictly finite partial model.*

Proof. In the proof of this fact we construct a model $\mathbf{A}_s$ which is useful in construction of a standard model in Section 6.

If $n$ is the number of all different variables in $s$, then we take any $n$-element set $A = \{a_1, ..., a_n\}$ and substitute univocally every $x_i$ by the element $a_i$. We describe on $A$ a partial model $\mathbf{A}_s$ of a language $\mathcal{L}$ as follows:

1. For every $f_i \in F$ let us determine $(a_{1_i}, ..., a_{n(f_i)}) \in dom f_i^{\mathbf{A}_s}$ and $f_i^{\mathbf{A}_s}(a_{1_i}, ..., a_{n(f_i)}) = a_{n(f_i)+1}$ iff $f_i(x_{1_i}, ..., x_{n(f_i)}) = x_{n(f_i)+1}$ is an atomic subformula of $s$

2. for every $r_j \in \Pi$ $\quad r_j^{\mathbf{A}_s}(a_{1_j}, ..., a_{n(f_j)})$ iff $\; r_j(x_{1_i}, ..., x_{n(f_i)})$ is an atomic subformula of $s$

For a model constructed in this way $s = s_{A_s}$ is a characterizing sentence. ■

FACT 5.5. *Let* $\mathbf{A}$ *be a strictly finite partial model of* $\mathcal{L}$. *Then*

1. *for any total model* $\mathbf{B}$ *of* $\mathcal{L}$ $\quad \mathbf{B} \models s_A$ *iff* $\mathbf{B} \in T(\mathbf{A})$
2. *for any partial model* $\mathbf{B}$ $\quad \mathbf{B} \models^p s_A$ *iff* $\mathbf{B} \in E(\mathbf{A})$
3. *every infallible sentence is a first order consequence of the characterizing sentence* $s_A$ *in* $\mathcal{L}$ *i.e.* $P_A = Cn(s_A)$

**Infinite case**

Recall (Example 4.8) that satisfiability of the infallible set of sentences for a given infinite partial model is not sufficient for embedding of this model.

In the infinite case the set $\Sigma_A$ is infinite and therefore infinite conjunctions are required. To omit this problem we give a set $S_A$ of sentences of the language $\mathcal{L}$ which characterizes the given not strictly finite partial model $\mathbf{A}$.

Let $\Phi^\star$ denote the set off all sentences of the $\star$-form from a given set $\Phi$ of sentences of a language $\mathcal{L}$.

DEFINITION 5.6. Let $s, \sigma$ be sentences of a language $\mathcal{L}$. We say that $s$ is a $\star$-consequence of $\sigma$ (and write $s \leq^\star \sigma$) if $\sigma$ is of the $\star$-form and $s \in (Cn(\sigma))^\star$. We say that $s$ is $\star$-equivalent to $\sigma$ (and write $s \equiv^\star \sigma$) if $s$ is logically equivalent to $\sigma$ and both are of the $\star$-form.

The next three theorems have a purely technical character.

THEOREM 5.7. *Let* $(\mathbf{B}_i)_{i \in I}$ *be a family of all strictly finite weak submodels of a given partial model* $\mathbf{A}$. *Let* $s_i$ *denote the characterizing sentence for* $\mathbf{B}_i$. *Then*

1. *If* $\mathbf{B}_i$ *is a weak submodel of* $\mathbf{B}_j$ *then* $s_i$ *is a* $\star$*-consequence of* $s_j$
2. *If* $s_i$ *is a* $\star$*-consequence of* $s_j$ *then* $\mathbf{A}_{s_i} \cong \mathbf{B}_i$ *and* $\mathbf{A}_{s_j} \cong \mathbf{B}_j$ *and* $\mathbf{B}_i$ *is a weak submodel of* $\mathbf{B}_j$.

THEOREM 5.8. *Let* $s, \sigma \in \{s_i : i \in I\}$. *Then*

1. *If* $|Var(s)| = k$ *and* $|Var(\sigma)| = m$ *then*
   $s \leq^\star \sigma$ *iff there exists an injective map* $\mu : Var(s) \to Var(\sigma)$ *such that if* $f(x_1, ..., x_{\eta(f)}) = x \in At(s)$ *then* $f(\mu(x_1), ..., \mu(x_{\eta(f)})) = \mu(x) \in At(\sigma)$
   *and if* $r(x_1, ..., x_{\eta(r)}) \in At(s)$ *then* $r(\mu(x_1), ..., \mu(x_{\eta(r)})) \in At(\sigma)$.

2. *$s \leq^\star \sigma$ and $\sigma \leq^\star s$ iff $s \equiv^\star \sigma$.*
3. *$s \leq^\star \sigma$ iff $\mathbf{A}_s$ embeds into $\mathbf{A}_\sigma$. By (1) above this embedding is uniquely determined by $\mu$ .*
4. *$\mathbf{A}_s \cong \mathbf{A}_\sigma$ iff $s \equiv^\star \sigma$*

PROOF. We give a proof of (*3.*) to show the induced embedding.
Let $s \leq^\star \sigma$. Then $A_s = \{v(x) : x \in Var(s)\}$ and $A_\sigma = \{w(x) : x \in Var(\sigma)\}$, where $v, w$ are injective valuations of variables. Let $\mu : Var(s) \to Var(\sigma)$ be a function as in (*1*). We define $h : A_s \to A_\sigma$ as $h(v(x)) = w(\mu(x))$. Then $h$ is injective because of the injectivity of $v, w, \mu$. It is a homomorphism since if $(v(x_1), ..., v(x_{\eta(f)})) \in dom f^{\mathbf{A}_s}$ and $f^{\mathbf{A}_s}(v(x_1), ..., v(x_{\eta(f)})) = v(x)$, then $f(x_1, ..., x_{\eta(f)}) = x \in At(s)$ and hence $f(\mu(x_1), ..., \mu(x_{\eta(f)})) = \mu(x) \in At(\sigma)$. In consequence $(w(\mu(x_1)), ..., w(\mu(x_{\eta(f)}))) \in dom f^{\mathbf{A}_\sigma}$ and $f^{\mathbf{A}_\sigma}(w(\mu(x_1)), ..., w(\mu(x_{\eta(f)}))) = w(\mu(x))$. ■

Notice that $\mu$ (and the induced homomorphism) is not unique.

THEOREM 5.9. *1. The relation of a $\star$-consequence is a directed partial order on the set $\{s_i : i \in I\}/\equiv^\star$*

*2. The relation of embedding induced by the $\star$-consequence relation is a directed partial order on the set $\{\mathbf{A}_{s_i} : i \in I\}/\cong$.*

DEFINITION 5.10. For a given partial model $\mathbf{A}$ let $(\mathbf{B}_i)_{i \in I}$ be the family of all strictly finite weak submodels of $\mathbf{A}$. *Characterizing set of sentences* is a set $S_A = \{s_i : i \in I\}/\equiv^\star$ of all characterizing sentences for all strictly finite weak submodels of $\mathbf{A}$ factored by the $\star$-equivalence relation.

FACT 5.11. *The characterizing sentence $s_A$ of a strictly finite partial model $\mathbf{A}$ is the greatest element in its characterizing set $S_A$. Moreover, $S_A = (Cn(s_A))^\star/\equiv^\star$.*

THEOREM 5.12. *For any partial model $\mathbf{A}$ of $\mathcal{L}$,*

$$Cn(S_A) = Cn(\Sigma_A) \cap Sent(\mathcal{L}) = P_A$$

## 6. Standard Models

Let us introduce the technical notion:

DEFINITION 6.1. A rank of a sentence $s$ of the $\star$-form is the natural number $r(s) = | Var(s) | + | At^+(s) |$ where $At^+(s)$ denotes the set of all atomic subformulas of $s$ that are not inequalities.

Fact 6.2. *1. If $s \leq^\star \sigma$ then $r(s) \leq r(\sigma)$.*

*2. $\sigma$ covers $s$ iff $s \leq^\star \sigma$ and $r(\sigma) = r(s) + 1$.*

Definition 6.3. *A standard model* for $\mathbf{A}$ is a partial model $\mathbf{A}^\varsigma$ such that $S_A = S_{A^\varsigma}$ and if there exists a partial model $\mathbf{B} \models^p S_A$ then $\mathbf{A}^\varsigma$ embeds into $\mathbf{B}$.

Example 6.4. 1. Every strictly finite partial model is a standard model for itself.

2. $\mathbf{A}'$ is a standard model for itself in the language $\mathcal{L}_A$.

Now we give a construction of a standard model for arbitrary partial model $\mathbf{A}$.

Let us take a family $(\mathbf{B}_s)_{s \in S_A}$ as in the proof of Fact 5.4. We are going to construct a direct limit of this family. $S_A$ with the relation $\leq^\star$ is a directed partial order. We define now a family of homomorphisms $h_{s\sigma} : \mathbf{B}_s \to \mathbf{B}_\sigma$ such that for any $s \leq^\star s' \leq^\star \sigma$ it holds that $h_{s\sigma} = h_{s'\sigma} \circ h_{ss'}$. In details, for every pair of sentences $s \leq^\star \sigma$ such that $r(\sigma) = r(s) + 1$ we take a function $\mu$ as in Theorem 5.7 (*1*), and then we take the homomorphism induced by $\mu$. The remaining injections are recursively defined as compositions. Let $\mathbf{A}^\star$ denote the direct limit of such a directed family.

Theorem 6.5. *Let $\mathbf{A}$ be a partial model. Then $\mathbf{A}^\star$ embeds into $\mathbf{A}$ and both have the same characterizing sets of sentences.*

Theorem 6.6. *If $\mathbf{B}^\star$ is the direct limit of a family $(\mathbf{B}_s)_{s \in S_A}$ with a family of injections $g_{s\sigma}$ given by another choice of $\mu$, then $\mathbf{A}^\star \cong \mathbf{B}^\star$.*

By above theorems for any partial model $\mathbf{A}$ the direct limit $\mathbf{A}^\star$ is a standard model for $\mathbf{A}$.

Corollary 6.7. *Let $\mathbf{A}$ be a partial model of $\mathcal{L}$*

1. *For every $\mathbf{A}$ there exists a unique (up to isomorphism) standard model, namely $\mathbf{A}^\star$.*
2. *If $\mathcal{L}$ is countable, then the standard model is countable, too.*
3. *The class of completions of the standard model for $\mathbf{A}$ is axiomatizable by $P_A$ and it is a closure of $T(\mathbf{A})$ under elementary equivalence.*

## 7. The infallible part of a theory

We have described the theory (the infallible set of sentences) of a partial model and constructed a standard model using a characterizing set of sentences. In this section we consider any consistent theory and we choose the maximal infallible part of this theory. We describe conditions for the existence of a standard model for the given theory or of a standard family, in the opposite case.

Let $\Phi$ be a consistent theory in the language $\mathcal{L}$ and take $\Phi^\star$. We decided to give the construction of required models in proofs of the following three theorems:

THEOREM 7.1. *If $\langle \Phi^\star / \equiv, \leq^\star \rangle$ is directed then there exists a unique (up to isomorphism) infinite partial model $\mathbf{A}$ such that $P_A = Cn(\Phi^\star)$. Moreover, if there exists the greatest element $\sigma$ in this partial order then this model is strictly finite.*

PROOF. The direct limit $\mathbf{A}^\Phi$ of the family $(A_s)_{s \in \Phi^\star/\equiv}$ is the required model. ∎

THEOREM 7.2. *If there exists an upper bound on the length of chains in $\langle \Phi^\star / \equiv, \leq^\star \rangle$, then there exists a unique (up to isomorphism) family of strictly finite partial models $\mathfrak{R}$ such that $P_{\mathfrak{R}} = Cn(\Phi^\star)$*

PROOF. By assumption every maximal chain is finite and has the greatest element. Let $(s_j)_{j \in J}$ be the set of these bounds. The required family $\mathfrak{R}$ is the family of all the $\mathbf{A}_{s_j}$'s for $j \in J$. Uniqueness follows from Theorem 7.1. Then for any $\mathbf{A}_{s_j}$ we have $P_{A_{s_j}} = Cn(s_j)$. Hence $Cn(\Phi^\star) = Cn(\bigcup (s_j)_{j \in J}) = Cn(\bigcup (P_{A_{s_j}})_{j \in J}) = P_{\mathfrak{R}}$. ∎

THEOREM 7.3. *For any order of $\langle \Phi^\star / \equiv, \leq^\star \rangle$ there exists a unique (up to isomorphism) family of partial models $\mathfrak{R}$ such that*
*$P_{\mathfrak{R}} = Cn(\Phi^\star)$.*

PROOF. The required family $\mathfrak{R}$ consists of all the direct limits for all maximal directed suborders. Every maximal directed suborder $\mathfrak{I}$ induces a partial model $A_{\mathfrak{I}}$ as in Theorem 7.1 such that $P_{A_{\mathfrak{I}}} = Cn((\Phi^\star) \cap \mathfrak{I})$. Thus $P_{\mathfrak{R}} = Cn(\Phi^\star)$. ∎

DEFINITION 7.4. *A standard family* of partial models for a theory $\Phi$ in a language $\mathcal{L}$ is a family $\mathfrak{R}$ such that $P_{\mathfrak{R}} = Cn(\Phi^\star)$ and if there is a family $\mathcal{R}$ satisfying $P_{\mathcal{R}} = Cn(\Phi^\star)$ then every model in $\mathfrak{R}$ embeds into a certain model in $\mathcal{R}$.

It follows from above four theorems that:

Corollary 7.5. *For every consistent theory there exists a unique (up to isomorphism ) standard family of partial models.*

## Some applications

### Scott's information systems

Definition 7.6. Scott's information system [6] is a structure:
$\mathbf{S} = (D, \Delta, Con, \vdash_S)$, where $D$ is a set (of data ) , $\Delta$ is a selected element from $D$ (the least information bit) , $Con$ is a certain family of finite subsets of $D$ (consistent finite sets), $\vdash_S$ is a binary entailment relation and any $\Phi, \Psi, \Gamma, \Phi', \Psi' \in Con$ satisfy the following conditions:

1. if $\Phi \subseteq \Psi \in Con$ then $\Phi \in Con$
2. $\{\varphi\} \in Con$ for every $\varphi \in D$
3. $\emptyset \vdash_S \{\Delta\}$
4. $\Phi \vdash_S \Phi$
5. $\Phi \vdash_S \Psi$ i $\Psi \vdash_S \Gamma$ imply $\Phi \vdash_S \Gamma$
6. $\Phi \subseteq \Phi'$, $\Phi \vdash_S \Psi$, i $\Psi' \subseteq \Psi$ imply $\Phi' \vdash_S \Psi'$
7. $\Phi \vdash_S \Psi$ i $\Phi \vdash_S \Psi'$ imply $\Phi \vdash_S \Psi \cup \Psi'$

Definition 7.7. A set $\Phi$ such that

1. all finite subsets of $\Phi$ are in $Con$
2. if $\Psi \subseteq \Phi$ and $\Phi \vdash_S \varphi$ then $\varphi \in \Phi$

is called *an element (partial)* of a system $S$.

Thus Scott's elements are all the sets of sentences closed under $\vdash_S$.

Using our terminology we construct a Scott's information system and then apply it to non-monotonic logics.

Let $\mathbf{A}$ be a partial model of a language $\mathcal{L}$. By Corollary 6.7 let $\mathbf{A}$ be a standard model for $P_A$ in $\mathcal{L}$.

Let the set of data be the set of all the sentences possible for $\mathbf{A}$ i.e. $M_A$. Let $Con$ be the set of all the finite possible sets for $\mathbf{A}$.

The least information bit is represented by the infallible set $P_A$. The entailment relation is defined as follows:
$\Phi \vdash_S \Psi$ iff $\Psi \subseteq Cn((Cn(P_A \cup \Phi))^\star)$.

THEOREM 7.8. $\mathbf{S_A} = (M_A,\ P_A, Con, \vdash_S)$ *is a Scott's information system.*

THEOREM 7.9. *If a set of sentences is Scott's element then it is the infallible set for a certain extension of* **A**.

PROOF. To prove this theorem assume that $\Phi = Cn((Cn(P_A \cup \Psi))^\star)$. Since $\Psi$ is possible for **A**, it is consistent with $P_A$ and $\Phi$ is satisfied in a certain completion **B** of **A**. By Theorem 7.3 there exists a family $\Re$ such that $P_\Re = \Phi$. Every model in $\Re$ is a weak submodel of **B**. Properties of the lattice of weak submodels imply that $\bigcup \Re$ is also a weak submodel of **B** and hence extends **A**. ■

COROLLARY 7.10. *A set of sentences is Scott's element iff it is the infallible set for an extension of* **A** *characterized by* $(Cn(P_A \cup \Psi))^\star$*, where* $\Psi$ *is a finite possible set for* **A**.

Thus the structure of Scott's elements with inclusion corresponds to the structure of extensions of **A** with $A$-embedding. It is a complete meet semi-lattice. Moreover if the union of two Scott's elements is consistent then there exists the supremum of these elements. Such structures are called *chopped lattices.*

**Default logics**

*A default theory* [5] is a pair $\triangle = \langle D, F\rangle$ such that
$F$ is a set of closed formulas (facts)
$D$ is a set of defaults of the form
$\frac{\alpha:\beta_1,...,\beta_m}{w}$, where $\alpha$ is any proven formula and $\beta_1, ..., \beta_m$ are formulas consistent with the knowledge we have. $w$ is called *the consequent* of the default.

We give an interpretation of default theories in our terminology for a fixed standard model **A** of a fixed language $\mathcal{L}$ .

Let $F = P_A$, let $D$ be a set of defaults
$\frac{\alpha:\beta_1,...,\beta_m}{w}$, where $\beta_1, ..., \beta_m$ are possible sentences for **A**.
$w$ is the consequent of the default $\frac{\alpha:\beta_1,...,\beta_m}{w}$
iff $w \in Cn((Cn(P_A \cup \{\alpha, \beta_1, ..., \beta_m\}))^\star)$

THEOREM 7.11. *Let* $\frac{\alpha:\beta_1,...,\beta_m}{w}$ *be a default of the default theory* $\langle D, P_A\rangle$ *introduced above. If* $\{\alpha, \beta_1, ..., \beta_m\}$ *is a set possible for* **A** *then the set of all the consequents of this default is equal to* $P_B$ *for a certain extension* **B** *of* **A**.

The proof is analogous to the proof of Theorem 7.9.

THEOREM 7.12. *The fixed point of a theory* $\langle D, P_A, \rangle$ *is* $P_\Re$*, where* $\Re$ *is the family of extensions of* **A** *obtained by the last theorem for all the defaults* *in $D$. An extension of this default theory is the infallible set for any family extending $\Re$.*

Introducing modal operators as in [8] i.e. if $\mathsf{L}, \mathsf{M}$ denote modal operators of necessity and possibility, respectively, then for every $\varphi \in Sent(\mathcal{L})$
$\models \mathsf{L}\varphi$ iff $\varphi \in P_A$
$\models \mathsf{M}\varphi$ iff $\varphi$ is possible for $\mathbf{A}$.
We use the introduction to autoepistemic logics by K.Konolige [4]. For any default we have corresponding $\mathsf{L}\alpha \wedge \mathsf{M}\beta_1 \wedge ... \wedge \mathsf{M}\beta_m \vdash w$ iff $w \in Cn((Cn(\alpha, \beta_1, ..., \beta_m))^\star)$. This description is adequate for the modal logic S5.

## References

[1] Bartol, W., 'Introduction to the Theory of Partial Algebras', *Lectures on Algebras, Equations and Partiality*, Univ. of Balearic Islands, Technical Report B-006, 1992: 36–71.

[2] Burmeister, P., *A Model Theoretic Oriented Approach to Partial Algebras*, Vol. 32 of Mathematical Research, Akademie-Verlag, 1986.

[3] Burris, S., and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, 1981.

[4] Konolige, K. 'Autoepistemic Logic', in D. M. Gabbay, C. J. Hogger, J. A. Robinson, (eds.), *Handbook of Logic in Artificial Intelligence and Logic Programming*, Volume 3, Oxford University Press, 1994, pp. 216–234.

[5] Poole, D., 'Default Logic', in D. M. Gabbay, C. J. Hogger, J. A. Robinson, (eds.), *Handbook of Logic in Artificial Intelligence and Logic Programming*, Volume 3, Oxford University Press, 1994, pp. 189–215.

[6] Scott, D., 'Domains For Denotational Semantics', in Proc. ICALP'82, *Lect. Notes in. Comp.Sc.* 140, 1982: 577–613.

[7] Shoenfield, J.R., *Mathematical Logic*, Addison-Wesley Publishing Company, 1967.

[8] Staruch, B., 'Derivation from partial knowledge in partial models', *Bulletin of the Section of Logic* 32, 2002: 75–84.

[9] Staruch, B. and B. Staruch, 'Possible sets of equations', *Bulletin of the Section of Logic* 32, 2002: 85–95.

[10] Staruch, B. and B. Staruch, 'Partial Algebras in Logic', submited to *Logika*, Acta Universitatis Vratislaviensis, (2002).

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